ANALYCITY PROPERTIES OF THE CHARACTERISTIC EXPONENTS OF RANDOM MATRIX PRODUCTS

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Introduction.

The "non-commutative ergodic theorem" of Oseledec [4] gives some sort of spectral theory for random matrix products. The eigenvalues are replaced by certain characteristic exponents which, in the case of a constant matrix, are the logarithms of the moduli of the eigenvalues. In the present paper we investigate the dependence of the characteristic exponents on the data of the problem and prove analyticity under certain conditions.

To be more specific, let (Ω,ρ) be a probability space and τ a measurable map $\Omega \to \Omega$ preserving the measure ρ. Let $T : \Omega \to \underline{M}_m(\mathbb{R})$ be a measurable function, with values in the real $m \times m$ matrices, such that *)

$$\log^+\|T(\cdot)\| \in L^1(\rho)$$

then

$$\lim_{n\to\infty} \frac{1}{n} \log\|T(\tau^{n-1}x)\dots T(\tau x)T(x)\|$$

exists ρ-almost everywhere. This is a theorem of Furstenberg and Kesten [2], slightly less powerful than that of Oseledec. If ρ is ergodic, the limit is almost everywhere equal to a constant $\chi(T,\rho)$, and we have

$$\chi(T,\rho) = \inf_n \frac{1}{n} \int \rho(dx) \log\|T(\tau^{n-1}x)\dots T(\tau x)T(x)\| \qquad (1.1)$$

Actually, $\chi(T,\rho)$ is the largest characteristic exponent determined by (Ω,ρ,τ,T) and we shall for the purposes of this introduction only concern ourselves with that characteristic exponent.

Suppose that Ω is compact, τ continuous and let $\mathfrak{J}$ be the Banach space of continuous maps $\Omega \mapsto \underline{M}_m(\mathbb{R})$ (use as norm the $\sup_x$ of the matrix norm). The function $\chi(\cdot,\rho)$ defined by (1.1) is upper semi-continuous on $\mathfrak{J}$. We can improve this result to one of real-analyticity of $\chi(\cdot,\rho)$ on a neighbor-

*) $\log^+ x = \max\{0, \log x\}$

hood of T_o if there is a proper closed convex cone $C \subset \mathbb{R}^m$ with apex at the origin such that

$$T_o(x)C \subset \{0\} \cup \text{int } C$$

for all x. This is the prototype of the results of this paper, and it applies for instance if T_o is constant*) and has a simple positive eigenvalue strictly larger than the moduli of the other eigenvalues. Actually it is useful and easy to consider the more general situation where C depends continuously on x and one assumes only

$$T_o(x)\ C(x) \subset \{0\} \cup \text{int } C(\tau x) \cup \text{int}(-C(\tau x))$$

Instead of using the same space $\mathbb{R}^m$ for each $x \in \Omega$ one can, as a further generalization consider a continuous vector bundle over Ω (see below for details). This extension (Theorem 3.1) allows to cover the case where Ω is a piece of differentiable manifold, τ a diffeomorphism of the manifold mapping Ω into itself and T, for instance, the tangent map ($T(x)$ maps the tangent space to the manifold at x to the tangent space at τx).

We shall see that considering the p-th exterior power $T^{\wedge p}$ of T gives information on the sum of the highest p characteristic numbers. (See Section 4.3). We shall also see that if the matrix $T(x)$ becomes complex, $\chi(T,\rho)$ is locally the real part of a complex analytic function of T (Section 4.7 and Proposition 4.8).

The methods of proof of the present paper are inspired by those used for differentiable dynamical systems (see Section 4.6).

The case of random products of matrices with positive entries is re-

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*) i.e. $T_o(x)$ is independent of x.

levant to the statistical mechanics of disordered one-dimensional spin systems. (One obtains for instance the analyticity of the free energy with respect to temperature for one-dimensional "spin glasses" with finite range interactions). On this case see also [6] Corollary 6.23.

An interesting question, not discussed here, is that of the nature of the singularities of $\chi(0,\rho)$. Can discontinuities occur ? This could be of interest for applications to physics (or ingeneering, etc.).

2. Continuous bundles.

Let Ω be a compact space, E a topological space and $\pi : E \mapsto \Omega$ a continuous surjection. We assume given open sets N_α covering Ω and homeomorphisms $\psi_\alpha : \pi^{-1}N_\alpha \mapsto N_\alpha \times \mathbb{R}^m$ such that $\psi_\alpha\xi = (\pi\xi, g_\alpha\xi)$ and $\psi_\alpha \circ \psi_\beta^{-1}(x,u) = (x, g_{\alpha\beta}(x)u)$ where $x \mapsto g_{\alpha\beta}(x)$ is continuous $N_\alpha \cap N_\beta \mapsto GL_m(\mathbb{R})$. These data define a continuous vector bundle over Ω which we denote by E or $(E(x))_{x\in\Omega}$, where $E(x) = \pi^{-1}x$ is a vector space isomorphic to $\mathbb{R}^m$. We shall call norm on E a continuous function $\|\cdot\| : E \mapsto \mathbb{R}$ such that its restriction to $E(x)$ is a norm for each x. It is clear how to define a continuous or a Borel measurable subbundle of E.

Let τ be a homeomorphism of Ω. A continuous vector bundle map T (of E) over τ is a continuous map $T : E \mapsto E$ such that $\pi\circ T = \tau\circ\pi$ and, if $T(x)$ is the restriction of T to $E(x)$, $T(x) : E(x) \mapsto E(\tau x)$ is linear. Such maps form a Banach space $\mathcal{J}$ with respect to the norm

$$\|T\| = \max_{x\in\Omega} \|T(x)\| \tag{2.1}$$

Different norms on E yield equivalent norms (2.1).

Given $T : E \mapsto E$, one defines readily its adjoint $T^* : E^* \mapsto E^*$

where E^* is the dual of E ; T^* is a continuous vector bundle map over τ^{-1}
One defines also $T^\wedge : E^\wedge \mapsto E^\wedge$ where $E^\wedge(x)$ is the exterior algebra of $E(x)$
$T^\wedge$ is a continuous vector bundle map over τ . If T is invertible, its inverse T^{-1} is a continuous vector bundle map over τ^{-1} .

We give now a version of the non-commutative ergodic theorem of Oseledec, which will be useful for our purposes.

2.1 Theorem. Let as above Ω be a compact space, $\tau : \Omega \to \Omega$ a continuous map, E a continuous m-dimensional vector bundle over Ω , and T a continuous vector bundle map of E over τ. Write

$$T_x^n = T(\tau^{n-1}x)\ldots\ T(\tau x)\ T(x)$$

and denote by I the set of τ-invariant probability measures on Ω .

There is a Borel subset Γ of Ω such that $\rho(\Gamma) = 1$ for every $\rho \in I$, and for each $x \in \Omega$ the following holds. There is a strictly increasing sequence of subspaces : $0 = V_x^{(o)} \subset V_x^{(1)} \subset \cdots \subset V_x^{(s(x))} = E(x)$ such that, for $r = 1,\ldots,s(x)$,

$$\lim_{n\to\infty} \frac{1}{n} \log \|T_x^n u\| = \lambda_x^{(r)} \text{ if } u \in V_x^{(r)} \setminus V_x^{(r-1)}$$

and $\lambda_x^{(1)} < \lambda_x^{(2)} < \cdots < \lambda^{s(x))}$; we may have $\lambda_x^{(1)} = -\infty$. [The $V_x^{(r)}$ and $\lambda_x^{(r)}$ are uniquely defined with these properties, and independent of the choice of norm on E]. The maps $x \mapsto s(x)$, $(V_x^{(1)},\ldots,V_x^{(s(n))})$, $(\lambda_x^{(1)},\ldots,\lambda_x^{(s(x))})$ are Borel. Furthermore

$$\lim_{n\to\infty} \frac{1}{n} \log\|T_x^n\| = \lambda_x^{(s(x))} \tag{2.2}$$

The $\lambda_x^{(r)}$ are the characteristic exponents of T at x .

2.2 Remarks. (a) If $\rho \in I$, define

$$\chi(T,\rho) = \int \rho(dx)\, \lambda_x^{(s(x))}$$

$$= \lim_{n\to\infty} \frac{1}{n} \int \rho(dx)\, \log\|T_x^n\|$$

Then, by a standard subadditivity argument

$$\chi(T,\rho) = \inf_n \frac{1}{n} \int \rho(dx)\, \log\|T_x^n\|$$

so that $\chi(\cdot,\rho)$ is upper semi-continuous, as noted in the introduction.

(b) One can show that the characteristic exponents of $T^{\wedge p}$ (the p-th exterior power) are all sums of p different characteristic exponents $\lambda_x^{(r)}$ of T , where these may be repeated with multiplicity $m_x^{(r)} = \dim V_x^{(r)} - \dim V_x^{(r-1)}$. In particular the subspace of $E(x)^{\wedge p}$ corresponding to the largest characteristic exponent of $T^{\wedge p}$ is spanned by p-vectors corresponding to the largest characteristic exponents of T .

2.3. Proofs.

The non-commutative ergodic theorem as formulated by Oseledec [4] assumes τ and T invertible, and the proof he gives is somewhat complicated. The principle of a simpler proof, based on a theorem by Furstenberg and Kesten [2], has been given by Raghunathan [5]. Using Raghunathan's method (unfortunately unpublished) one obtains readily theorem 2.1, and Remark 2.2 (b). Actually, the main results of the present paper (Section 3) do not use the full strength of Theorem 2.1, the theorem of Furstenberg and Kesten, in the form 2.2, being sufficient.

3. Bundle maps preserving a family of cones.

The main result of this paper is the following.

3.1. Theorem. Let Ω be a compact space, τ a homeomorphism of Ω, E a continuous m-dimensional vector bundle over Ω, and $\mathfrak{J}$ the Banach space of continuous vector bundle maps $T : E \to E$ over τ, with the norm (2.1).

Let $\mathcal{P}$ be the open subset of $\mathfrak{J}$ consisting of those T such that for each $x \in \Omega$ there is a proper closed convex cone $C(x) \subset E(x)$ for which $C(x) \cup (-C(x))$ depends continuously on x, and

$$TC(x) \subset \{0_{\tau x}\} \cup \text{int } C(\tau x) \cup \text{int}(-C(\tau x))$$

[$C(x)$ has its apex at the origine 0_x of $E(x)$].

Then for every $\rho \in I$, the function $\chi(\cdot,\rho)$ is real analytic on $\mathcal{P}$

Notice that $\mathcal{P}$ is in general not dense in $\mathfrak{J}$ [counterexample : take Ω reduced to a point and T the rotation by $\frac{\pi}{2}$ in $\mathbb{R}^2$].

Our proof will be based on use of the implicit function theorem. Another method of proof is given below (Theorem 4.8). Explicit forms for $\chi(\cdot,\rho)$ and its derivatives will be indicated in Section 4.1. We start with two propositions which are of interest in their own right.

3.2. Proposition.

(a) If $T \in \mathcal{P}$, there is a unique T-invariant vector subbundle F of E such that for each x

$$F(x) \subset C(x) \cup (-C(x))$$

Furthermore $F(x)$ is one-dimensional and depends continuously on x and T.

(b) <u>A subbundle</u> F^* <u>of</u> E^* <u>is similarly defined with respect to</u> T^* <u>and the cones</u>

$$C^*(x) = \{\eta \in E(x)^* : (\eta,\xi) \geq 0 \quad \text{for all} \quad \xi \in C(x)\}$$

(c) <u>If</u> $T_o \in P$, <u>there exist a neighborhood</u> $\mathcal{O}$ <u>of</u> T_o , $\alpha < 1$ <u>and</u> C <u>such that for all</u> $n \geq 0$, <u>and</u> $x \in \Omega$

$$\|T^n\eta\| \leq C\alpha^n\|T^n\xi\| \tag{3.1}$$

<u>whenever</u> $T \in \mathcal{O}$ <u>and</u> ξ,η <u>are unit vectors respectively in</u> $F(x)$ <u>and orthogonal to</u> $F^*(x)$.

Let Ω' be the set of pairs (x,ε) where $x \in \Omega$, $\varepsilon = \pm C(x)$ and let $\tau'(x,\varepsilon) = (\tau x, \tau\varepsilon)$. By considering bundles and bundle maps over Ω' and τ' instead of Ω and τ , one reduces the proof of the proposition to the case when $C(x)$ depends continuously on x and $TC(x) \subset \{0_{\tau x}\} \cup \text{int } C(\tau x)$.

We may then choose a_x belonging to the interior of $C(x)$, depending continuously on x , and such that $\|a_x\| = 1$.

The convex compact set

$$\prod_{x\in\Omega} \{\eta_x \in C(x)^* : (\eta_x, a_x) = 1\}$$

in the topological vector space $\Pi_{x\in\Omega} E(x)^*$ is mapped into itself continuously by T' :

$$(T'(\eta_x))_y = \frac{T^*\eta_{\tau y}}{(\eta_{\tau y}, Ta_y)}$$

It has therefore a fixed point (a_x^*) by Leray-Schauder (notice that $x \mapsto a_x^*$ is not a priori continuous).

Taking $b_x^* = a_x^*/\|a_x^*\|$ we have

$$T^* b^*_{\tau x} / \|T^* b^*_{\tau x}\| = b^*_x$$

Notice that, since $b^*_x \in T^* C(\tau x)^*$, there is $\epsilon > 0$ independent of x such that the ball of radius ϵ centered at b^*_x is contained in $C(x)^*$. Dually to the existence of (b^*_x), one proves the existence of (b_x) such that $b_x \in C(x)$ and

$$Tb_x / \|Tb_x\| = b_{\tau x}$$

Define now

$$K(x) = (\{\xi \in E(x) : (b^*_x, \xi) = 0 \text{ and } b_x + \xi \in C(x)\}$$

and

$$T''\xi = T\xi / \|Tb_x\| \quad \text{if} \quad \xi \in E(x) .$$

Since $TC(x) \subset \operatorname{int} C(\tau x) \cup \{0_{\tau x}\}$ and since the ball of radius ϵ centered at b^*_x is contained in $C(x)^*$, there is $\alpha < 1$ such that

$$T''K(x) \subset \alpha K(\tau x) \tag{3.2}$$

for all x. Therefore we have

$$\lim_{n \to \infty} \operatorname{diam} T''^n K(x) = 0$$

exponentially fast and uniformly in x. This implies that $T^n C(\tau^{-n} x)$ tends to the half line along b_x and that

$$\lim_{n \to \infty} \frac{T^n a_{\tau^{-n} x}}{\|T^n a_{\tau^{-n} x}\|} = b_x \tag{3.3}$$

uniformly in x. Therefore $x \to b_x$ is continuous.

If T is allowed to vary in a small open set $\mathcal{O}$ we can assume that a_x is continuous with respect to $(x,T) \in \Omega \times \mathcal{O}$, and that the convergence of (3.3) is uniform on that set. Therefore b_x is continuous with respect to

$(x,T) \in \Omega \times \mathcal{P}$.

If $F(x)$ is the one-dimensional space spanned by b_x , the subbundle F clearly satisfies part (a) of the Proposition. Part (b) follows from part (a), noting that

$$T^* C^*(\tau x) \subset \{0_x\} \cup \text{int } C^*(x)$$

Finally, (3.1) follows from (3.2), proving part (c) of the proposition.

3.3. Proposition. The continuous sections of the bundle of one-dimensional subspaces of E (resp. E^*) form in a natural manner a real analytic Banach manifold $\mathcal{G}$ (resp. $\mathcal{G}^*$) .

The map $T \mapsto F$ (resp. $T \to F^*$) defined by Proposition 3.2 is real analytic $\mathcal{P} \mapsto \mathcal{G}$ (resp. $\mathcal{P} \mapsto \mathcal{G}^*$) .

The real analytic structure on $\mathcal{G}$, $\mathcal{G}^*$ is described in Bourbaki [1] (§15, first footnote).

For $G \in \mathcal{G}$ and G sufficiently close to F we can define $\theta_T G \in \mathcal{G}$ by

$$(\theta_T G)\ (x) = T(G(\tau^{-1}x))$$

If F_o is the subbundle corresponding to $T_o \in \mathcal{P}$ in Proposition 3.2 (a), we have $\theta_{T_o} F_o = F_o$, and there are neighborhoods $\mathcal{O}$ of T_o in $\mathcal{P}$ and $\mathcal{U}$ of F_o in $\mathcal{G}$ such that $(T,G) \to \theta_T F$ is real analytic : $\mathcal{O} \times \mathcal{U} \to \mathcal{G}$. Furthermore the tangent map *) $t_{F_o} \theta_{T_o} : t_{F_o} \mathcal{G} \to t_{F_o} \mathcal{G}$ has spectral radius < 1 [actually (3.1) shows that the spectral radius of $t_{F_o} \theta_{T_o}$ is $\leq \alpha$]. Therefore $t_{F_o} \theta_{T_o}$ - id is invertible and there is, by the implicit function theorem (see

*) To avoid confusion we use t to denote tangents to $\mathcal{G}$.

[1] § 5.6.7), a function φ real analytic in a neighborhood of T_o with values in $\mathcal{G}$ such that $\varphi(T_o) = F_o$ and $\theta_T\varphi(T) = \varphi(T)$.

In view of Proposition 3.2, these conditions imply that $\varphi(T) = F$, and thus that $T \mapsto F$ is real analytic. Similarly $T \to F^*$ is real analytic.

3.4. Proof of the theorem.

Let $T_o \in \mathcal{P}$ and let $a_o(x)$, $a_o^*(x)$ respectively by unit vectors in the bundles F_o, F_o^* of Proposition 3.2 (a) and (b).

Define

$$f_n(T) = \int \rho(dx)\, \frac{1}{n} \log|(a_o^*(\tau^n x) \ , \ T^n a_o(x))|$$

From Proposition 3.2 and the definition of $\chi(T,\rho)$ (Remark 2.2) we obtain

$$\lim_{n\to\infty} f_n(T) = \chi(T,\rho) \tag{3.4}$$

uniformly for T in some neighborhood of T_o . The function f_n has a derivative

$$[Df_n(T)]\,(U) = \int \rho(dx)\, \frac{1}{n} \sum_{k=1}^{n} \frac{(a_o^*(\tau^n x), T^{n-k} U T^{k-1} a_o(x))}{(a_o^*(\tau^n x),\ T^n a_o(x))}$$

and therefore, again by Proposition 3.2,

$$\lim_{n\to\infty} [Df_n(T)]\,(U) = \int \rho(dx)\, \frac{(a^*(\tau x),\ Ua(x))}{(a^*(\tau x), Ta(x))}$$

where $0 \neq a(x) \in F(x)$, $0 \neq a^*(x) \in F^*(x)$. Clearly, the normalization of a, a^* is without importance. The limit (3.5) is again uniform in a neighborhood of T_o , therefore (3.4) yields

$$[D\chi(T,\rho)]\,(U) = \ \rho(dx)\, \frac{(a^*(\tau x),\ Ua(x))}{(a^*(\tau x), Ta(x))}$$

We shall now see that the map

$$(T,U) \mapsto \left[x \mapsto \frac{(a^*(\tau x),\ Ua(x))}{(a^*(\tau x),\ Ta(x))} \right]$$

is real analytic in T,U and linear in U from $\mathcal{P} \times \mathcal{J}$ to $C(\Omega)$ [the real continuous functions on Ω]. It suffices to verify this locally with respect to x . Now since $T \to F, F^*$ are real analytic by Proposition 3.3, we can take locally $T \to a(x)$, $a^*(x)$ real analytic (with values in continuous sections of E, E^*) and we have proved what we announced. From (3.6) we find thus that $T \to D\chi(T,\rho)$ is real analytic (with values in the dual of $\mathcal{J}$). Therefore, finally, $T \mapsto \chi(T,\rho)$ is real analytic on $\mathcal{P}$.

4. Remarks and complements

4.1. Expressions for $\chi(T,\rho)$ and its derivatives.

From Remark 2.2 (a), we have

$$\chi(T,\rho) = \int \rho(dx) \quad \frac{\|Ta(x)\|}{\|a(x)\|}$$

if $0 \neq a(x) \in F(x)$. If $0 \neq a^*(x) \in F^*(x)$ we have also, according to (3.6),

$$[D\chi(T,\rho)]\ (U) = \int \rho(dx) \quad \frac{(a^*(\tau x),\ Ua(x))}{(a^*(\tau x),\ Ta(x))}$$

Let

$$c_k(U,V) = \frac{(a^*(\tau^k x),\ UT^{k-1}V\ a(x))}{(a^*(\tau^k x),\ T^{k+1}\ a(x))}$$

$$- \frac{(a^*(\tau^k x),\ Ua(\tau^{k-1} x))}{(a^*(\tau^k x),\ Ta(\tau^{k-1} x))} \times \frac{(a^*(\tau x),\ Va(x))}{(a^*(\tau x),\ Ta(x))}$$

then

$$[D^2\chi(T,\rho)](U,V) = \sum_{k=1}^{\infty} [c_k(U,V) + c_k(V,U)]$$

The easy proof is left to the reader (proceed as for (3.6)) . Higher derivatives can be computed similarly.

4.2. Proposition. If $T \in \mathfrak{I}$, a necessary and sufficient condition for $T \in \mathcal{P}$ is that E and E^* respectively have T- and T^*-invariant continuous one-dimensional subbundles F and F^* such that $F(x)$ and $F^*(x)$ are not orthogonal for any $x \in \Omega$, and there exist $\alpha < 1$ and C real such that

$$\|T^n\eta\| \leq C\alpha^n \|T^n\xi\| \qquad (4.1)$$

whenever ξ, η are unit vectors respectively in $F(x)$ and orthogonal to $F^*(x)$, $n \geq 0$, $x \in \Omega$.

We have already shown that (4.1) holds if $T \in \mathcal{P}$ (Proposition 3.2 (c)) . It remains to prove that (4.1) implies $T \in \mathcal{P}$. Let ξ,η be the components of $\zeta \in E$ respectively along F and orthogonal to F^* ; assume $\xi \neq 0$ and define

$$n(\zeta) = \sum_{k=0}^{n-1} \frac{\|T^k\eta\|}{\|T^k\xi\|}$$

which is a convex function of $\eta/\|\xi\|$. Then, if n is chosen such that $a < C^{-1/n}$,

$$n(T\zeta) = n(\zeta) + \frac{\|T^n\eta\|}{\|T^n\xi\|} - \frac{\|\eta\|}{\|\xi\|}$$

$$\leq n(\zeta) - (1-C\alpha^n) \frac{\|\eta\|}{\|\xi\|} < n(\zeta)$$

and we can take

$$C(x) \cup [-C(x)] = \{0_x\} \cup \{\zeta \in E(x) : n(\zeta) \leq 1\}$$

4.3. Exterior powers.

Define

$$\chi_p(T,\rho) = \lim_{n\to\infty} \frac{1}{n}\int \rho(dx)\ \log \|(T_x^n)^{\wedge p}\|$$

Then, according to the Remark 2.2 (6), $\chi_p(T,\rho)$ is the integral with respect to ρ, of the sum of the largest p different characteristic exponents $\lambda_x^{(r)}$ of T, counted with multiplicity $m_x^{(r)} = \dim V_x^{(r)} - \dim V_x^{(r-1)}$. We may thus apply Theorem 3.1 to prove analyticity of $T \to \chi_p(T,\rho)$ if $T^{\wedge p} \in \rho^{(p)}$. Here $\rho^{(p)}$ consists of those bundle maps $T^{(p)} : E^{\wedge p} \mapsto E^{\wedge p}$ over τ such that for each $x \in \Omega$ there is a proper closed convex cone $C^{(p)}(x) \subset E(x)^{\wedge p}$ for which $C^{(p)}(x) \cup (-C(x))$ depends continuously on x, and

$$T^{(p)}\ C^{(p)}(x) \subset \{0_{\tau x}\} \cup \text{int}\ C^{(p)}(\tau x) \cup \text{int}(-C^{(p)}(\tau x))$$

The following result reduces to Proposition 4.2 when $p = 1$.

4.4. Proposition. If $T \in \mathfrak{T}$, a necessary and sufficient condition for $T^{\wedge p} \in \rho^{(p)}$ is the following.

There are continuous T-invariant subbundles E_+, E_- of E with $\dim E_+ = p$, $\dim E_- = m-p$, and $E = E_+ \oplus E_-$ and there exist $\alpha < 1$ and A real such that

$$\|T^n \xi_-\| \le A\alpha^n \|T^n \xi_+\| \qquad (4.2)$$

whenever ξ_+, ξ_- are unit vectors in $E_+(x)$, $E_-(x)$ respectively, $n \ge 0$, $x \in \Omega$.

In view of Proposition 4.2, the condition $T^{\wedge p} \in \rho^{(p)}$ is equivalent to (4.1) with T replaced by $T^{\wedge p}$. We shall prove the equivalence with (4.2) using the same α. First notice that we can (without changing α) assume that the norm on E is Euclidean. By diagonalizing $(T^n)^* T^n$ one can find orthogonal unit vectors $\xi_1,\dots,\xi_m \in E(x)$ and $\eta_1,\dots,\eta_m \in E(\tau^n x)$ such that $T^n\xi_k = a_k\eta_k$ where $a_1 \geq a_2 \geq \cdots \geq a_m$. If n is such that $A\alpha^n < 1$, (4.2) means that $\xi_1,\dots,\xi_p \in E_+(x)$, $\xi_{p+1},\dots,\xi_m \in E_-(x)$ and $a_{p+1} \leq (A\alpha^n)a_p$. On the other hand, if n is such that $C\alpha^n < 1$, (4.1) holds for $T^{\wedge p}$ when the one-dimensional bundle is that generated by $\xi_1 \wedge \dots \wedge \xi_p$, and

$$a_1 a_2 \cdots a_{p-1}\, a_{p+1} \leq (C\alpha^n)\, a_1 \dots a_p$$

Therefore, for sufficiently large n, (4.2) is the same as (4.1) applied to $T^{\wedge p}$.

4.5. Corollary. (a) The set $\mathcal{P}_p$ of those $T \in \mathcal{T}$ such that property (H_p) holds is open in $\mathcal{T}$, and $E_+(x)$, $E_-(x)$ depend continuously on $(x,T) \in \Omega \times \mathcal{P}_p$.

(b) If $T_o \in \mathcal{P}_p$, one can choose $\alpha < 1$, A real and a neighborhood $\mathcal{O}$ of T_o such that (4.2) holds uniformly for $(x,T) \in \Omega \times \mathcal{O}$.

4.6. Hyperbolic splittings.

Let E_+, E_- be continuous T-invariant subbundles of E, with $\dim E_+ = p$, $\dim E_- = m-p$, and $E = E_+ \oplus E_-$. Suppose that $\beta_\pm$, $B_\pm$ are such that $\beta_- < \beta_+$, $B_+ > 0$, and if $n \geq 0$

$$\|T^n\xi_-\| \leq B_-\beta_-^n\|\xi_-\| \quad \text{if} \quad \xi_- \in E_- \tag{4.3}$$

$$\|T^n\xi_+\| \geq B_+\beta_+^n\|\xi_+\| \quad \text{if} \quad \xi_+ \in E_+$$

Then (H_p) holds with $A = B_-/B_+$, $\alpha = \beta^-/\beta^+$. Thus, if T' is sufficiently close to T, there will be T'-invariant subbundles $E'_\pm$ close to $E_\pm$ and it is readily seen that inequalities corresponding to (4.3) will hold with constants

$B'_{\pm}$, $\beta'_{\pm}$ close to $B_{\pm}$, $\beta_{\pm}$. This is true in particular for <u>hyperbolic</u> splittings of E , i.e. when one can take $\beta_- < 1 < \beta_+$. Those T for which E has a hyperbolic splitting thus form an open subset of $\mathcal{T}$.

Hyperbolic splittings have been studied mostly when Ω is a differentiable manifold or part of it, τ a diffeomorphism, and $T = T\tau$ the tangent to the diffeomorphism. See in particular Smale [7], Moser [3].

4.7. <u>Complex matrices</u>.

Let Ω, τ be as before, and $E_{\mathbb{C}}$ a continuous complex vector bundle over Ω . The continuous complex vector bundle maps (of $E_{\mathbb{C}}$) over τ form a complex Banach space $\mathcal{T}_{\mathbb{C}}$. If $E_{\mathbb{C}}$ has complex dimension m' , there is an underlying structure of m-dimensional real bundle E on $E_{\mathbb{C}}$, where $m = 2m'$. The space $\mathcal{T}_{\mathbb{C}}$, considered as real Banach space becomes then a closed subspace of the space $\mathcal{T}$ of real vector bundle maps of E over τ . We define $\mathcal{P}_{\mathbb{C}}$ to consist of those $T \in \mathcal{T}_{\mathbb{C}}$ such that $E_{\mathbb{C}}$ and $E^*_{\mathbb{C}}$ respectively have T- and T^*-invariant continuous one-dimensional complex subbundles F and F^* such that $F(x)$ and $F^*(x)$ are not orthogonal for any $x \in \Omega$ and there exist $\alpha < 1$ and C real such that

$$\|T^n\eta\| \le C\alpha^n \|T^n\xi\|$$

whenever ξ, η are unit vectors respectively in $F(x)$ and orthogonal to $F^*(x)$, $n \ge 0$, $n \in \Omega$.

Corollary 4.5 applies to the present situation with $p = 2$ and E_+, E_- are complex subbundles when $T \in \mathcal{T}_{\mathbb{C}}$. In particular $\mathcal{P}_{\mathbb{C}}$ is open in $\mathcal{T}_{\mathbb{C}}$.

If $T \in \mathcal{T}_{\mathbb{C}}$ then the largest two characteristic exponents of T , considered as element of $\mathcal{T}$, are equal. Thus, with the notation of section 4.3,

$$\chi(T,\rho) = \tfrac{1}{2}\chi_2(T,\rho)$$

In particular $\chi(\cdot,\rho)$ is real analytic on $\mathcal{P}_{\mathbb{C}}$ for the structure of real Banach space on $\mathcal{P}_{\mathbb{C}}$. We have however the following more precise result.

4.8. Proposition. The function $\chi(\cdot,\rho)\to\mathbb{R}$ on $\mathcal{P}_{\mathbb{C}}$ is locally the real part of a complex analytic function.

Let indeed $T_o \in \mathcal{P}_{\mathbb{C}}$ and let $a_o(x)$, $a_o^*(x)$ respectively be unit vectors in the complex bundles F_o and F_o^* in the definition of $\mathcal{P}_{\mathbb{C}}$. Define

$$f_n(T) = \chi(T_o,\rho) + \int\rho(dx)\,\frac{1}{n}\log\,\frac{(a_o^*(\tau^n x),T^n a_o(x))}{(a_o^*(\tau^n x),T_o^n a_o(x))}$$

Using Corollary 4.5, we find $\epsilon > 0$ such that $(a_o^*(\tau^n x),\, T^n a_o(x)) \neq 0$ for all n and $T \in \mathcal{O} = \{T \in \mathfrak{T}_{\mathbb{C}} : |T-T_o| < \epsilon\}$. Clearly

$$T \to \left[x \mapsto \frac{(a_o^*(\tau_x^n),\; T^n a_o(x))}{(a_o^*(\tau^n x,\; T_o^n a_o(x))}\right]$$

is holomorphic from $\mathcal{O}$ to the complex continuous functions on Ω and the same is therefore true of

$$T \to \left[x \to \log\frac{(a_o^*(\tau^n x),\; T^n a_o(x))}{(a_o^*(\tau^n x),\; T_o^n a_o(x))}\right]$$

provided the log is defined by continuity with the value 0 for $T = T_o$. Therefore f_n is holomorphic on $\mathcal{O}$.

We have

$$\mathrm{Re}\, f_n(T) = \chi(T_o,\rho) + \int\rho(dx)\,\frac{1}{n}\log\frac{|(a_o^*(\tau^n x),\; T^n a_o(x))|}{|(a_o^*(\tau^n x),\; T_o^n a_o(x))|}$$

From the definitions and Corollary 4.5 we obtain

$$\lim_{n\to\infty} \mathrm{Re}\, f_n(T) = \chi(T,\rho)$$

uniformly for T in some neighborhood - say $\mathcal{O}$ - of T_o. To conclude the proof of the proposition it suffices to show that the f_n tend to a limit uniformly in $\mathcal{O}$. Equivalently it suffices to prove this for the derivatives Df_n :

$$[Df_n(T)](U) = \int \rho(dx)\, \frac{1}{n} \sum_{k=1}^{n} \frac{(a_o^*(\tau^n x),\, T^{n-k}\, UT^{k-1} a_o(x))}{(a_o^*(\tau^n x),\, T^n a_o(x))}$$

This follows readily from Corollary 4.5.

4.9. Abstract measure theory.

Suppose τ is a measure preserving map of the probability space (Ω,ρ). The various theorems of this paper have analogs where functions in $L^\infty(\rho)$ occur instead of continuous functions on Ω. This is because of the canonical isomorphism of the C^*-algebra $L^\infty_{\mathbb{C}}(\rho)$ with the algebra of complex continuous functions on its spectrum. We leave the details to the reader.

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