SIMPLE MATHEMATICAL MODELS OF SYMMETRY BREAKING. APPLICATION TO PARTICLE PHYSICS.

Louis MICHEL

Institut des Hautes Etudes Scientifiques

91440 - Bures-sur-Yvette, France

Conference given on March 26, 1974 at the Warsaw Symposium in Mathematical Physics.

Simple Mathematical Models of Symmetry Breaking . Application to Particle Physics.

0.-INTRODUCTION

There are many approximate symmetries in particle physics. It is tempting to consider them as broken higher symmetries. There has been many simplified models along this lines ; in a large subset of them, tentative explanations of the Cabibbo angle are suggested. They are not very convincing.

In this lecture I will first present some mathematical facts relevant to symmetry breaking pertaining to two different approaches. Their application to particle physics seems rather suggestive.

1.- WHAT IS A SPONTANEOUSLY BROKEN SYMMETRY?

The general expression "Symmetry breaking" covers different physical phenomena. So it might be worthwhile to precise which aspect of symmetry breaking is considered here. We will not consider what could be qualified as "apparent symmetry breaking". This is the case for instance of classical systems in the neighbourhood of an unstable equilibirum which possesses a symmetry group G . Such systems, very near from each other[1)], (and not invariant under G) may then evolve to very different states, transformed into each other by G ; they may also tend to the same state invariant only under the subgroup H of G . In both cases however the symmetry of each system has increased rather than decreased since the initial state had a smaller symmetry[2)].

[1)] Even if the experimentalist prepares them as similar as possible, there are fluctuations which make them different.

[2)] Apparent symmetry breaking does also raise interesting problems. R. Thom [1] in his lecture will also distinguish several types of symmetry breaking and study interesting cases which I do not consider here.

The problem which interests us is of a broader nature. For instance, in the preceeding example, we would ask the question : "why there exist stable equilibria with symmetry group H , strictly smaller than G ?" To take a concrete example : although interactions between atoms or ions are invariant under translations and rotations i.e. G is the Euclidean group, at some temperature and pression the lowest energy state might be a crystal ; its state is invariant under a crystallographic group H , strict subgroup of G . By Euclidean transformations, this state is transformed into other states of the same crystal ; the complete set of transforms by G is called an orbit of G . The interesting problem is not to explain which state of the orbit will appear (this might be due to any heterogeneity such as crystal seed, etc...), but why crystals exist ? More generally, which subgroups H of the Euclidean group G can be symmetry groups of equilibrium states ? As we will see, one can answer this question.

To summarize, we say that a symmetry is spontaneously broken when for a physical problem invariant under a group G there exist solutions (which can be grouped into orbits of G) which are only invariant under a strict subgroup of G . We shall omit from now on the adverb "spontaneously!"

The mechanism of symmetry breaking is well understood ; its appears in statistical mechanic when one goes to the thermodynamics limit for systems for which one has rigourous solutions, it also appear in quantum field when one performs the renormalization. For quantum field theory we refer to early examples with perturbative renormalization [2] and recent examples in the lectures of Glimm and Jaffé at this conference [3] . There are even more examples in statistical mechanics, e.g. models of spontaneous magnetization [4] . It is also a criterion that broken symmetries are well understood when one can predict that they cannot occur, as Dobrushin and Schlossmann [5] have recently proven for a large class of 2-dimensional lattice models invariant[1)] under a compact connected Lie group G .

The description of broken symmetry is very simple and natural when one uses the mathematical frame of C^* algebra. This frame covers both classical and quantum statistical mechanics, and quantum mechanics and quantum field theory. The physical states are positive linear forms on A ; in the dual A^* of A , they form a convex set whose extremal points are the pure states. Let G be a locally compact group of automorphisms of A . Let Φ be a G-invariant state and $\mathcal{H}_\Phi$, π_Φ the corresponding Hilbert space and representation of A obtained by the Gelfand-Naimark-Segal construction. When Φ is not a pure state, π_Φ is reducible . From the assumption of asymptotic abelianness[2)] one proves [7,8] that Φ is an integral over a subset θ of the pure states

$$\Phi = \int_\theta \Psi \, d\mu(\Psi) \tag{1}$$

where $d\mu(\Psi)$ is a G-invariant measure (normalised to $\int_\theta d\mu = 1$) and that the representation π_Φ is factorial (i.e. $\mathcal{H}_\Phi = \int_\theta^\oplus \mathcal{H}_\Psi \, d\mu(\Psi)$ and all irreducible representaions π_Ψ on $\mathcal{H}_\Psi$ are unitary equivalent) . The symmetry is broken. Indeed each state Ψ is only invariant under a strict subgroup G_Ψ of G , and the automorphisms $g \in G$, $g \notin G_\Psi$ of A are not unitarily

---

1) For more details : Define an action of G on a manifold X . So G acts (diagonally) on X^{Z2} . The potential U is bounded, has finite range and is invariant under G . and there are some conditions of non-degeneraty. The proof use the limit theorem for random variables on Lie groups.

2) This assumption, introduced in [5] , is :
$\forall a,b \in A$, $\forall \Phi \in A^*$ $\Phi([a,\alpha_t(b)]) \to 0$ as $G \ni t \to \infty$ e.g. if G is the Euclidean group, t is a translation going to infinity).

implementable on $\mathcal{H}_\psi$. Finally one also proves that the only G-invariant subsets of θ are either of μ-measure one or of μ-measure zero.

When there are G-invariant μ-measure zero subsets, Φ is a ergodic transitive state. The classification of such sets for the Euclidean group is still to be done. When there are no G-invariant μ-measure zero subsets, Φ is called a transitive state. The classification of such states is obtained by finding all isomorphie classes of G-orbits carrying a finite G-invariant measure. This has been done in [7] for the Euclidean group. Outside the crystallographic groups, in three dimensions (230), and those in two dimensions to which are added continuous translations in the third direction (17), and the extension of translations by discrete subgroups of SO(3) (oriented homogeneous material, as ferromagnet ; infinite number), there are two infinite classes of helicoïdal symmetries with rational or irrational rotation angle (helimagnetic states, cholesteric liquids, etc...)[1)]

It is more simple to classify "possible symmetry breakings" than to prove that they dynamically occur (or do not occur); for a given system the latter requires the study of the action of G on the set of pure states. However there also exists in physics simplified models which predict correctly the possible symmetry breaking ; I think particularly of the Landau theory of phase transtisions[2)]. Since these transitions are reversible it deals not only with symmetry breaking, but also with enlargement of summetry. However enlargements of symmetry seemed more natural than symmetry breaking (cf the

---

1) Of course not all these symmetries classes, in infinite number, are known to occur in Nature, this is already the case of some of 230 crystallographic classes. But it seems to me interesting that man has for instance make borane molecules with the dodecahedron, icosahedron symmetry (e.g. anion $B_{12}H_{12}^{2-}$) which has never been observed in natural molecules .

2) It is succently exposed in the Landau-Lifschitz [9] and the Lubarskii [10] text books. For improvements and an up to dated review see Birman [11].

pionnier work of Curie in this domain [12]).

It is time now to expose the work I did with L.A. Radicati these last few years on two mathematical models of symmetry breaking.

2.- TWO MATHEMATICAL MODELS OF SYMMETRY BREAKING.

2.1.- Smooth action[1)] of a compact Lie group G on a manifold M.

When a group G acts on a set M, we denote by G_m the isotropy group (or little group) of $m \in M$:

$$G_m = \{y \in G \text{ , } g.m = m\} \tag{2}$$

and by $G(m)$ the G-orbit of m :

$$G(m) = \{ m' \in M, \exists g \in G \text{ , } m' = gm\} \text{ .} \tag{3}$$

The isotropy groups of elements of the same orbit are conjugated $m' = g.m \Rightarrow G_{m'} = g\, G_m\, g^{-1}$.

There is a natural definition of isomorphy class of G-orbit ; these classes are in bijective correspondance with conjugation classes of subgroups of G. In the action of G in M we call layer the union of all isomorphic orbits : elements $m' \in M$ whose isotropy group is conjugated to G_m form the layer $S(m)$. By inclusion up to a conjugation there is a natural order on the conjugation classes of subgroups of G (we denote by $\{H\}$ the class of $H \subset G$) ; it is customary to use the reverse order on the isomorphy classes of

1) For classical reviews of the subject see [13] . To avoid details we assume smothness i.e. C^∞ ; most results are valid with weaker hypothesis ; Mostow, Palais theorem use C' . Palais [14] proved that for a compact M, C' action is equivariant to C^∞ action.

orbits. Indeed, when G is a Lie group:

$$\dim G = \dim G_m + \dim G(m) \qquad (4)$$

so the smaller is the subgroup G_m the largest is the orbit $G(m)$. To be on the same orbit is an equivalence relation for the elements of M ; the quotient is called the orbit space ; we denote it by M/G and by π the canonical projection $M \xrightarrow{\pi} M/G$.

For smooth action of compact Lie groups G on manifolds everything is beautiful. The isotropy groups are closed Lie subgroups. Orbits and layers are submanifolds, layers are strata (see Thom's lecture), the continuous map π is open, closed, proper. There is a maximal layer S_0 ($\sim$ minimal isotropy group) which is open dense.[1)]

Consider some examples : a) M is the five dimensional phase space of three distincts particle with fixed total energy momentum p; G is the little group of p for the Lorentz group, G is isomorphic to O(3) ; M/G is the Dalitz plot ; there are two layers, whose images by π are the interior and the boundary of the Dalitz plot.

b) M is S_2 and G is O(2) which includes the inversion through the origin ; the rotation axis defines poles and equator on the sphere.
The orbits of the open dense layer are the two paralleles of same N-S latitude $\lambda : 0 < \lambda < 90°$.
There are two other layers of one orbit each : the poles and the equator.

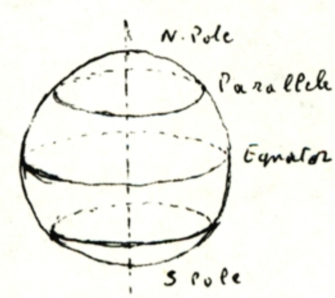

Figure 1

1) Remarkable results, not used here, are those of Mostow. For C^1 action, if M is compact, the number of layers is finite ; if the number of layers is finite, there is an embedding of M is a finite dimensional vector space on which the action of G is linear orthogonal.

c) M is S_7 , the unit sphere of the octet space, i.e. the space of the adjoint representation of $SU(3)$; $G = \text{Aut } SU(3)$, isomorphic to the semi direct product $SU(3) \square Z_2$; in physics, outer automorphisms of $SU(3)$ an generated by charge conjugation. We can use figure 1 again ; the orbit of the open dense layers contain 2 connected components of dimension 6 . The equator is a connected orbit of dimension 6 ; the orbit represented by the two poles contain 2 connected components of dimension 4, the corresponding isotropy group is $U(2)$.

There is a G-invariant Riemann metric on M (take any Riemann metric on M , average it by the group with the Haar measure) and M/G is a metric space. For any globally G-invariant submanifold Ω of M , there is a tubular neighbourhood $U \supset \Omega$, such that $\forall m' \in U$, there exists a unique $m \in \Omega$ such that distance $m\, m'$ is minima. The retraction map $r(m') = m$ is equivariant : $\forall g \in G$, $r(g.m') = g . r(m') = g.m$; hence $G_{m'} \subset G_m$. This prove the : (take $\Omega = G(m)$)

<u>Theorem 1</u> : " For every $m \in M$, there is a neighbourhood U such that $\forall m' \in U$, $G_{m'}$ is larger (not necessarily strictly!) than G_m."

Taking local geodesic coordinates in m , one sees that the local action of G_m is linear : this is also the linear orthogonal representation D_m of G_m on $T_m(M)$ the tangent vector space at m ; D_m is orthogonal and fully reducible ; $T_m(S(m))$ and $T_m(G(m))$, the tangent spaces to the layer and the orbit are invariant subspaces. Theorem 1 proves that the subspace of fixed points of G_m on $T_m(m)$ is included in $T_m(S(m))$. For a G-invariant vector field $m \to v_m \in T_m(M)$, $\forall g \in G_m$, $g\, v_m = v_m$; hence [16]:

<u>Theorem 2</u> : " A G-invariant vector field on M is at each m tangent to the layer $S(m)$" .

Consider the set $\mathfrak{F}$ of all G-invariant real valued smooth functions on M . i.e. $f \in \mathfrak{F} \Rightarrow \forall g \in G$, $f(g.m) = f(m)$: f is constant on each orbit ; so its gradient is orthogonal to the orbit and if the orbit is isolated in its layer (i.e. $\pi(G(m))$ is an isolated point of $\pi(S(m))$) then grad $f = 0$. The convex is also true ; hence if we call "critical orbits" those orbits on which the gradient of every $f \in \mathfrak{F}$ vanishes, we have [16] .

Theorem 3."The critical orbits are the orbits isolated in their layer"[1)]

The interest of this theorem for the study of symmetry breaking is obvious. If the physical G-invariant problem is a variational problem, whatever the function to be varied, the symmetry is broken on critical orbits which are not fixed points, whatever the function to be varied.

While our first mathematical model deals with the smooth action of compact Lie groups on real manifolds, the second model consider linear action of any group on real or complexe finite dimensional vector spaces.

2.2.- G-invariant algebras.

Consider two linear representation D and D_1 on two vector spaces $\mathcal{E}$ and $\mathcal{E}_1$. We denote by $Hom(\mathcal{E}, \mathcal{E}_1)$ the set of linear maps from $\mathcal{E}$ to $\mathcal{E}_1$ (it is a vector space) and $Hom(\mathcal{E},\mathcal{E}_1)^G$ those maps invariant by G :

$$\forall g \in G \quad f \circ D(g) = D_1(g) \circ f \quad ; \qquad (5)$$

$\dim H(\mathcal{E},\mathcal{E}_1)^G > 0$ if D and D_1 have in common some irreducible representations in their reduction (one says they are not "disjoint"). As a

1) We can also give conditions implying that all G-invariant vector fields vanish on a orbit isolated in its layer. The points of $G(m)$ which have G_m as isotropy group form a submanifold diffeomorphic to the group $H_m = N(G_m)/G_m$ where $N(G_m) = \{h \in G, hG_m h^{-1} = G_m\}$ is the normalizer of G_m in G . Let H_m^0 the connected component which contains the identity So two such conditions are :

1°) $H_m^0 = G_m$

2°) The Euler characteristic of the orbit $\chi(G(m)) \neq 0$.

particular case, consider the action of G on $\mathcal{E}_1 = \mathcal{E} \otimes \mathcal{E}$ defined by the tensor representation $D_1 = D \otimes D$; then each map $f \in \mathrm{Hom}(\mathcal{E} \otimes \mathcal{E}, \mathcal{E})^G$ defines an algebra on $\mathcal{E}$ for which G is a group of automorphism. (This algebra is generally not associative). On can decompose the tensor product $\mathcal{E} \otimes \mathcal{E}$ into its symmetric and antisymmetric part $\mathcal{E} \otimes \mathcal{E} = (\mathcal{E} \overset{S}{\otimes} \mathcal{E}) \oplus (\mathcal{E} \overset{A}{\otimes} \mathcal{E})$; the corresponding algebras $\in \mathrm{Hom}(\mathcal{E} \overset{S}{\otimes} \mathcal{E}, \mathcal{E})^G$ and $\in \mathrm{Hom}(\mathcal{E} \overset{A}{\otimes} \mathcal{E}, \mathcal{E})^G$ are respectively symmetric and antisymmetric . Ex : G is a simple Lie group of the type B_n, C_n, D_n ; D is the adjoint representation ; then $\dim \mathrm{Hom}\,(\mathcal{E} \overset{A}{\otimes} \mathcal{E}, \mathcal{E})^G = \ell = 1$ and the corresponding algebra is the Lie algebra, while $\dim \mathrm{Hom}(\mathcal{E} \overset{S}{\otimes} \mathcal{E}, \mathcal{E})^G = d = 0$. For the Lie algebra of the type $A_n (n \geq 2)$ (e.g. $SU(n+1), SL(n+1)$), $d = 1$ and the corresponding symmetric algebra for $SU(3)$ is well known to physicists. In all physical examples we shall meet below $\ell \leq 1$, $d \leq 1$ so the corresponding algebra is unambiguously defined ; e.g. the symmetric algebra for the $(3, \bar{3}) \oplus (\bar{3}, 3)$ representation of $SU(3) \times SU(3)$ is studied in details in [17] . When we need to consider one of these algebra we shall denote by $_T$ the corresponding law i.e. :

$$f(a \otimes b) = a \,_T\, b \quad . \tag{6}$$

In quantum mechanics or quantum field theory, invariance under a group G leads to consider "Tensor Operators" [1)] , $T \in \mathrm{Hom}(\mathcal{E}, \mathcal{L}(\mathcal{H}))^G$ where $\mathcal{H}$ is the Hilbert space of states, $\mathcal{L}(\mathcal{H})$ the vector space of linear operators on $\mathcal{H}$, and $\mathcal{E}$ is a finite dimensional vector space on which acts a representation D of G . When D is irreducible, T is an irreducible tensor operators . $\mathcal{E}$ is called the variance of the tensor operator. Let $a, b \in \mathcal{E}$, $T_1, T_2 \in \mathrm{Hom}(\mathcal{E}, \mathcal{L}(\mathcal{H}))^G$, then

$$a \mapsto T_1(a) + T_2(a) \quad \text{is a tensor operator of variance } \mathcal{E} \tag{7}$$

---

1) Which are not operators on the Hilbert space of states ! but linear functions with operator value .

$$a \otimes b \mapsto T_1(a)\ T_2(b) \quad \text{is a tensor operator of variance } \mathcal{E} \otimes \mathcal{E}. \tag{8}$$

Any polynomial equation

$$P(T_1,\ldots,T_2) = 0 \tag{9}$$

involving tensor operators of the same variance $\mathcal{E}$ defines a polynomial equation in the tensorial algebra $\mathfrak{T}$ on $\mathcal{E}$ (i.e. $\mathfrak{T} = \bigoplus_{n=0}^{\infty} (\overset{n}{\otimes} \mathcal{E})$) and also an equation $\mathcal{P} = 0$ on the quotient $\mathfrak{U}$ of $\mathfrak{T}$ modulo the equivalence relations of the type (6) due to the G-invariant algebras. (When this algebra is the Lie algebra, $\mathfrak{U}$ is its envelloping algebra). Generally we want equation $P = 0$ to be satisfied for any $a \in \mathcal{E}$; however for the solutions of $\mathcal{P}(a) = 0$, $a \in \mathcal{E}$, equation $P = 0$ is trivially satisfied and this leads to symmetry breaking "along the direction" a and the symmetry group is reduced to the isotropy group G_a . For quadratic equations $P = 0$, corresponding for example to simple bootstrap models, and often, for higher degree equations, the idempotent $(\alpha \neq 0)$ or nilpotent $(\alpha = 0)$ of the algebra

$$a\, {}_T\, a = \alpha a \tag{10}$$

are solutions of $\mathcal{P}(a) = 0$.

This part 2.2. can be generalized to include multilinear algebras ; for example trilinear algebras with G as group of automorphisms are elements of $\mathrm{Hom}(\mathcal{E} \otimes \mathcal{E} \otimes \mathcal{E}\ ,\ \mathcal{E})^G$. In some physical applications discussed in ref [27] , idempotents and nilpotents of trilinear algebras appear

3.- APPLICATION TO PARTICLE PHYSICS.

Since the SU(3) symmetry for hadrons was recognized by Gell-Mann and Ne'eman, we know that the hypercharge Y , the electric charge Q , the Cabibbo direction of weak-current $C_{\pm} = C_1 \pm iC_2$, the weak hypercharge[1] Z define directions (i.e. unit vectors) in the octet space : the space of the adjoint representation of SU(3). It is to be noted ([18] and earlier papers quoted there) that the unit vectors y, q, z belong to the critical orbit represented by the poles of Figure 1 while c_1 and c_2 belong to the critical orbit of the equator. We also remark that y, q, z are idempotents and c_+ , c_- are nilpotents of the (complex) SU(3) invariant symmetrical algebra defined on the octet space [2)3)] .

These remarks extend naturaly to

$$G = (SU(3) \times SU(3)) \,\square\, (Z_2(P) \times Z_2(C)) \qquad (11)$$

where the charge conjugation C is an outer automorphism for each SU(3) factor while the parity operation P exchanges them (this is an interesting interplay between geometrical invariance in space time and internal

---

1) It is generally admitted that non-leptonic weak interaction is invariant under a U(2) group, corresponding to the Q-spin of Cabibbo's original paper.

2) Gell-Mann has denoted d_{ijk} the structure constants of this symmetrical algebra.

3) It is not clear that physicists should use only the compact form SU(3) and not its complexified form SL(3,C). Indeed the weak currents are not Hermitian; we also know that complexification of the Poincaré group has been fruitful for the study of analytic properties of field theory and CPT invariance. In the adjoint action of SL(3,C) the set $\mathcal{S}$ of semi-simple elements contains two orbits, one open dense in $\mathcal{S}$ and one exceptional which contains y, q, z ; the set $\mathfrak{n}$ of nilpotent elements also contains two orbits, one open dense in $\mathfrak{n}$ and the exceptional one which contains c_+, c_- [19]

hadronic symmetry). In the action of G on S is the unit sphere of the adjoint representation space, there are 12 layers ; five of them contains only critical orbits ; one contains y, q, a second contains z , a third contains c_+, c_- , a fourth contains what is likely to be the direction of the CP - breaking interaction in K^0 decay ; what is the role of the fifth one ! [18]

G-invariance is broken by semi-strong interaction in a different linear representation space ; this might be the $(3,\overline{3}) \oplus (\overline{3}, 3)$ representation 1) as first suggested in [20] . There are groups containing P and C whose connected component is SU(3) (the diagonal SU(3)) or $SU(2)\times SU(2)\times U(1)$: chiral invariance for pions). They correspond to each of the two types of critical orbits, and also respectively to an idempotent and to a nilpotent of the symmetric algebra [17,18]. However, in nature, semi-strong interactions break G up to the subgroup $H = (U(2) \square Z_2(C)) \times Z_2(P)$. (It is not the isotropy group of the open dense layer) 2). One can also extend the study of critical orbits and idempotents or nilpotents for the reducible representations interesting for the semi strong symmetry breaking. Pegoraro, Subba Rao [21] and then Darzens [22] have shown that the physically interesting schemes appear when H is the isotropy group of a critical orbit.

More recently, the group G has been extended to $(SU(4) \times SU(4)) \square (Z_2(P) \times Z_2(C))$ either for having a unified theory of weak and electromagnetic interactions [23] , to explain the absence of strangeness changing neutral currents [24] or to include the four basics leptons in the scheme [24, 25, 26] . In a recent preprint Mott [27] has shown that all physical directions of symmetry breaking in these papers correspond both to critical orbits and to idempotents or nilpotents of the involved symmetrical algebra. 3)

---

see foot notes page 13

CONCLUSION.

Isospin was introduced by Heisenberg [27] in 1932, immediatly after the discovery of the neutron. For the last forty years, "internal symmetry" have appeared richer and richer and the interplay of the different types of interactions with internal symmetry is fascinating. Of course I believe that complete understanding of "internal symmetry" breaking will require to solve the hard dynamical problems of the particle interactions. The concepts of 1) critical orbits, 2) idempotents and nilpotents of canonical algebra, seem very useful for the study of this subject. At least they show that many simplified models made to understand the subject do not shed much light on it, because their predictions are not at all specific but are mere consequences of general theorems. Finally we remark that in particle physics, the symmetry breaking is never maximal (i.e. symmetry is not broken up to the minimal isotropy group of the open dense layer). When the symmetry breaking occurs along a direction of critical orbit, it is minimal (isotropy groups of critical orbits are largest of any completely ordered chain of conjugated classes of isotropy group which contains them).

FOOT NOTES PAGE 12

1) Remark that the representation is irreducible in the real and it is the real canonical algebra which is consider here.

2) Indeed the minimal subgroup is $U(1) \times U(1)$.

3) Similarly, the semi-strong breaking, which is in a small strata, is near a critical orbit corresponding to chiral symmetry and to a nilpotent of a symmetric trilinear algebra (there is no bilinear algebra associated to the $(4,\bar{4}) \oplus (\bar{4},4)$ representation).

Note added on proof.

After this symposium has appeared a preprint (Scuola Normale Superiore, Pisa, Italy) from F. PEGORARO with the title "Three applications to SO(4) invariant systems of a theorem of L. Michel relating extremal points to invariance properties".

This author applies the concept of critical orbits to

1) the mixed linear Stark Zeeman effect in a hydrogen atom,

2) perturbations of a finite Robertson Walker metric in general relativity, 3) in hydrodynamics, to gas evolutions preserving angular momentum and vorticity.

REFERENCES

[1] R. THOM, These Proceedings

[2] Earliest references are B.W. LEE, Nucl. Phys. B9, 649 (1969),
B.W. LEE and J.L. GERVAIS, Nucl. Phys. B12, 627 (1969)
K. SYMANZIK, Comm. Math. Phys. 16 48, (1970) and for an early review
Lectures at the Cargese Summer School 1970 (included acknowledgements
to R. STORA) (GORDON and BREACH).

[3] J. GLIMM, These Proceedings ; A. JAFFE, ibid.

[4] An historical reference is R. PEIERLS, "On Ising model of Ferromagnetism"
Proc. Camb. Phil. Soc. 32, 477(1936). I am not competent for giving the
the large list of publications for the last ten years.

[5] Unpublished ; private communication of Iu. SUKHOV.

[6] S. DOPLICHER, D. KASTLER, D.W. ROBINSON, Comm. Math. Phys. 3, 1,
(1966).
R. HAAG, D. KASTLER, L. MICHEL, Marseille preprint (1969).

[7] D. KASTLER, G. LOUPIAS, M. MEBKHOUT, L. MICHEL, Comm. Math. Phys. 27, 195-222 (1972)
Pionnier works on the same subject are :
D. KASTLER, D.W. ROBINSON Comm. Math Phys. 3, 151 (1966)
D.W. ROBINSON and D. RUELLE, Ann. Inst. Henri Poincaré A6, 299 (1967)
D. RUELLE, J. Functional Analysis 6, 116 (1970).
S. DOPLICHER, D. KASTLER, E. STØRMER, J. Functional Analysis 3, 419 (1969).

[8] A. GUICHARDET and D. KASTLER, J. Math Pures et Appliquées 49, 349 (1970).

[9] L.D. LANDAU and E.M. LIFSCHITZ, statistical Physics § 136.
Quantum Mechanics § 136 .

[10] C. Ya LUBARSKII : Application of Group theory in Physics, Chap VII

[11] J.L. BIRMAN, IInd Colloquium on group Theory Methods in Physics
(Nijmegen 1973).

[12] P. CURIE, Journal de Physique 3ème série 3, 393 (1894).

[13] R.S. PALAIS, Memoirs Ann. Math. Sre n° 36, (1960) ,
D. MONTGOMMERY "Differential Analysies" n° 43, Bombay Colloquium (1964).

[14] R.S. PALAIS, Ann. J. Math. 92, 748 (1970).

[15] G.D. MOSTOW, Ann. Math 65. 513 and 432 (1957).

[16] L. MICHEL, C.R. Acad. Sc. Paris 272, 433 (1971) ; for more details nc: Proceeding 3rd GIFT Seminar in Theoretical Physics p. 49-131, Madrid (1972)

[17] L. MICHEL and L. RADICATI, "Breaking of the SU(3) x (SU(3) symmetry in Hadronic Physics" p. 191 "Evolution of particle physics" (dedicated to E. Amaldi) Acad. Press New York (1970).

[18] L. MICHEL and L. RADICATI, "Properties of the Breaking of Hadronic internal Symmetry" Ann. of Phys. 66, 758-783 (1971), and earlier reference quoted there.
The canonical symmetric algebra of the $(6,\bar{6}) \oplus (\bar{6},6)$ and the (8,8) are studied by A. CHAKRABARTY and C. DARZENS, Ann. Phys. 69,193 (1972).

[19] L. ABELLANAS, J. Math. Phys, 13, 1064 (1972).

[20] M. GELL-MANN, R.S. OAKES, B. RENNER, Phys. Rev. 175, 2195 (1968).

[21] PEGORARO and J. SUBBA RAO, Nucl. Phys. B44, 221 (1972).

[22] C. DARZENS, Ann. Phys. 76, 236 (1973).

[23] J.D. BJORKEN and C.H. LLEWELLYN SMITH, Phys. Rev. D7, 887 (1973).

[24] S.L. GLASHOW, J. ILIOPOULOS and L. MAIANI, Phys. Rev. D2, 1285 (1970).

[25] B.J. BJORKEN and S.L. GLASHOW, Phys. Lett. 11, 255 (1964).

[26] P. DITTNER, A. ELIEZER and T.K. KUO, Phys. Rev. Lett. 30, 1274 (1973).

[27] R.E. MOTT "Algebraic properties of SU(4) x SU(4) symmetry breaking" preprint, Queen Mary College London.

[28] W. HEISENBERG, S. Phys. 77, 1 (1932).