PROBABILITY ESTIMATES FOR CONTINUOUS SPIN SYSTEMS

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Abstract. Probability estimates for classical systems of particles with superstable interactions [1] are extended to continuous spin systems.

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1. Notation and assumptions.

On a lattice $\mathbb{Z}^\nu$ we consider continuous d-dimensional spins. A spin configuration in $\Lambda \subset \mathbb{Z}^\nu$ is thus a function $s_\Lambda : \Lambda \mapsto \mathbb{R}^d$; its value at $x \in \Lambda$ will be denoted by s_x.

If $x = (x^1,\dots,x^\nu) \in \mathbb{Z}^\nu$, we write $|x| = \max_i |x^i|$. If $s = (s^1,\dots,s^d) \in \mathbb{R}^d$, we write $|s| = (\sum_i (s^i)^2)^{1/2} = \sqrt{s^2}$.

A measure $\mu \geq 0$ on $\mathbb{R}^d$ is given such that

$$\int \mu(ds)\, e^{-\alpha s^2} < +\infty$$

if $\alpha > 0$, and μ is not identically 0 .

We shall call interaction a real function U on all configurations in all finite $\Lambda \subset \mathbb{Z}^\nu$ satisfying the following conditions.

(a) U is $\otimes^\Lambda \mu$-measurable on $(\mathbb{R}^d)^\Lambda$ and invariant under translations of $\mathbb{Z}^\nu$.

(b) Superstability. There exist $A > 0$, $C \in \mathbb{R}$ such that if $s_\Lambda \in (\mathbb{R}^d)^\Lambda$ is a configuration on any finite Λ , then

$$U(S_\Lambda) \geq \sum_{x\in\Lambda} [A\, s_x^2 - C]$$

(c) Regularity. There exists a decreasing positive function Ψ on the natural integers such that

$$\sum_{x\in \mathbb{Z}^\nu} \Psi(|x|) < +\infty \quad .$$

Furthermore if Λ_1 , Λ_2 are disjoint finite subsets of $\mathbb{Z}^\nu$ and s_{Λ_1} , s_{Λ_2} the restrictions to Λ_1, Λ_2 of a configuration $s_{\Lambda_1 \cup \Lambda_2}$ on $\Lambda_1 \cup \Lambda_2$, then

$$|W(s_{\Lambda_1 \cup \Lambda_2})| \le \sum_{x\in\Lambda_1} \sum_{y\in\Lambda_2} \Psi(|y-x|)\tfrac{1}{2}(s_x^2 + s_y^2)$$

where we have written

$$U(s_{\Lambda_1 \cup \Lambda_2}) = U(s_{\Lambda_1}) + U(s_{\Lambda_2}) + W(s_{\Lambda_1}, s_{\Lambda_2}) \quad .$$

Condition (c) implies the following

(d) There are $r > 0$ and $\lambda > 0$ such that for all finite $\Lambda \subset \mathbb{Z}^\nu$

$$\int_{\Sigma^\Lambda} (\prod_{x\in\Lambda} \mu(ds_x)) \exp[-U(s_\Lambda)] > \lambda^{-\mathrm{card}\,\Lambda}$$

where $\Sigma = \{s \in \mathbb{R}^d : |s| \le r\}$. This is because, using (c), we have

$$U(s_\Lambda) \le \sum_{x\in\Lambda} U(s_x) + (\sum_{x\in\Lambda} s_x^2) \sum_y \Psi(|y|)$$

and, for sufficiently large r , $\int_{|s|\le r} \mu(ds) > 0$.

Notice also that if there are $\epsilon > 0$, $B \in \mathbb{R}$ such that

$$U(s_\Lambda) \ge \sum_{x\in\Lambda} [(A+\epsilon)s_x^2 - B|s_x|]$$

then (b) holds with $C = B/4\epsilon$.

2. Probability estimates.

Let $\Delta \subset \Lambda \subset \mathbb{Z}^\nu$, Λ finite. We denote by s_Δ the restriction to Δ of a configuration s_Λ on Λ , and write

$$\rho_\Delta^{(\Lambda)}(s_\Delta) = Z_\Lambda^{-1} \int (\prod_{x\in\Lambda\setminus\Delta} \mu(ds_x)) \exp[-U(s_\Lambda)] \qquad (1)$$

where

$$Z_\Lambda = \int (\prod_{x\in\Lambda} \mu(ds_x)) \exp[-U(s_\Lambda)]$$

The probability estimates of this section are bounds on $\rho_\Delta^{(\Lambda)}$, given in Theorem 2.2. below. To obtain these bound we imitate the arguments of [1]. That paper in effect treats a special case of the problem considered here, where $d = 1$ and μ is carried by the natural integers. In [1], the probability estimates are obtained on the basis of technical results, which carry over immediately to the present case if the variable n is allowed to vary in $\mathbb{R}^d$ rather than take natural integer values. As an example we transcribe below (Proposition 2.1) the main technical estimate of [1].

Given $\alpha > 0$, we can choose an integer $P_o > 0$ and for each $j \geq P_o$ an integer $\ell_j > 0$ such that

$$\left| \frac{\ell_{j+1}}{\ell_j} - (1+2\alpha) \right| < \alpha \quad .$$

We use the notation

$$[j] = \{x \in \mathbb{Z}^\nu : |x| \leq \ell_j\} \quad , \quad V_j = (2\ell_j+1)^\nu$$

2.1. <u>Proposition</u>. <u>Let</u> $\epsilon > 0$ <u>and</u> $C \geq 0$ <u>be given</u>, and let Ψ be a decreasing <u>positive function on the natural integers such that</u>

$$\sum_{x\in\mathbb{Z}^\nu} \Psi(|x|) < +\infty \quad .$$

<u>If</u> α <u>is sufficiently small one can choose an increasing sequence</u> (ψ_j) <u>such that</u> $\psi_j \geq 1$, $\psi_j \to \infty$, <u>and fix</u> $P > P_o$ <u>so that the following is true</u>.

<u>Let</u> $n(\cdot)$ <u>be a function from</u> $\mathbb{Z}$ <u>to the reals</u> ≥ 0 . <u>Suppose that there</u>

<u>exists</u> q <u>such that</u> $q \geq P$ <u>and</u> q <u>is the largest integer for which</u>

$$\sum_{x\in[q]} n(x)^2 \geq \psi_q V_q \quad .$$

Then

$$\sum_{x\in[q+1]} C + \sum_{x\in[q+1]} \sum_{y\notin[q+1]} \Psi(|y-x|)\, \tfrac{1}{2}(n(x)^2 + n(y)^2)$$

$$\leq \epsilon \sum_{x\in[q+1]} n(x)^2 \quad .$$

This differs from Proposition 2.1 of [1] mostly by the fact that $n(\cdot)$ has real rather than integer values. Lemmas 2.2, 2.3, 2.4, and Proposition 2.5 of [1] similarly carry over to the present case.

To adapt Proposition 2.6 of [1] to $\rho_\Delta^{(\Lambda)}$ some care is needed because we do not have in general $\rho(\{0\}) > 0$. Since however we have (d) and the regularity condition (c) (rather than only lower regularity in [1]), we can write $\rho_\Delta^{(\Lambda)}(s_\Delta) = \rho' + \rho''$ where (3.30) and (3.31) of [1] are replaced (see Appendix) by

$$\rho' \leq C' \exp\Big[\sum_{y\in\mathbb{Z}^\nu} \Psi(|y|) - A)s_x^2\Big] \cdot \rho^{(\Lambda)}_{\Delta\setminus\{x\}}(s_{\Delta\setminus\{x\}}) \tag{2}$$

$$\rho'' \sum_{q\geq P} e^{-C''\psi_{q+1} V_{q+1} + D'' V_{q+1}} \,.\, \exp \sum_{x\in[q+1]\cap\Lambda} [-(A-3\epsilon)s_x^2]$$

$$\rho^{(\Lambda)}_{\Delta\setminus[q+1]}(s_{\Delta\setminus[q+1]}) \tag{3}$$

with some constants C', C'', D'' . Therefore, by induction on $\operatorname{card} \Delta$,

$$\rho_\Delta^{(\Lambda)}(s_\Delta) \leq \exp \sum_{x\in\Delta} (Es_x^2 + F) \tag{4}$$

with some constants E, F .

We show now, following Proposition 2.7 of [1], that for any $\epsilon > 0$ one can choose δ independent of (Λ) , Δ , s_Δ such that

$$\rho_\Delta^{(\Lambda)}(s_\Delta) \leq \exp \sum_{x\in\Delta} [-(A-3\epsilon)\, s_x^2 + \delta] \tag{5}$$

We may assume $A > 3\epsilon$. Let $\delta = (E+A-3\epsilon)\, \psi_P V_P + F$. If $|s_x| \leq (\psi_P V_P)^{1/2}$ for each $x \in \Delta$, then (5) follows from (4). If $|s_x| > (\psi_P V_P)^{1/2}$ for some x , we put x at the origin by a translation. Then $\rho' = 0$, and $\rho_\Delta^{(\Lambda)}(s_\Delta) = \rho''$ so that, using (3) and induction,

$$\rho_\Delta^{(\Lambda)}(s_\Delta) \leq \exp \sum_{x\in\Delta} [-(A-3\epsilon)\, s_x^2]$$

$$\sum_{q\geq P} e^{-C''\psi_{q+1} V_{q+1} + DV_{q+1}} e^{\delta\, \mathrm{card}(\Delta\setminus[q+1])}$$

$$\leq \exp \sum_{x\in\Delta} [-(A-3\epsilon)s_x^2] \,.\, e^{\delta\, \mathrm{card}(\Delta\setminus[q+1]) + F}$$

and (4) follows. We have proved the following

2.2. Theorem. Let $\rho_\Delta^{(\Lambda)}(s_\Delta)$ be defined by (1) for an interaction U satisfying (a), (b), (c). Given $A^* < A$, there exists δ independent of Λ , Δ , s_Δ such that

$$\rho_\Delta^{(\Lambda)}(s_\Delta) \leq \exp \sum_{x\in\Delta} [-A^*\, s_x^2 + \delta]$$

2.3. Corollary. Let $\gamma \geq 2$, and suppose that the superstability condition is strengthened to

$$U(s_\Lambda) \geq \sum_{x\in\Lambda} [A|s_x|^\gamma - C] \ .$$

Then the conclusion of Theorem 2.2 can be strengthened to

$$\rho_\Delta^{(\Lambda)}(s_\Delta) \le \exp \sum_{x\in\Delta} [-A^*|s_x|^\gamma + \delta]$$

Define $F : \mathbb{R}^d \mapsto \mathbb{R}^d$ by

$$Fs = \begin{cases} s & \text{if } |s| \le 1 \\ (|s|^{2/\gamma - 1})s & \text{if } |s| \ge 1 \end{cases}$$

and write $F(s_x)_{x\in\Lambda} = (Fs_x)_{x\in\Lambda}$.

Let $\tilde{\mu}$ be the image by F of the measure μ , and let $\tilde{U}(s_\Lambda) = U(Fs_\Lambda)$. Then $\tilde{U}$ is an interaction satisfying the conditions of Section 1 with respect to the measure $\tilde{\mu}$. In particular

$$\tilde{U}(s_\Lambda) = U(Fs_\Lambda) \ge \sum_{x\in\Lambda} [A|Fs_x|^\gamma - C]$$

$$\ge \sum_{x\in\Lambda} [As_x^2 - A - C]$$

and

$$|\tilde{W}(s_{\Lambda_1 \cup \Lambda_2})| \le \sum_{x\in\Lambda_1} \sum_{y\in\Lambda_2} \Psi(|y-x|)\tfrac{1}{2}(|Fs_x|^2 + |Fs_y|^2)$$

$$\le \sum_{x\in\Lambda_1} \sum_{y\in\Lambda_2} \Psi(|y-x|)\tfrac{1}{2}(s_x^2 + s_y^2) \quad .$$

Therefore

$$\rho_\Delta^{(\Lambda)}(s_\Delta) = \tilde{\rho}_\Delta^{(\Lambda)}(F^{-1}s_\Delta) \le \exp \sum_{x\in\Delta} [-A^*|F^{-1}s_x|^2 + \delta]$$

$$\le \exp \sum_{x\in\Delta} [-A^*|s_x|^\gamma + \delta] \quad .$$

2.4. <u>Corollary</u>. <u>Suppose that</u>

$$U(s_\Lambda) = \tilde{U}(s_\Lambda) + \sum_{x\in\Lambda} V(s_x)$$

and that $\tilde{U}$ is an interaction satisfying the conditions of Section 1 with respect to the measure $\tilde{\mu} = e^{-V}\mu$. Then Theorem 2.2 can be replaced by

$$\rho_{\Delta}^{(\Lambda)}(s_{\Delta}) \le \exp \sum_{x\in\Delta} [-A^*|s_x|^{\gamma} + \delta - V(s_x)| \quad .$$

This is because

$$\rho_{\Delta}^{(\Lambda)}(s_{\Delta}) = \exp[-\sum_{x\in\Delta} V(s_x)]\, \tilde{\rho}_{\Delta}^{(\Lambda)}(s_{\Delta})$$

where $\tilde{\rho}$ is defined by (1) with μ, U replaced by $\tilde{\mu}$, $\tilde{U}$.

Appendix.

We sketch here the proofs of (2) and (3), using notation which is either that of [1], or has obvious meaning.

Proof of (2).

$$\rho' = Z_{\Lambda}^{-1}\int_R \mu^{\Lambda\setminus\Delta}(ds_{\Lambda\setminus\Delta})\exp[-U(s_x)-U(s_{\Lambda\setminus\{x\}})-W(s_x,s_{\Lambda\setminus\{x\}})]$$

$$\le e^{-U(s_x)}Z_{\Lambda}^{-1}\int_R \mu^{\Lambda\setminus\Delta}(ds_{\Lambda\setminus\Delta})\exp[-U(s_{\Lambda\setminus\{x\}})-W(s'_x,s_{\Lambda\setminus\{x\}})]$$

$$\times \exp[(\tfrac{1}{2}\sum_y \Psi(|y|))(s_x^2+s_x'^2) + 2D']$$

$$\le \lambda\, e^{2D'}\exp[-As_x^2 + C + (\tfrac{1}{2}\sum_y \Psi(|y|))s_x^2]$$

$$\times \sup_{s'_x\in\Sigma} \exp[(\tfrac{1}{2}\sum_y \Psi(|y|))s_x'^2]$$

$$\times Z_{\Lambda}^{-1}\int_{\Sigma}\mu(ds'_x)\int_R \mu^{\Lambda\setminus\Delta}(ds_{\Lambda\setminus\Delta})\exp[-U(s_{\Lambda}^{*})]$$

$$\le C' \exp[(\sum_y \Psi(|y|)-A)s_x^2]\cdot \rho_{\Delta\setminus\{x\}}^{(\Lambda)}(s_{\Delta\setminus\{x\}})$$

<u>Proof of (3)</u>.

$$\rho'' = \sum_{q\geq P} Z_\Lambda^{-1} \int_{R_q} \mu^{\Lambda\setminus\Delta}(ds_{\Lambda\setminus\Delta})\exp(-U(s_{[q+1]\cap\Lambda}))$$

$$\exp(-W(s_{[q+1]\cap\Lambda}, s_{\Lambda\cap[q+1]}))\exp(-U(s_{\Lambda\setminus[q+1]}))$$

$$\leq \sum_{q\geq P} Z_\Lambda^{-1} \int_{R_q} \mu^{\Lambda\setminus\Delta}(ds_{\Lambda\setminus\Delta})\exp \sum_{x\in[q+1]\cap\Lambda} [-As_x^2 + C]$$

$$\exp \sum_{x\in[q+1]\cap\Lambda} \sum_{y\in\Lambda\,[q+1]} \Psi(|y-x|)\tfrac{1}{2}(s_x^2 + s_y^2)$$

$$\exp \sum_{x\in[q+1]\cap\Lambda} \sum_{y\in\Lambda\setminus[q+1]} \Psi(|y-x|)\tfrac{1}{2}(s_x'^2 + s_y^2)$$

$$\exp[-W(s'_{[q+1]\cap\Lambda}, s_{\Lambda\setminus[q+1]})-U(s_{\Lambda\setminus[q+1]})]$$

$$\leq \sum_{q\geq P} Z_\Lambda^{-1} \int_{R_q} \mu^{\Lambda\setminus\Delta}(ds_{\Lambda\setminus\Delta})$$

$$\exp[-(A-3\epsilon) \sum_{x\in[q+1]\cap\Lambda} s_x^2 - C''\Psi_{q+1}V_{q+1}]$$

$$\exp(\tfrac{1}{2} \sum_y \Psi(|y|) \sum_{x\in[q+1]\cap\Lambda} s_x'^2$$

$$\exp[-W(s'_{[q+1]\cap\Lambda}, s_{\Lambda\setminus[q+1]})-U(s_{\Lambda\setminus[q+1]})]$$

$$\leq \sum_{q\geq P} \exp \sum_{x\in[q+1]\cap\Delta} [-(A-3\epsilon)s_x^2]$$

$$e^{-C''\Psi_{q+1}V_{q+1}} \left[\int \mu(ds)e^{-(A-3\epsilon)s^2}\right]^{|[q+1]\cap\Lambda\setminus\Delta|}$$

$$(\sup_{s'\in\Sigma} \exp[(\tfrac{1}{2} \sum_y \Psi(|y|))s'^2])^{|[q+1]\cap\Lambda|}\ \lambda^{|[q+1]\cap\setminus\Delta|}$$

$$Z_\Lambda^{-1} \int_{\Sigma[q+1]\cap\Lambda} \mu^{[q+1]\cap\Lambda}(ds'_{[q+1]\cap\Lambda}) \int \mu^{\Lambda\setminus[q+1]}(ds_{\Lambda\setminus[q+1]\setminus\Delta})e^{-U(s^*_\Lambda)}$$

$$\leq \sum_{q\geq P} \exp \sum_{x\in[q+1]\cap\Lambda} [-(A-3\epsilon)s_x^2]$$

$$e^{-C''\Psi_{q+1}V_{q+1} + D''V_{q+1}}\ \rho^{(\Lambda)}_{\Delta\setminus[q+1]}(s_{\Delta\setminus[q+1]})$$

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Reference.

[1] D. Ruelle. Superstable interactions in classical statistical mechanics Commun. math. Phys. 18, 127-159(1970).

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