Complete description of the Voronoï cell of the Lie algebra A_n weight lattice. On the bounds for the number of d-faces of the n-dimensional Voronoï cells.

Louis MICHEL

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Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

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Complete description of the Voronoï cell of the Lie algebra $\mathbf{A_n}$ weight lattice. On the bounds for the number of d-faces of the n-dimensional Voronoï cells.

Louis MICHEL, IHES, 91440 Bures-sur-Yvette, France, michel@ihes.fr

Abstract. Denoting these bounds by $N_d(n)$, $0 \le d \le n$, we prove that $N_d(n)/(n+1)!$ is a polynomial $p_d(n)$ of degree d with rational coefficients. We give explicitly the polynomials for $d \le 5$. The proof uses the fact that these bounds $N_d(n)$ are also the number of d-faces of the Voronoï cell of the weight lattice of the Lie algebra A_n (it is also the Cayley diagram of the symmetric group $\mathcal{S}_{n+1}$ which is isomorphic to the Weyl group of A_n). Each d-face of this cell is a zonotope which can be defined by a symmetry group $\sim G_d(\alpha)$, d-dimensional reflection subgroup of the A_n Weyl group. We show that for a given d and n large enough, all such subgroups of A_n are represented and we compute explicitly $N(G_d(\alpha), n)$, the number of d-faces of type $G_d(\alpha)$ in the Voronoï cell of $L = A_n^w$. The final result is obtained by summing over α. That also yields the simple expression: $N_d(n) = (n+1-d)!\,\mathbf{S}_{n+1}^{(n+1-d)}$ where the last symbol is the Stirling number of second kind.

§1. *Introduction.*

The proximity cell of a lattice of points was defined and studied by (Lejeune-) Dirichlet for 2-dimensional lattices (= 2 variable quadratic forms) and also by Hermite. The 3-dimensional case was thoroughly treated in the book [FED885] of Fedorov: *An introduction to the theory of figures* [1] . The last Voronoï memoir is the fundamental study for arbitrary dimension; it appeared in two parts [VOR08], [VOR09], the second has been printed after Voronoï's death at the age of forty. The proximity cells are usually called [2] now Voronoï cells. We whall give their definition in (1abc) after introducing the necessary notations.

Let E_n be a n-dimensional real orthogonal vector space whose scalar product is denoted by $(\vec{x}, \vec{y})$; we define $N(\vec{x}) = (\vec{x}, \vec{x})$. A lattice $L \subset E_n$ is the set of vectors generated from an E_n basis by the addition of vectors. It is a group $L \sim Z^n$, free Abelian group of rank n, closed subgroup of R^n. Three equivalent definitions of the Voronoï cell $D(L)$ of L are:

$$a)\ D(L) = \{\vec{x} \in E_n;\ \forall \ell \in L, N(\vec{x}) \le N(\vec{x} - \vec{\ell})\};$$

$$b)\ D(L) = \{\vec{x} \in E_n;\ \forall \ell \in L, 2(\vec{\ell}, \vec{x}) \le N(\vec{\ell})\};$$

$$c)\ D(L) = \{\vec{x} \in E_n;\ \vec{x} \text{ is a shortest vector in the } \vec{x} + L \text{ coset of } R^n\}. \qquad (1abc)$$

If $\{\vec{b}_j\}$, $1 \le j \le n$, is a basis of L, any other basis is of the form $\sum_j m_{ij}\vec{b}_j$, $m \in GL_n(Z)$. So $|\det \vec{b}_j|$ is an invariant of L that we denote simply $\mathrm{vol}(L)$. By definition, two lattices are isomorphic if they can be transformed into each other by an orthogonal transformation.

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[1] E.S. Fedorov wrote this book between the age of 16 and 26, while serving in the army, or studying medecine, chemistry and physics. Then he became a mineralogist and six years later his book was accepted for publication in a crystallography series. No translation in a Western language is known. There exists a detailed analysis of it in [SEN84].

[2] They are often called Wigner and Seitz cells for crystals; that of the dual of the crystal lattice is called Brillouin zone. Indeed these scientists introduced their use in physics at the beeginning of the thirty's.

So a class of isomorphism depends only on the Gram matrices $q_{ij} = (\vec{b}_i, \vec{b}_j)$; each one is a symmetric positive $n \times n$ matrix which defines a positive quadratic form $\sum_{ij} q_{ij} x_i x_j$. Since the sum of two positive quadratic forms is a positive quadratic form, the set of positive quadratic forms form a convex cone that we denote by $\mathcal{C}_+(\mathcal{Q}_n)$ in the $n(n+1)/2$-dimensional vector space of n variable quadratic forms. To the change of basis of a lattice $\vec{b}_i \mapsto \sum_j m_{ij}\vec{b}_j$ by the matrix $m \in GL_n(Z)$ corresponds the transformation $q \mapsto mqm^\top$ for the corresponding quadratic forms. So there is a natural bijective map between the set of isomorphic classes of lattices and the orbit space $\mathcal{C}_+(\mathcal{Q}_n) \,|\, GL_n(Z)$. The generic lattices (represented by an open dense set in the $n(n+1)/2$ dimensional manifold $\mathcal{C}_+(\mathcal{Q}_n)$) and their Voronoï cells are studied in [VOR08], [VOR09]where they are called *primitive*. These cells are combinatorialy equivalent for $n = 2, 3$. Voronoï established that they form three combinatorialy distinct classes for $n = 4$. In [VOR09] he also gave an expression for the bounds $N_d(n)$. The aim of this paper is to give a different and more explicit expression.

§ **2.** *The expression of the bounds $N_d(n)$ obtained by Voronoï .*

That expression is given in (11), at the end of the section. Before, we recall the fundamental concepts and objects introduced by Voronoï . Instead of giving a summary of his papers, we shall introduce a more modern (and faster) presentation of the basic facts on lattice and their Voronoï cells. This can be found for instance in [DGS], [E], [CS]. Here we shall follow [MS], a monograph in preparation with M. Senechal.

The symmetry point group $P \subset O_n$ of the lattice is the symmetry group of $D(L)$; hence the origin o is its symmetry center. Definition (1b) shows that $D(L)$ is convex since it is the interesection of half spaces bounded by hyperplanes. Let $\mathcal{E}_n$ be the Euclidean space built from E_n. Its points are $x = o + \vec{x}$, the translate of o by $\vec{x} \in E_n$. Conversely, any pair of points $x, y \in \mathcal{E}_n$ defines a vector $y - x \equiv \vec{xy} \in E_n$. The set of translates of the Voronoï cell by all the vectors $\vec{\ell} \in L$, i.e. $\{D(L) + \vec{\ell} = D_{o+\vec{\ell}}(L);\ \ell \in L\}$ form a face to face paving of the space $\mathcal{E}$. We notice that $D(L)$ is a fundamental domain of L, so vol$(D(L))$ =vol(L).

Following [MS], we say that the cells which have a contact with $D_o(L)$ form its *corona* and the centers of these cells define the *corona vectors*. Their set is

$$C = \{\vec{c} \in L; D_o(L) \cap D_c(L) \neq \emptyset\} = L \cap \partial D_o(2L). \tag{2}$$

It follows from (1c) that the corona vectors are shortest in their $L/2L$ cosets, i.e.

$$\vec{c} \in C: \ \forall \vec{\ell} \in L, \quad N(\vec{c} + 2\vec{\ell}) - N(\vec{c}) \geq 0 \Leftrightarrow (\vec{c}, \vec{\ell}) + N(\vec{\ell}) \geq 0. \tag{3}$$

The converse also holds. Note that $\vec{c} \in C \Rightarrow -\vec{c} \in C$. Let us replace 2 by $m > 2$ in (3):

$$\vec{c} \in C, \forall \vec{\ell} \neq 0, \vec{\ell} \in L, \quad m^{-1}(N(\vec{c} + m\vec{\ell}) - N(\vec{c})) = 2\big((\vec{c}, \vec{\ell}) + N(\vec{\ell})\big) + (m-2)N(\vec{\ell}) > 0. \tag{4}$$

This proves ([MS]) that *a corona vector is* **the** *shortest vector in its coset* L/mL when $m > 2$. So we obtain

$$|C| \leq 3^n - 1. \tag{5}$$

This result was first obtained by Minkowski [MIN07] for the more general cases of lattice packings of any convex domain (instead of the Voronoï cell). Equation (5) implies that the number of supporting hyperplanes of the faces of $D_0(L)$ is finite. Since $\mathrm{vol}(D_0(L))$ finite, they cannot be all parallele to a direction, so $D(L)$ *is a polytope*. To simplify notation when there is no ambiguity, we will use D for $D(L)$ from now on. Since $\forall \vec{\ell} \in L$, $\frac{1}{2}\,\ell = o + \frac{1}{2}\,\vec{\ell}$ is a symmetry center of L, then $\frac{1}{2}\,c$ is symmetry center of the face $\mathcal{D}_o \cap \mathcal{D}_c$ (that face is convex and that symmetry exchanges the points o, c, so it transforms this face in itself). From now on will we call the (n-1)-dimensional faces of $D(L)$ *facet*. The set of *facet vectors* $F \subset C$ is

$$F = \{\vec{f} \in C; \quad \dim(D_o \cap D_f) = n - 1\}. \tag{6}$$

One easily proves the

Proposition 1. *$\vec{f} \in L$ is a facet vector if and only if $\pm\vec{f}$ are strictly shorter than the other vectors of their $L/2L$ coset.*

That proposition was first proven in [VOR09] §55, p.67–69. As a corollary, combined with (5), we obtain:

$$2n \leq |F| \leq 2(2^n - 1) \leq |C| \leq 3^n - 1. \tag{7}$$

The second inequality is in [VOR09] p.70; it was also proven before in [MIN897].

At least n facets meet at each vertex of D. Each vertex v belongs to n_v Voronoï cells D_{o_α}, $1 \leq \alpha \leq n_v > n$ and the o_α are on a sphere of center v. Voronoï defined ([VOR09] §60 p.74) and used the polytope that is the convex hull of the points o_α; we denote it Δ_v because it was thoroughly studied by Delone and his school. In dimension n a sphere is determined by $n+1$ points in general position: i.e. they are the vertices of a simplex. So for the generic lattices (forming an open dense set in $\mathcal{C}_+(\mathcal{Q}_n)$, see [VOR08]) $n_v = n + 1$ for each vertex. Each Voronoï cell containing a vertex v meet at v the n others, along n facets. Voronoï called such cells *primitive*; we call here the corresponding lattices *primitive*. It is easy to prove ([VOR08] §17, p.228–9)

Proposition 2. *$D(L)$ primitive $\Leftrightarrow$ each of its d-faces is the intersection of $n+1-d$ cells, for $d = 0, 1, 2, \ldots, n$.*

For $d = n - 1$, we have as a corollary

$$L \text{ is a primitive lattice } \Leftrightarrow \forall v,\ \Delta_v \text{ is a simplex}\ . \tag{8}$$

From propositions 1 and 2 we obtain:

$$L \text{ primitive lattice } \Rightarrow |F| = |C| = 2(2^n - 1). \tag{9}$$

The converse of (9) is not true for $n \geq 4$. All 4-dimensional lattices which are tensor product of two primitive 2-dimensional lattices, have the same combinatorial type of Voronoï cells with $|F| = |C| = 30$ but those cells, which are studied thoroughly in [MS], are not primitive: among their 102 vertices, 12 of them are the intersection of 5 facets [3].

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[3] In dimension 5 [ENG95] has found 225 combinatorial types of non primitive Voronoï cells with the greatest possible number of facets, i.e. 62.

Let $\vec{f}_i(v)$, $1 \le i \le n$ be the facet vectors of the n facets meeting at the vertex v of the Voronoï cell D of a primitive n-dimensional lattice L; these vectors form a basis of the space. The lattice generated by these n linearly independent $\vec{f}_i$ may be only a sublattice of L of index ω_v; so

$$\omega_v \operatorname{vol}(L) = \det(\vec{f}_i)| = n! \operatorname{vol}(\Delta_v), \tag{10}$$

where the last equality is obtained from the known formula giving the volume of a simplex. Let us study the sets $V(\mathcal{D})$ of vertices of the Voronoï cell $\mathcal{D}$. If $v \in V(\mathcal{D})$ the n points $v_i = v - \vec{f}_i$ are also vertices of $\mathcal{D}$ and one verifies that the set $V(\mathcal{D})$ is the disjoint union of subsets, each one containing exactly $n+1$ vertices of $\mathcal{D}$ which are obtained from each other by translations of L ([VOR09] p. 71). Since the set of all Delone cells of L pave the Euclidean space $\mathcal{E}_n$, by choosing a representative v_ρ for each of these subsets, $\cup_\rho \Delta(v_\rho)$ is a fundamental domain of L; so

$$\sum_{v \in V(\mathcal{D})} \operatorname{vol}(\Delta_v) = (n+1) \operatorname{vol}(L). \tag{10'}$$

From this equation and the sum of (10) over the vertices, we obtain

$$|V(\mathcal{D})| \le \sum_{v \in V(\mathcal{D})} \omega_v = (n+1)!; \tag{10''}$$

the equality holds if and only if $\omega_v = 1$ at each vertex. As we have seen, that means that at *at each vertex v of a primitive Voronoï cell, the corresponding n facet vectors $\vec{f}_i$ generate the full lattice*; we call these lattices and their cells, *principal primitive*.

The proof of (10") is only a particular case of the proof Voronoï gave in [VOR09], §63-66, p. 78–83, of the theorem:

Theorem (Voronoï). *The number of d-faces of a principal primitive n-dimensional Voronoï cell is*

$$0 \le d < n, \quad N_d(n) = (n+1-d) \sum_{\ell=0}^{n-d} (-1)^{n-d-\ell} \binom{n-d}{\ell} (1+\ell)^n. \tag{11}$$

Voronoï proof uses a discretisation of the space and finite differences [4] . In [VOR09] §101 p. 136 Voronoï proved that the $N_d(n)$ are upper bounds for the number of d-faces of **any** n-dimension Voronoï cell. That was a remarkable achievement.

Finally Voronoï gave for all dimensions an open set in $\mathcal{C}_+(\mathcal{Q}_n)$, of primitive principal lattice and he called *type I* the combinatorial type of their Voronoï cells [5] . These lattices

[4] At the top of p. 82, there is a misprint in the equation which defines the finite difference operator (replace the last symbol μ by k). At the bottom of the page Voronoï equation (11) (the first of §66) is equivalent to (11) above.

[5] As we told at the end of the introduction, Voronoï [VOR09] showed that there are 3 types of primitive lattices (all principal) in dimension 4 and, in a side remark (p. 84), he stated that he found some non principal primitive lattices in dimension 5. In that dimension, with the correction by Engel [ENG95] of the Baranovskii and Ryshkov result [BAR37], it is now known that there are 222 types of primitive lattices including 21 non principal ones. Voronoï did not introduce a word for the concept of principal primitive lattices.

are described by the quadratic forms

$$Q_{\lambda_{ij}}(x_i) = \sum_i \lambda_{ii} x_i^2 + \sum_{i,j,i<j} \lambda_{ij}(x_i - x_j)^2 \equiv \sum_{ij} q(\lambda)_{ij} x_i x_j; \quad \lambda_{ij} > 0. \tag{12}$$

Voronoï showed that each one of these quadratic forms has a formal symmetry $\mathcal{S}_{n+1}$, the permutation group of $n+1$ objects; it was a generalisation to dimension n of the Selling's study [SEL874] of the 2 and 3 variable quadratic forms. In the particular case where all the λ's are equal, $\mathcal{S}_{n+1}$ becomes a geometric symmetry [6] . In that case, Voronoï computed the coordinates of the $(n+1)!$ vertices (see (27)) and showed that they form a principal orbit of $\mathcal{S}_{n+1}$. At that time it was not known that, up a dilation of scale, these lattices were A_n^w, the weight lattices of the simple Lie algebra A_n. Moreover, the Voronoï cells $\mathcal{D}(A_n^w)$ are the *Cayley graphs* of the permutation groups $\mathcal{S}_{n+1}$ (see [COX80], p. 65–66). That is one more incentive to study them in detail in the next section.

§ **3.** *Detailed description of the Voronoï cells of the A_n^w lattices.*

A representative quadratic form q of A_n^w is obtained from (12) with $\lambda_{ij} = (n+1)^{-1}$; so

$$q = I - \frac{1}{(}n+1)J, \text{ with } I_{ij} = \delta_{ij},\ J_{ij} = 1. \tag{13}$$

From $J^2 = nJ$ one obtains easily q^{-1}, the quadratic form of the dual lattice $A_n^r = (A_n^w)^*$, the root lattice of the simple Lie algebra A_n:

$$q^{-1} = I + J. \tag{14}$$

To write explicitly the corresponding bases of this pair of dual lattices we introduce the following notation. Let $\mathcal{N}_n$ and $\mathcal{N}_n^+$ be the sets of integers m satisfying respectively $0 \le m \le n$ and $1 \le m \le n$. To exploit easily the action of the symmetry group [7] $A_n \sim \mathcal{S}_{n+1}$, it is usual to consider the $n+1$ dimensional space E_{n+1} with the orthonormal basis and the vector $\vec{e}$:

$$\alpha, \beta \in \mathcal{N}_n, \quad (\vec{e}_\alpha, \vec{e}_\beta) = \delta_{\alpha\beta}, \quad \vec{e} = \frac{1}{(}n+1)\sum_\alpha \vec{e}_\alpha, \quad \text{so } (\vec{e},\vec{e}) = (\vec{e},\vec{e}_\alpha) = \frac{1}{(}n+1). \tag{15}$$

The group $\mathcal{S}_{n+1}$ is the group of permutations of the $n+1$ vectors $\vec{e}_\alpha$; it leaves $\vec{e}$ fixed. We denote by $\mathcal{H}_e$ the hyperplane (= vector subspace) orthogonal to $\vec{e}$. It contains the vectors

$$(\vec{e})^\perp \equiv \mathcal{H}_e \ni \vec{u}_\alpha; \quad u_\alpha = \vec{e}_\alpha - \vec{e},\ (\vec{u}_\alpha, \vec{u}_\beta) = \delta_{\alpha\beta} - \frac{1}{(}n+1), \quad \sum_\alpha \vec{u}_\alpha = 0. \tag{16}$$

The vectors of $\mathcal{H}_e$ are those of E_{n+1} with a vanishing sum of coordinates. $\mathcal{S}_{n+1}$ acts linearly (and irreducibly) on $\mathcal{H}_e$ and is the group of permutations of the $n+1$ vectors $\vec{u}_\alpha$.

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[6] As shown between (15) and (17); see also (29). In fact these lattices are the primitive ones with the largest symmetry; see [MS].

[7] We use the same notation A_n for the simple Lie algebra and its Weyl group, but to avoid any ambiguity we shall always precise if A_n denotes the group or the Lie algebra.

Those vectors generate the weight lattice A_n^w. From the last equality of (16) one can take as a basis of A_n^w the n vectors

$$i \in \mathcal{N}_n^+, \quad \{\vec{u}_i = \vec{e}_i - \vec{\epsilon}\} \text{ basis of } A_n^w; \quad (\vec{u}_i, \vec{u}_j) = \delta_{ij} - \frac{1}{(} n+1). \tag{17}$$

The last equality shows that its Gram matrix is q of (13).

We denote by $\mathcal{A}$ a non empty proper subset of $\mathcal{N}_n$ and its complement by $\bar{\mathcal{A}}$, in order to define the set W of lattice vectors of A_n^w:

$$\emptyset \neq \mathcal{A} \subset \mathcal{N}_n, \quad W = \{\vec{w}_{\mathcal{A}} = \sum_{\alpha \in \mathcal{A}} \vec{u}_\alpha\}, \quad \vec{w}_{\mathcal{A}} + \vec{w}_{\bar{\mathcal{A}}} = 0; \quad |W| = 2(2^n - 1). \tag{18}$$

From the defining quadratic form (13) Voronoï proved ([VOR09] §102, p.137–139) that for each $L/2L$ coset of A_n^w, the shortest vectors form a pair: $\vec{w}_{\mathcal{A}}, \vec{w}_{\bar{\mathcal{A}}} = -\vec{w}_{\mathcal{A}}$; by proposition 1 we see that W is the set of the facet vectors of A_n^w:

$$W = F(A_n^w). \tag{19}$$

Another proof is given in [MS]).

We now use well known properties of the A_n Lie algebra which were not all discovered at Voronoï's time. The roots of the Lie algebra A_n form the set R of $n(n+1)$ vectors:

$$R = \{\vec{r}_{\alpha\beta} = \vec{e}_\alpha - \vec{e}_\beta = \vec{u}_\alpha - \vec{u}_\beta\}, \qquad N(\vec{r}_{\alpha\beta}) = 2. \tag{20}$$

These vectors generate the the root lattice A_n^r. One can choose for its basis:

$$\{\vec{r}_{i0}\} \text{ basis of } A_n^r; \quad \text{its vectors satisfy : } (\vec{r}_{i0}, \vec{u}_j) = \delta_{ij}. \tag{21}$$

This proves the duality $A_n^w = (A_n^r)^*$. The weights of the Lie algebra A_n are the non zero vectors whose scalar products with every root are ± 1 or 0; they form the set $W = F(A_n^w)$ of facet vectors of the weight lattice.

For the study of Lie algebras the traditional basis of the root lattice A_n^r is

$$\{\vec{r}_i = \vec{r}_{i-1,i} = \vec{e}_{i-1} - \vec{e}_i = \vec{u}_{i-1} - \vec{u}_i\} \text{ basis of } A_n^r. \tag{22}$$

Notice that

$$(\vec{r}_i, \vec{r}_j) = \begin{cases} 2 & \text{when } i = j, \\ -1 & \text{when } |i-j| = 1, . \\ 0 & \text{when } |i-j| > 1. \end{cases} \tag{23}$$

The corresponding quadratic form is called the Cartan matrix of A_n^r. The dual basis of (22) has to be made of weights:

$$\{\vec{w}_k = \sum_{\alpha=0}^{k-1} \vec{u}_\alpha\} \text{ is the basis of } A_n^w \text{ satisfying } (\vec{r}_i, \vec{w}_k) = \delta_{ik}. \tag{24}$$

Notice also that

$$k \le \ell, \quad (\vec{w}_k, \vec{w}_\ell) = \frac{k(n+1-\ell)}{n+1} > 0. \tag{25}$$

The inequality is a necessary condition for the n facet vectors $\vec{w}_k$ to be those of the facets meeting at a vertex v. Here this condition is sufficient since there is a unique vector v satisfying the system of linear equations:

$$(\vec{w}_k, \vec{v}) = \frac{1}{2} N(\vec{w}_k). \tag{26}$$

The solution

$$\vec{v} = \frac{1}{(} n+1)(\frac{n}{2}\,\vec{e} - \sum_{\alpha=0}^{n} \alpha \vec{e}_\alpha); \qquad N(\vec{v}) = \frac{n(n+2)}{12(n+1)}. \tag{27}$$

was obtained by Voronoï ([VOR09] §103, p. 140–143) who remarked that all coordinates of v are distinct, so the vertices of $\mathcal{D}(A_n^w)$ form at least a $\mathcal{S}_{n+1}$ orbit of $(n+1)!$ vertices. Since this value attains the bound (11), there are no other vertices and (10") proves that A_n^w is a principal primitive lattice. No other symmetry arguments were used by Voronoï ; it is time to produce more.

In an orthogonal vector space, we denote by $\sigma(\vec{r})$ the reflection through the hyperplane $\mathcal{H}_r$ orthogonal to $\vec{r}$. Its action on the vectors is given by

$$\sigma(\vec{r}).\vec{x} = \vec{x} - 2(\vec{x}, \vec{r}) N(\vec{r})^{-1} \vec{r}. \tag{28}$$

The Weyl group A_n is the group generated by the reflections $\sigma(\vec{r}_{\alpha\beta})$ with $\vec{r}_{\alpha\beta} \in R$. Since the roots $\vec{r}_{\alpha\beta}$ generate A_n^r, the Weyl group is symmetry group of this lattice. That group leaves the vector $\vec{e}$ of (15) fixed and acts irreducibly on the n-dimensional subspace $\mathcal{H}_e$. Applying the reflection $\sigma(\vec{r}_{\alpha\beta})$ to the basis vectors $\vec{e}_\gamma$ of E_{n+1} one verifies that the $\vec{e}_\gamma$'s, $\alpha \neq \gamma \neq \beta$ stay fixed while the vectors $\vec{e}_\alpha$ and $\vec{e}_\beta$ are exchanged. This verifies the isomorphism of the Weyl group $A_n \sim \mathcal{S}_{n+1}$, the group of permutations of the $n+1$ coordinates of the vectors $\in E_{n+1}$ in the basis (15). With the use of the classical notation by cycles for the permutations, this isomorphism corresponds to:

$$\sigma(\vec{r}_{\alpha\beta}) \sim (\alpha\beta). \tag{29}$$

It is well known that the permutation group S_{n+1} is generated by the n involutive permutations $(i-1, i)$. They are represented by the n reflections $\sigma(\vec{r}_i)$ and from (23) and (28) one obtains the relations between these generators for the group A_n

$$1 \le i \le n;\ \sigma(\vec{r}_i)^2 = 1;\ |i-j| = 1, (\sigma(\vec{r}_i)\sigma(\vec{r}_j))^3 = 1;\ |i-j| > 1, (\sigma(\vec{r}_i)\sigma(\vec{r}_j))^2 = 1. \tag{30}$$

Remark that the first equality implies that the last one is equivalent to

$$|i-j| > 1; \quad \sigma(\vec{r}_i)\sigma(\vec{r}_j) = \sigma(\vec{r}_j)\sigma(\vec{r}_i). \tag{30'}$$

The Coxeter diagram for the group A_n has n vertices labelled by $i, 1 \le i \le n$, each representing the reflection $\sigma(\vec{r}_i)$, and the $n-1$ edges joining the pair of vertices $i, i+1$.

Coxeter diagram of A_{10} :

A pair of vertices not joined by an edge represents commuting reflections. As we have seen, this orthogonal n-dimensional representation of A_n is also the group of permutations of the vectors $\vec{u}_\alpha$, generators of A_n^w. So the Weyl group is symmetry group of the lattices A_n^w, A_n^r. It is not their full symmetry group because, for $n > 1$, it does not contain $-I_n$, the symmetry through the origin on the n-dimensional vector space $\mathcal{H}_e$. It will be more convenient here to use only the symmetry A_n.

We have proven after (27) that the group $A_n \sim \mathcal{S}_{n+1}$ acts transitively on the set V of vertices. We leave as an exercise the computation of the action of some of its elements on the vertex v (given in (27)):

$$\sigma(\vec{r}_{\alpha\beta}).\vec{v} \equiv (\alpha\beta).\vec{v} = \vec{v} + \frac{1}{n+1}(\alpha-\beta)\vec{r}_{\alpha\beta}. \tag{31}$$

$$\left(\sigma(\vec{r}_1)\sigma(\vec{r}_2)\sigma(\vec{r}_3)\ldots\sigma(\vec{r}_n)\right)^k.\vec{v} \equiv (1,2,\ldots,n,n+1)^k.\vec{v} = \vec{v} - \vec{w}_k. \tag{32}$$

For $1 \le k \le n$ those are the n other vertices of $D_o(A_n^w)$ obtained from $\vec{v}$ by translations of the lattice; it could have been obtained geometrically that these n translations are the opposite of the n facet vectors $\{\vec{w}_k\}$ (24) meeting at v. Any edge from the vertex v is the intersection of $n-1$ facets meeting at v, so it is orthogonal to $n-1$ of the facet vectors $\vec{w}_k$. This shows that edges meeting at v are parallele to the roots of the basis $\{\vec{r}_i\}$ defined in (22) which is the dual basis of $\{\vec{w}_k\}$. From (22) and (27) we obtain $(\vec{v},\vec{r}_i) = 1/(n+1)$. With the symmetry $A_n \sim \mathcal{S}_{n+1}$, this shows that any vertex is equidistant from the n walls of the Weyl chamber which contains it. As a particular case of (31) we obtain:

$$\sigma(\vec{r}_i).\vec{v} \equiv (i,i+1).\vec{v} = \vec{v} - \frac{1}{(}n+1)\vec{r}_i. \tag{33}$$

That shows the the neighbouring vertices of v are obtained by reflection through the Weyl chamber walls. Given another vertex v', there exists a unique $g \in A_n \sim \mathcal{S}_{n+1}$ transforming v into $v' = g.v$; it transforms roots into roots and $\sigma(g.\vec{r}_i) = g\sigma(\vec{r}_i g^{-1})$. Hence the

Proposition 3. *All edges of the Voronoï cell of A_n^w are parellel to the roots ($\in R$ defined in (20)) and have the same length $\sqrt{2}/(n+1)$.*

In [COX72] at the end of §6.2 p. 65–66, it is explained that this cell is combinatorially equivalent to the Cayley diagram of $\mathcal{S}_{n+1}$. The proof is easily obtained from the isomorphism based on (29) and from (30). The vertex v of (27) can be chosen to represent $1 \in \mathcal{S}_{n+1}$ so the n vertices of (33) represent the n group generators $\sigma(\vec{r}_i)$. Repeating the action of the group generators on the newly obtained vertices one defines the correspondence between the $(n+1)!$ elements of $\mathcal{S}_{n+1}$ and the $(n+1)!$ vertices of $\mathcal{D}(A_n^w)$.

Proposition 4. *The Voronoï cell $\mathcal{D}(A_n^w)$ is a zonotope.*

Proof: We must recall first the definition of the *vector sum* [8] $\dot{+}$ of two convex polytopes P', P'' in the Euclidean space $\mathcal{E}_n$. We follow here [GRU67], beginning of Chapter 15 (which has been written by G.C. Shephard). The following two equivalent definitions are given:

$$P' \dot{+} P'' = \{x' + x'' \mid x' \in P',\ x'' \in P''\}; \tag{34}$$

[8] It is also called Minkowski sum.

With $\{v_i'\}$ and $\{v_j''\}$ the sets $V(P'), V(P'')$ of vertices of P', P'', the second definition is:

$$P' \dot{+} P'' = \text{convex hull}(\{v_i' + v_j''\}), \; 1 \le i \le |V(P')|, \; 1 \le j \le |V(P'')|. \tag{35}$$

A change of origin translates the vector sum; so it should be considered as an binary operator on the classes of translated polytopes. It is straightforward to prove that this sum is commutative and associative. A trivial case of such sum is the parallelipipede built on a basis of vectors (identified with 1-dimensional polytopes=line segment). In the general case of eventually non linearly independent vectors we have

Definition: A zonotope is the vector sum of a finite number of segments.

For $\mathcal{D}(A_n^w)$, these segments are defined by the roots. Following the literature on semi simple Lie algebras, we choose as the set R_+ of *positive roots* the $n(n+1)/2$ roots $r_{\alpha\beta}$ satisfying $\alpha < \beta$. To conclude the proof one verifies that by adding to v, for each subset of R_+, the sum of its elements, one obtains the set of vertices

$$V(A_n^w) = \{v + \frac{1}{n+1} \sum_{0 \le \alpha < \beta \le n} \eta_{\alpha\beta} \vec{r}_{\alpha\beta} | \; \eta_{\alpha\beta} = 0 \text{ or } 1\}. \tag{36}$$

Notice that most vertices can be expressed by different sums.

Let us recall well known properties of zonotopes. A zonotope has a symmetry center; for the Voronoï cell A_n^w it is

$$v + \frac{1}{n+1} \vec{\rho}, \text{ with } \vec{\rho} = \frac{1}{2} \sum_{0 \le \alpha < \beta \le n} \vec{r}_{\alpha\beta}; \tag{37}$$

(this half sum of all positive roots is often used in the study of semi simple Lie algebras). The faces of zonotopes are zonotopes; so they have a symmetry center. Notice that it is generally not true for the d-faces, $1 < d < n-1$, of Voronoï cells. That is the case of the two other types of primitive Voronoï cells in dimension 4; they have pentagons among their 2-cells, e.g. [ENG92].

It is well known that the number of edges of a primitive principal Voronoï cell is $N_1(n) = (n+1)!n/2$ (indeed n edges meet at any vertex and each edge has two vertices). Let us apply to this elementary case $d = 1$ the method we shall develop for computing the $N_d(n)$'s. Since the stabiliser of any vertex v is trivial, the stabilizer of an edge is the A_n subgroup $A_1 \sim \mathcal{S}_2$ which permutes the two vertices of an edge. Note that this group leaves fix the middle of the edge, so it is a stabilizer of the Weyl group acting on E_n. There are $|A_n|/|A_1| = (n+1)!/2!$ edges in each A_n orbit. From the fact that the stabiliser of a vertex is trivial, no element of the group A_n can be a permutation of the edges meeting at the vertex. Hence the n edges meeting at v (or at any vertex) belong to the n different orbits of edges. Moreover each orbit can be associated with a $\vec{r}_i$, that is the position of A_1 at one of the n vertex of the Coxeter diagram of the reflection group A_n. In (10") and now, we have determined two polynomials defined in the abstract:

$$p_0(n) = 1, \qquad p_1(n) = \frac{n}{2}. \tag{38}$$

We now study the general case of a d-face Φ_d^α of the Voronoï cell $\mathcal{D}$ of A_n^w. It is defined by its set of vertices $V(\Phi_d^\alpha) \subset V$. This d-face defines its supporting d-plane $\mathcal{H}(\Phi_d^\alpha)$ (the affine span of $V(\Phi_d^\alpha)$) and $V(\Phi_d^\alpha) = V(\mathcal{D}) \cap \mathcal{H}(\Phi_d^\alpha)$. So the edges of Φ_d^α are parallele to the roots parallele to the d-plane $\mathcal{H}(\Phi_d^\alpha)$. Since this d-plane does not contain the origin (center of $\mathcal{D}$), its normaliser in the group A_n (i.e. the subgroup of A_n made of all elements which transform $\mathcal{H}(\Phi_d^\alpha)$ into itself) is a reflection group generated by the reflections $\sigma(\vec{r}_{\alpha\beta})$ whose roots $\vec{r}_{\alpha\beta}$ are parallele to the d-plane. From proposition 3 and (31), this group is also the normaliser in A_n of the d-face Φ_d^α and it acts transitively on its vertices. We will denote this reflection group (or anyone of its conjugacy class) simply by $G_d^{(\alpha)}$. All the d-dimensional reflection subgroups of A_n are known. Their Coxeter graph is obtained from that of A_n by removing n-d vertices and the edges issued from them; in general it is not connected but is the disjoint union of diagrams A_{m_i} with $d = \sum_i m_i$. From (30'), these distinct subgroups A_{m_i} commute between each other. Hence the group $G_d^{(\alpha)}$ is a direct product:

$$G_d^{(\alpha)} = \times_i A_{m_i}, \quad \sum_i m_i = d \tag{39}$$

It is convenient to write the direct product of s isomorphic group A_m as A_m^s. So the general preceeding equation can be written

$$G_d^{(\alpha)} = \times_i A_{m_i}^{s_i}, \quad \sum_i m_i s_i = d; \tag{39'}$$

when $s_i = 1$, we write simply A_{m_i}. Conversely, given a d-dimensional reflection group of type (39) (i.e. direct product of irreducible reflection groups A_{m_i}), it defines a zonotope with all edges equal which, for n large enough, is a d-face of a Voronoï cell $\mathcal{D}(A_n^w)$. Each factor A_{m_i} defines the Voronoï cell $D(A_{m_i}^w)$. The direct product of factors define a particular case of the vector sum of the cells; indeed these polytopes are in linearly independent (in our case, orthogonal direct sum of) vector subspaces [9]. We will use a different notation, $P' * P"$, for this particular case of the polytope vector sum defined in (34),(35) because it has richer properties. For example, with the notation $N_d(P_n)$ for the number of d-faces of the n-dimensional convex polytope P_n and the conventions $N_d(P_n) = 1$ for $d = n$ and 0 for $d > n$, we have:

$$N_{d'+d''}(P'_{n'} * P''_{n''}) = N_{d'}(P'_{n'})N_{d''}(P''_{n''}) \;\Leftrightarrow \tag{40}$$

$$\Leftrightarrow\; N_d(P'_{n'} * P''_{n''}) = \sum_{d=d'+d''} N_{d'}(P'_{n'})N_{d''}(P''_{n''}). \tag{40'}$$

In this paper we replace the symbol P_n by its symmetry group: for instance A_1 is a line segment, $N_0(A_1) = 2$; then A_1^s represents the s-dimensional hypercube and we obtain immediatly the well known formula

$$0 \le d \le s, \quad N_d(A_1^s) = \binom{s}{d} 2^{s-d}. \tag{41}$$

[9] This particular vector sum of polytopes is studied in [COX63], p. 123–124; it was called "rectangular product" by Pólya.

Similarly we obtain immediately the number of vertices of the d-faces labelled by the general reflection group (39'); it is exactly its number of elements:

$$N_0(\times_i A_{m_i}^{s_i}) = |\times_i A_{m_i}^{s_i}| = \prod_i \left((m_i+1)!\right)^{s_i}. \tag{41'}$$

It is important to emphasize that, as A_n was not the full symmetry point group of the lattices A_n and of their Voronoï cell (see end of paragraph containing (30')), $\times_i A_{m_i}^{s_i}$ is not the full symmetry group of the d-face type it labels. For instance the symmetry of the hypercube is the wreath product [10] $A_1 \uparrow s = O_s(Z) = B_s$; it is again a group generated by reflection, hence the notation B_s of the Weyl group of the simple Lie algebra $B_s \sim O_{2s+1}$. Note that for $d > 1$, $s > 1$, $A_d \uparrow s$ is not a group generated by reflection; note also that it is not a subgroup of A_{sd} but it is a subgroup of $GL_{sd}(Z)$.

We shall denote by $Q_d(\times_i A_{m_i}^{s_i}, n)$, $\sum_i s_i m_i = d$, the number of such d-faces of $\mathcal{D}(A_n^w)$ per orbit of the group A_n:

$$Q_d(\times_i A_{m_i}^{s_i}, n) = \frac{(n+1)!}{\prod_i \left((m_i+1)!\right)^{s_i}}. \tag{42}$$

Then the number of d-faces of $\mathcal{D}(A_n^w)$ for a given symmetry group of (39') is

$$N_d(\times_i A_{m_i}^{s_i}, n) = Q_d(\times_i A_{m_i}^{s_i}, n) K_d(\times_i A_{m_i}^{s_i}, n), \tag{43}$$

where $K_d(\times_i A_{m_i}^{s_i}, n)$ is the number of d-face of symmetry $\times_i A_{m_i}^{s_i}$ at the vertex v (or at any vertex); that is also the number of different Coxeter subdiagrams (of the diagram of A_n) corresponding to the group isomorphism class $\times_i A_{m_i}^{s_i}$. For instance there are $n+1-d$ possible positions of the Coxeter diagram of A_d in that of A_n. Hence the number of $\mathcal{D}(A_n^w)$ d-faces which are $D(A_d^w)$ zonotopes is:

$$N_d(A_d, n) = (n+1)!\,\frac{n+1-d}{(d+1)!}; \quad \text{e.g. } N_2(A_2, n) = (n+1)!\,\frac{n-1}{6}, \quad N_1(A_1, n) = (n+1)!\,\frac{n}{2}. \tag{44}$$

The last expression is the maximum number of edges of an n-dimensional Voronoï cell; we have already computed it before (38). The 2-faces of the Voronoï cell $\mathcal{D}(A_n^w)$ are either regular hexagons, with symmetry A_2, or squares, with symmetry A_1^2; so $N_2(A_2, n)$ is the number of hexagonal 2-faces.

By recursion with all m_i different we find

$$\neq m_i,\ 1 \le i \le k,\ d = \sum_i m_i, \quad K_d(\times_i A_{m_i}) = \prod_{\ell=0}^{k-1}(n+1-d-\ell) = k!\binom{n+1-d}{k}. \tag{45}$$

The vanishing of this expression when $d+k \le n$ corresponds to the necessity, for obtening the diagram of a direct product of k factors as a subdiagram of A_n, that one has to remove

[10] The group $G \uparrow s = G^s \rtimes S_s$ is generated by the direct product G^s and the group of permutations of the s factors of this direct product. So $|G \uparrow s| = |G|^s\, s!$.

at least $k-1$ vertices in order to separate the subdiagrams of the A_{m_i}'s. When, in the direct product A^s_m the factors are identical, one cannot distinguish between their diagrams in that of A_n so one must divides the expression of K by the factorial of the number of factors:

$$d = ms,\ K_d(A^s_m) = (s!)^{-1} \prod_{\ell=0}^{s-1} (n+1-d-\ell) = \binom{n+1-d}{s}. \tag{46}$$

Using this expression for the particular case of the d-dimensional hypercube, $m=1$, $d=s$ and (42-43) we obtain for the number of d-hypercube in $\mathcal{D}(A^w_n)$:

$$N_d(A^d_1) = (n+1)!\ 2^{-d} \binom{n+1-d}{d}; \quad \text{e.g. } N_2(A^2_1, n) = (n+1)!\ (n-1)(n-2)/8. \tag{47}$$

Adding this expression with the middle expression of (44), we obtain:

$$N_2(n) = (n+1)!\ p_2(n) \quad \text{with} \quad p_2(n) = (n-1)(3n-2)/24. \tag{48}$$

The general expression of K_d for a general reflection group of (39') is:

$$d = \sum_i m_i s_i,\ k = \sum_i s_i, \qquad K_d(\times_i A^{s_i}_{m_i}, n) = \frac{\prod_{\ell=0}^{k-1}(n+1-d-\ell)}{\prod_i s_i!} = \frac{k!}{\prod_i s_i!} \binom{n+1-d}{k}. \tag{49}$$

With (42-43) we finally obtain for $N_d(\times_i A^{s_i}_{m_i}, n)$ with $d = \sum_i m_i s_i$, $k = \sum_i s_i$:

$$N_d(\times_i A^{s_i}_{m_i}, n) = (n+1)! \frac{\prod_{\ell=0}^{k-1}(n+1-d-\ell)}{\prod_i s_i!\big((m_i+1)!\big)^{s_i}} = \frac{(n+1)!k!}{\prod_i s_i!\big((m_i+1)!\big)^{s_i}} \binom{n+1-d}{k}. \tag{50}$$

Note that d is the dimension of the face of symmetry type $G^{(\alpha)}_d$ and k is the number of factors of the direct product group $G^{(\alpha)}_d$ (we always denote by n the dimension of the space spanned by the lattice).

To summarize this section, we have obtained a detailed knowledge of the combinatorial structure of the Voronoï cell of the A^w_n lattice by giving:
i) the list of the d-dimensional reflection subgroups of A_n; it describes the different types (α) of the d-faces. Each one is either the Voronoï cell of dimension d or the $*$-sum of such cells of smaller dimension.
ii) the number of (α)-cells at each vertex; it is given by $K_d(G^{(\alpha)}_d)$ (49).
iii) the total number of (α)-cells; it is given by $N_d(G^{(\alpha)}_d)$ (50).

As a last example, let us study the symetry types and the number of facets of the Voronoï cell A^w_n. Their symmetry type are A_{n-1}, $A_{n-2}\times A_1$, $A_{n-3}\times A_2$, $A_{n-4}\times A_3$, ...(This structure of the facets was given by theorem 2 of [RYŠ62] and its theorem 3 gives informations on the structure of the d-faces). From (49), it is easy to complete the proof of

d	$G_d^{(\alpha)}$	$K_d(G_d^{(\alpha)})$	$N_d(G_d^{(\alpha)})$
0	1=vertex	1	$(n+1)!$
1	A_1=edge	n	$(n+1)!\,\frac{n}{2}$
2	A_2=hexagon	$n-1$	$(n+1)!\,\frac{n-1}{6}$
	A_1^2=square	$\frac{(n-1)(n-2)}{2}$	$(n+1)!\,\frac{(n-1)(n-2)}{8}$
3	A_3=3-(Vcell)	$n-2$	$(n+1)!\,\frac{n-2}{24}$
	$A_2 \times A_1$ prism	$(n-2)(n-3)$	$(n+1)!\,\frac{(n-2)(n-3)}{12}$
	A_1^3=3-cube	$\frac{(n-2)(n-3)(n-4)}{6}$	$(n+1)!\,\frac{(n-2)(n-3)(n-4)}{48}$
4	A_4=4-(Vcell)	$n-3$	$(n+1)!\,\frac{n-3}{120}$
	$A_4 \times A_1$ prism	$(n-3)(n-4)$	$(n+1)!\,\frac{(n-3)(n-4)}{48}$
	A_2^2	$\frac{(n-3)(n-4)}{2}$	$(n+1)!\,\frac{(n-3)(n-4)}{72}$
	$A_2 \times A_1^2$	$\frac{(n-3)(n-4)(n-5)}{2}$	$(n+1)!\,\frac{(n-3)(n-4)(n-5)}{48}$
	A_1^4=4-cube	$\frac{(n-3)(n-4)(n-5)(n-6)}{24}$	$(n+1)!\,\frac{(n-3)(n-4)(n-5)(n-6)}{384}$
5	A_5=5-(Vcell)	$n-4$	$(n+1)!\,\frac{(n-4)}{720}$
	$A_4 \times A_1$ prism	$(n-4)(n-5)$	$(n+1)!\,\frac{(n-4)(n-5)}{240}$
	$A_3 \times A_2$	$(n-4)(n-5)$	$(n+1)!\,\frac{(n-4)(n-5)}{144}$
	$A_3 \times A_1^2$	$\frac{(n-4)(n-5)(n-6)}{2}$	$(n+1)!\,\frac{(n-4)(n-5)(n-6)}{192}$
	$A_2^2 \times A_1$	$\frac{(n-4)(n-5)(n-6)}{2}$	$(n+1)!\,\frac{(n-4)(n-5)(n-6)}{144}$
	$A_2 \times A_1^3$	$\frac{(n-4)(n-5)(n-6)(n-7)}{6}$	$(n+1)!\,\frac{(n-4)(n-5)(n-6)(n-7)}{288}$
	A_1^5=5-cube	$\frac{(n-4)(n-5)(n-6)(n-7)(n-8)}{120}$	$(n+1)!\,\frac{(n-4)(n-5)(n-6)(n-7)(n-8)}{3840}$
n-1	A_{n-1}=(n-1)-(Vcell)	2	$2(n+1)$
	$A_m \times A_{n-m-1}$	$2\ ,\ 1 \leq m < \frac{n-1}{2}$	$2\binom{n+1}{m+1}$
	A_m^2	$1\ ,\ n = 2m+1$	$\binom{n+1}{m+1}$
n	A_n=n-(Vcell)	1	1

Table 1. The d-faces of the Voronoï cell $\mathcal{D}(A_n^w)$.
Column 2: k-(Vcell)=A_k^w Voronoï cell. $G_k^{(\alpha)} \times A_1$ is a prism of basis $G_k^{(\alpha)}$.
Column 3: number of $G_k^{(\alpha)}$ faces per vertex. Column 4: total number of $G_k^{(\alpha)}$ faces.
The last line is a natural convention.

Proposition 5. *At each vertex meet 2 facets of each of the* $[\frac{n}{2}]$ *symmetry types* $A_m \times A_{n-1-m}$ *with* $m \in Z$, $0 \le m < (n-1)/2$ *and, when* n *is odd, 1 facet of type* $A^2_{(n-1),'2'}$.

Hence, the total numbers of facets of the different symmetry types are (with m integer and A_0 the trivial group):

$$0 \le m < \frac{n-1}{2}, \qquad N_{n-1}(A_m \times A_{n-1-m}, n) = 2\binom{n+1}{m+1}, \tag{51}$$

$$\text{and for odd } n,\ m = \frac{n-1}{2}, \qquad N_{n-1}(A^2_m, n) = \binom{n+1}{m+1}. \tag{51'}$$

Adding these numbers for all values of m yields $N_{n-1}(n) = 2(2^n - 1)$, already given in (9).

Table 1 summarizes the nature of d-faces for $0 \le d \le 5$ and for $d = n-1$ and n (i.e. the cell itself), and for each symmetry type, their number per vertex and their total number. Some results on the structure of the Voronoï cell of the lattice A^w_n have been given in [CS], [CON91]. A general algorithm for studying the structure of the Voronoï cells of root and weight lattices is established in [MOO95].

§**4.** *The new explicit expression of the bounds* $N_d(n)$.

When d is given, (50) shows that $N_d(G^{(\alpha)}_d/(n+1)!$ is a polynomial of n of degree $k \le d$; the maximum value of the degree is reached for $G^{(\alpha)}_d = A^d_1$ (the d-dimensional hypercube). To find the expression of $N_d(n)$, which is both the total number of d-faces of the Voronoï cell $\mathcal{D}(A^w_n)$ and the upper bound of the number of d-faces of any n-dimensional Voronoï cell, we just add the polynomials $N_d(G^{(\alpha)}_d, n)$ for all d-dimensional reflection subgroups $G^{(\alpha)}_d$ of A_n. That proves the result announced in the abstract: for a given d, $N_d(n)/(n+1)!$ is a polynomial in n of degree d. Moreover the coefficients of these polynomial are rational numbers.

We can predict an upper bound of the smallest common multiple of the denominators of these coefficients (in the reduce form). These denominators contain two types of terms. One of them appears in the number of elements of an orbit: $Q_d(\times_i A^{s_i}_{m_i})$ in (42); indeed it is the quotient $|A_n|/|\times_i A^{s_i}_{m_i}|$. Since we factorize $(n+1)!$ in the final expression of $N_d(n)$, the number of elements of all reflection subgroups of A_n are in the denominators and their smallest common multilple is $(n+1)! = |A_n|$. For the direct product of identical reflection groups, as in A^s_m another type of terms appears in denominator: $s!$ in the middle of (46). This number of permutations was introduced because one cannot distinguish between the different factors A_m of A^s_m in the Coxeter diagram of the later group. As we pointed it (after (41)) for the hypercube (case $m = 1$), that was another way to state that the symmetry of the face A^s_m is the wreath product $A_m \uparrow s$ with

$$|A_m \uparrow s| = \big((m+1)!\big)^s (s!). \tag{52}$$

For $m > 1$, A^s_m is no longer a group generated by reflection but, as for A^s_1, it is also a subgroup of $GL_d(Z)$ with $n \ge ms$.

To summarize: the denominators of the rational coefficients of the polynomial $p_d(n)$ are the orders of different finite subgroups of $GL_n(Z)$. Minkowski [MIN890] has computed the *smallest common multiple of all finite subgroups of* $GL_d(Z)$ and has denoted it by $\overline{d|}$:

$$\overline{d|} = \prod_{q \text{ is prime}} q^{\sum_{k=0}^{\infty}\left[\frac{d}{q^k(q-1)}\right]}; \qquad \Rightarrow \overline{2d+1|} = 2.\overline{2d|}. \tag{53}$$

He also proved (B_d is the dth Bernouilli number):

$$b_d = \text{denominator of}(B_d/d), \quad \overline{2d|} = 2b_d.\overline{2d-1|} \Leftrightarrow \overline{d|} = 2^d \prod_{k=1}^{[d/2]} b_k. \tag{54}$$

From $\overline{1|} = 2$ with this equation and the last equality of (53), it is easy to compute:

$$1 \le d \le 7 \qquad \overline{d|} = 2,\ 24,\ 48,\ 5\,760,\ 11\,520,\ 2\,903\,040,\ 5\,806\,080. \tag{55}$$

Then

$$d > 0, \quad N_d(n) = (n+1)!(n+1-d)(\overline{d|})^{-1}P_d(n). \tag{56}$$

where $P_d(n)$ is a polynomial in n of degree d-1 and integer coefficients. For $1 \le d \le 5$ the polynomials $P_d(n)$ can be computed by adding the expressions of the last column of table 1 corresponding to a given d. These $P_d(n)$ are listed in table 2.

$P_1(n) = 1$
$P_2(n) = 3n - 2$
$P_3(n) = (n-1)(n-2)$
$P_4(n) = 15n^3 - 105n^2 + 230n - 152$
$P_5(n) = 3n^4 - 38n^3 + 173n^2 - 330n + 216.$

Table 2. The polynomials $P_d(n)$ for $1 \le d \le 5$.

§ **5.** *Expression of $N_d(n)$ as multiple of a Stirling number of second kind.*

In this section we first give a reinterpretation of the general formula (50). We consider first the cases where all $s_i = 1$. We have studied in (51) the simplest of these cases with $k = n+1-d = 2$; we can generalize it to other values of k small enough compared to n (e.g. $k^2 < (n+1-k)$); then $G_d^{(o)} = \times_{i=1}^{k} A_{m_i}$ with $1 \le m_i$, the m_i's are all different. With these conditions (50) becomes:

$$k = n+1-d, 1 \le i \le d, 1 \le m_i \in Z,\ m_i \text{ all different}, \quad N_d(n) = k! \sum_{\sum m_i = d} \frac{(n+1)!}{\prod_{i=1}^{k}(m_i+1)!}. \tag{57}$$

With the change of notation $N_d(n) = N'_k(n)$, $n_i = m_i + 1$ and ordering the integers n_i's by decreasing values, this last equation can also be written:

$$k = n+1-d,\ 1 \le i \le k,\ n_i > n_{i+1},\ n_k \ge 2, \quad N_d(n) \equiv N'_k(n) = k! \sum_{\sum n_i = n+1} \frac{(n+1)!}{\prod_{i=1}^{k} n_i!}. \tag{57'}$$

To obtain the equation equivalent to the general form of (50), we have to relax the condition $n_i > n_{i+1}$ to $n_i \geq n_{i+1}$ and count the contiguous numbers of = :

$$s_1, s_2, \ldots, s_\alpha, \ldots \text{ are the numbers of contiguous } = \text{ in the sequence } n_i \geq n_{i+1}. \tag{58}$$

Then (50) becomes, with $k = n + 1 - d$:

$$1 \leq i \leq k,\ n_i > n_{i+1},\ n_k \geq 2, N_d(n) \equiv N'_k(n) = k! \sum_{\sum n_i = n+1} \frac{(n+1)!}{\prod s_\alpha \prod_{i=1}^{k} n_i!}. \tag{59}$$

Finally we have to relax the condition k small compared to n and to consider the cases where $k < n + 1 - d$ (as is the case for instance for the facet of symmetry A_{n-1}). For that we complete the sequence $n_1 \geq \ldots \geq n_k \geq 2$ by adding to it $(n + 1 - d - k)$ "1" 's; this add a new s_α whose value is $n + 1 - d - k$. Then (59), which is equivalent to (50), is transformed into:

$$\ell = n + 1 - d, 1 \leq i \leq \ell, n_i > n_{i+1},\ n_\ell \geq 1,\ N_d(n) \equiv N'_\ell(n) = \ell! \sum_{\sum n_i = n+1} \frac{(n+1)!}{\prod s_\alpha \prod_{i=1}^{\ell} n_i!}. \tag{60}$$

This is a new general expression for $N_d(n)$. It is easy to verify that for $1 \leq d \leq 5$ and for the different partitions of $n+1$ into $\ell = n+1-d$ terms, the different terms in the the sum over the partitions in (60) have exactly the form given in the last colummn of table 1.

The expression

$$1 \leq i \leq \ell,\ n_i > n_{i+1},\ n_\ell \geq 1, \quad \sum_{n_i = n+1} \frac{(n+1)!}{\prod s_\alpha \prod_{i=1}^{\ell} n_i!}. \tag{61}$$

is the number of ways of partitioning a set of $n+1$ elements into ℓ non empty subsets. By definition this number is the Stirling number of second kind; it is denoted $\mathbf{S}_{n+1}^{(\ell)}$ in [ABR64]. Thus we have establish that (it is also valid for $d = 0$ and $d = n$):

$$0 \leq d \leq n, \quad N_d(n) = (n + 1 - d)!\ \mathbf{S}_{n+1}^{(n+1-d)}. \tag{62}$$

The right hand side corresponds to the ordered partitions: it includes the permutations of the different non empty subsets which define the partition. Through the use of (62) and (56), any relation between Stirling numbers of second kind yields a relation between the polynomials $P_d(n)$ (defined in (56)). For instance:

$$\mathbf{S}_{n+1}^{n+1-d} = (n + 1 - d)\ \mathbf{S}_n^{n+1-d} + \ \mathbf{S}_n^{n-d} \tag{63}$$

yields:

$$(n + 1)P_d(n) - (n - d)P_d(n - 1) = (n + 1 - d) \frac{\overline{d|}}{\overline{d-1|}} P_{d-1}(n - 1). \tag{64}$$

This gives another possibility for computing the $P_d(n)$ by recursion.

One closed expression given in [ABR64] for the Stirling numbers of the second kind is:

$$\mathbf{S}_n^{(m)} = (m!)^{-1} \sum_{k=1}^{m} (-1)^{m-k} \binom{m}{k} k^n. \tag{65}$$

That yields

$$N_d(n) = \sum_{k=1}^{n+1-d} (-1)^{n+1-d-k} \binom{n+1-d}{k} k^{n+1}. \tag{66}$$

With the change of dumb index $k \mapsto \ell + 1$, one obtains the expression (11) given by Voronoï .

So we can either consider that we have given another proof of the Voronoï formula (11) or, most modestly, that (62) is a simplest form of the Voronoï expression. Equations (62) and (56) show that the Stirling numbers of second kind can be expressed as a one family of polynomials; that was known long ago: e.g. [JOR60], §58. However the only reference I know [11] to a form similar to

$$\mathbf{S}_{n+1}^{(n+1-d)} = (\overline{d}|)^{-1} P_d(n) \prod_{\ell=0}^{d} (n+1-\ell), \tag{67}$$

is in [GRA88], their equations (6.45),(6.50). These authors define the "Stirling polynomials" $\sigma_d(n+1)$ from a formula identical to (67), but written for the Stirling numbers of the first kind [12] . The relation between the two families of polynomials is:

$$P_d(n) = (-1)^{d+1} \overline{d}| \sigma_d(-(n+1-d)). \tag{68}$$

Using this relation, the recursion (64) yields that to be proven in [GRA88] exercise 6.18.

§ 6. *Final remarks.*

As we noted, we have also studied the Cayley graph of the symmetric group $\mathcal{S}_{n+1}$. The use of its natural realisation (29) as n-dimensional reflection group imbeds this graph as a polytope of the orthogonal vector space E_n. That introduces the roots, their orthogonal hyperplanes and the Weyl chambers. Then we can define the set of vertices of the Cayley diagram, not as an abstract principal orbit of $\mathcal{S}_{n+1}$, but as an orbit of a point in the interior of a Weyl chamber and equidistant to its walls. The Coxeter elements of this reflection group are the permutations corresponding to a cycle of length $n + 1$. They play an important role (see equation (32) which involves the weigths = facet vectors of the Cayley graph). Then the theorem 1.7.7 of [KER96] which in somme cases leads to set ordered partitions, is probably the shortest way to obtain (62). So one could have started the study of Cayley graphs of symmetric groups without any reference to their

[11] I am very grateful to Dr Thomas Scharf (in Bayreuth) who pointed out to me this reference.

[12] Their Stirling numbers of first kind do not change of sign, so they are those of [ABR64] multiplied by $(-1)^d$. In their book, the denominators of the σ_d polynomials are not recognized as Minkowski numbers.

identification (given e.g. in [COX63]) with the Voronoï cells of the lattices A_n^w. But in this paper, we wanted first to describe the structure of the Voronoï cell of A_n^w.

§7. *References.*

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