CONSTRAINTS ON SPIN ROTATION PARAMETERS DUE TO ISOSPIN CONSERVATION

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Abstract. We made a complete study of the relations between the three cross sections and the three sets of spin rotation parameters P,A,R for three reactions related by internal symmetry via two channels.

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1. The transition matrix T of a reaction involving spin 0 and spin $\frac{1}{2}$ particles :

$$0 + \tfrac{1}{2} \longrightarrow 0' + \tfrac{1}{2}' \, , \qquad (1)$$

can be written $f + ig\,\vec{\sigma}.\vec{n}$ where f and g are respectively the non-spin-flip and the spin-flip amplitudes. Such reactions (e.g. πN,KN) are going through two channels of isospin [1] ; hence, for three reactions which differ only by the isospin components of the particle isospin multiplets, the transition matrices satisfy a linear relation :

$$\Sigma_{\alpha=1}^{3} \gamma_\alpha T_\alpha = 0 \qquad (2)$$

where each γ_α is a homogeneous fourth degree polynomial of Clebsch-Gordan coefficients.

In this letter we derive all relations imposed by (2) on the cross-sections and on the spin rotation parameter A , P , R [2,3,4]. It is convenient to consider $\gamma_\alpha f_\alpha$ and $\gamma_\alpha g_\alpha$ as the components of an element $|\alpha>$ of a two dimensional Hilbert space. Then, denoting by σ_α the cross-section, one has ($\vec{\sigma}$ is the set of the three Pauli matrices)

$$M_\alpha = |\alpha><\alpha| = \frac{s_\alpha}{2}\,(1+\vec{\zeta}_\alpha.\vec{\sigma}) \qquad (3)$$

where

$$s_\alpha = <\alpha|\alpha> = \gamma_\alpha^2\,\sigma_\alpha > 0 \qquad (4)$$

and

$$\vec{\zeta}_\alpha = \frac{1}{s_\alpha} < \alpha|\vec{\sigma}|\alpha > = (A_\alpha, P_\alpha, R_\alpha) \qquad (5)$$

i.e. the components of $\vec{\zeta}$ are the spin rotation parameters of the reaction α . They satisfy

$$\vec{\zeta}_\alpha^{\,2} = 1 = A_\alpha^2 + P_\alpha^2 + R_\alpha^2 \, . \qquad (6)$$

The vector $\vec{\zeta}$ will be called the <u>spin rotation vector</u>.

2. From now on, the three indices α, β, γ represent any permutation of 1, 2, 3 . The linear relation on the vectors $|\alpha>$, corresponding to Eq. (2), is

$$|\alpha> + |\beta> + |\gamma> = 0 \tag{7}$$

Each $|\alpha>$ with spin rotation vector $\vec{\zeta}_\alpha$ has an orthogonal element $|\alpha^\perp>$ with same s_α and spin rotation vector $-\vec{\zeta}_\alpha$. The scalar product of equation (7) with $<\alpha^\perp|$ gives

$$<\alpha^\perp|\beta> = -<\alpha^\perp|\gamma> \tag{8}$$

from

$$|<\alpha^\perp|\beta>|^2 = \mathrm{tr}\, M_{\alpha^\perp} M_\beta = \mathrm{tr}\, M_\alpha M_{\beta^\perp} = |<\alpha|\beta^\perp>|^2 \tag{9}$$

and from (8) we obtain

$$2\mathrm{tr}\, M_{\alpha^\perp} M_\beta = s_\alpha s_\beta (1 - \vec{\zeta}_\alpha . \vec{\zeta}_\beta) = \tfrac{1}{2} H \geq 0 \tag{10}$$

where H is a constant independent of α, β, γ . Since the $\vec{\zeta}$ have unit length, we can write :

$$0 \leq (\vec{\zeta}_\alpha \times \vec{\zeta}_\beta . \vec{\zeta}_\gamma)^2 \leq 1 \quad ; \tag{11}$$

with the use of (10), Eq. (11) is equivalent to :

$$0 \leq H \leq -\Delta(s_\alpha, s_\beta, s_\gamma) \leq 4(s_\alpha s_\beta s_\gamma)^2 H^{-2} + H \tag{12}$$

where

$$\Delta(s_\alpha, s_\beta, s_\gamma) \equiv s_\alpha^2 + s_\beta^2 + s_\gamma^2 - 2s_\alpha s_\beta - 2s_\beta s_\gamma - 2s_\gamma s_\alpha \tag{13}$$

The last inequality in (12) is always satisfied because $s_\alpha > 0$ and $H \geq 0$; the equality holds only when $s_\alpha = s_\beta = s_\gamma$. We denote by $\theta_{\alpha\beta}$ the angle between $\vec{\zeta}_\alpha$ and $\vec{\zeta}_\beta$; so

$$0 \leq \theta_{\alpha\beta} \leq \pi \quad ; \quad \cos\theta_{\alpha\beta} = \vec{\zeta}_\alpha . \vec{\zeta}_\beta \quad . \tag{14}$$

In the following we will say that $\vec{\zeta}$ is described equivalently by a unit vector or a point on the unit sphere.

3. Equations (10) and (12) allow to study the following experimental situations.

(i) One knows only s_α, s_β.

So $0 \le H \le 4\, s_\alpha s_\beta$ and from (12) the cross sections $\sigma_\alpha = s_\alpha \gamma_\alpha^{-2}$ must satisfy :

$$\Delta(s_\alpha, s_\beta, s_\gamma) \le 0 \qquad (14)$$

This is the well-known condition that the three $\sqrt{s_\alpha}$ must form a triangle. This condition gives the bounds for s_γ :

$$|s_\gamma - s_\alpha - s_\beta| \le 2\sqrt{s_\alpha s_\beta} \qquad (14')$$

(ii) One knows $s_\alpha, s_\beta, \vec{\zeta}_\alpha, \vec{\zeta}_\beta$.

Better bounds on s_γ are given by (12) : $-\Delta \ge H$; they are

$$|s_\gamma - s_\alpha - s_\beta| \le 2\sqrt{s_\alpha s_\beta (1 + \vec{\zeta}_\alpha \cdot \vec{\zeta}_\beta)/2} = 2\sqrt{s_\alpha s_\beta}\ \cos \tfrac{1}{2}\theta_{\alpha\beta} \qquad (15)$$

This condition (15) is always stricter than condition (14'), except in the case $\vec{\zeta}_\alpha = \vec{\zeta}_\beta$; then $H = 0$ and $\vec{\zeta}_\gamma = \vec{\zeta}_\alpha = \vec{\zeta}_\beta$; this happens when the transition matrix of one of the two isospin channels vanishes.

Equation (15) can also be written in the two equivalent forms

$$|\cos \omega_{\alpha\beta}| \le \cos \tfrac{1}{2}\theta_{\alpha\beta}\ , \qquad (15')$$

$$0 \le \tfrac{1}{2}\theta_{\alpha\beta} \le \omega_{\alpha\beta} \le \pi - \tfrac{1}{2}\theta_{\alpha\beta}\ , \qquad (15'')$$

where $\omega_{\alpha\beta}$ is the angle between the sides $\sqrt{s_\alpha}, \sqrt{s_\beta}$ of the triangle defined by (14).

(iii) One knows $s_\alpha, s_\beta, s_\gamma$ satisfying (14) and $\vec{\zeta}_\alpha$.

The equations (15) give the domain of $\vec{\zeta}_\beta$; it is, on the unit sphere, a circular portion whose center is $\vec{\zeta}_\alpha$; its aperture is $\theta_{\alpha\beta}$ such that

$$0 \leq \theta_{\alpha\beta} \leq \mathrm{Min}(2\omega_{\alpha\beta}\, , \, 2(\pi-\omega_{\alpha\beta})) \tag{16}$$

Note that there is no restriction on $\theta_{\alpha\beta}$ when $\omega_{\alpha\beta} = \frac{\pi}{2}$.

(iv) <u>One knows</u> $s_\alpha s_\beta s_\gamma$ <u>satisfying</u> (14) <u>and</u> $\vec{\zeta}_\alpha, \vec{\zeta}_\beta$ <u>satisfying</u> (15).

The point on the unit sphere which defines $\vec{\zeta}_\gamma$ must be, according to Eq. (10), at the intersection of the two circles whose centers and apertures are :

$$\vec{\zeta}_\alpha \quad , \quad \theta_{\alpha\gamma} = \cos^{-1}(1 - (s_\beta/s_\gamma)(1 - \vec{\zeta}_\alpha.\vec{\zeta}_\beta)) \tag{17a}$$

$$\vec{\zeta}_\beta \quad , \quad \theta_{\beta\gamma} = \cos^{-1}(1 - (s_\alpha/s_\gamma)(1 - \vec{\zeta}_\alpha.\vec{\zeta}_\beta)) \tag{17b}$$

That these two circles intersect is a consequence of (14) and (15). In general, they have two common points, representing two distinct values of $\vec{\zeta}_\gamma$.

These two values become equal when the equalities hold in Eq. (15) and (15').

There are two exceptional cases when the two circles coïncide ; this happens when they have same axis i.e. $\vec{\zeta}_\alpha = \pm \vec{\zeta}_\beta$. Equation (10) shows that when $\vec{\zeta}_\alpha = \vec{\zeta}_\beta$ the two circles reduce to one point i.e. $\vec{\zeta}_\alpha = \vec{\zeta}_\beta = \vec{\zeta}_\gamma$. When $\vec{\zeta}_\alpha + \vec{\zeta}_\beta = 0$, Eq. (15) reads $s_\gamma = s_\alpha + s_\beta$ which, with Eq. (17), yields

$$\cos\,\theta_{\alpha\beta} = \frac{s_\alpha - s_\beta}{s_\alpha + s_\beta} = -\cos\,\theta_{\beta\gamma} \tag{18}$$

This completely defines the common circle.

4. For each one of the three reactions, the measurement of the cross section and of the spin rotation parameters determine the scattering amplitude f_α and g_α , up to an unobservable common phase factor $e^{i\varphi_\alpha}$:

$$\begin{aligned} \gamma_\alpha f_\alpha &= e^{i\varphi_\alpha}\, [\tfrac{1}{2}\, s_\alpha(1+R_\alpha)]^{\frac{1}{2}} \\ \gamma_\alpha g_\alpha &= e^{i\varphi_\alpha}\, [\tfrac{1}{2}\, s_\alpha(1-R_\alpha)]^{\frac{1}{2}}\, e^{i\chi_\alpha} \end{aligned} \tag{19}$$

with $$\chi_\alpha = \tan^{-1}(-P_\alpha/R_\alpha) \tag{20}$$

We consider the angles $\varphi_{\alpha\beta}$ defined by

$$e^{i\varphi_{\alpha\beta}} = \frac{<\alpha|\beta>}{|<\alpha|\beta>|} \tag{21}$$

They satisfy the relations

$$\varphi_{\alpha\beta} + \varphi_{\beta\alpha} = 0 , \tag{22a}$$

$$e^{i(\varphi_{\alpha\beta}+\varphi_{\beta\gamma}+\varphi_{\gamma\alpha})} = \frac{1+\vec{\zeta}_\alpha\cdot\vec{\zeta}_\beta + \vec{\zeta}_\beta\cdot\vec{\zeta}_\gamma + \vec{\zeta}_\gamma\cdot\vec{\zeta}_\alpha + i(\vec{\zeta}_\alpha \times \vec{\zeta}_\beta\cdot\vec{\zeta}_\gamma)}{[2(1+\vec{\zeta}_\alpha\cdot\vec{\zeta}_\beta)(1+\vec{\zeta}_\beta\cdot\vec{\zeta}_\gamma)(1+\vec{\zeta}_\gamma\cdot\vec{\zeta}_\alpha)]^{\frac{1}{2}}} \tag{22b}$$

and they are related to the relative phases $\varphi_\beta-\varphi_\alpha$ of the amplitudes by

$$\varphi_{\alpha\beta} = \varphi_\beta - \varphi_\alpha + \mathrm{Arg}\left[1+\sqrt{\frac{(1-R_\alpha)(1-R_\beta)}{(1+R_\alpha)(1+R_\beta)}}\; e^{i(\chi_\beta-\chi_\alpha)}\right] \tag{23}$$

Let us show that by using the isospin conservation condition (3) one can determine the angles $\varphi_{\alpha\beta}$ and hence that the phases between the amplitudes of different reactions are observable. Equation (7) can be written

$$- |\gamma> = |\alpha> + |\beta> \quad .$$

Multiplying this equation by its Hermitian conjugate we obtain, when [5] $\mathrm{tr}\, M_\alpha M_\beta \neq 0$:

$$M_\gamma = M_\alpha + M_\beta + (X_{\alpha\beta})^{-\frac{1}{2}}(M_\alpha M_\beta\, e^{i\varphi_{\alpha\beta}} + M_\beta M_\alpha\, e^{i\varphi_{\beta\alpha}}) \tag{24}$$

with $$X_{\alpha\beta} = \mathrm{tr}(M_\alpha M_\beta) = \tfrac{1}{2}\, s_\alpha s_\beta(1+\vec{\zeta}_\alpha\cdot\vec{\zeta}_\beta) \tag{25}$$

Using the explicit form (3) of M_α, the trace of (23) yields

$$- \cos\omega_{\alpha\beta} = \cos\tfrac{1}{2}\theta_{\alpha\beta}\cos\varphi_{\alpha\beta} \tag{26}$$

and the trace with $\vec{\sigma}$ gives the vector relation

$$s_\gamma \vec{\zeta}_\gamma = s_\alpha \vec{\zeta}_\alpha + s_\beta \vec{\zeta}_\gamma + 2(s_\alpha s_\beta)^{\frac{1}{2}} \cos \varphi_{\alpha\beta}\, \hat{\ell}_{\alpha\beta} + (H)^{\frac{1}{2}} \sin \varphi_{\alpha\beta}\, \hat{k}_{\alpha\beta} \qquad (27)$$

where H is defined in (10) and

$$\hat{\ell}_{\alpha\beta} = \frac{\vec{\zeta}_\alpha + \vec{\zeta}_\beta}{|\vec{\zeta}_\alpha + \vec{\zeta}_\beta|} \quad , \quad \hat{k}_{\alpha\beta} = \frac{\vec{\zeta}_\alpha \times \vec{\zeta}_\beta}{|\vec{\zeta}_\alpha \times \vec{\zeta}_\beta|} \qquad (28)$$

If the cross section $s_\alpha, s_\beta, s_\gamma$ and the spin rotation vectors $\vec{\zeta}_\alpha, \vec{\zeta}_\beta$ are known, Eq.(26) allows to determine $\cos \varphi_{\alpha\beta}$. If furthermore $\vec{\zeta}_\gamma$ is known, Eq.(27) yields the sign of $\sin \varphi_{\alpha\beta}$; indeed the scalar product of (27) with $\hat{k}_{\alpha\beta}$ gives

$$\text{sign}(\sin \varphi_{\alpha\beta}) = \text{sign}(\vec{\zeta}_\alpha \times \vec{\zeta}_\beta \cdot \vec{\zeta}_\gamma) \qquad (29)$$

Note that all solutions to the problems settled in section 3 can be obtained from Eq.(26) and (27). For instance, if one knows $s_\alpha, s_\beta, \vec{\zeta}_\alpha$ and $\vec{\zeta}_\beta$, these equations show that the values of s_γ and ζ_γ depend only on one parameter which is the angle $\varphi_{\alpha\beta}$.

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FOOTNOTES

1. Those reactions commonly go also through two channels of U-spin and V spin, and in some cases, such as π^+p^+, through two channels of the full unitary spin. The considerations of this letter can be extended to these invariances and to the cases when several 0-spin particles are produced or when the 0-spin particles are replaced by unpolarized particles.

2. L. MICHEL solved this problem for P alone : N. Cim. 22, 203 (1961) and "1965-Brandeis Summer Institute in Theoretical Physics" Vol. I p.347. Gordon and Breach, New York 1966. R.J.N. PHILLIPS (N. Cim. 26, 103 (1962) remarked that this work can be extended to A or R separately and M. KORKEA-AHO and N. TÖRNQVIST, preprint Helsinki 1971, remarked that it is also valid for any linear combination $\alpha P + \beta A + \gamma R$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$.

3. Some comparisons of results of ref. 2 with data have been made by O. KAMEI and S. SASAKI, N. Cim. 1969 59 (535) and by G.V. DASS, J. FROYLAND, F. HALZEN, A. MARTIN, C. MICHAEL and S.M. ROY, Phys. Lett. 36B, 339 (1971).

4. F. HALZEN and C. MICHAEL Phys. Lett. 36 B, 367 (1971) study a case of partial information on P and R only and make a comparison with data. Although we do not treat this case implicitely it is implicitely contained in this letter. A complete study for arbitrary spin of the relations between internal symmetry and polarization will appear as a forthcoming issue of our work "Polarization density matrix".

5. When $\operatorname{tr} R_\alpha R_\beta = |\langle\alpha|\beta\rangle|^2 = 0$ then, from (7), $\langle\alpha|\gamma\rangle = -\langle\alpha|\alpha\rangle \neq 0$ and $\langle\beta|\gamma\rangle = -\langle\beta|\beta\rangle \neq 0$, so $\varphi_{\alpha\gamma} = \varphi_{\gamma\alpha} = \pi = \varphi_{\beta\gamma} = \varphi_{\gamma\beta}$. This exceptional case was already met in § 3 (iv). For the pairs β,γ or γ,α of indices, (26) gives $s_\gamma = s_\alpha + s_\beta$ and (27) gives the circle of solutions for $\vec{\zeta}_\gamma$. The angle $\varphi_{\alpha\beta}$ parametrizes this circle.

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