TOPOLOGICAL INVARIANTS OF DYNAMICAL SYSTEMS AND SPACES OF HOLOMORPHIC MAPS

Part I

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§ 0. Introduction

0.1. From $\underline{X}$ to $\underline{X}^\Gamma$. Start from some category of spaces $\underline{X}$ and maps between them. These can be bare sets with no additional structure and all maps, topological spaces and continuous maps, smooth manifolds, algebraic varieties, linear or affine spaces, etc. Then, given a group Γ, we have a functionally defined Γ-*space* X, i.e. a space with a Γ-action, namely the Cartesian power $\underline{X}^\Gamma$ thought of as the space of $\underline{X}$-valued functions on Γ. Here the action of Γ on $\underline{X}^\Gamma$ is induced by the left action Γ on Γ,

$$\gamma' x(\gamma) = x(\gamma'\gamma) .$$

This action is called the *shift* and $\underline{X}^\Gamma$ is called the *(full) shift space over* Γ *with the alphabet* $\underline{X}$, where the basic example is $\Gamma = \mathbb{Z}$ and $\underline{X}$ consisting of finitely many elements, called *letters*.

0.1.A. Maps of finite type. There are by far more Γ-*maps*, i.e. Γ-equivariant maps $\underline{X}^\Gamma \to \underline{Y}^\Gamma$, than those coming from maps $\underline{X} \to \underline{Y}$ if our category admits *finite* Cartesian products. In fact, every map φ from the finite Cartesian power $\underbrace{\underline{X} \times \underline{X} \times \ldots \times \underline{X}}_{d}$ to $\underline{Y}$ defines a Γ-map $\underline{X}^\Gamma \to \underline{Y}^\Gamma$ determined by the choice of a finite subset $D = \{\delta_1, \delta_2, \ldots, \delta_d\} \subset \Gamma$ as follows: each function $x(\gamma)$ goes to $y(\gamma)$, $\gamma \in \Gamma$, by the usual "finite difference operator" recipe,

$$y(\gamma) = \varphi\left(x(\gamma\delta_1),\ x(\gamma\delta_2), \ldots, x(\gamma\delta_d)\right) .$$

In other words, $y(\gamma)$ for each $\gamma \in \Gamma$ is determined by the value of φ on the restriction of x to the γ-translate of D,

$$y(\gamma) = \varphi(x|\gamma D) ,$$

where φ is interpreted as a map from $\underline{X}^{\gamma D} = \underline{X}^d$ to Y. We say in this case that our map $x \mapsto y$ is *based on* D and/or *defined by* φ.

Notice that this construction can be used as a *definition* of Γ-morphisms f over categories of certain spaces $\underline{X}$, e.g. for algebraic varieties. But for the topological category there are additional *continuous* Γ-maps $\underline{X}^\Gamma \to \underline{Y}^\Gamma$, not coming by the way of $\varphi : \underline{X}^D \to \underline{Y}$. In fact, every continuous map $\varphi : \underline{X}^\Gamma \to \underline{Y}$ (which may essentially depend on infinitely many $x(\gamma)$, $\gamma \in \Gamma$) defines $f = f_\varphi : \underline{X}^\Gamma \to \underline{Y}^\Gamma$ by the same rule $f : x \to y$ for $y(\gamma) = \varphi(\gamma x)$.

0.2. Subspaces in $\underline{X}^\Gamma$. The simplest Γ-invariant subset in $\underline{X}^\Gamma$ consists of the *fixed point set* $\operatorname{Fix}\Gamma \subset \underline{X}^\Gamma$ which obviously identifies with $\underline{X}$ itself realized by construct maps $\Gamma \to \underline{X}$.

More interesting subspaces in $X = \underline{X}^\Gamma$ appear as pull-backs of fixed points in Y by Γ-maps $f : X \to Y$. One can think of such a subspace $X_0 = f^{-1}(y_0)$, $y_0 \in Y$, as the set of solutions to the "difference" equation $f(x) = y_0$ and if $f = f_\varphi$ for $\varphi : \underline{X}^D \to \underline{Y}$ with $D = \{\delta_1, \dots, \delta_d\}$ as earlier, then this equation turns into the following system of algebraic equations denoted (φ_γ), $\gamma \in \Gamma$,

$$\varphi\left(x(\gamma\delta_1),\ x(\gamma\delta_2), \dots, x(\gamma\delta_d)\right) = \underline{y}_0 \in \underline{Y}\,. \tag{φ_γ}$$

In fact, one can drop $\underline{Y}$ from this definition and start with an arbitrary subset $L \subset \underline{X}^D = \underline{X}^d$, $d = \operatorname{card} D$ (corresponding to $\varphi^{-1}(\underline{y}_0)$ in the previous setting). Then $X_0 = X_0(L) \subset \underline{X}^\Gamma$ is defined as the space of functions $x : \Gamma \to \underline{X}$ such that the restriction of x to each translate γD is contained in L, where we identify $\underline{X}^{\gamma D}$ with $\underline{X}^D$ via the correspondence $\gamma\delta \leftrightarrow \delta$, $\delta \in D$, and where we view $\underline{X}^{\gamma D}$ as the space of functions $\gamma D \to \underline{X}$. These $X_0 = X_0(L) \subset \underline{X}^\Gamma$ are called *subshifts of finite type* in $\underline{X}^\Gamma$ (where "finite" refers to finiteness of $D \subset \Gamma$) and L is regarded as a "*law*" distinguishing "*legal*" function on Γ.

0.2.A. Remark on quotient spaces. Besides taking subspaces one may consider various Γ-equivariant quotient spaces of $\underline{X}^\Gamma$ and of the above $X_0 \subset \underline{X}^\Gamma$ where the most attractive ones are defined by Γ-invariant equivalence relations on $\underline{X}^\Gamma$ (or on $\underline{X}_0 \subset \underline{X}^\Gamma$) of *finite type*. The simplest example is the quotient space $\underline{X}^\Gamma / \operatorname{Fix}\Gamma$, where the fixed point set $\operatorname{Fix}\Gamma \subset \underline{X}^\Gamma$ is shrunk to a single (fixed) point. Finding more interesting Γ-equivariant equivalence relations

$$R \subset X_0 \times X_0 \subset \underline{X}^\Gamma \times \underline{X}^\Gamma = (\underline{X} \times \underline{X})^\Gamma$$

of finite type is a non-trivial matter which we shall not discuss at this stage.

0.3. From geometry to dynamics. Given our category of spaces $\underline{X}$, take an invariant (property, theory) in this category and try to extend it to a class of Γ-spaces including $\underline{X}^\Gamma$ and subshifts of finite type in X^Γ. Our extension must satisfy

$$\operatorname{Inv}_\Gamma \underline{X}^\Gamma = \operatorname{Inv} \underline{X}$$

and the essential (formal functorial) properties of $\operatorname{Inv}_\Gamma$ must be similar to those of Inv. Besides, we want our new invariant $\operatorname{Inv}_\Gamma$ to be "dynamical" which expresses a vague idea of Inv_Γ depending on the overall behavior of the Γ-orbits. For example, we wish

$$\mathrm{Inv}_\Gamma(\underline{X}^\Gamma / \operatorname{Fix}\Gamma) = \mathrm{Inv}_\Gamma \underline{X}^\Gamma ,$$

so that the "few" fixed points of Γ should not matter.

Here is a specific example indicating what we have in mind.

(A) Embedding problem. Let X and Y be topological spaces where we have a non-trivial obstruction for the existence of a topological embedding $X \to Y$, e.g. $S^1 \not\subset \mathbb{R}^1$ or $\mathbb{R}P^2 \not\subset \mathbb{R}^3$. Does this obstruction translate to a dynamical language and yield a non-embedding result for the Γ-spaces $\underline{X}^\Gamma$ and $\underline{Y}^\Gamma$ with their respective product topologies and the shift actions of Γ?

Of course, every Γ-embedding $\underline{X}^\Gamma \to \underline{Y}^\Gamma$ automatically embeds $\underline{X} = \mathrm{Fix}_\Gamma \subset \underline{X}^\Gamma$ to $\underline{Y}^\Gamma = \mathrm{Fix}_\Gamma \subset \underline{Y}^\Gamma$ and so we have a trivial "yes" to our question. But if we take $X_\bullet = \underline{X}^\Gamma / \mathrm{Fix}_\Gamma$ and $Y_\bullet = \underline{Y}^\Gamma / \mathrm{Fix}_\Gamma$, then the non-existence of a Γ-equivariant embedding $X_\bullet \to Y_\bullet$ does not (seem to) immediately follow from what we know for maps $\underline{X} \to \underline{Y}$. And, truly, what we want to show is that every continuous Γ-map $\underline{X}^\Gamma \to \underline{Y}^\Gamma$ identifies "many" pairs of points (and thus of Γ-orbits) in $\underline{X}^\Gamma$.

(A′) Subexample. Let $\Gamma = \mathbb{Z}$ and observe that every $\mathbb{Z}$-embedding between $\mathbb{Z}$-spaces, say $f : X \to Y$, sends the *periodic points* of X to those of Y, i.e. the subset $\mathrm{Per}_n(X) \overset{def}{=} \mathrm{Fix}(n\mathbb{Z}) \subset X$ goes to $\mathrm{Per}_n(Y) = \mathrm{Fix}(n\mathbb{Z}) \subset Y$ for each $n \in \mathbb{N}$ where, obviously, *all* periodic points are dense in $\underline{X}^\mathbb{Z}$. In particular, there is no Γ-embedding from $X_\bullet = \underline{X}^\Gamma / \operatorname{Fix}\Gamma$ to $Y_\bullet = Y^\Gamma / \operatorname{Fix}\Gamma$. All this is obvious and trivially extend to all *residually finite* (see 1.3) groups Γ but more general Γ provide many challenging problems as we shall see later on.

(B). Among Γ-embeddings $X \to Y$ one distinguishes Γ-*homeomorphisms* and, actually, when we speak of Γ-invariants one means invariance under Γ-homeomorphisms. Here one may have an extra structure on our spaces (e.g. a measure, symplectic structure, complex structure etc.) and one wishes to study (groups of) Γ-homeomorphisms preserving such a structure.

(C). The Γ-topology can be naturally relaxed to Γ-*homotopy* with many standard invariants (such as homology) passing from $\underline{X}$ to $X = \underline{X}^\Gamma$. For example, the global homological dimension becomes $\operatorname{Fildim}(X:\Gamma)$ in this framework (see 1.1.F).

0.4. Mean entropy and mean dimensions. The simplest non-embedding theorem is the *pigeon hole principle*: there is no embedding $\underline{X} \to \underline{Y}$ if $\operatorname{card}\underline{X} > \operatorname{card}\underline{Y}$ for finite sets $\underline{X}$ and $\underline{Y}$. The dynamical version of the cardinality, or rather of the entropy $\overset{def}{=} \log(\text{cardinality})$, is the *mean topological entropy* defined for arbitrary compact (and sometimes non-compact) topological Γ-*spaces* X (i.e. with continuous actions of groups Γ) denoted $\operatorname{ent}(X:\Gamma)$ (see 1.7 for definition). If Γ is an *amenable* group (see 1.3) then, not surprisingly,

$$\operatorname{ent}(\underline{X}^\Gamma:\Gamma) = \operatorname{ent}\underline{X}^\Gamma/\operatorname{Fix}\Gamma = \operatorname{ent}\underline{X} \qquad (\text{ent} =)$$

for all finite sets $\underline{X}$. This is a common knowledge. (Probably, something like this must be true for sets $\underline{X}$ of *infinite cardinality* where the interesting Γ's are those with $\operatorname{card}\Gamma > \operatorname{card}\underline{X}$.) Also, one knows that

$$\operatorname{ent}(X:\Gamma) \le \operatorname{ent}(Y:\Gamma) \qquad (\text{ent} \le)$$

if X admits a topological Γ-embedding to Y or, more generally, if there is a *finite-to-one* Γ-map $f: X \to Y$, (i.e. $\operatorname{card} f^{-1}(y) < \infty$ for all $y \in Y$). It follows that there is no finite-to-one (not even countable to one) map $f: \underline{X}^\Gamma \to \underline{Y}^\Gamma$ if $\operatorname{card}\underline{X} > \operatorname{card}\underline{Y}$ and the group Γ is amenable. (It is clear for all Γ that there is no Γ-embedding $f: \underline{X}^\Gamma \to \underline{Y}^\Gamma$ as this would embed $\underline{X} = \operatorname{Fix}_\Gamma \subset \underline{X}^\Gamma$ to $\underline{Y} = \operatorname{Fix}_\Gamma \subset \underline{Y}^\Gamma$ but I do not see how to exclude Γ-embeddings $\underline{X}^\Gamma/\operatorname{Fix}\Gamma \to \underline{Y}^\Gamma/\operatorname{Fix}^\Gamma$ for general groups Γ.)

Now, let us replace the cardinality by the *topological dimension* of underlying space $\underline{X}$ which we assume at the moment being a compact metric space with *finite* topological dimension. One can mimic the way one goes from $\operatorname{ent}\underline{X}$ to $\operatorname{ent}(X:\Gamma)$ and define the *mean dimension* $\dim(X:\Gamma)$ in the spirit of Lebesgue (with Lebesgue number of an ε-covering replacing log card (covering) appearing with the entropy) for all topological Γ-spaces (see 1.5). Here again

$$\dim(\underline{X}^\Gamma:\Gamma) = \dim(\underline{X}^\Gamma/\operatorname{Fix}\Gamma:\Gamma) = \dim\underline{X} \qquad (\dim =)$$

for most reasonable (see 1.1.E) spaces $\underline{X}$ and amenable groups Γ. Furthermore,

$$\dim(X:\Gamma) \le \dim(Y:\Gamma) \qquad (\dim \le)$$

if X admits a Γ-embedding to Y or, more generally, a Γ-map $f: X \to Y$ with $\dim f^{-1}(y) \le d < \infty$ for all $y \in Y$. This leads to non-existence of such a map f from $\underline{X}^\Gamma$ to $\underline{Y}^\Gamma$ if $\dim X > \dim Y$ and Γ is amenable. Also, one sees this way that if $\dim \underline{X} > \dim \underline{Y}$, then $\underline{X}^\Gamma / \operatorname{Fix}\Gamma$ does not Γ-embed to $\underline{Y}^\Gamma / \operatorname{Fix}_\Gamma$ for amenable groups Γ where the case of general Γ remains unclear.

0.5. Smooth subshifts of finite type. Let $\underline{X}$ be a smooth manifold and $L \subset \underline{X}^D$, $D \subset \Gamma$, a smooth submanifold, or more generally, a stratified subset, e.g. an analytic subvariety in $\underline{X}^D$. One thinks of such L as the zero set of $r = \operatorname{codim} L$ (sufficiently generic) equations $\varphi_j(x_1, \dots, x_d) = 0$, $d = \operatorname{card} D$, $j = 1, \dots, r$ and then $X_0 = X_0(L)$ is given by r Γ-invariant systems of equations. So the *expected* mean dimension of this X_0 is

$$\dim(X_0 : \Gamma) \stackrel{?}{=} \dim \underline{X} - \operatorname{codim} L\,. \tag{?}$$

Problem. Find specific sufficient conditions on L which would guarantee the above equality.

Example. Let $\underline{X}$ be the complex projective space $\mathbb{C}P^n$ and $L \subset (\mathbb{C}P^n)^D = (\mathbb{C}P^n)^d$, where $D = \{\delta_1, \dots, \delta_d\} \subset \Gamma$, be a complex algebraic subvariety. We shall show in 2.6.F (using positivity of the cycle represented by L) that

$$\dim(X_0(L) : \Gamma) \geq \dim \underline{X} - \operatorname{codim} L \tag{$\geq$}$$

for all L. Then we prove that the equality holds for (suitably understood) *generic* L.

Remark. Evaluating $\dim(X_0 : \Gamma)$ and, in particular, verifying (?) is not a trivial matter even for $\underline{X} = \mathbb{R}^s$ and *linear* laws $L \subset (\mathbb{R}^s)^D$ since the γ-translates of the linear equations

$$\varphi_j\,(x(\delta_1), \dots, x(\delta_d)) = 0,\ j = 1, \dots, r\,,$$

may develop unexpected linear relations. These are easy to control for such groups as $\Gamma = \mathbb{Z}$ for instance and, up to some extent, for more general *unique product* groups (see 2.2.B) but the general case seems rather subtle.

0.6. Spaces of harmonic maps and minimal varieties. The most interesting spaces from our point of view appear as solutions of elliptic differential equations over manifolds V with groups Γ acting on V. A basic example is the space of *harmonic maps* $V \to \underline{X}$ between Riemannian manifolds V and $\underline{X}$, where V is noncompact, $\underline{X}$ is compact and where V comes along with an isometry group Γ, such that V/Γ is compact. For instance, one may take $V = \mathbb{R}^n$, where Γ is either taken to be all $\mathbb{R}^n$ or some lattice $\Lambda \subset \mathbb{R}^n$.

The full space of the harmonic map $V \to \underline{X}$ is too big and usually has infinite mean dimension but it has interesting Γ-invariant subspaces where the mean dimension is finite and, sometimes, different from zero. A particular space of this kind, denoted X_c, is distinguished by the pointwise bound on the differential of such maps $x : V \to \underline{X}$, namely

$$\|Dx\| \leq c < \infty\,. \tag*{$(*)_c$}$$

0.6.A. Upper bound on the mean dimension of X_c (see 3.4). *If Γ is amenable then*

$$\dim(X_c : \Gamma) < \infty \tag*{$(\star)$}$$

for all $c \geq 0$. Furthermore, if c is sufficiently large, $c \geq c_0(V, X)$, then

$$\dim(X_c : \Gamma) \leq bc^n \tag*{$(\star)_\infty$}$$

for $n = \dim V$ and some constant $b = b(V, \underline{X}, \Gamma)$, where $b = a(V, \underline{X})\operatorname{vol}(V/\Gamma)$ for discrete groups Γ. Moreover,

$$\dim(X_c : \Gamma) \to 0 \quad \text{for } c \to 0\,. \tag*{$(\star)_\circ$}$$

Remark. If $V = \mathbb{R}^n$, then $(\star)_\infty$ holds true for *all* $c \geq 0$ as follows by an obvious scaling argument. Probably, this remains valid for non-flat metrics on $\mathbb{R}^n$ invariant under $\mathbb{Z}^n$, but, in general, the asymptotics of $\dim(X_c : \Gamma)$ for $c \to 0$ should depend on the growth rate of the group Γ.

0.6.B. Non-vanishing of $\dim(X_c : \Gamma)$ and instantons. We shall prove in 3.6 the following

Theorem. *Let V be a complex manifold, where an amenable group Γ acts discretely by complex analytic transformations, such that the quotient V/Γ is a projective algebraic variety. Then the space X_c of complex analytic maps $x : V \to \mathbb{C}P^N$ with $\|Dx\| \leq c$ satisfies for all $N \geq \dim_{\mathbb{C}} V$, and all $c \geq c_0 = c_0(V, \underline{X}) > 0$,*

$$\dim(X_c : \Gamma) \geq b'c^n\,, \tag*{$(\star\star)$}$$

for $n = \dim_{\mathbb{R}} V$ and some positive constant $b' = b'(V, \underline{X}, \Gamma)$, which is of the form $a'(V, \underline{X})\operatorname{vol}(V/\Gamma)$ for discrete Γ.

Remarks. (a) If V is Kähler, then holomorphic maps are harmonic and so $\dim_c(X:\Gamma)$ is also bounded from above according to $(\star)_\infty$ and $(\star)_0$. In fact, these bounds remain valid without V being Kähler as we shall see in 3.4.

(b) It seems that the strict inequality $\dim(X_c:\Gamma)>0$ manifesting the abundance of our maps $V\to\underline{X}$ is intimately linked to the *bubbling phenomenon*, i.e. the presence of *instantons*, highly localized solutions of our elliptic equations. Here is a specific

0.6.B′. Conjecture. Let $\underline{X}$ be a complex projective manifold and look at the space X_c of holomorphic maps $x:\mathbb{C}\to\underline{X}$ with derivatives bounded by some $c>0$. *Then* $\dim(X_c:\mathbb{C})>0$, *if and only if* $\underline{X}$ *contains a rational curve.*

Here (for holomorphic maps of $\mathbb{C}$) the "if" part of the conjecture follows from above and also can be derived by a simple interpolation argument. On the other hand the "only if " claim (which parallels Lang's conjecture on hyperbolicity of $\underline{X}$) requires a study of "normal" deformations of holomorphic curves in $\underline{X}$ which we postpone till the second part of this paper. (At the moment, I worked out the proof only under rather unpleasant technical assumptions.)

Remark on continuity of $\dim(X_c:\Gamma)$. It is easy to see in many cases that the mean dimension $\dim(X_c:\Gamma)$ is continuous in $c\in\mathbb{R}_+$ and whenever it is positive, it is also *non-constant* as a function of c. Thus we get Γ-spaces with mean dimension taking continuous spectra of values. To see it clearer, take the case of *meromorphic functions*, i.e. *holomorphic maps* $x:\mathbb{C}\to P^1$ where we bound the (spherical) derivative by one, i.e. take $X=X_1=\{x\mid\|Dx\|\le 1\}$. Then consider a lattice $\Lambda=\lambda\mathbb{Z}^2\subset\mathbb{C}$, for $\lambda\in\mathbb{C}^\times$, and observe (this is nearly obvious) that

$$\dim(X:\Lambda)=|\lambda|^2\dim(X:\mathbb{C})\,, \qquad (+)$$

as $|\lambda|^2$ equals the volume (area in this case) of the fundamental domain of Λ in $\mathbb{C}$. Thus by varying Λ with λ we get a continium of mean dimensions of Λ-spaces.

Next we observe that the restriction map $\rho_\lambda:X\to(P^1)^\Lambda$, where we evaluate our maps $x:\mathbb{C}\to P^1$ at the points $z\in\Lambda$, is *injective* for all sufficiently small λ. In fact this follows from the Cauchy inequality and yields the finiteness property $(\star)$ for the present case as

$$\dim(X_1:\mathbb{C})=|\lambda|^{-2}\dim(X:\Lambda)\le|\lambda|^{-2}\dim\left((P^1)^\Lambda:\Lambda\right)=2|\lambda|^{-2}$$

(see 3.4). Now, our space X is embedded into the shift space $(P^1)^\Lambda = (P^1)^{\mathbb{Z}^2}$, where $\Lambda = \lambda\mathbb{Z}^2$ and $\lambda \in \mathbb{C}^\times$ is small, with continuously varying mean dimension of the image $X_\lambda = \rho_\lambda(X_1) \subset (P^1)^\Lambda = (P^1)^{\mathbb{Z}^2}$. Actually, $\dim(X_\lambda : \mathbb{Z}^2)$ varies in the interval $(0, 2]$, since for large λ, where the lattice $\Lambda = \lambda\mathbb{Z}^2$ is sufficiently rare, the restriction map $\rho_\lambda : X_1 \to \Lambda$ becomes *onto* as every map $\Lambda \to P^1$ can be extended (interpolated) to a holomorphic map $X : \mathbb{C} \to P^1$ with $\|Dx\| \leq 1$ (see 3.6, where such an interpolation is used to show that $\dim(X_1 : \mathbb{C}) > 0$).

Remark on the bound $\|Dx\| \leq 1$. This may look quite restrictive but, in fact, harmonic (holomorphic) maps x with $\|Dx\| \leq 1$ often give a fair representation of all harmonic (holomorphic) maps. For example, if we deal with holomorphic (or pseudoholomorphic) maps x of $\mathbb{C}$, then the $\operatorname{Aff}\mathbb{C}$-*orbit of every* $x_0 : \mathbb{C} \to \underline{X}$ *for compact* X *contains, in its closure, a non-constant holomorphic map* x *with* $\|Dx_0\| \leq 1$, where $\operatorname{Aff}\mathbb{C}$, where the group of transformations $z \mapsto \lambda z + \mu$ of $\mathbb{C}$ naturally acts on the spaces of holomorphic maps of $\mathbb{C}$. This simple remarkable dynamical property of spaces of holomorphic maps, called *Bloch-Brody principle*, will be expanded further in the second part of this paper.

0.6.C. About residual dimension. Let $\Gamma_i \subset \Gamma$, $i = 1, \ldots,$ be a decreasing sequence of subgroups of finite index $\to \infty$, where we emphasize the case $\bigcap_i \Gamma_i = \{\mathrm{id}\}$ (which makes Γ *residually finite*). Then we consider subspaces $X_i \subset X$ of Γ_i-*invariant* (holomorphic, harmonic etc.) maps $V \to \underline{X}$ which correspond to maps from $V_i = V/\Gamma_i$ to $\underline{X}$. In our case (when we deal with harmonic maps, holomorphic maps, etc.) the ordinary dimensions of these X_i are finite and, moreover, are bounded by $\operatorname{const}\operatorname{card}(\Gamma/\Gamma_i)$ (see 3.4.C), but it is unclear when the limit $\lim\limits_{i\to\infty} \dim X_i / \operatorname{card}(\Gamma/\Gamma_i)$ exists. If it does, it can be called the *residual dimension* $\operatorname{resdim}(X : \Gamma)$ and it is tempting to conjecture it equals the mean dimension $\dim(X : \Gamma)$ in many interesting cases.

Example. Let $V = \mathbb{C}^n$, $\Gamma_i = i\mathbb{Z}^{2n}$, $i = 1, 2, \ldots,$ and $\underline{X}$ be a projective algebraic variety, e.g. $\underline{X} = \mathbb{C}P^N$, $N \geq n$. If $X = X_c$ consists of holomorphic maps $x : \mathbb{C}^n \to \underline{X}$ with $\|Dx\| \leq c$, then $X_i = X_{c,i}$ are made of such maps x_i from the tori $\mathbb{C}^n / i\mathbb{Z}^{2n}$ to $\underline{X}$. The bound $\|Dx\| \leq c$ obviously implies that the volumes of the images of these maps counted with multiplicities (as well as the volumes of their graphs in $\mathbb{C}^n / i\mathbb{Z}^{2n} \times \underline{X}$) are bounded by $d = \operatorname{const}(\underline{X})(ci)^{2n}$. With this in mind, we define the space A_d^n of "Abelian subvarieties in $\underline{X}$ of degree $\leq d$", i.e. of pairs (A, x) where A is an n-dimensional Abelian variety and $x : A \to X$ is a holomorphic map with n-dimensional image whose volume counted with multiplicity is bounded by d. It is rather obvious (see 3.4.C) that $\dim A_d^n \leq d \operatorname{const}(\underline{X})$ and probably it is not hard to prove the existence of the limit $a_n = a_n(\underline{X}) = \lim\limits_{i\to\infty} d^{-1} \dim A_d^n$. Then we define the corresponding space Y_d of holomorphic maps $x : \mathbb{C}^n \to \underline{X}$ by requiring that their graphs $G_x : \mathbb{C}^n \to \mathbb{C}^n \times X$ have

$$\operatorname{vol} G_x(B) \leq d \operatorname{vol} B$$

for all unit balls $B \subset \mathbb{C}^n$. (Actually, it would be more logical to require $\operatorname{Vol} x(B) \leq d$ but then one must be more careful in compactifying the resulting space of maps.) The space Y_d admits a natural $\mathbb{C}^n$-invariant compactification, say $\overline{Y}_d$ with the mean dimension bounded by $d \operatorname{const}(\underline{X})$. (This bound follows from the first main theorem of the Nevanlinna theory as was pointed out to me by Alex Eremenko.) It is not hard to show that the limit $\lim\limits_{d\to\infty} d^{-1} \dim(Y_d : \mathbb{C}^n)$ exists but it appears more difficult to show this limit equals the above number $a_n(\underline{X})$. Observe that a rough bound on $\dim(Y_d : \mathbb{C}^n)$ in terms of a_n for $n = 1$ would solve conjecture 0.6.A$'$. On the other hand, 0.6.A$'$ is vacuous for such spaces as $\underline{X} = \mathbb{C}P^N$, for instance, but the *equality* between the two dimensions, one referring to all maps $\mathbb{C} \to \mathbb{C}P^N$ and the other to $i\mathbb{Z}^2$-invariant maps, does not seem obvious even for $N = 1$. (Actually, the easiest case concerns not maps of $\mathbb{C}$ but rather of $\mathbb{C}/\mathbb{Z}$ versus maps of $\mathbb{C}/\mathbb{Z} \oplus i\sqrt{-1}\mathbb{Z}$, $i = 1, 2, \dots$, to $\mathbb{C}P^1$.)

0.6.D. Spaces of subvarieties. Take a Riemannian manifold W and consider the space X of all closed subsets $M \subset W$ with the Hausdorff convergence topology on compact parts of W. Clearly, X is compact. Notice that each isometry group Γ of W continuously acts on X, where, obviously, $\dim(X : \Gamma) = \infty$ unless X/Γ is finite.

The subsets in W worth looking at are those coming from some class of n-dimensional subvarieties $M \subset W$ which satisfy an elliptic equation, (e.g. being minimal, complex analytic, etc.) and furthermore are locally bounded in a suitable sense. Then the space $\mathcal{M}$ of such M's is expected to have $\dim(\mathcal{M} : \Gamma) < \infty$, for a cocompact amenable isometry group of W, and this dimension should be positive in significantly many examples. Here is a specific

Theorem. *Let W be a Hermitian manifold isometrically acted upon by a cocompact amenable group Γ. Denote by $\widetilde{\mathcal{M}}_d$ the space of n-dimensional* *complex subvarieties $M \subset W$, such that the intersection of M with every unit ball B in W satisfies*

$$\mathrm{Vol}_{2n}(M \cap B) \leq d$$

for a given $d \geq 0$. Then

$$\dim(\widetilde{\mathcal{M}}_d : \Gamma) \leq \text{const} < \infty$$

for some $\text{const} = \text{const}(W, \Gamma, d)$. *Furthermore if Γ is discrete and the quotient space W/Γ is projective algebraic then, for $0 \leq n = \dim_{\mathbb{C}} M < \dim_{\mathbb{C}} W$, one has*

$$\dim(\widetilde{\mathcal{M}}_d : \Gamma) \geq \text{const}' \, d^{n+1} ,$$

for all sufficiently large $d \geq d_0(W)$ and some positive constant $\text{const}' = \text{const}'(W, \Gamma) > 0$.

Example. The above applies to complex subvarieties $M \subset \mathbb{C}^N$ with $\Gamma = \mathbb{Z}^{2N}$ and implies, for instance, that there is *no* $\mathbb{Z}^{2N}$-equivariant topological embedding from $\mathcal{M}_d$ to $\mathcal{M}_{d'}$ if d is much (?) larger than d'.

Remark. This example should be taken with a grain of salt as our proof of the lower bound on $\dim(\widetilde{\mathcal{M}}_d : \Gamma)$ is based on a Γ-embedding of $\widetilde{\mathcal{M}}_d$ to $([0,1]^{N_1})^\Gamma$ while the lower bound exploits an embedding $([0,1]^{N_2})^\Gamma \to \widetilde{\mathcal{M}}_d$.

0.6.D$'$. Subvarieties in compact spaces and residual dimension. Along with the mean dimension one considers the residual dimension of X refering, for example, to subvarieties in the tori $\mathbb{R}^n/i\Lambda$ for a lattice $\Lambda \subset \mathbb{R}^n$ and $i \to \infty$ (see 4.2).

About this paper. The present notion of mean dimension(s) arose from my attempts to geometrize the algebraic and model theoretic conception of dimension over difference fields. It was gratifying to see that the mean dimension distinguishes certain spaces of holomorphic maps thus rekindling my hopes of setting some branches of the Nevanlinna theory into a dynamical casting. I could not trace this definition in the literature and, apparently, this did not come up in the dynamical systems, as was confirmed to me by Benjy Weiss with whom I was fortunate to discuss the subject matter. Benjy encouraged me by showing his interest in the mean dimension (actually, it was Benjy who suggested the "mean dimension" terminology) and he immediately generated a flow of dynamical ideas, including several conjectures relating the mean dimension and entropy. Many of his conjectures have already turned into theorems which appear along with many other results in [Lin-Wei] and [Lin]. Then I had an opportunity to discuss the holomorphic part of this paper with Mario Bonk and Alex Eremento. Alex explained to me several essential points on normal spaces and professionally sharpened the inequalities on the dimension of the spaces of meromorphic maps (see his survey paper [Ere]). More recently, I had a pleasure of talking to Michael McQuillan about the problems related to Lang's conjecture which made me more confident in my mean-dimensional version of it.

Part I of our paper focuses on elementary properties of the mean dimension and on illustrative examples. More technical discussion is postponed till Part II.

§ 1. Mean dimension in various categories of Γ-spaces.

1.1. Width and dimension. A map $f : X \to P$, where X is a metric space, is called an *ε-embedding* if f does not identify points in X with distances $> \varepsilon$. In other words

$$\operatorname{Diam} f^{-1}(p) \le \varepsilon \text{ for all } p \in P\,.$$

Then, following Uryson, we define $\operatorname{Widim}_\varepsilon X$ as the minimal number k, such that X admits a continuous ε-embedding to a k-dimensional polyhedron P. Clearly, $\operatorname{Widim}_\varepsilon$ is monotone decreasing in ε.

1.1.A. The basic example of evaluation of this ε-dimension is the following

Lebesgue Lemma. *The unit cube* $[0,1]^N \subset \mathbb{R}^N$ *has*

$$\operatorname{Widim}_\varepsilon [0,1]^N = N \textit{ for all } \varepsilon < 1\,.$$

Consequently

$$\operatorname{Widim}_\varepsilon \mathbb{R}^N = N \textit{ for all } \varepsilon > 0\,.$$

Here is a more general (and slightly less precise)

1.1.B. Widim inequality. *Let B be the unit ball in an N-dimensional Banach space. Then*

$$\operatorname{Widim}_\varepsilon B = N \textit{ for all } \varepsilon < 1\,. \qquad (*)$$

Proof. The inequality $\operatorname{Widim}_\varepsilon B \le N-1$ trivially implies that $\operatorname{FilRad}(\partial B) \le \varepsilon/2$ (compare App. 1 in $[\mathrm{Gro}]_{\mathrm{FRM}}$). On the other hand, the boundary sphere $S^{N-1} = \partial B$ with the induced metric has

$$\operatorname{FilRad} S^{N-1} \ge 1/2$$

by the argument in 1.2.C of $[\mathrm{Gro}]_{\mathrm{FRM}}$ since every k-tuple of points in this S^{N-1} with mutual distances < 1 canonically (and obviously) spans a $(k-1)$-simplex in S^{N-1}. Q.E.D.

Remark. The above will be used in 2.4 for evaluating the mean dimension of (sub)-linear subshifts $Y \subset \underline{B}^\Gamma \subset (\mathbb{R}^s)^\Gamma$, where we shall need another

1.1.C. Trivial Lemma. *Let Y be a closed subset in a Banach space X and let $p : X \to \mathbb{R}^N$ be a bounded linear operator. Then, for arbitrary metrics on Y and on $p(Y) \subset \mathbb{R}^N$ compatible with their topologies, one has*

$$\operatorname{Widim}_\varepsilon Y \geq \operatorname{Widim}_\varepsilon p(Y)$$

for all $\varepsilon > 0$.

Proof. As the fibers of the map $p : Y \to p(Y)$ are all non-empty convex, there is a continuous section, i.e. a map $q : p(Y) \to Y$ such that $p \circ q = \mathrm{Id} : Y \to Y$. Thus one has

$$\operatorname{Widim}_\varepsilon Y \geq \operatorname{Widim}_\varepsilon qp(Y) \geq \operatorname{Widim}_\varepsilon p(Y)\,.$$

Q.E.D.

1.1.D. Open questions. The Widim inequality allows a lower bound on $\operatorname{Widim}_\varepsilon$ of the intersection of a linear subspace Y in a Banach space X with the unit ball,

$$\operatorname{Widim}_\varepsilon Y \cap B \geq \dim Y \text{ for } \varepsilon < 1 \qquad (*)$$

(compare 2.6). Then we wish to have a similar inequality for non-linear subvarieties $Y \subset X$. For example:

Does $(*)$ hold true for $X = \mathbb{C}^N$ and Y being a *complex analytic* subvariety passing through the origin?

We would not mind $(*)$ with a slightly smaller $\varepsilon > 0$ but the answer is not even known for $\varepsilon = \varepsilon_N > 0$. On the other hand it is not hard to prove $(*)$ with ε depending on the degree of Y in the case Y is complex algebraic. In fact, $(*)$ holds true with $\varepsilon = \varepsilon\left(\mathrm{Vol}_d(Y \cap 2B)\right)$, $d = \dim_{\mathbb{R}} Y$ for all minimal subvarieties in $\mathbb{R}^{2N}$ by the usual compactness argument. It would be interesting to make such an argument work uniformly for all dimensions and thus applicable for evaluating of the mean dimension of (local) algebraic subvarieties in $(\mathbb{C}^1)^\Gamma$ (compare 2.5). On the other hand, one may ask on the possible range of $\operatorname{Widim}_\varepsilon$ for given ε on a given class of subvarieties and then one is tempted to extend this question to other “slicing invariants” of $Y \cap B$ defined in App. 1 of $[\mathrm{Gro}]_{\mathrm{FRM}}$.

It seems $\mathrm{Widim}_\varepsilon$ has not been evaluated even for simple convex subsets in $\mathbb{R}^n$, e.g. for the simplex $\Delta^{n-1} = \{x_i \geq 0,\ \Sigma x_i = 1\}$, where one expects (maybe too navely) that

$$\mathrm{Widim}_\varepsilon \Delta^n \sim \mathrm{const}_\varepsilon\, n\,.$$

Another interesting example is the Euclidean ball $B_{\ell_2} = \left\{ \sum_{i=1}^{n} x_i^2 \leq 1 \right\}$ whose $\mathrm{Widim}_\varepsilon$ is to be measured with respect to the sup-product metric (with the corresponding norm $\|x\|_{\ell_\infty} = \sup_{i=1,\dots,n} |X_i|$). More generally, one asks, what is $\mathrm{Widim}_\varepsilon B_{\ell_p}$ with respect to the ℓ_q-norm in $\mathbb{R}^n$?

1.1.E. It is clear that $\mathrm{Widim}_\varepsilon X < \infty$ for all *compact* metric spaces X and all $\varepsilon > 0$ but it may become infinite for non-compact spaces X (where, in fact the definition must be modified by replacing $\mathrm{Diam}\, f^{-1}(p)$ by $\limsup_{U \to p} \mathrm{Diam}\, f^{-1}(U)$ where U runs over the neighbourhoods of U in P) and this inequality is *strict*. It is also clear that Cartesian product $X_1 \times X_2$ with the *sup-product metric*, that is

$$\mathrm{dist}\left((x_1, x_2), (x_1', x_2')\right) = \max\left(\mathrm{dist}(x_1, x_1'),\ \mathrm{dist}(x_2, x_2')\right)\ ,$$

satisfies *the product inequality*

$$\mathrm{Widim}_\varepsilon(X_1 \times X_2) \leq \mathrm{Widim}_\varepsilon X_1 + \mathrm{Widim}_\varepsilon X_2\,.$$

It follows, that $\mathrm{Widim}_\varepsilon$ is also subadditive for taking maxima of metrics on the same space X,

$$\mathrm{Widim}_\varepsilon(X, \mathrm{dist}) \leq \mathrm{Widim}_\varepsilon(X, \mathrm{dist}_1) + \mathrm{Widim}_\varepsilon(X, \mathrm{dist}_2)$$

for $\mathrm{dist} = \max(\mathrm{dist}_1, \mathrm{dist}_2)$.

Warning. One should be careful with the additivity of $\mathrm{Widim}_\varepsilon$ for Cartesian products. In fact, even the Lebesgue dimension is not always additive, but the extent of the non-additivity is completely clarified by the work of Dranishnikov (see [Dra]), who, kindly explained this to me.

1.1.F. Remarks on cov_ε **and** $\mathrm{Fildim}_\varepsilon$**.** The ε-dimension $\mathrm{Widim}_\varepsilon X$, as a function of ε carries the same information about the geometry of X as the totality of its *Uryson's widths* (see $[\mathrm{Gro}]_{\mathrm{NLS}}$). A more traditional and essentially equivalent definition of ε-dimension is the *Lebesgue covering* *number* $\mathrm{Leb}_\varepsilon X$, that is the minimal intersection multiplicity of the ε-covers of X minus one. We prefer $\mathrm{Widim}_\varepsilon$ as this leads to interesting variations of the theme in the spirit of metric geometry such as the global ε-dimension $\mathrm{Fildim}_\varepsilon X$. The latter is defined as the maximal dimension of cycles $C \subset X$ with $\mathrm{FilRad}\, C \geq \varepsilon$, i.e. non-bounding in any metric extension $Y \supset X$ with $\sup\limits_{y \in Y} \mathrm{dist}(y, X) \leq \varepsilon$ (compare [Gro]$_{\mathrm{FRM}}$).

1.2. Definition of $\mathrm{Widim}_\varepsilon(X : \{\Omega_i\})$ **and** $\mathrm{Widim}_\varepsilon(X : \Gamma)$**.** Let X be a metric space and a group Γ act on X. We assume Γ is given a *proper* left invariant metric, where "proper" means that the balls $B(\gamma, R) = \{\gamma' \in \Gamma \mid \mathrm{dist}(\gamma', \gamma) \leq R\}$ are compact for all $R < \infty$. Also, we fix a left invariant (Haar) measure on Γ, denote $|\Omega| = \text{measure}\ (\Omega)$ and observe that $|\Omega| < \infty$ for all bounded (with respect to the metric) domains Ω in Γ.

Our basic examples are Lie groups, such as $\Gamma = \mathbb{R}^n$ with the usual metric and measure, as well as discrete finitely generated groups Γ with given generators, $\gamma_j, \dots, \gamma_i, \dots, \gamma_k$ where the *word metric* is defined by setting $\mathrm{dist}(id, \gamma)$ equal the length of the shortest words in γ_i representing γ and where $|\Omega| = \mathrm{card}\, \Omega$.

We denote by $|x - x'|_\gamma$ the γ-translate of the original metric on X, denoted $|x - x'|$, and assume that the identity map $(X, |x - x'|_{\gamma_1}) \to (X, |x - x'|_{\gamma_2})$ is uniformly continuous for all $\gamma_1, \gamma_2 \in \Gamma$ where the implied continuity modulus depends only on $\mathrm{dist}(\gamma_1, \gamma_2)$. In other words, the action of Γ is assumed uniformly continuous on X. We define the metrics $|x - x'|_\Omega$ on X for all bounded $\Omega \subset X$ as

$$|x - x'|_\Omega = \sup_{\gamma \in \Omega} |x - x'|_\gamma$$

and let $X_\Omega = (X, |x - x'|_\Omega)$. Then we look at $\mathrm{Widim}_\varepsilon X_\Omega$ as a function on bounded subsets $\Omega \subset \Gamma$ and observe that this function is *subadditive* according to the inequalities in 1.1.E. This implies, for amenable (see below) sequences $\Omega_i \subset \Gamma$, that the limit

$$\mathrm{Widim}_\varepsilon(X : \{\Omega_i\}) = \lim |\Omega_i|^{-1}\, \mathrm{Widim}_\varepsilon\, \Omega_i$$

exists and does not depend on a sequence Ω_i (see 1.3.C), exactly as it happens to the entropy (see [Orn-Weis]). Then we use this limit for the definition of $\mathrm{Widim}_\varepsilon(X : \Gamma)$ (see 1.4).

1.3. Amenability. Given a subset $\Omega \subset \Gamma$ we define its ρ-boundary $\partial_\rho \Omega \subset \Gamma$ for all $\rho > 0$ as the set of those $\gamma \in \Gamma$ for which the ball $B(\gamma, \rho)$ intersects Ω as well as the complement $\Gamma \backslash \Omega$. Then a sequence $\Omega_i \subset \Gamma$ is called amenable (or Fölner), if $|\partial_\rho \Omega_i|/|\Omega_i| \to 0$ for $i \to \infty$ and each $\rho > 0$. In other words, the ρ-boundary of Ω_i is "asymptotically negligible". Notice that, on the one hand, this definition uses no group structure but rather the metric and the measure on Γ. On the other hand, the amenability of a sequence does not depend on the choice of a Haar measure and of (proper left invariant) metric on Γ.

A group Γ is called *amenable* if it admits an amenable sequence $\Omega_i \subset \Gamma$. (If Γ is discrete or, more generally, unimodular, this equivalent to the classical definition of amenability where every continuous action of Γ on a compact space is required to have an invariant measure. Actually all amenable groups we consider in this paper may be assumed unimodular and so one should not be bothered by the discrepancy between the two definitions.)

1.3.A. Ornstein-Weiss Lemma (see [Orn-Weis]). *Let $h(\Omega)$ be a positive function defined on bounded subsets $\Omega \subset \Gamma$ such that*

(a) h is subadditive, i.e.

$$h(\Omega_1 \cup \Omega_2) \le h(\Omega_1) + h(\Omega_2) \tag{$*$}$$

for all pairs of bounded subsets Ω_1 and Ω_2 in Γ.

(b) h is invariant under Γ,

$$h(\gamma\Omega) = h(\Omega), \ \text{for all } \gamma \in \Gamma .$$

Then the limit

$$\lim_{i \to \infty} h(\Omega_i)/|\Omega_i| \tag{$\star$}$$

exists for every amenable sequence $\Omega_i \subset \Gamma$.

Remark. (a) Clearly the existence of the limit for *all* amenable sequences implies its independence of a choice of a sequence.

(b) if $h(\Omega)$ is monotone increasing for $\Omega' \supset \Omega$, then it suffices to assume $(*)$ only for *disjoint* subsets Ω_1 and Ω_2.

Sketch of the Proof. Take two subsets Ω_0 and Ω in Γ, where Ω will be eventually taken much larger than Ω_0, and consider some translates $\gamma_i\Omega_0 \subset \Gamma$, $i = 1, 2, \dots$, such that:

(a) all $\gamma_i\Omega_0$ are contained in Ω;

(b) the intersection of $\gamma_i(\Omega_0)$ with the union $U_0^{i-1} = \bigcup_{j=1}^{i-1} \gamma_j \Omega_0$ satisfies

$$\left|(\gamma_i \Omega_0) \cap U_0^{i-1}\right| \leq \varepsilon |\Omega_0| \qquad (*)_\varepsilon$$

for a given $\varepsilon > 0$.

We take a *maximal* sequence of translates $\gamma_i \Omega_0$, $i = 1, \ldots, k$, satisfying the *ε-packing conditions* $(*)_\varepsilon$ for all i and estimate from below the measure of the resulting union $U_0^k \subset \Omega$ as follows. Denote by ρ_0 the diameter of Ω_0, i.e. $\sup \operatorname{dist}(\delta, \delta')$ for δ, $\delta' \in \Omega_0$ and let α denote the relative amenability constant, i.e. $\alpha_0 = \alpha(\Omega, \Omega_0) = |\partial_{\rho_0} \Omega| / |\Omega|$. We claim that

$$|U_0^k| / |\Omega| \geq \varepsilon (1 - 2\alpha_0). \qquad (+)_\varepsilon$$

To see this, let $\Omega^+ \subset \Gamma$ consist of those γ for which the intersection $\gamma\Omega_0 \cap \Omega$ is non-empty and $\Omega^- \subset \Gamma$ consist of γ where $\gamma\Omega_0 \subset \Omega$. It is convenient to assume at this point that $id \in \Omega_0$. Then Ω^+ is *contained* in the ρ_0-neighbourhood of Ω, i.e. in $\Omega \cup \partial_{\rho_0} \Omega$, while Ω^- *contains* the ρ_0-interior of Ω, i.e. the complement $\Omega \backslash \partial_{\rho_0} \Omega$. Thus $|\Omega^-| / |\Omega^+| \geq 1 - 2\alpha_0$.

On the other hand, obviously,

$$\int_{\Omega^+} |U_0^k \cap \gamma\Omega_0| d\gamma = |U_0^k| \, |\Omega_0|$$

and so

$$\begin{aligned} |\Omega^-|^{-1} \int_{\Omega^-} |U_0^k \cap \gamma\Omega_0| d\gamma &\leq |U_0^k| \, |\Omega_0| \, |\Omega^+|^{-1} (1 - 2\alpha)^{-1} \\ &\leq |U_0^k| \; |\Omega_0| \, |\Omega|^{-1} (1 - 2\alpha_0)^{-1}. \qquad (1) \end{aligned}$$

Next, by the maximality of k, $(*)_\varepsilon$ must be violated for all $\gamma \in \Omega^-$, i.e.

$$|U_0^k \cap \gamma\Omega_0| \geq \varepsilon |\Omega_0|$$

for all $\gamma \in \Omega^-$ and thus

$$|\Omega_-|^{-1} \int_{\Omega_-} |U_0^k \cap \gamma\Omega_0| d\gamma \geq \varepsilon |\Omega_0|.$$

Hence

$$\varepsilon \leq |U_0^k| \, |\Omega|^{-1} (1 - 2\alpha|^{-1},$$

and $(+)_\varepsilon$ is proven.

Now we are ready to prove the existence of the limit ($\star$) by adopting the classical (and trivial) argument establishing convergence of $h(t)/t$ for sublinear functions $h(t)$. Denote by ℓ_- the lower limit

$$\liminf_{i\to\infty} h(\Omega_i)/|\Omega_i|\,,$$

and take some Ω_{i_1}, $\Omega_{i_2}, \dots \Omega_{i_s}$ among Ω_i such that

(a) the ratios $h(\Omega_i)/\Omega_i$ are all close to ℓ_-, say

$$h(\Omega_i)/|\Omega_i| \le \ell_- + \varepsilon$$

for a given $\varepsilon > 0$;

(b) the relative amenability constants $\alpha(\Omega_{i_\mu}, \Omega_{i_\nu})$ are very small compared to ε^s for all $i_\mu < i_\nu$;

(c) the number s is very large.

Then we bound the ratio $h(\Omega)/\Omega$ for all sufficiently large Ω where the relative amenability constants $\alpha(\Omega, \Omega_{i_\mu})$ are small. To do this we start with the above "ε-packing" of Ω by Ω_{i_s} (playing the role of Ω_0). The remaining part $\Omega' = \Omega \backslash \cup \gamma_i \Omega_{i_s}$ has measure $\approx (1-\varepsilon)\Omega$ and its ρ-boundary equals the union of these of Ω and the translates $\gamma_i \Omega_{i_s}$. Thus the relative amenability constants $\alpha(\Omega', \Omega_{i_\mu})$ remain small for $\mu < s$ and we can "ε-pack" Ω' by translates of $\Omega_{i_{s-1}}$. We keep doing this and finally cover much of Ω by translates of Ω_{i_μ}, namely the union of all these translates has total measure at least $(1-(1-\varepsilon)^s)(1-2\alpha)^s|\Omega|$, where α is the upper bound on the relative amenability constants. Since (we may assume) α is much smaller than ε^s, we cover almost all of Ω. On the other hand, our covering is $(1-\varepsilon)$-efficient, i.e. the total measure of our translates does not exceed $(1+\varepsilon)\Omega$ according to $(*)_\varepsilon$. Thus the union of all our translates, say $U \subset \Omega$ has $h(U)$ bounded by something of the order $\ell_+ + 2\varepsilon$. On the other hand, the complement $\Omega\backslash U$ has small measure and retains some "amenability" having $|(\Omega\backslash U) \cup \partial_1(\Omega\backslash U)|$ also small, say $\le \varepsilon$. It follows, by subadditivity of h, that $h(\Omega\backslash U)$ is bounded by something of the order of $\varepsilon|\Omega|$ and $h(\Omega)$ is bounded by $\ell_-|\Omega| + O(\varepsilon)|\Omega|$. This yields the Ornstein-Weiss lemma.

1.3.A$'$. Euclidean example. Let $\Gamma = \mathbb{R}^n$ and Ω_i be Euclidean i-balls for $i = 1, 2, \dots$ Then the above somewhat simplifies as large balls can be efficiently packed by smaller ones *without* any overlaps at all. (This is especially useful when we deal with superadditive functions such as maximal degrees of 1-Lipschitz maps $\Omega_i \to S^n$, see [G-L-P], § 2.)

1.4. Existence of $\mathrm{Widim}_\varepsilon(X:\Gamma)$ **for amenable and non-amenable** Γ**.** We continue 1.1.E and 1.3.A and define

$$\mathrm{Widim}_\varepsilon(X:\Gamma) = \lim_{i\to\infty} \mathrm{Widim}_\varepsilon(X:\{\Omega_i\})$$

with any amenable sequence $\Omega_i \subset \Gamma$.

In general, if we do not assume amenability, we set

$$\mathrm{Widim}_\varepsilon(X:\{\Omega_i\}) \stackrel{def}{=} \liminf_{i\to\infty} |\Omega_c|^{-1}\,\mathrm{Widim}_\varepsilon X_{\Omega_i}$$

for all sequences $\Omega_i \subset \Gamma$ with $\mu(\Omega_i) \to \infty$. And if we want to eliminate Ω_i we consider all exhaustions $\{\Omega_i\}$ of Γ and take the infimum of $\mathrm{Widim}_\varepsilon(X:\{\Omega_i\})$ over all exhaustions. This can be regarded as $\mathrm{Widim}_\varepsilon(X:\Gamma)$ which equals to the above if Γ is amenable as a simple reasoning shows. But we are not seriously concerned with keeping our definition independent of Ω_i as all our considerations are as good for one sequence of Ω's as for another.

1.5. Letting $\varepsilon \to 0$ **and defining** $\dim(X:\Gamma)$**.** The above mean ε-dimensions $\mathrm{Widim}_\varepsilon(X:\{\Omega_i\})$ and $\mathrm{Widim}_\varepsilon(X:\Gamma)$ are monotone decreasing in ε. Thus we can go to the limit and set

$$\dim(X:\{\Omega_i\}) = \mathrm{Widim}(X:\{\Omega_i\}) = \lim_{\varepsilon\to 0} \mathrm{Widim}_\varepsilon(X:\{\Omega_i\})$$

and

$$\dim(X:\Gamma) = \mathrm{Widim}(X:\Gamma) = \lim_{\varepsilon\to 0} \mathrm{Widim}_\varepsilon(X:\Gamma)$$

if we want to be Ω_i-free.

Also we observe that this definition makes sense for every (not necessarily invariant) subset $Y \subset X$ (as we may work with the metrics $|x-x'|_\Omega$ restricted to Y) and we shall be using this for *compact* subsets $Y \subset X$.

If X is itself a *compact* metric space, then the above definition of Widim does not depend on the original metric $|x-x'|$ in X. In general, one could make things invariant by first taking $\sup_Y \mathrm{Widim}(Y:\{\Omega_i\})$ over all compact $Y \subset X$ and then taking infimum over all metrics $|x-x'|$ on X compatible with the topology of X and such that the action of Γ on X is *uniformly* continuous. (We shall return to this later on when it becomes relevant.)

1.5.A. Topological invariance of mean dimension. If X is a compact space then, clearly, the mean dimension $\mathrm{Widim}(X:\{\Omega_i\})$ does not depend on the choice of the original metric $|x-x'|$ in X. In fact, continuity of the identity map $(X, |x-x'|^{\mathrm{old}}) \to (X, |x-x'|^{\mathrm{new}})$ implies uniform continuity for the metrics $|x-x'|^{\text{old}}_\gamma$ and $|x-x'|^{\text{new}}_\gamma$ simultaneously for all $\gamma\in\Gamma$ and consequently for $|x-x'|^{\text{old}}_\Omega$ and $|x-x'|^{\text{new}}_\Omega$. This gives a bound on $\text{Widim}^{\text{new}}_\varepsilon$ in terms of $\text{Widim}^{\text{old}}_\delta$ for some $\delta=\delta(\varepsilon)$ and as $\varepsilon\to 0$ we arrive at the equality $\text{Widim}^{\text{new}}=\text{Widim}^{\text{old}}$ in the limit, since $\delta(\varepsilon)\to\varepsilon$ for $\varepsilon\to 0$.

1.5.B. Monotonicity of Widim. Clearly every Γ-invariant subspace $Y\subset X$ has $\text{Widim}(Y:\{\Omega_i\})\leq\text{Widim}(X:\{\Omega_i\})$. In fact, as we mentioned earlier, $\text{Widim}(Y:\{\Omega_i\})$ makes sense for arbitrary, not necessarily invariant, subsets $Y\subset X$ as all we need are our metrics $|y-y'|_\gamma$ on Y and these come by just restricting the metrics $|x-x'|_\gamma$ from X to $Y\subset X$ for all $\gamma\in\Gamma$. Then obviously,

$$\text{Widim}(Y_1:\{\Omega_i\})\leq\text{Widim}(Y_2:\{\Omega_i\})$$

for all $Y_1\subset Y_2\subset X$ and all sequences $\Omega_i\subset\Gamma$. In particular,

$$\dim(Y:\Gamma)\leq\dim(X:\Gamma)$$

if Y admits a Γ-equivariant embedding to Γ.

1.6. On isometric actions on Banach spaces. There are certain topological spaces X, which admit *weak compactification*, i.e. a compact topological space $X_\bullet$ along with a bijective continuous map $e:X\to X_\bullet$. For example, the unit ball $\{\|X\|\leq 1\}$ in each Banach space is like that. Clearly, if $X_\bullet$ exists it is unique up to homeomorphism.

Now, let X come along with an action of Γ and let weak compactification refer to a compact Γ-space $X_\bullet$ with a bijective continuous Γ-equivariant map $X\to X_\bullet$. This $(X_\bullet,\Gamma)$ is also (obviously) unique, if it exists, and its Γ-invariants, such as $\dim(X_\bullet:\Gamma)$ can be regarded as invariants of (X,Γ).

Basic example. Let Γ isometrically act on a Banach space and thus on the unit ball X in this space. A Γ-invariant weak compactification is obvious for reflexible spaces and it also exists for some (all?) other examples, such as $\ell_\infty(\Gamma)$. Then one may speak of

$$\dim(X:\Gamma)\underset{def}{=}\dim(X_\bullet:\Gamma)\,.$$

It is clear, that

$$\dim(X:\Gamma)=s$$

for X being the unit ball in the ℓ_∞-space of bounded functions $\Gamma\to\mathbb{R}^s$ (Γ is discrete here) and that this dimension $\leq n$ for all other ℓ_p-spaces. But I could not decide if it is actually positive for $p < \infty$ (where the problem is related to evaluation of ℓ_∞-width of ℓ_p-balls, compare 1.1.D) and non-trivially depends on p. (If so, this would imply the spaces (X, Γ) are mutually Γ-non-homeomorphic for different p, which, I guess, is unknown for infinite groups Γ.) This problem on one hand and the idea of the Von Neumann dimension on the other hand lead to the following modification of our $\dim(X : \Gamma)$ (see 1.12 – 1.12 A″).

1.6.A. Definition of $\dim(X : \Gamma)_{\ell_p}$**.** Let us replace the sup-product distance $|x - x'|_\Omega$ from ... by the ℓ_p-distance, $|x-x'|_{\Omega,\ell_p} = \left(\int\limits_\Omega |x - x'|^p_\gamma d\gamma\right)^{\frac{1}{p}}$ and then repeat everything with $|x - x'|_{\Omega,\ell_p}$ instead of $|x - x'|_\Omega$. Notice that the resulting dimension is *not* a topological Γ-invariant, it is only a Lipschitz invariant (and Hölder "covariant" in an obvious sense). This is not so bad if we speak of isometric actions on (balls in) Banach spaces (where even the linear Lipschitz invariance is a non-trivial issue) but our definition needs an adjustment to this case. It seems reasonable, to consider all compact *convex* metric Γ-spaces $X_\bullet$ admitting bijective (surjective?) Lipschitz *linear* Γ-maps $X \to X_\bullet$, and take $\sup \dim(X_\bullet : \Gamma)_{\ell_p}$ over all such $X_\bullet$ "under" X. (And as the discussion became linear, one might try more manageable linear widths instead of the topological one.)

1.7. Remarks about entropy, coverings etc. Our definition of the mean dimension mimics that of the topological entropy where instead of our $\operatorname{Widim}_\varepsilon X$ one uses $\operatorname{ent}_\varepsilon X = \log \operatorname{cov}_\varepsilon X$ where $\operatorname{cov}_\varepsilon$ is the minimal number of the open subsets in X of diameter $\leq \varepsilon$ needed to cover X. In fact, one can avoid any metric in the definition of both invariants $\operatorname{ent}(X : \{\Omega_i\})$ and $\dim(X : \{\Omega_i\})$ by a direct appeal to (sufficiently fine) finite open covers of X, say $X = \bigcup\limits_\nu U_\nu$ and the associated covers by the intersections $\bigcap\limits_{\gamma\in\Omega} \gamma(U_\nu)$. This definition of the mean dimension has an advantage of being applicable to non-metrizable spaces and it is adopted in [Lin-Wei]. We choose here Widim as it is easier on the level of notations and also more flexible when it comes to generalizations. For example, our definition does not truly need any action: every family Δ of metrics $|x - x'|_\delta$, $\delta \in \Delta$, on X will do. Such situation naturally comes up in the study of spaces X of $\underline{X}$-valued functions over a given *background space* Δ replacing Γ in the example of $X = \underline{X}^\Gamma$. Here each point $\delta \in \Delta$ gives rise to a metric on functions $x(\delta)$ via some *weight function* $w(\delta, \delta_1)$ on $\Delta \times \Delta$ by the formula

$$|x - x'|_\delta = \sup_{\delta_1 \in \Delta} w(\delta, \delta_1)|x(\delta_1) - x'(\delta_1)|_{\underline{X}}$$

where $|\underline{x} - \underline{x}'|_{\underline{X}}$ refers to a preassigned metric on $\underline{X}$. Typically, Δ itself is a metric space (e.g. a graph as in $[\mathrm{Gro}]_{\mathrm{ESAV}}$), and

$$w(\delta, \delta_1) = \exp -\beta \operatorname{dist}_\Delta(\delta, \delta_1)\,.$$

"Microscopic" observations. One can think of a subset $\Omega \subset \Gamma$ (or more generally $\Omega \subset \Delta$) as a "microscope" applied to the metric space $X = (X, |x - x'|)$ and enlarging its visual image to the greater size $X_\Omega = (X, |x - x'|_\Omega)$, where the resolving power of Ω depends on the presence of transformations $\gamma : X \to X$, $\gamma \in \Omega$, which expand the original metric in X. This expansion brings invisibly small geometric details of $X = (X, |x - x'|)$ to the observable scale ε where we have a variety of "macroscopic" geometric techniques at our disposal (see $[\mathrm{Gro}]_{\mathrm{FRM}}$, $[\mathrm{Gro}]_{\mathrm{AI}}$ and $[\mathrm{Gr}]_{\mathrm{PCMD}}$). The magnification may be highly non-uniform in different directions and so when we eventually send $\varepsilon \to 0$ we arrive at a new "non-isotropic" image of X quite different from the original $(X, |x - x'|)$ (compare §4.10 in $[\mathrm{Gro}]_{\mathrm{CC}}$). Thus various "macroscopic" invariants discussed in the above cited papers (e.g. $\mathrm{Widim}_\varepsilon X$, $\mathrm{Fildim}_\varepsilon X$, etc.) are getting transported from the geometric realm to the domain of topological dynamics.

1.8. Mean Minkowski dimension. This dimension is defined for invariant sub-spaces Y of a topological Γ-space X with a Borel measure μ on X as follows.

Let $U \supset Y$ be a (non-invariant!) neighbourhood of Y in X and consider the intersection of the γ-translates of U, say $U_{\Omega_i} = \bigcap_{\gamma \in \Omega_i} \gamma U$. Then pass to the limit

$$M_U = \limsup_{i \to \infty} (\mu(U_{\Omega_i}))^{1/|\Omega_i|} \qquad (*)$$

and finally let

$$\operatorname{Min\,dim}(Y : \{\Omega_i\}) = \inf_U M_U$$

where $U \subset X$ runs over all neighbourhoods of $Y \subset X$.

1.8.A. Motivating example. Let $(\underline{X}_1, \mu_0)$ be a compact probability space and $\underline{Y} \subset \underline{X}$ be a closed subset. Then the subset $Y = \underline{Y}^\Gamma \subset X = \underline{X}^\Gamma$ has $\operatorname{Min\,dim}(Y : \{\Omega_i\}) = \mu_0(\underline{Y})$ for amenable sequences Ω_i. This directly follows from the definitions.

1.8.B. Measuring non-invariant subsets $Y \subset X$. Instead of translating U we may transport a given metric $|x - x'|$ on X and define $U^{\varepsilon}_{\Omega_i}$ as the intersection of the ε-neighbourhoods of Y with respect to the metrics $|x-x'|_\gamma$ for $\gamma \in \Omega_i$. Then we take the limit M_ε with $U^{\varepsilon}_{\Omega_i}$ substituting U_{Ω_i} in $(*)$ and finally let $\varepsilon \to 0$. The resulting version of the Minkowski dimension (obviously) reduces to the above Min dim for closed *invariant* subsets in *compact* probability spaces X.

1.8.C. Variation. Rather than intersecting the ε-neighbourhoods for the metrics $|x - x'|_\gamma$, one could take the ε-neighbourhood with respect to the metric $|x - x'|_{\Omega_i} = \sup\limits_{\gamma \in \Omega_i} |x - x'|_\gamma$. This may be only smaller than $U^{\varepsilon}_{\Omega_i}$ and so the resulting dimension is smaller than Min dim. (Probably, there are easy examples where it is *strictly* smaller.)

1.8.D. Smooth remark. If X is a compact smooth manifold with a Γ-action then one can apply the above to a smooth (not necessarily invariant) measure μ on X. In particular, one may speak of $\operatorname{Min\,dim}(\{x\} : \{\Omega_i\})$ for all points $x \in X$ and observe that the topological entropy is (obviously) constrained by the numbers

$$\begin{aligned} M_+ &= \sup_{x \in X} \operatorname{Min\,dim}(\{x\} : \{\Omega_i\}) \text{ and} \\ M_- &= \inf_{x \in X} \operatorname{Min\,dim}(\{x\} : \{\Omega_i\}) \text{ as follows,} \\ &- \log M_+ \leq \operatorname{topent}(X : \{\Omega_i\}) \leq -\log M_- \,. \end{aligned}$$

1.8.E. Minkowski dimension and coentropy. In many examples where μ is an *invariant* measure of the *maximal* entropy and the topological entropy is finite, the Minkowski dimension equals $\exp(\operatorname{topent}(X : \Gamma) - \operatorname{topent}(X : \Gamma))$. Furthermore, there are easy examples where $\operatorname{topent}(X : \Gamma) = \infty$ but (X, Y, Γ) can be approximated by actions with bounded entropy, say (X_i, Y_i, Γ), such that

$$\exp\left(\operatorname{topent}(Y_i : \Gamma) - \operatorname{topent}(X_i : \Gamma)\right) \underset{i \to \infty}{\to} \operatorname{Min\,dim}\left(Y : \Gamma\right),$$

where the notation $\operatorname{Min\,dim}(Y : \Gamma)$ refers to a suitable exhaustion $\{\Omega_i\}$ of Γ.

1.8.F. About examples. The mean Minkowski dimension is invariant under measure preserving continuous maps, e.g. homeomorphisms, $\alpha : X' \to$

X where $Y' = \alpha^{-1}(Y)$. Constructing such maps is an interesting problem which makes sense in every geometric category of $\underline{X}$'s, where one is especially interested in the structure of the group Aut $\underline{X}^\Gamma$ consisting of invertible maps $\underline{X}^\Gamma \to \underline{X}^\Gamma$ of finite type with inverse also being of finite type. Besides right translations by Γ and automorphism in Aut $\underline{X}$ acting on $\underline{X}^\Gamma$ in an obvious way one has two general possibilities.

I. *Triangular maps.* The simplest instance of this appears where $\underline{X}$ is split, say $\underline{X} = \underline{Y} \times \underline{Z}$. Here every map $\varphi : \underline{Z}^\Gamma \to (\text{Aut}\,\underline{Y})^\Gamma$ of finite type defines an automorphism of $\underline{X}^\Gamma$ by $(y, z) \mapsto (\varphi(z)(y),\ z)$.

II. *Markers.* The idea is similar to the above with Aut $\underline{Y}$ replaced by Aut $\underline{Y}^D$ for a finite subset $D \subset \Gamma$ (or a finite collection of these). Such D, as well as its translates in Γ, are distinguished by insisting on certain values of z on these D's. If these D's happen to be mutually disjoint, then suitable automorphisms of $\underline{Y}^D$ parametrized by z give us automorphisms of $\underline{X}^\Gamma$. All this has been carefully studied for shifts $S^{\mathbb{Z}}$ and finite S (see [Hed]) and we shall return to the general case in the second part of this paper.

III. Sometimes one can ensure invertibility of a map by an implicit function argument but then the resulting inversion is, typically, of infinite type.

IV. If $\underline{X}$ is a smooth manifold, one may speak of Γ-invariant vectorfields on $\underline{X}^\Gamma$ of finite type and study the corresponding flows (which may be only partially defined). For instance, if $\underline{X}$ is a symplectic manifold, then every function (local Hamiltonian) $h : \underline{X}^D \to R$ defines such a flow. (We shall return to this and will study the corresponding symplectic geometry in the second part of the paper.)

1.9. Projective and legal dimension in $\underline{X}^\Gamma$. Consider a subspace $Y \subset X = \underline{X}^\Gamma$ and define its dimension using natural projections $\underline{X}^\Gamma \to \underline{X}^\Omega$ $\Omega \subset \Gamma$, (corresponding to restriction of functions from Γ to subsets $\Omega \subset \Gamma$) as follows. Let $Y|\Omega_i$ denote the image of Y under our projection $\underline{X}^\Gamma \to X^{\Omega_i}$ and set

$$\text{prodim}(Y : \{\Omega_i\}) = \underset{i\to\infty}{\text{liminf}}\,\text{dim}(Y|\Omega_i)/|\Omega_i|$$

for every sequence of bounded subsets $\Omega_i \subset \Gamma$ with $|\Omega_i| \to \infty$.

This projective dimension looks more approachable than $\dim(Y : \Gamma) = \text{Widim}(Y : \Gamma)$ and sometimes the two dimensions are known to be equal. In any case we have the following

1.9.A. Pro-mean Inequality. *If $\underline{X}$ is compact, then every closed (not necessarily invariant) subset $Y \subset \underline{X}^{\Gamma}$ satisfies*

$$\dim(Y : \{\Omega_i\}) \leq \operatorname{prodim}(Y | \{\Omega_i\})$$

for every amenable sequence $\Omega_i \subset \Gamma$.

Proof. The projection from Y to the $\underline{X}^{\Omega_i + \rho}$ (where $\Omega_i + \rho \subset \Gamma$ denotes the ρ-neighbourhood of Ω_i) is an ε-embedding with $\varepsilon = \varepsilon(\rho) \to 0$ for $\rho \to \infty$ and, clearly,

$$\dim Y \mid \Omega_i \geq (\dim Y \mid \Omega_i + \rho) - |\partial_\rho \Omega_i| \dim \underline{X}\,.$$

Q.E.D.

1.9.B. Legal dimension. Let $Y \subset \underline{X}^{\Gamma}$ be an invariant subset of finite type defined by a law $L \subset \underline{X}^D$, $D \subset \Gamma$. Consider all translates γD in Γ which are contained in a given subset Ω and let $L_\Omega \subset \underline{X}^{\Omega}$ consist of *L-legal* functions on Ω, i.e. of those $x : \Omega \to \underline{X}$ where the restriction of x to each $\gamma D \subset \Omega$ is contained in L (where, as earlier, γD is identified with D and $\underline{X}^{\gamma D}$ with $\underline{X}^D$). Then define

$$\operatorname{legdim}(Y : \{\Omega_i\}; L) = \liminf_{i \to \infty} \dim L_{\Omega_i} / |\Omega_i|\,,$$

where Ω_i is a sequence of subsets in Γ (which is assumed amenable in most applications).

1.9.C. On non-topological spaces. If Γ is a discrete group where bounded subsets D are finite, then the definition of legdim makes sense in every category with (finite!) Cartesian products and a notion of dimension (or rank). For example, this applies to linear and affine spaces over an arbitrary field and up to a certain extent to moduli over more general (commutative and noncommutative) rings. Also, one may use this definition for (pro)-algebraic varieties over an arbitrary field and also for analytic varieties over a local field.

1.9.D. On subspaces $Y \subset \underline{X}^{\Gamma}$ of infinite type for metric spaces $\underline{X}$. If Y is a subshift of infinite type, the projections $Y \to \underline{X}^{\Omega}$ may be easily *onto* (an open subset in $\underline{X}^{\Omega}$) even for relatively small Y (e.g. for $\underline{X} = \mathbb{R}^s$ and Y being a generic infinite dimensional linear subspace) and so the ordinary dimensions of the images do not tell us much. It is more useful to take the $\mathrm{Widim}_\varepsilon$ of these images $Y \mid \Omega$ which works well for example, for the space $\ell_p(\Gamma, \mathbb{R}^s)$ and $Y = B \cap Y_0$, where Y_0 is a Γ-invariant linear subspace in our ℓ_p and B is the unit ℓ_p-ball. Here it seems reasonable to evaluate $\mathrm{Widim}_\varepsilon Y \mid \Omega$ with respect to the ℓ_p-norm on $\ell_p(\Omega, \mathbb{R}^s)$ as is suggested by the ℓ_2-case where this leads to the *Von Neumann dimension* (see 1.12).

1.9.E. On invariance of legdim **and introduction of** stablegdim**.** The definition of legdim depends not only on $Y = Y(L)$ but also on the defining law $L \subset \underline{X}^D$ although in most cases the dependence on L is illusory.

In fact legdim = prodim in many cases (see §2) and it is useful to bring in an intermediate notion of *stable legal dimension*, denoted $\mathrm{stablegdim}(Y : \{\Omega_i\})$, where Ω_i is increasing sequence of subsets in Γ. To define this we project L_{Ω_j} to $\underline{X}^{\Omega_i}$ for all $j \geq i$ and let $L_{i_j} \subset \underline{X}^{\Omega_i}$ denote the images of these projections. Then we set $\mathrm{stablegdim}(Y : \{\Omega_i\}; L) = \liminf\limits_{i\to\infty} \lim\limits_{j\to\infty} \dim L_{i_j}/|\Omega_{i_j}|$. Observe that this stablegdim extends to nontopological categories in most cases where it is possible for legdim and this sometimes allows such an extension for prodim (see below).

1.9.F. Elementary inequalities. It is clear that

$$\mathrm{prodim} \leq \mathrm{stablegdim} \leq \mathrm{legdim}\,,$$

(where, recall, $\mathrm{meandim} \leq \mathrm{prodim}$ for compact Y).

Also observe that the intersection $L_{i_\infty} \overset{def}{=} \bigcap\limits_{j\geq i} L_{i_j}$ equals the projection $Y \mid \Omega_i$ of Y to Ω_i for compact subspaces $Y \subset \underline{X}^\Gamma$. Hence,

$$\mathrm{prodim}\, Y = \mathrm{stablegdim}\, Y\,, \tag{$*$}$$

provided the dimension is stable under countable intersections of subsets in our category. This is so, for instance, for *compact complex analytic varieties* by the Noether intersection property: *every decreasing family of compact complex spaces stabilizes.*

Thus we have the following simple

Proposition. *Let X be a complex analytic variety and $L \subset \underline{X}^D$ be a compact subvariety. Then $Y = Y(L) \subset \underline{X}^\Gamma$ satisfies the above equality* $(*)$.

Remark. The point of this is our evaluation of some dimension of a "transcendental" object, our Y, in terms of "elementary" ones, i.e. L_{i_j}.

On extension of prodim to non-topological categories. The equality $Y \mid \Omega = L_{i_\infty}$ remains valid in many algebraic categories, (e.g. for complex algebraic varieties and saturated models of first order theories in general) and if we have a notion of dimension in our category which passes to countable intersections of varieties (as it happens, for instance, to countable intersections of constructible subsets in K^N for an uncountable algebraically closed field K), then we can define $\operatorname{prodim} Y(L)$ for laws $L \subset \underline{X}^D$ in our category.

1.9.G. On stable laws. A law $L \subset \underline{X}^D$ is called *stable* if there exists ρ_0, such that the image of the projection from $L_{\Omega+\rho}$ to L_Ω does not depend on ρ for all $\rho \geq \rho_0$ and all bounded $\Omega \subset \Gamma$, where, recall, $\Omega + \rho \subset \Gamma$ denotes the ρ-neighbourhood of Ω in Γ. Clearly, if L is stable, then

$$\operatorname{prodim}(Y : \{\Omega_i\}) = \operatorname{legdim}(Y : \{\Omega_i\}; L) = \operatorname{stablegdim}(Y : \{\Omega_i\}; L)$$

for all amenable sequences $\Omega_i \subset \Gamma$.

Problem. Find less restrictive conditions ensuring the above equalities between different dimensions. (See §2 for practical results in this direction.)

1.10. Residual dimension. Given a discrete subgroup $\Gamma_0 \subset \Gamma$, we consider the fixed point set $\operatorname{Fix}\Gamma_0 \subset X$ in a given Γ-space X. For example, if $X = \underline{X}^\Gamma$, this $\operatorname{Fix}\Gamma_0$ consists of all Γ_0-invariant functions $\Gamma \to \underline{X}$ which can be identified with functions $\Gamma/\Gamma_0 \to \underline{X}$.

We are especially interested in the case where Γ_0 is of finite covolume, i.e. when the Haar measure $|\Gamma/\Gamma_0|$ is finite. In this case we may expect $\dim \operatorname{Fix}\Gamma_0 < \infty$ and so we set

$$\operatorname{resdim} X/\Gamma_i = \liminf_{i\to\infty} (\dim \operatorname{Fix}\Gamma_i)/|\Gamma/\Gamma_i|$$

for every sequence of discrete subgroups $\Gamma_i \subset \Gamma$ of finite covolumes with $|\Gamma/\Gamma_i| \to \infty$.

The most interesting case is where the spaces Γ/Γ_i *converge* to Γ, i.e. if for each bounded subset $\Omega \subset \Gamma$ the intersection $\Omega \cap \Gamma_i$ consists of $\{id\}$ for all $i \geq i_0 = i_0(\Omega)$. Recall that a *discrete* group Γ admitting such a sequence of Γ_i is called *residually finite*, and many residual finite groups are far from being amenable. Such are the free groups and most finitely generated subgroups in the linear group $GL_n\mathbb{R}$.

What may limit the applicability of the residual dimension is absence of a sufficient amount of periodic (i.e. Γ_i-fixed) points. However, if $L \subset \underline{X}^D$ is a *strongly stable* law (see 7.E.‴ in $[\mathrm{Gro}]_{\mathrm{ESAV}}$ and below) *then periodic points are dense in* $Y(L) \subset X$ *for residually finite groups* Γ *and* resdim = procim *if* Γ *is also amenable.* This follows by the argument in 7.E.″ in $[\mathrm{Gro}]_{\mathrm{ESAV}}$.

Definition of strong stability. Call L strongly stable if there exists $\rho_0 > 0$, such that the following condition (loc_{ρ_0}) is sufficient for extendability of a function $X_0 : \Omega_0 \to \underline{X}$ to our $x : \Gamma \to \underline{X}$ belonging to $Y(L) \subset \underline{X}^\Gamma$,

(loc_{ρ_0}). For every ρ_0-ball $B \subset \Gamma$ the restriction $x_0 \mid \Omega \cap B$ is extendable to an *L-legal* function x_1 on B, i.e. the restriction of x_i to each translate of D inside B must be in L.

Remarks (a). Besides the limit of $\dim \operatorname{Fix} \Gamma_i/(\Gamma/\Gamma_i)$ the totality of the numbers $\dim \operatorname{Fix} \Gamma_i$ for all lattices $\Gamma_i \subset \Gamma$ carries an interesting information about (X, Γ). For example, if $\Gamma = \mathbb{Z}$ and $\Gamma_i = i\,\mathbb{Z}$, this information is encoded in the generating function $\sum\limits_i t^i \dim \operatorname{Fix} \Gamma_i$ which we shall study in the second part of this paper.

(b) One can make the above definition of resdim more robust by using *δ-fixed* points $\operatorname{Fix}_\delta \Gamma_i$, i.e. moved by judiciously chosen generators of Γ_i by at most δ. Also, one may use $\operatorname{Widim}_\varepsilon \operatorname{Fix} \Gamma_i$ for the metric $\sup\limits_{\gamma \in \Gamma} |x - x'|_\gamma$ on $\operatorname{Fix} \Gamma_i$ instead of $\dim \operatorname{Fix} \Gamma_i$, where eventually $\delta, \varepsilon \to 0$.

1.10.A. Residual amenability. This signifies the existence of a decreasing sequence of normal subgroups $\Gamma_i \subset \Gamma$ with amenable quotient groups Δ_i and with trivial intersections, $\bigcap\limits_i \Gamma_i = \{id\}$. Now, for each Δ_i, we may have some notion of mean (legal, projective, etc.) dimension which passes to Γ as we apply it Δ_i acting on $\operatorname{Fix} \Gamma_i$ and let $i \to \infty$. Alternatively, one may take an amenable sequence $\Omega_i \subset \Delta_i$ and use $\operatorname{Fix}_\delta \Gamma_i$ with respect to the metric $\sup\limits_\gamma |x - x'|$ for $\gamma \in \Gamma$ summing over the pull-back of Ω_i under the quotient map $\Gamma \to \Delta_i$.

In fact, the natural class of groups where this idea works consists of all *initially subamenable groups* (essentially introduced in [Ve-Go] and used in $[\mathrm{Gro}]_{\mathrm{ESAV}}$) generalizing residually amenable groups.

1.11. Linear laws and mean dimension over amenable algebras. Given an arbitrary field K, one may take a vector space $\underline{X}$ over K, e.g. $X = K^s$, and speak of linear laws (i.e. subspaces) $L \subset \underline{X}^D$. Then, if Γ is an amenable group, we have our (mean) projective dimension

$$\operatorname{prodim}(Y:\Gamma) \qquad \text{for } Y=Y(L)\subset \underline{X}^\Gamma$$

defined with an amenable exhaustion of Γ.

Remark on finite fields K. If K is finite, then $X=\underline{X}^\Gamma$ is compact (totally disconnected) for the product topology and $Y\subset X$ is a closed (and so also compact) subspace in X. Then the basic topological invariant of the action of Γ on Y, *the topological entropy*, is (obviously) related to the mean dimension by the equality

$$\operatorname{topent}(Y:\Gamma)=\operatorname{prodim}(Y:\Gamma)\log|K|\,,$$

for $|K| \underset{def}{=} \operatorname{card} K$. (See §2 for continuation of this discussion.)

Replacing $(K^s)^\Gamma$ by $K^s(\Gamma)$ and passing to (group) algebras. Instead of the space $(K^s)^\Gamma$ of all functions $\Gamma\to K^s$ one can look at the dual space denoted $K^s(\Gamma)$ which can be identified with the space of functions with finite support on Γ. Then each linear law $L\subset (K^s)^D$ defines a subspace $Y_0=Y_0(L)\subset K^s$, namely $Y_0=Y(L)\cap K^s(\Gamma)$ for the obvious embedding $K^s(\Gamma)\subset (K^s)^\Gamma$ and, clearly, $\operatorname{prodim} Y_0=\operatorname{prodim} Y$. Then we observe that $K^s(\Gamma)$ can be identified with the free module of rank s over the *group algebra* $K(\Gamma)$ where the (Γ-invariant!) subspaces $Y_0\subset K^s(\Gamma)$ are just submoduli in $K^s(\Gamma)$.

Now we generalize everything to an arbitrary K-algebra A in place of $K(\Gamma)$. We say A is *amenable* if it admits *an amenable exhaustion* by K-linear subspaces $\mathcal{A}_i\subset A$, $i=1,2,\dots$, where amenability of $\{\mathcal{A}_i\}$ signifies that $\mathcal{A}_i$, for large i are "almost invariant" under right multiplication in A, i.e.

$$(\dim_K \mathcal{A}_i+\mathcal{A}_i a)/\dim_K\mathcal{A}_i \underset{i\to\infty}{\to} 1$$

for each $a\in A$.

Next, given a finitely generated *left* module B over A, we define its dimension relative to $\{\mathcal{A}_i\}$ as follows. Take some finite K-dimensional linear subspace $B_0\subset B$ generating B over A and set

$$\dim_A B\mid\{\mathcal{A}_i\}=\liminf_{i\to\infty}\dim\mathcal{A}_iB_0/\dim\mathcal{A}_i\,.$$

Clearly, this dimension does not depend on the choice of B_0 and it gives the "right" number for free moduli: $\dim_A A^s=s$ for all amenable exhaustions.

Furthermore, if A equals the group ring $K(\Gamma)$ of some Γ, this reduces to the notion of legal (or stable) dimension over Γ, but I do not know if the existence of the limit $\lim\limits_{i\to\infty} \mathcal{A}_i B_0 / \dim \mathcal{A}_i$ holds in full generality.

Remark (made by Ofer Gabber). Since $\liminf$ is non-additive, we cannot claim the additivity

$$\dim_A B_1 \oplus B_2 = \dim_A B_1 \oplus \dim_A B_2$$

prior to proving the existence of the limit. Yet we always can take some generalized limit (the best here, I think, is an ultralimit) and thus recapture the additivity. Eventually we shall be interested in additivity of $\dim_A$ for exact sequences, $0 \to B_1 \to B \to B_2 \to 0$, where some extra problems arise (as was also pointed out to me by Ofer).

Let us relax the assumption of B being finitely generated over A by giving B a topology where the action of A is continuous and such that B admits a dense finitely generated submodule B'. (For example, if $A = K(\Gamma)$, one can take B equal the space of all functions $\Gamma \to K^s$ with the product topology in this $B = (K^s)^\Gamma$, where K^s comes with the discrete topology. Clearly, the finitely generated module $B' = K^s(\Gamma)$ densely embeds to this B.) Then we can define $\dim B$ as $\dim B'$ or (which is essentially equivalent) by approximating the above B_0 by some B_ε and taking

$$\liminf_{\varepsilon\to 0} \lim_{i\to\infty} \mathcal{A}_i B_\varepsilon / \dim \mathcal{A}_i \,.$$

The major drawback of all this is the amenability assumption on A. This can be overcome in the context of the *Von Neumann algebras*, e.g. for the rings $\mathbb{R}(\Gamma)$ for arbitrary countable groups Γ. Here $K = \mathbb{R}$ and the relevant modules are those of ℓ_2-functions $\Gamma \to \mathbb{R}^s$ as well as their submodules and factor modules (compare 1.11.A. below). The resulting *Von Neumann dimension* $\dim_{\ell_2} B$ is well defined for all Γ and if Γ is amenable it equals the above $\dim_A B$ as an easy argument shows (explained to me by Alain Connes about 20 years ago and exposed in the case of ℓ_2-cohomology in [Dod-Mat]).

1.12. Von Neumann dimension. Let $Y \subset (\mathbb{R}^s)^\Gamma_{\ell_2} \subset (\mathbb{R}^s)^\Gamma$ be a Γ-invariant Hilbert space inside $(\mathbb{R}^s)^\Gamma_{\ell_2} = \ell_2(\Gamma, \mathbb{R}^s)$, the space of the square summable functions $\Gamma \to \mathbb{R}^s$. Then for every subset Ω we define the restriction operator (map) $R_\Omega : Y \to (\mathbb{R}^s)^\Omega_{\ell_2}$ for $R_\Omega(x) = x|\Omega$ and let $R^*_\Omega : (\mathbb{R}^s)^\Omega_{\ell_2} \to Y$ be the adjoint operator. The Γ-invariance of Y (trivially) implies that $\frac{\operatorname{trace} R_\Omega R^*_\Omega}{\operatorname{trace} R_{\Omega'} R^*_{\Omega'}} = \frac{\operatorname{card}\Omega}{\operatorname{card}\Omega'}$ for all non-empty finite subsets Ω

and $\Omega' \subset \Gamma$ (where $(\mathbb{R}^s)^{\Omega}_{\ell_2} = (\mathbb{R}^s)^{\Omega}$) and one defines the Von Neumann dimension of Y as

$$\dim_{\ell_2}(Y : \Gamma) \overset{\text{def}}{=} |\Omega_i^{-1}| \operatorname{trace} R_\Omega R_\Omega^* \tag{+}$$

for some (and so for each) finite subset $\Omega \subset \Gamma$, where $|\Omega| \overset{\text{def}}{=} \operatorname{card} \Omega$ (see [Con], $[\text{Gro}]_{\text{AI}}$, [Lück] and references therein).

Remark. In what follows we use standard embeddings $(\mathbb{R}^s)^{\Omega} \to (\mathbb{R}^s)^{\Omega'}$ for all $\Omega' \supset \Omega$ where we just extend functions by zero outside Ω. In particular, we embed $(\mathbb{R}^s)^{\Omega} \subset (\mathbb{R}^s)^{\Gamma}$ and observe that $R_\Omega^* = R_\Gamma^* | (\mathbb{R}^s)^{\Omega}$, and so we abbreviate R_Ω^* to simple R^* for all $\Omega \subset \Gamma$.

To see this more geometrically in the case of an amenable group Γ we indicate the following (well known, I believe)

1.12.A. Proposition. *Let $\Omega_i \subset \Gamma$, $i = 1, \dots$ be an amenable exhaustion of Γ by finite subsets Ω_i and let $n_i[a,b]$ denote the number of the eigenvalues of the operator $R_{\Omega_i} R^*$ in the interval $[a,b]$. Then, if $0 < a \leq b < 1$,*

$$n_i[a,b]/|\Omega_i| \to 0 \text{ for } i \to \infty\,,$$

(while $n_i[0,1] = s|\Omega_i|$, of course). *In other words the majority of eigenvalues is concentrated near the ends of the α-interval $[0,1]$.*

Proof. Let $x : \Omega \to \mathbb{R}^s$ be an *approximate λ-eigenfunction* of $R_\Omega R^*$ for some $\lambda \in [0,1]$ in the sense that

$$\|R_\Omega R^*(x) - \lambda x\| \leq \alpha x \tag{α}$$

and assume that the restriction of $R^*(x)$ to the complement of Ω is β-small, i.e.

$$\|R^*(x)|\Gamma \backslash \Omega\| \leq \beta \|x\|\,. \tag{$(\beta)_{\Gamma\backslash\Omega}$}$$

We claim that for small α and β the number λ must be close to zero or one. Namely

$$\lambda(1-\lambda) \leq 2\alpha + \beta\,. \tag{$\star$}$$

Q.E.D.

Proof. Write $(\beta)_{\Gamma\backslash\Omega}$ as

$$\|R^*(x) - R_\Omega R^*(x)\| \leq \beta \|x|$$

and obtain with (α),

$$\|R^*(x) - \lambda x\| \leq (\alpha + \beta)\|\alpha\| .$$

Since $\|R^*\| \leq 1$ and $R^*R^* = R^*$, we have

$$\|R^*(x) - \lambda R^*(x)\| \leq (\alpha + \beta)\|x\|$$

and

$$\|R_\Omega R^*(x) - \lambda R_\Omega R^x_x(\dot{x})\| \leq (\alpha + \beta)\|x\| ,$$

i.e.

$$(1 - \lambda)\|R_\Omega R^*_\Omega(x)\| \leq (\alpha + \beta)\|x\| .$$

Now use (α) again and conclude

$$\lambda(1 - \lambda)\|x\| \leq (\alpha + \beta + \alpha(1 - \lambda))\, \|x\|$$

and, finally,

$$\lambda(1 - \lambda) \leq \alpha + \beta + \alpha(1 - \lambda) .$$

In particular we get $(\star)$ as well as the relations,

$$\lambda(1 - \lambda) = 0(\alpha + \beta)$$

and

$$1 - \lambda = 0\left(\frac{\alpha + \beta}{\lambda}\right) .$$

Next let $\Omega^{-\rho} < \Omega$ be the ρ-interior of Ω, i.e. $\gamma \in \Omega^{-\rho}$ iff the ρ-ball $B(\gamma, \rho) \subset \Omega$ for a given $\rho \geq 0$. We claim that the majority of functions $x : \Omega^{-\rho} \to \mathbb{R}^s$ satisfy $(\beta)_{\Gamma\backslash\Omega}$ with some $\beta = \beta(\rho) \to 0$ for $\rho \to \infty$, at least for *finite* subsets $\Omega \subset \Gamma$. To say it precisely, we denote by $S_\rho : (\mathbb{R}^s)^{\Omega^{-\rho}} \to (\mathbb{R}^s)^{\Gamma\backslash\Omega}$ the operator $R_{\Gamma\backslash\Omega}R^*$ on $(\mathbb{R}^s)^{\Omega^{-\rho}} \subset (\mathbb{R}^s)^\Omega$ and show that

$$\operatorname{trace} S^*_\rho S_\rho \leq \beta(\rho)|\Omega^{-\rho}| \tag{$(\beta)_\rho$}$$

where $\beta(\rho) = \beta(\rho, \Gamma, s) \to 0$ for $\rho \to \infty$ and where $S^*_\rho : (\mathbb{R}^s)^{\Gamma\backslash\Omega} \to (\mathbb{R}^s)^{\Omega_0}$ is the adjoint to S_ρ.

In fact every δ-function $x = x_\gamma$ on Γ concentrated at some $\gamma \in \Gamma$ satisfies $\|R^*x_\gamma\| \in 1$ since $\|R^*\| \leq 1$. It follows, that the restriction of $\|R^*x_\gamma\|$ to the complement of the ball $B(\gamma, \rho)$ has norm $\leq \beta(\rho)$ for $\beta(\rho) \underset{\rho\to\infty}{\to} 0$. Therefore, $\|S_\rho(x_\gamma)\| \leq \beta(\rho)$ for all $\gamma \in \Omega^{-\rho}$ as well, because $\Gamma\backslash\Omega$ is contained in the complement of the balls $B(\gamma, \rho)$ for $\gamma \in \Omega^{-\rho}$. Then the same inequality is clearly satisfied by $S_\rho^* S_\rho : (\mathbb{R}^s)^{\Omega^{-\rho}} \to (\mathbb{R}^s)^{\Omega^{-\rho}}$,

$$\|S_\rho^* S_\rho(x_\gamma)\| \leq \beta(\rho)$$

that implies $(\beta)_\rho$, since the δ-function make an orthonormal basis in $(\mathbb{R}^s)^{\Omega^{-\rho}}$.

Now we prove our proposition by first evaluating $n_i[a, b]$ for small intervals $[a, b]$, namely for those where $|a - b| = \alpha$ for some $\alpha > 0$ specified later on. We denote by $X_i = X_{i,a,b} \subset (\mathbb{R}^s)^{\Omega_i}$ the span of the λ-eigenfunctions of $R_{\Omega_i} R^*$ for $\lambda \in [a, b]$ and observe that all $x \in X_i$ are α-approximate λ-eigenfunctions for every $\lambda \in [a, b]$.

Next we consider those $x \in X_i$ which vanish on the ρ-boundary of Ω_i i.e. $x \in X_i^\rho \overset{def}{=} X_i \cap (\mathbb{R}^s)^{\Omega_i^{-\rho}}$ and observe that

$$|\dim X_i^\rho - \dim X_i|/|\Omega_i| \underset{i \to \infty}{\to} 0$$

for every fixed ρ, a and b by the amenability of $\{\Omega_i\}$. Thus the estimate for $\dim X_i$ reduces to that for X_i^ρ. Then we take the intersection of $X_i^{-\rho}$ with the span of the eigenfunctions of $S_\rho^* S_\rho$, (with Ω_i in place of Ω) corresponding to the eigenvalues $\leq \beta^2$. We denote this by $X_i^{\rho,\beta} \subset X_i^\rho$ and notice that the operator $S_\rho : (\mathbb{R}^s)^{\Omega_i^{-\rho}} \to (\mathbb{R}^s)^{\Gamma \setminus \Omega_i}$ has norm $\leq \beta$ on $X_i^{\rho,\beta}$. Furthermore, according to $(\beta)_\rho$, the dimension of $X_i^{\rho,\beta}$ is rather close to that of X_i^ρ for large i and ρ. Namely, $\forall \beta > 0,\ \varepsilon > 0 \exists \rho$, s.t.

$$\limsup_{i \to \infty} (\dim X_i^\rho - \dim X_i^{\rho,\beta})/|\Omega| \leq \varepsilon\,.$$

Thus all we need is to estimate the dimension of $X_i^{\rho,\beta}$. To do this we invoke $(\star)$, and apply it to $\lambda = a \in [a, b]$ with $a - b = \alpha$ and get

$$a(1 - a) \leq 2(a - b) + \beta\,,$$

provided some space $X_i^{\rho,\beta}$ has positive dimension. In other words, the inequality

$$|a - b| < (a(1 - a) - \beta)\,/2$$

forces $\dim X_i^{\rho,\beta} = 0$; consequently

$$\limsup_{i \to \infty} \dim X_i^\rho / |\Omega_i| \leq \varepsilon$$

for sufficiently large $\rho = \rho(\beta, \varepsilon)$ and then also

$$\limsup_{i \to \infty} X_i / |\Omega_i| = 0$$

since $\varepsilon \to 0$ for $\rho \to \infty$. Thus we proved our propositions for all intervals $[a, b]$, where

$$|a - b| < a(1 - a)/2\,, \tag{$*$}$$

since $\beta > 0$ can be chosen arbitrarily small. Finally, we cover an arbitrary interval lying strictly within $[0, 1]$ by those satisfying $(*)$ and thus conclude the proof.

1.12.A$'$. Corollary. *Let $E_i \subset (\mathbb{R}^s)^{\Omega_i}_{\ell_2}$ be the image of the unit ℓ_2-ball in a Γ-invariant Hilbert subspace $Y \subset (\mathbb{R}^s)^{\Gamma}_{\ell_2}$ under the restriction map R_{Ω_i}. Then*

$$\lim(\mathrm{Widim}_\varepsilon B_i)/|\Omega_i| \to \dim_{\ell_2}(Y : \Gamma)$$

for each ε in the interval $0 < \varepsilon < 1$.

In fact, E_i is a full ellipsoid in the Euclidean space $(\mathbb{R}^s)^{\Omega_i}_{\ell_2}$ where the majority of the principal semiaxes λ_ν, $\lambda = 1, \dots |\Omega_i|$, is concentrated at zero or at one. It follows, the average of λ_ν, eventually defining the Von Neumann dimension $\dim_{\ell_2}(Y : \Gamma)$, is essentially determined by λ's close to one and our claim follows since $\mathrm{Widim}_\varepsilon$ of an ellipsoid E with semiaxes λ_ν equals the number of λ_ν's greater than ε' for some ε' in the interval $\varepsilon \le \varepsilon' \le 2\varepsilon$.

Remark. It is obvious that the number $n(\varepsilon')$ of $\lambda_\nu \ge \varepsilon'$ satisfies

$$n(\varepsilon') \ge \mathrm{Widim}_\varepsilon E \text{ for } \varepsilon' = 2\varepsilon$$

while the inequality

$$n(\varepsilon) \le \mathrm{Widim}_\varepsilon E$$

trivially follows from 1.1.B. Probably, it is not hard to evaluate the critical ε' for which $n(\varepsilon') = \mathrm{Widim}_\varepsilon E$.

1.12.A$''$. The restriction maps $R_{\Omega_i} : Y \to (\mathbb{R}^s)^{\Omega_i}$ arise from the evaluation map $R_e : Y \to \mathbb{R}^s$ for $y \mapsto y(e)$ for the identity element $e \in \Gamma$. Now, let $R : Y \to \mathbb{R}^N$ be an arbitrary bounded operator and let $R_i : Y \to (\mathbb{R}^N)^{\Omega_i}$ be the orthogonal sum of the γ-translates of R for γ ranging over Ω_i. We define $E_i \subset (\mathbb{R}^N)^{\Omega_i}$ as above with R_i in place of R_{Ω_i} and let

$$d_R = \lim_{\varepsilon \to 0} \lim_{i \to \infty} \mathrm{Widim}_\varepsilon E_i\,.$$

Then a straightforward generalization of the above arguments shows that the supremum of d_R over all operators $R : Y \to \mathbb{R}^N$, $N = 1, 2, \dots$, equals the Von Neumann dimension $\dim_{\ell_2}(Y : \Gamma)$.

ℓ_p**-remark.** The above definition of $\dim_{\ell_2}(Y:\Gamma)$ via d_R makes sense for an arbitrary Banach space Y with a Γ-action. Here one can make some modifications, e.g. by using the ℓ_p-norm in the Cartesian power $(\mathbb{R}^N)^{\Omega_i}$ for $p \neq 2$ (compare 1.6.A) and/or to allow more general (linear and non-linear) maps R from Y to suitable spaces. Eventually this line of thought converges to the discussion in 1.6.A.

Question. What is $\dim_{\ell_p}(\mathbb{R}^s)^\Gamma_{\ell_q}$ for amenable groups Γ? One may (?) expect that $\dim_{\ell_p}(\mathbb{R}^s)^\Gamma_{\ell_p} = s$ for all p in the interval $1 < p < \infty$, where the major issue is the inequality $\dim_{\ell_p}\mathbb{R}^\Gamma_{\ell_p} < \infty$. This would imply, in particular, that the ℓ_p-spaces $(\mathbb{R}^{s_1})^\Gamma_{\ell_p}$ and $(\mathbb{R}^{s_2})^\Gamma_{\ell_p}$ are not Γ-isomorphic for $s_1 \neq s_2$, at least for amenable groups Γ. This seems to be unknown even for $\Gamma = \mathbb{Z}$.

1.12.B. Non-linear Von Neuman. The classical definition of the Von Neumann dimension extends to certain infinite dimensional smooth manifolds with invariant measures where the tangent bundles admit Γ-invariant Hilbert structures. More generally, let Γ act on a compact space X with a probability measure μ and let $T \to X$ be a Hilbert bundle, such that the action of Γ lifts to T and preserves the Hilbert norm in the fibers $T_x \subset T$, $x \in X$. Then we take the space $\mathcal{X}$ of $L_2(\mu)$-sections $s : X \to T$ acted upon by the *Von Neumann* algebra A generated by all $\gamma \in \Gamma$ acting on the sections and by the operators $s \mapsto fs$ for all continuous functions $f : X \to \mathbb{R}$. With all this, one has a bona fide Von Neumann dimension $\dim_A \mathcal{X}$. (If the measure is concentrated at a single fixed point $x_0 \in X$, then $\dim_A \mathcal{X} = \dim_{\ell_2}(T_{x_0} : \Gamma)$.) And if Γ is trivial the above becomes the ordinary rank of T, i.e. $\dim T_x$.

Remark. While our mean dimension parallels the topological entropy, the above Von Neumann dimension is reminiscent of the metric entropy. This may suggest the following questions. Which (infinite dimensional) Γ-manifolds X have $\dim_A \chi \leq \dim(X:\Gamma)$ and when does $\sup\limits_{\mu} \dim_A \chi = \dim(X:\Gamma)$ for μ running over all invariant probability measures on X? However, we do not expect the positive answer, unless the definitions are modified in some (?) way (compare 2.1).

1.13. Transcendence degree for Γ-fields. Let F be an extension of a given field K and let Γ act by automorphisms of F fixing K, i.e. we are given a homomorphism $\Gamma \to \mathrm{Gal}(F/K)$.

Basic Example. Consider independent variables x_γ associated to all $\gamma \in \Gamma$ and take F equal the field of rational K-fractions (functions) in these variables. In other words, F equals the field of rational functions on K^Γ viewed as a proalgebraic variety.

In general, we assume that F is finitely Γ-generated over K, i.e. there exists a subfield $F_0 \subset F$ whose Γ-translates generate F (as the above $K(x_{id})$ Γ-generates $F = K\{x_\gamma\}$) and then define Γ-*transcendence degree* of F *over* K with a given amenable exhaustion $\{\Omega_i\}$ of Γ as follows. Let $F_i \subset F$ be generated by γF_0 for all $\gamma \in \Omega_i$ and a given F_0 Γ-generating F. Then

$$\operatorname{trandeg}(F : \{\Omega_i\}) \underset{def}{=} \liminf_{i\to\infty} |\Omega_i|^{-1} \operatorname{trandeg} F_i/K\,.$$

We shall not pursue this algebraic line of thinking anymore but shall return to proalgebraic varieties Y in 2.7 where their dimension will be studied in a topological framework. (If Y is *irreducible* then we can pass to the function field F and define $\dim(Y : \Gamma)$ as $\operatorname{trandeg}(F : \Gamma)$. But isolating an irreducible component in (reducible) Y may cause a problem as this may be not Γ-invariant.)

1.14. Γ-spaces with slowly growing $\dim Y \mid \Omega_i$**.** If our $Y \subset \underline{X}^\Gamma$ is given by a *balanced* (or determined) law $L \subset \underline{X}^D$, i.e. L has $\operatorname{codim} L = \dim \underline{X}$, then we expect that the spaces L_Ω of L-legal functions on $\Omega \subset \Gamma$ (see 0.2) have dimensions much smaller than $|\Omega|$ and this is even more likely to happen for the *overdetermined* case where $\operatorname{codim} L > \dim \underline{X}$ (here we deal with discrete groups Γ and finite subsets $D \subset \Gamma$) for "sufficiently amenable" Ω, i.e. having relatively small boundary $\partial_\rho \Omega$. For example, if $\Gamma = \mathbb{Z}^n$ and Ω_i are the i-balls in Γ, then we expect that $\dim L_{\Omega_i}$ is asymptotic to $i^{n-1}(\approx |\Omega_i|^{\frac{n-1}{n}})$ rather than to $i^n(\approx |\Omega_i|)$, since solutions of balanced difference equation should be determined by their values on a suitably "Cauchy (hyper)surface" in $\mathbb{Z}^n$, e.g. on $\mathbb{Z}^{n-1} \subset \mathbb{Z}^n$.

The above suggests a modification of our definitions of various dimensions $(X : \{\Omega_i\})$ where the cardinality $|\Omega_i|$ is replaced by $|\Omega_i|^\beta$ for some $\beta < 1$ or by a more general function $\alpha(|\Omega_i|)$. Then one can speak of the *critical exponent*, that is the maximal (or, rather supremal) β, such that the β-dimension is infinite for all sequences $\Omega_i \subset \Gamma$ with $|\Omega_i| \to \infty$. Next, one may try to compute the β_{crit}-dimension with some "most amenable" exhaustion $\{\Omega_i\}$ of Γ. This will be done for some examples in the second part of this paper. Here we only observe that for *linear* laws L the following three conditions are equivalent:

(1) $\operatorname{prodim}(Y : \{\Omega_i\}) > 0$, i.e. $\liminf\limits_{i\to\infty} \dim L_{\Omega_i}/|\Omega_i| > 0$;

(2) $\liminf_{i\to\infty} \dim L_{\Omega_i}/|\partial_\rho \Omega_i| = \infty$ for all $\rho > 0$;

(3) there exists a non-zero function $y : \Gamma \to \underline{X}$ from $Y = Y(L)$ with finite support.

Indeed, obviously, (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) for all amenable sequences $\{\Omega_i\}$ in Γ.

Notice that (3) says in effect that the implied homomorphism $K^s(\Gamma) \to K^r(\Gamma)$ (for $K^s = \underline{X}$, compare 1.11) is non-injective. Also, one can replace "with finite support" in (3) by $y \in \ell_2(\Gamma)$ in the case $K = \mathbb{R}$.

1.15. Mean Poincaré polynomial. Next topological invariant coming after dimension is the *Poincaré polynomial* of a metric space X encoding its Betti numbers. This can be modified to $\mathrm{Poincar}_\varepsilon X$ by factoring away *ε-fillable* classes in $H_*(X)$, i.e. realizable by cycles C with $\mathrm{FilRad}\, C \leq \varepsilon$ for the metric on C induced from X (compare $[\mathrm{Gro}]_{\mathrm{FRM}}$). Thus the degree of $\mathrm{Poincar}_\varepsilon X$ equals the filling dimension mentioned in 1.1.F. Then we enlarge the metric $|x - x'|$ in a Γ-space X to $|x - x'|_\Omega$ for $\Omega \subset \Gamma$ (see 1.2) and define the mean Poincaré polynomial as a limit of suitably normalized polynomials $\mathrm{Poincar}_\varepsilon\ X_{\Omega_i}$, for $X_{\Omega_i} \overset{def}{=} (X, |x - x'|_{\Omega_i})$. Namely, we take

$$\lim_{\varepsilon\to 0}\lim_{i\to\infty} \ (\text{Poincaré}_\varepsilon X_{\Omega_i})^{1/|\Omega_i|}\,. \tag{$*$}$$

For example, the zero degree term of this limit equals $\exp(\mathrm{topextent}(\mathrm{comp}\, Y : \{\Omega_i\})$ where $\mathrm{comp}\, Y$ denotes the space of connected components of Y.

Remark. The above "normalization" by the $|\Omega_i|^{-1}$ exponent is motivated by the exponential bound on the Poincaré polynomial for algebraic laws. Namely, if $L \subset (\mathbb{R}^k)^D$ is an algebraic variety, then the Poincaré polynomial $P_\Omega(t)$ of the space L_Ω of L-legal functions on Ω is bounded by $\exp C_t|\Omega|$ as follows from Petrovski-Thom-Milnor inequality. But the behavior of coefficients of fixed degree (i.e. of individual Betti numbers) (as well as the convergence in $(*)$ for $i \to \infty$ with suitable Ω_i) is a more delicate matter which we do not study in this part of the paper.

§ 2. Evaluation of the mean dimension for subshifts of finite type.

We exhibit in this section a variety of examples, where the dimension of a subshift $Y \subset \underline{X}^\Gamma$ equals $\dim \underline{X}$ minus the number of (difference) equations defining Y.

2.1. Prodim and legdim in the linear category. Let $\underline{X}$ be a finite dimensional vector space over a field K, say $\underline{X} = K^s$, (e.g. $K = \mathbb{R}$) and take a subshift $Y = Y(L) \subset \underline{X}^\Gamma$ defined by a linear law $L \subset \underline{X}^D$, $D \subset \Gamma$ (see 1.8.B). We observe that *the projective dimension of such a linear Y equals its legal dimension, i.e.*

$$\operatorname{prodim}(Y|\{\Omega_i\}) = \operatorname{legdim}(Y : \{\Omega_i\}) \tag{$\circ$}$$

for all amenable sequences $\Omega_i \subset \Gamma$. (See §0 and 1.9 for notations.)

Proof. Let $L_\Omega \subset \underline{X}^\Omega$ be the space of legal K^s-valued functions on Ω, denote by $M^\rho_\Omega \subset L_\Omega$ the subspace of functions $\Omega \to K^s$ vanishing on the ρ-boundary of Ω, i.e. on $\Omega \cap \partial_\rho\Omega$, and observe that $\dim M^\rho_\Omega \geq \dim L_\Omega - \operatorname{card}\partial_\rho\Omega$. On the other hand, if ρ is sufficiently large, i.e. if the ρ-ball in Γ around the identity contains our $D \subset \Gamma$, then M^ρ_Ω naturally embeds into $Y = Y(L)$, where each function $x : \Omega \to K^s$ extends by the identical zero on $\Gamma\backslash\Omega$ and where the inclusion $D \subset \rho$-ball ensures the L-legality of such extension. Now, if Ω_i is an amenable sequence, then $(\operatorname{card}\Omega_i)^{-1}\dim M^\rho_{\Omega_i}$ and $(\operatorname{card}\Omega_i)^{-1}\dim L_{\Omega_i}$ have the same asymptotice behavior for $i \to \infty$ by the above inequality and our claim follows. Q.E.D.

2.1.A. Remarks. (a) Nonlinearity. If L is a non-linear law it makes little sense to make $y \in Y = Y(L) \subset \underline{X}^\Gamma$ vanish at infinity but instead of this one can look at the pairs (y_1, y_2). $y_i \in Y$, $i = 1, 2$, such that $y_1(\gamma) = y_2(\gamma)$ for all but finitely many $\gamma \in \Gamma$. Then one can easily show for many non-linear subshifts of *finite type* that such pairs are abundant in $Y \times Y$ if a suitable mean dimension of Y is positive.

(b) Summability. A more interesting generalization concerns the linear case where $\underline{X} = \mathbb{R}^s$ and $Y \subset \underline{X}^\Gamma$ is a *weakly closed* linear subspace, i.e. it is closed for the weak convergence in $\underline{X}^\Gamma$ where $x_i \to x$ iff $x_i(\gamma) \to x(\gamma)$ for each $\gamma \in \Gamma$. Here one cannot guarantee that the inequality prodim$(Y\{\Omega_i\}) > 0$ implies the existence of non-zero $y \in Y$ with finite support in Γ, but one can ensure the existence of a *square summable* $y : \Gamma \to \mathbb{R}$ in Y. *In fact, for every* $p \geq 2$,

$$\operatorname{prodim}(Y_{\ell_p}|\{\Omega_i\}) = \operatorname{prodim}(Y|\{\Omega_i\}),$$

where Y *is a weakly closed linear subspace in* $\underline{X}^\Gamma = (\mathbb{R}^s)^\Gamma$, *where* $Y_{\ell_p} \underset{def}{=} Y \cap \ell_p(\Gamma, \mathbb{R}s)$ *and* $\Omega_i \subset \Gamma$ *is an amenable sequence.*

Idea of the proof. If a linear space of functions on a finite set Ω, e.g. our $Y|\Omega$ has dimension close to card Ω, then it contains many sharply localized (concentrated) functions y where $(\sum_{w\in\Omega} y^2(w))^{1/2}$ is of the order $\sup_{w\in\Omega} |y(w)|$. Furthermore, one can find many, about dim(our space), such y's, which vanish on a given subset in Ω provided this subset has relatively small cardinality (such as $\partial_\rho\Omega_i$ in Ω_i for large i). All this follows by simple-minded linear algebra and, when applied to $Y|\Omega_i$, yields in the limit for $i \to \infty$ "many" non-zero functions $y \in Y_{\ell_2}$ and thus in all Y_{ℓ_p} for $p \geq 2$. Actually there is the following standard trick of doing this very quickly. Let P_i denote the normal projection from $\ell_2(\Gamma; \mathbb{R}^s)$ to the space of functions $Y \mid \Omega_i$ extended by zero outside $\Omega_i \subset \Gamma$. We think of these operators on $\ell_2(\Gamma; \mathbb{R}^s)$ as matrices indexed by Γ with entries in $GL_s\mathbb{R}$, written $P_i(\gamma, \gamma')$, and observe that

$$\dim Y|\Omega_i = \sum_{\gamma\in\Omega_i} \operatorname{trace} P_i(\gamma, \gamma).$$

Next we observe that the functions $P_i(\gamma, \gamma')$ on $\Gamma \times \Gamma$ weakly converge for $i \to \infty$ to some $P = P(\gamma, \gamma')$ which is invariant under the diagonal action of Γ on $\Gamma \times \Gamma$. Clearly, the image $P(\ell_2(Y; \mathbb{R}^s))$ is contained in Y_{ℓ_2} and its projective dimension with respect to $\{\Omega_i\}$ equals that of Y. In fact, this argument shows that the *Von Neumann dimension* of Y_{ℓ_2} equals the projective dimension of Y.

Notice that ℓ_2-functions produced by this method appear as normal projections of δ-functions to Y_{ℓ_2},

$$\delta(\gamma) = \begin{cases} 1 \text{ at a given } \gamma \in \Gamma \\ 0 \text{ for } \gamma' \neq \gamma \end{cases}$$

and of more general functions ρ with compact supports on Γ. Such a projection can be obtained in certain cases as the limit $(t \to \infty)$ of the heat flow which suggests a possible (not the only one) non-linear generalization of the ℓ_2-story and which shall be discussed further in the second part of this paper.

2.1.B. Comparison between $\dim_{\ell_1}$, $\dim_{\ell_2}$ and $\dim_{\ell_\infty}$. The above suggests that the mean dimension of $Y_{\ell_\infty} \overset{def}{=} Y \cap \ell_\infty(\Gamma; \mathbb{R}^s)$ minorizes the Von Neumann dimension of Y_{ℓ_2}. Here $\ell_\infty(\Gamma; \mathbb{R}^s)$ denotes the space of bounded functions on Γ with the sup-norm and the mean dimension refers to that of $Y_{\ell_\infty} \cap B^\Gamma$ for a ball $B \subset \mathbb{R}^s$. I do not know if this actually is true but a similar result is valid with our topological widim_ε replaced by its linear counterpart, denoted $\dim_\varepsilon^- A$. This is defined for (centrally symmetric) subsets A in a Banach space L as the maximal dimension n of a linear subspace $L_0 \subset L$, such that the intersection $A \cap L_0$ contains the ε-ball in L_0 around the origin. Notice that $\dim_\varepsilon^- A \leq \text{widim}_\varepsilon A$ according to 1.1.B but it remains unclear when $\dim_\varepsilon^- A \geq \text{widim}_{\varepsilon'} A$.

Next, given a Γ-invariant subspace $X \subset \ell_\infty(\Gamma; \mathbb{R}^s)$, we take its intersection with the unit ball in $\ell_\infty(\Gamma; \mathbb{R}^s)$, call this $X_1 \subset X$ and project it to the spaces $\ell_\infty(\Omega_i; \mathbb{R}^s)$ by just restricting functions from Γ to $\Omega_i \subset \Gamma$ as we always do. We look at the images of X_1, call them $X_1|\Omega_i \subset \ell_\infty(\Omega_i; \mathbb{R}^s)$, and set

$$\dim_{\ell_\infty}(X : \{\Omega_i\}) = \lim_{\varepsilon \to 0} \lim_{i \to \infty} \sup \dim_\varepsilon^- (X_1|\Omega_i)/|\Omega_i| \,.$$

Now, the argument in 1.12 shows that *if the space X_1 is weakly closed in $\ell_\infty(\Gamma; \mathbb{R}^s)$, then*

$$\dim_{\ell_2}(X_{\ell_2} : \Gamma) \geq \dim_{\ell_\infty}(X : \{\Omega_i\}) \tag{$*$}$$

for every amenable exhaustion $\{\Omega_i\}$ of Γ. In particular, if X_1 is weakly closed and $\dim_{\ell_\infty}(X : \{\Omega_i\} > 0$ for *some amenable exhaustion $\{\Omega_i\}$ then X contains a non-zero ℓ_2-function $\Gamma \to \mathbb{R}^s$.*

2.1.B′. Remarks. (a) The present condition of weak closeness is by far less demanding than the one in 2.1. In particular, the above $(*)$ applies to the spaces coming from solutions of linear elliptic P.D.E.

(b) Every ℓ_2-function on Γ is bounded. Furthermore every non-zero $x \in X \cap \ell_2(\Gamma; \mathbb{R}^s)$ gives rise to many functions in $X \subset \ell_\infty(\Gamma : \mathbb{R}^s)$ by taking sums $\sum\limits_{\gamma \in \Gamma} c_\gamma \gamma x$ for *square summable* (i.e. ℓ_2) functions $\gamma \mapsto c_\gamma \in \mathbb{R}$ on Γ. But it is unclear if

$$\dim_{\ell_2}(X_{\ell_2} : \Gamma) \geq \dim_{\ell_\infty}(X : \{\Omega_i\}) \,.$$

It is not even clear what kind of ℓ_2-condition ensures the positivity of $\dim_{\ell_\infty}$ and/or of the mean dimension. On the other hand, *if X contains a single non-zero ℓ_1-function, then $\dim_{\ell_\infty}(X : \{\Omega_i\})$ is positive* for every amenable *exhaustion $\{\Omega_i\}$* of Γ. In fact, given an $x \in X \cap \ell_1(\Gamma; \mathbb{R}^s)$, we get lots of bounded functions in X by taking sums $\sum_{\gamma\in\Gamma} e_\gamma\lambda(x)$ for *bounded* functions $\gamma \mapsto c_\gamma \in \mathbb{R}$ on Γ. These suffice to prove that $\dim_{\ell_\infty} > 0$ and, probably, to show that $\dim_{\ell_1} \le \dim_{\ell_\infty}$ for a suitably defined dimension $\dim_{\ell_1} = \dim_{\ell_1}(X_{\ell_1} : \{\Omega_i\})$.

Squaring ℓ_2-functions. Suppose we are given a bilinear map $\mathbb{R}^s \otimes \mathbb{R}^s \to \mathbb{R}^{s'}$, denoted $(x, y) \mapsto x\bullet y$ and observe that so defined product of ℓ_2-function $\Gamma \to \mathbb{R}^s$ lands in $\ell_1(\Gamma, \mathbb{R}^{s'})$. Denote by $X^2_{\ell_2} \subset \ell_1(\Gamma, \mathbb{R}^{s'})$ the set of the products of all $x, y \in X^2_{\ell_2}$, take the linear span of X_{ℓ_2} and let $\ell_\infty(X^2_{\ell_2})$ be the closure of this span in $\ell_\infty(\Gamma : \mathbb{R}^{s'})$. If our product is sufficiently non-degenerate, then

$$\dim_{\ell_2}(X_{\ell_2} : \Gamma) \neq 0 \Rightarrow \dim(\ell_\infty(X^2_{\ell_2}) : \Gamma) \neq 0\,.$$

Question. When does one have the inequality

$$\dim(\ell_\infty(X^2_{\ell_2}) : \Gamma) \ge \dim_{\ell_2}(X_{\ell_2} : \Gamma)\,?$$

For example, is this true for $\mathbb{C}^s$-valued functions with the component-wise product $\mathbb{C}^s \otimes \mathbb{C}^s \to \mathbb{C}^s$?

2.2. Genericity and Γ-transversality. Denote by $\Omega \div D$ the set of $\gamma \in \Gamma$, such that $\gamma D \subset \Omega$. In other words, $\Omega \div D$ is the maximal subset Ω^- in Γ such that $\Omega^- D \subset \Omega$. Clearly, the cardinality of this subset $\Omega \div D$ in Γ satisfies

$$|\Omega \div D| \le |\Omega|$$

and

$$|\Omega_i \div D| / |\Omega_i| \to 1$$

for every finite subset D and each amenable sequence $\Omega_i \subset \Gamma$. (Notice that if Γ has no torsion, then $|\Omega \div D| \le |\Omega| - |D| + 1$.) It is equally clear that the subspace $L_\Omega \subset \underline{X}^\Omega$ of L-legal functions on Ω (i.e. those $x : \Omega \to \underline{X}$ whose restriction to every translate $\gamma D \subset \Omega$ is contained in L) has

$$\operatorname{codim} L_\Omega \le |\Omega \div D| \operatorname{codim} L$$

and so

$$\operatorname{legdim}(Y : \{\Omega_i\}) \ge \dim \underline{X} - \operatorname{codim} L \tag{+}$$

for all amenable Ω_i in Γ.

Ω-Transversality. Denote by $\tilde{L} \subset \underline{X}^\Gamma$ the pull-back of $L \subset \underline{X}^D$ under the restriction map (projection) $\underline{X}^\Gamma \to \underline{X}^D$ and say that L is *Ω-transversal*, for a given subset $\Omega \subset \Gamma$, if the translates $\gamma\tilde{L} \subset \underline{X}^\Gamma$ are all simultaneously transversal for γ running over $\Omega \div D$. This makes sense, strictly speaking, only for finite subsets $\Omega \subset \Gamma$; if Ω is infinite this is understood as Ω_0-transversality for all finite subsets $\Omega_0 \subset \Omega$.

It is clear that

$$L_\Omega = \bigcap_{\gamma\in\Omega\div D} \gamma\tilde{L} \quad \text{and } Y(L) = \bigcap_{\gamma\in\Gamma} \gamma\tilde{L}\,.$$

Thus Ω-transversality implies that

$$\operatorname{codim} L_\Omega = |\Omega \div D| \operatorname{codim} L$$

and Γ-transversality yields the "expected" identity

$$\operatorname{legdim}(Y : \{\Omega_i\}) = \dim \underline{X} - \operatorname{codim} L \tag{=}$$

for all amenable $\Omega_i \subset \Gamma$.

Now we want to decide how generic is the Γ-transversality assumption. To do this we represent $L \subset \underline{X}^D$ by r independent linear equations with $r = \operatorname{codim} L$, i.e. we make $L = \operatorname{Ker} \underline{\alpha}$ for some linear map $\underline{\alpha} : \underline{X}^D \to K^r$, let $\alpha : \underline{X}^\Gamma \to (K^r)^\Gamma$ be the corresponding difference operator and $\alpha' : K^r(\Gamma) \to K^s(\Gamma)$ be the dual $K(\Gamma)$-morphism for $s = \dim \underline{X}$ (and $K^s = \underline{X}$). It is obvious that

the morphism α' is one-to-one if and only if L is Γ-transversal.

Thus the Γ-transversality problem and issuing relation (=) reduce to deciding when α' is injective.

2.2.A. It is notationally convenient to interchange r and s and look at maps α from $K^r(\Gamma) \subset (K^r)^\Gamma$ to $K^s(\Gamma) \subset (K^s)^\Gamma$ defined by $\underline{\alpha} \in$ Maps $((K^r)^D \to K^s)$. We denote by $\underline{\operatorname{In}}_\Omega \subset$ Maps $((K^r)^D \to K^s)$, $\Omega \in \Gamma$ the subset of those $\underline{\alpha}$ for which the kernel of $\alpha : K^r(\Gamma) \to K^s(\Gamma)$ contains *no* function $x : \Gamma \to \underline{X} = K^s$ with support in Ω (where we view elements in $K^s(\Gamma)$ as K^s-valued functions on Γ with finite supports). Clearly, if Ω is finite, this is a Zariski open subset and if $s \geq r$ it is *non-empty* as it contains an injective $\underline{\alpha} : K^r(\Gamma) \to K^s(\Gamma)$ corresponding to an embedding $K^r \to K^s$. The subset we really want to understand is $\underline{\operatorname{In}} \underset{def}{=} \underline{\operatorname{In}}_\Gamma$ corresponding to *injective* α and this equals intersection of $\underline{\operatorname{In}}_D$ over all finite $D \subset \Gamma$. We see from the above that this $\underline{\operatorname{In}} \subset$ Maps $((K^r)^D \to K^s) = K^{rs|D|}$ equals the intersection of a countable family of Zariski open subsets and therefore it is rather large, at least for uncountable fields K. In fact, it is clear that

if $s \geq r$, then every $\underline{\alpha}$, whose all $sr|D|$ components are algebraically independent over the prime field $K_0 \subset K$, gives rise to an injective α, where the corresponding (dual) L satisfies $(=)$.

2.2.B. Suppose $K = \mathbb{R}$ and show that

$$\underline{\mathrm{In}} \subset \mathbb{R}^{sr|D|} \textit{ contains a non-empty open subset for } s \geq r\,.$$

To see this let first $s = r$ and observe that every operator of the form $\mathbb{1} + \varepsilon : \mathbb{R}^s(\Gamma) \to \mathbb{R}^s(\Gamma)$ is injective if the sup-norm of ε is < 1, since the equation $\varepsilon(x) = x$ has no nontrivial solution for $\|\varepsilon\| < 1$ (where one may allow non-linear operators ε as well). Now, if ε comes from $\underline{\varepsilon} : \mathbb{R}^{s|D|} \to \mathbb{R}^s$, then the condition $\|\varepsilon'\| < 1$ is ensured by the inequality $\|\underline{\varepsilon}\| < 1$, where the norm of $\underline{\varepsilon}$ can be taken relative to an arbitrary norm on $\mathbb{R}^s$ and the corresponding sup-norm on $\mathbb{R}^{s|D|} = (\mathbb{R}^s)^{|D|}$. This yields our claim for $s = r$ and the case $s > r$ trivially follows.

(a) One can relax the condition $\|\underline{\varepsilon}\| < 1$ to $\|\underline{\varepsilon}\| \leq 1$ provided the equality $\|\underline{\varepsilon}(\underline{x})\| = \|(\underline{x})\|$, for $\underline{x} = (\underline{x}_1, \underline{x}_2, \dots, \underline{x}_{|D|})$, $\underline{x}_i \in \mathbb{R}^s$, is possible only for $\underline{x}_1 = \underline{x}_2 = \dots \underline{x}_{|D|}$. This situation arises, for example, for diffusion operators on $\mathbb{R}(\Gamma)$.

Remarks. (b) The above applies to every normed field K, e.g. to $\mathbb{Q}$ with a p-adic norm. For example, if $\underline{\varepsilon}$ is given by an $(s|D| \times s)$ matrix with integer entries divisible by a prime number, then *the corresponding morphism $\mathbb{1} + \varepsilon : \mathbb{Q}^s(\Gamma) \to \mathbb{Q}(\Gamma)$ is injective.*

In fact, $\mathbb{1} + \varepsilon$ remains injective if we replace $\mathbb{Q}$ by an arbitrary field of characteristic zero. More generally, let λ be an arbitrary complex number and let all entries of the above matrix $\underline{\varepsilon}$ be integer polynomials in λ without constant terms (i.e. divisible by λ). Then *the corresponding morphism $\mathbb{1} + \varepsilon : \mathbb{C}^s(\Gamma) \to \mathbb{C}^s(\Gamma)$ is injective, provided either λ is transcendental or an algebraic integer which is not a root of unity.*

In fact, the field $\mathbb{Q}(\lambda)$ obviously admits a norm making $\|\underline{\varepsilon}\| < 1$ under the above assumptions.

(c) Our (implicite function) argument, shows that an injective morphism $\alpha : K^r(\Gamma) \to K^s(\Gamma)$ remains injective under small perturbations if it admits a right inverse, i.e. a morphism $\beta : K^s(\Gamma) \to K^r(\Gamma)$, such that $\beta \circ \alpha : K^r(\Gamma) \to K^r(\Gamma)$ equals $\mathbb{1}$. Such β obviously exists (and this was used above) for α induced by an embedding $K^r \to K^s$, but it is unclear how typical such invertibility is for general α. To get a perspective, let $\Gamma = \mathbb{Z}^n$ and $K = \mathbb{C}$. Then $K(\Gamma) = \mathbb{C}(\mathbb{Z}^n)$ identifies with the ring of regular functions on the torus $(\mathbb{C}^\times)^n$ and morphisms $\mathbb{C}^r(\mathbb{Z}^n) \to \mathbb{C}^s(\mathbb{Z}^n)$ become homomorphisms from the trivial vector bundle of rank r over this torus to such bundle of rank s. Then injectivity of α translates to injectivity of the vector bundle homomorphism on *some* fiber, while invertibility of α amounts to injectivity on *all* fibers. Thus we see that those $\underline{\alpha}$ for which α is injective (i.e. those fom $\underline{\mathrm{In}}$) constitute a *non-empty Zariski open subset in* $\mathbb{C}^{sr|D|}$ for $s \geq r$ and all finite subsets $D \subset \mathbb{Z}^n$, while $\underline{\alpha}$ corresponding to invertible α have a similar property only for $s \geq r+n$.

(d) If one replaces the space $\mathbb{R}^s(\Gamma)$ by its dual $(\mathbb{R}^s)^\Gamma$, then the corresponding implicite function argument yields *surjectivity* of maps $1\!\!1 + \varepsilon$ for suitably contracting (possibly non-linear) maps $\varepsilon : (\mathbb{R}^s)^\Gamma \hookleftarrow$ of finite type (i.e. defined via $\underline{\varepsilon} : (\mathbb{R}^s)^D \to \mathbb{R}^s$ for finite $D \subset \Gamma$). In fact, such an argument yields *bounded* (and also ℓ_p for $p < \infty$) solutions to the equation $x + \varepsilon(x) = y$ for $y \in \mathbb{R}^s(\Gamma) \subset \ell_\infty(\Gamma; \mathbb{R}^s) \subset (\mathbb{R}^s)^\Gamma$ and then one uses density of $\mathbb{R}^s(\Gamma)$ in $(\mathbb{R}^s)^\Gamma$. Notice that all this applies to $(K^s)^\Gamma$ for all complete normed fields K as well as some subvarieties in these $(K^s)^\Gamma$.

(e) Another class of injective examples is provided by positive selfadjoint operators $\mathbb{R}^s(\Gamma) \hookleftarrow$. For example, if $\beta : \mathbb{R}^s(\Gamma) \to \mathbb{R}^t(\Gamma)$ is injective, then $\beta^*\beta : \mathbb{R}^s(\Gamma)$ is positive selfadjoint and so injective.

2.2.C. Let us give a combinatorial condition on $D \subset \Gamma$ which provides a *non-empty Zariski open* subset in $\underline{\mathrm{In}} = \underline{\mathrm{In}}_\Gamma \subset \mathrm{Maps}\,((K^r)^D \to K^s)$.

Definition. A collection $\mathcal{E}$ of non-empty subsets $E \subset D$ is called an *extremal family* if for every non-empty finite subset $\Omega \subset \Gamma$ there exists $\gamma \in \Gamma$, such that $(\gamma\Omega) \cap D \in \mathcal{E}$.

Examples. (a) If $D \subset \mathbb{Z}^n \subset \mathbb{R}^n$, then the collection of the extremal points of the convex hull of D makes an extremal family as an obvious argument shows. The same is true for free groups realized as vertex sets in trees, where the convex hulls are defined as the spanning subtrees.

Next, for an arbitrary collection $\mathcal{E}$ of subsets in D define $\underline{\mathrm{In}}_\mathcal{E} \subset \mathrm{Maps}\,((K^r)^D \to K^s)$ as the set of those $\alpha : (K^s)^D \to K^r$ which are injective on the subspaces $(K^s)^E$ for all $E \in \mathcal{E}$, where $(K^s)^E$ embeds to $(K^s)^D$ by extending functions by zero on $D \backslash E$. Clearly, $\underline{\mathrm{In}}_\mathcal{E}$ is Zariski open and it is non-empty if

$$\sup_{E \in \mathcal{E}} \mathrm{card}\, E \leq s/r\,.$$

On the other hand, if $\mathcal{E}$ is an extremal family, then

$$\underline{\mathrm{In}}_{\mathcal{E}} \subset \underline{\mathrm{In}}_{\Gamma}\,.$$

In fact, if $\alpha(x) = 0$, then the support $\Omega \subset \Gamma$ of $x : \Gamma \to K^r$ must be empty as trivially follows from the above definitions. Thus we obtain a non-empty open subset in $\underline{\mathrm{In}}_{\Gamma}$, provided D admits an extremal family with

$$\sup \operatorname{card} E \leq s/r\,.$$

Example. A group Γ is called *D-uncoiled* if D admits an extremal system $\mathcal{E}$ with $\sup \operatorname{card} E = 1$. We say Γ is *uncoiled* (traditionally, *a unique product group*) if it is D-uncoiled for all finite subsets $D \subset \Gamma$. (Notice that free groups are uncoiled by the above remark and extensions of uncoiled groups by uncoiled are, obviously, uncoiled. All this is well known, see [Pass]. Also many hyperbolic groups are uncoiled, see [Delz].) For such groups *we have our non-empty Zariski open subset in $\underline{\mathrm{In}}$ for all D, provided* $s \geq r$ (which extend the solution of the *Kaplansky problem* for these groups, i.e. non-vanishing of α for $s = r = 1$ and $\underline{\alpha} \neq 0$), see [Pass]).

Next we want to make examples of injective α for $s < r$, i.e. we want to embed $K^r(\Gamma)$ to $K^s(\Gamma)$ for $s < r$.

Definition. Call a subset $D \subset \Gamma$ *tree-like* if for every finite subset Ω, there exist γ and $\gamma' \neq \gamma$ in Γ and $w \in \Omega$, such that

$$\gamma\Omega \cap D = \{\gamma w\} \text{ and } \gamma'\Omega \cap D = \{\gamma' w\}\,.$$

Basic example. Let $a,\ b \in \Gamma$ be freely independent. Then the subset $\{a,\ b,\ a^{-1}\} \subset \Gamma$ is tree-like. Indeed this reduces to the free case for $\Gamma = F(a, b)$, where everything is clear with $w \in \Omega$ being an extremal point of the spanning tree (convex hull) of Ω.

Now, define a subset $T_2 \subset \mathrm{Maps}((\mathbb{R}^r)^D \to \Gamma^s)$ as follows. For a pair of points $\delta,\ \delta' \in D$, consider the subspaces $\mathbb{R}^r_\delta,\ \mathbb{R}^r_{\delta'} \subset (\mathbb{R}^r)^D$ consisting of functions $D \to \mathbb{R}^r$ concentrated at δ and δ' correspondingly and identify them with the standard $\mathbb{R}^r$. Then each map $\underline{\alpha} : (\mathbb{R}^r)^D \to \mathbb{R}^s$ restricted to δ and δ', gives us a map of this $\mathbb{R}^r$ to $\mathbb{R}^{2s}$ and we declare $\underline{\alpha} \in T_2$ if this map is injective for every pair of distinct points in D. Clearly T_2 is Zariski open and it is non-empty for $2s \geq r$. On the other hand, our previous argument shows that if D is star-like, then $T_2 \subset \underline{\mathrm{In}}$ and so we obtain, in particular, an *injective $K(\Gamma)$-morphism from $K^2(\Gamma)$ to $K^1(\Gamma)$, provided* Γ *admits a star-like subset,* e.g. *if* Γ *contains a free non-Abelian subgroup.*

Remark. If the group Γ is amenable, then an easy argument shows that there is no injective morphism $K^r(\Gamma) \to K^s(\Gamma)$ for $r > s$. On the other hand such maps may exist for *all* non-amenable goups. Also one can ask which non-amenable groups admit star-like subsets where the picture is unclear, for example, for torsion groups with sufficiently rare sets of relations. (It is easy to construct a *non-Γ-invariant* embedding $K^2(\Gamma) \to K^1(\Gamma)$ of finite type for every non-amenable group using a bounded measure contracting "vector field" or such Γ.)

Application to the entropy. Let Γ be amenable and the subset $\underline{\mathrm{In}} \subset \mathrm{Maps}((K^r)^D \to K^s)$ be non-empty Zariski open, e.g. Γ is polycyclic torsion free, $D \subset \Gamma$ is an arbitrary finite subset and $r \leq s$. If the field K is finite, this $\underline{\mathrm{In}}$ may be still rather small but it increases as we pass to finite extensions $K_\bullet$ of K. In fact it covers almost all space $K_\bullet^{rs|D|} = \mathrm{Maps}((K_\bullet^r)^D \to K_\bullet^s)$ for large $K_\bullet$, i.e.

$$\mathrm{card}(\underline{\mathrm{In}}(K_\bullet)) / \mathrm{card}\, K_\bullet^{rs|D|} \to 1$$

for $\mathrm{card}\, K_\bullet \to \infty$.

Next we observe that the topological entropy of the space $Y = Y(L(K_\bullet'))$ for a given law $L \subset (K\bullet)^D$ obviously equals $\mathrm{prodim}(Y : \Gamma) \log \mathrm{card}\, K_\bullet$. Thus we obtain many examples of subshifts of finite type where we know what the entropy is,

$$\mathrm{topent}(Y : \Gamma) = (s - r) \log |K_\bullet| \,.$$

Observe that $\mathrm{topent}(Y : \Gamma)$ is notoriously difficult to compute for subshifts of finite type over Γ non-commensurable to $\mathbb{Z}$ and the above algebraic systems constitute the bulk of available examples (compare [Schm]). We conclude by noticing that the above applies to uncoiled groups and it seems harder to generate such examples for groups which contain coils or where uncoilness is unknown.

2.3. Stability and transversality. A (possibly non-linear) law $L \subset \underline{X}^D$ is called *ρ-stable* on $\Omega \subset \Gamma$ if the *legal* extendability of functions x from $\Omega_0 \subset \Omega$ (i.e. of $x \in (\underline{X})^{\Omega_0}$) to Ω, i.e. to functions $y \in L_\Omega \subset (\underline{X})^\Omega$, is equivalent to legal extendability to the ρ-neighbourhood $\Omega_0 + \rho$ of Ω_0, where we require this property for all Ω_0 in Ω, such that $\Omega_0 + \rho \subset \Omega$ (compare $[\mathrm{Gro}]_{\mathrm{ESAV}}$). It is easy to see that linear laws $L \in \mathrm{Gr}_r(K^s)^D$ which are ρ-stable on a finite subset Ω make a constructible subset in $\mathrm{Gr}_r(K^s)^D$ (i.e. a union of intersections of Zariski closed and Zariski open subsets). Also, if $\Omega_0 + 3\rho \subset \Omega$, and L is ρ-stable on Ω, then every legal function x on $\Omega_0 + 3\rho$ can be modified to x_0, such that $x_0 \mid \Omega_0 = x \mid \Omega_0$ and such that x_0 vanishes outside $\Omega_0 + 2\rho$. Thus, for $\rho > \operatorname{Diam} D$, one can extend such x_0 to a legal function on all of Γ vanishing outside $\Omega_0 + 2\rho$.

Lemma. *If L is ρ-stable on 10ρ-ball Ω_ρ in Γ for $\rho > \operatorname{Diam} D$ and L is Ω_ρ-transversal, then it is Γ-transversal.*

Proof. Suppose L is Ω-transversal and prove it is Ω_1-transversal for $\Omega_1 = \Omega \cup \{\gamma_1\}$ and some $\gamma_1 \in \Gamma \backslash \Omega$. Denote by $L_1^\perp \subset (K^s)^{\Omega_1}$ the intersection of those $L^\gamma \subset (K^s)^{\Omega_1}$ for which $\gamma_1 \in \gamma D \in \Omega_1$ and observe that all we need to prove is the transversality of $L_1^\perp$ with L_Ω. Then we take the 2ρ-ball $B_1 = B(\gamma_1, 2\rho)$ around γ_1 and project $(K^s)^{\Omega_1}$ to $(K^s)^{B_1 \cap \Omega_1}$. We denote by $\underline{L}_1^\perp$ and $\underline{L}_\Omega$ the images of $L_1^\perp$ and of L_Ω under this projection and observe that the transversality of these images is equivalent to the transversality of the original spaces L_Ω and $L_1^\perp$. But in the stable case these images are the same for smaller Ω, namely for $\Omega' = \Omega \cap B(\gamma_1, 5\rho)$, where the transversality follows from our assumptions. Thus the proof follows by induction on $\operatorname{card} \Omega$. Q.E.D.

Corollary. *The intersection of the subset* $\mathrm{Tran}_\Gamma \subset \mathrm{Gr}(K^s)^D$ *with the set of ρ-stable laws on Ω_ρ, say* $\mathrm{Tran}_\Gamma \cap St_\rho$*, is Zariski open in St_ρ.*

Question. Under which assumptions on Γ, D, ρ does $St_\rho \subset \mathrm{Gr}_r(K^s)^D$ contain a non-empty Zariski open subset ?

2.3.A. Disjoint transversality. This property is very close to the above stability and it expresses the idea of non-interaction between subsets in Γ separated by distances $\geq \rho$, where the space $Y = L_\Gamma \subset X = (K^s)^\Gamma$ serves as the medium of such intersection. Namely, we say that the space $Y \subset X$ is *ρ-disjointly transversal* if for every finite system of finite subsets $\Omega_1, \ldots, \Omega_i, \ldots, \Omega_m$ in Γ with $\operatorname{dist}(\Omega_i, \Omega_j) \geq \rho$, $1 \leq i < j \leq m$, the m subspaces Y_i consisting of $y \in Y$ satisfying $y \mid \Omega_i = 0$, $i = 1, 2, \ldots, m$, are transversal in Y. Then *disjoint transversality* means ρ-disjoint transversality for some $\rho > 0$.

It is clear that

$$\text{stability} \Rightarrow \text{disjoint transversality}$$

(where "stability" means "ρ-stability on Γ for some ρ") and in many cases (e.g. for groups with bounded asymptotic dimensions, see [Gro]$_{\mathrm{ESAV}}$) the disjoint transversality implies the existence of a *stable* sublaw $L' \subset L_{D'} \subset$

$(K^s)^{D'}$ for some finite subset $D' \subset \Gamma$, such that $L'_\Gamma = Y = L_\Gamma$. On the other hand, disjoint transversality of the spaces L_Ω for *all* finite $\Omega \subset \Gamma$ (with an obvious modification of the definition where Ω takes the role of Γ) is equivalent to the stability as an easy argument shows.

2.3.B. Open problem. The old unsolved question concerns the possible values of prodim$(Y : \Gamma)$. The above considerations indicate many examples, where this prodim is an *integer* for certain (torsion free) groups and it is quite easy to make examples where it takes rational values for groups with torsion. (See [Lück] for further discussion of this problem for the Von Neumann dimension.)

As we have seen above, the integrality of prodim follows from the Γ-transversality of a given presentation (i.e. a law L) of our $Y \subset (K^s)^\Gamma$, and one may ask for which Γ every $Y \subset (K^s)^\Gamma$ of finite type admits a presentation (possibly in some $(K^{s'})^\Gamma$ for $s' > s$) with Γ-transversal L. This can be, probably, expressed with a suitable Grothendieck group $\mathbb{K}_0(\Gamma) = \mathbb{K}_0(K(\Gamma))$ of finitely generated moduli over $K(\Gamma)$ (or a given amenable algebra A in general). Our prodim should give us a homomorphism, say $d : \mathbb{K}_0(\Gamma) \to \mathbb{R}$, and we also have a homeomorphism $i : \mathbb{Z} \to \mathbb{K}_0(\Gamma)$ where each $s \in \mathbb{Z}$ goes to $[K^s(\Gamma)] \in \mathbb{K}_0(\Gamma)$. Now the basic questions read: What is the image of d? When does it equal to $\mathbb{Z} \subset \mathbb{R}$ or is contained in $\mathbb{Q}$? What is there in $\mathbb{K}_0(\Gamma)/i(\mathbb{Z})$? Do the subgroups $i(\mathbb{Z})$ and $\ker d$ generate $\mathbb{K}_0(\Gamma)$?

Apparently, all this is well-known for *polycyclic* and, moreover, for *elementary amenable* groups, where $\mathbb{K}_0(\Gamma)$ tends to be quite small (as was pointed out to me by Ofer Gabber, also see [Lück]).

Another kind of a transversality question is as follows.

Given submoduli $Y \subset K^s(\Gamma)$ and $Y_0 \subset K^{s_0}(\Gamma)$, can one find a $K(\Gamma)$-morphism $\rho : K^s(\Gamma) \to K^{s_0}(\Gamma)$, such that

$$\operatorname{prodim}(Y \cap \rho^{-1}(Y_0)) \leq \delta$$

for a given $\delta > 0$? More specifically, when is this possible with $\delta = \operatorname{prodim} Y - s_0 + \operatorname{prodim} Y_0$? Or, even better, when can one find ρ mapping Y Γ-transversally (in an obvious sense) to Y_0? For example, when does, for a given $Y \subset K^s(\Gamma)$, there exist $\rho : K^s(\Gamma) \to K(\Gamma)$, such that the kernel of ρ is Γ-transversal to Y? Also, observe that the dimension type invariants of moduli lead to norms on $\mathbb{K}_0(\Gamma)$ and $\mathbb{K}_0(\Gamma)/i\,\mathbb{Z}$ (see $[\text{Gro}]_{\text{PCMD}}$). Finally, notice that the K-theoretic point of view does not do justice to such moduli as $K^s(\Gamma)/\rho K^s(\Gamma)$ for embeddings $\rho : K^s(\Gamma) \to K^s(\Gamma)$ (describing determined systems of independent difference equations).

2.4. Mean dimension of sub-linear subshifts. Take a linear subshift $Y_0 \subset \underline{X}^\Gamma$ for $\underline{X} = \mathbb{R}^s$ and let $\underline{B} \subset \mathbb{R}^s$ be a compact subset containing the origin in its interior. Then, $Y = Y_0 \cap \underline{B}^\Gamma \subset \underline{B}^\Gamma$ makes a closed Γ-invariant subshift (which can be called "sublinear") in the compact (full shift) space $\underline{B}^\Gamma$ where one may speak of our mean dimension $\dim(Y : \{\Omega_i\})$.

2.4.A. Proposition. *The mean dimension of $Y = Y(L) \subset \underline{B}^\Gamma$ equals the projective dimension of $Y_0 \subset \underline{X}^\Gamma$*

$$\dim(Y : \{\Omega_i\}) = \operatorname{prodim}(Y_0 \mid \{\Omega_i\})$$

for all amenable sequences $\Omega_i \subset \Gamma$. Consequently,

$$\dim(Y : \{\Omega_i\}) = \operatorname{legdim}(Y_0 : \{\Omega_i\})\,.$$

Proof. The upper bound on $\dim(Y : \{\Omega_i\})$ follows from 1.9 and we concentrate on the lower bound. We observe that the projection $p : \underline{B}^\Gamma \to \underline{B}^\Omega$ is distance decreasing for the metric $|x - x'|_\Omega$ on $\underline{B}^\Gamma$ and the sup-product metric on $\underline{B}^\Omega$. Then we assume without loss of generality that $\underline{B}$ equals a small ball in $\mathbb{R}^s$ around the origin and then, by applying 1.1.B, obtain the following inequality for the image $Y \mid \Omega \underset{def}{=} p(Y) \subset \underline{B}^\Omega$ with the sup-product metric

$$\operatorname{Widim}_\varepsilon Y_\Omega \geq \operatorname{Widim}_\varepsilon Y \mid \Omega$$

where $Y_\Omega = (Y, |x - x'|_\Omega)$ as earlier. Q.E.D.

Now the proposition is reduced to the following

Lemma. *There exists $\varepsilon = \varepsilon(\underline{X}) > 0$, such that*

$$\operatorname{Widim}_\varepsilon(B^\Omega \cap M) = \dim M$$

for all finite subsets $\Omega \subset \Gamma$ and every linear subspace $M \subset (\mathbb{R}^s)^\Omega \supset \underline{B}^\Omega$, (where, we use the sup-product metric in $\underline{B}^\Omega$).

Proof. Everything trivially reduces to $\underline{B}$ being a ball in $\mathbb{R}^h$ around the origin and then $\underline{B}^\Omega \cap M$ appears as the unit ball with respect to the Banach norm in M induced from the norm in $(\mathbb{R}^s)^\Omega$ with the unit ball $\underline{B}^\Omega$. Then 1.1.B applies and the proof follows. Q.E.D.

2.4.B. Corollary. *If $s \geq r$, then the subset of the laws L's in $\mathrm{Gr}_r(\mathbb{R}^s)^D$ giving Y of mean dimension $s - r$ is residual. Furthermore, it always contains a non-empty open subset and, if Γ is uncoiled, then "open" can be strengthened to "Zariski open".*

2.5. On local dimension of Γ-subvarieties. Let X be an algebraic variety (over some field, e.g. $\mathbb{R}$ or $\mathbb{C}$) or an analytic space (over $\mathbb{R}$ or over $\mathbb{C}$) or a smooth manifold (where, more generally, we may allow stratified spaces, e.g. polyhedra). We are interested in subsets $Y \subset X = \underline{X}^\Gamma$ defined by a law which is a subvariety $L \subset \underline{X}^D$ in our category. Here, as earlier, one may first look at the legal and projective dimensions and then try to prove that the mean dimension of Y intersected with a bounded (and weakly compact) part $B \subset \underline{X}^\Gamma$ equals the projective dimension.

Conjecture. If Y is defined by a generic law $L \subset \underline{X}^D$ of codimension r, then

$$\mathrm{legdim}(Y : \Gamma) = \mathrm{prodim}(Y : \Gamma) = \dim(Y \cap B : \Gamma) = s - r \qquad (=)$$

for $s = \dim \underline{X}$ and $B = \underline{B}^\Gamma$, where $\underline{B} \subset \underline{X}$ is a sufficiently large compact subset.

Notice that the major difficulty in proving that

$$\dim(Y \cap B : \Gamma) \geq \mathrm{prodim}(Y : \Gamma) \qquad (*)$$

stems from the problem of bounding from below $\mathrm{Widim}_\varepsilon\, Y \cap B$ in the finite dimensional case (see 1.1.D). Yet, even without resolving the finite dimensional problem from 1.1.D, one expects $(*)$ in many cases, e.g. for $Y = Y(L)$ where $L \subset \underline{X}^D$ is real analytic.

Another (rather technical) issue, which comes about in the smooth category, is a possible bad behavior of the singularities of $Y = Y(L)$ and of the intermediate finite dimensional spaces $L_\Omega \subset \underline{X}^\Omega$, $\Omega \subset \Gamma$. It is unclear if L_Ω can be as bad as any other closed subset but, in any case, one can rule out major (?) pathologies by imposing genericity assumptions on L, making all L_Ω stratified subsets (and often just smooth submanifolds) in $\underline{X}^\Omega$ (compare 2.7).

2.5.A. Many examples of $Y = Y(L) \subset \underline{X}^D$, where $\dim(Y : \Gamma) = s - r$, e.g. for $\underline{X} = \mathbb{R}^s$, can be obtained with the implicit function theorem (see 2.2.B). Namely, we start with some $L_0 \subset \underline{X}^\Gamma$, where the equality $\dim(Y(L_0) : \Gamma) = s - r$ is known for some reason (e.g. $L_0 \subset (\mathbb{R}^s)^D$ is a generic linear law of codimension r) and then apply a difference operator $A : \underline{X}^\Gamma \to \underline{X}^\Gamma$ which is sufficiently close to the identity, e.g. $A = Id + \varepsilon :$

$(\mathbb{R}^s)^\Gamma \to (\mathbb{R}^s)^\Gamma$, where the implied $\underline{\varepsilon} : (\mathbb{R}^s)^D \to R^s$ is bounded and has small differential. Then $Y = A^{-1}(Y(L_0)) \subset \underline{X}^\Gamma$ will have the same mean dimension as Y_0 by the discussion in 2.2. For example, if we start with $L_0 \subset (\mathbb{R}^s)^D$ of codimension $r \leq s$ represented as the kernel of a linear map $(\mathbb{R}^s)^D \to \mathbb{R}^r$ factoring through a coordinate projection $(\mathbb{R}^s)^D \to \mathbb{R}^s$, then every small smooth C^1-perturbation L_ε of L_0 in $(\mathbb{R}^s)^D$ gives us $Y_\varepsilon = Y(L_\varepsilon) \subset (\mathbb{R}^s)^\Gamma$ with $\dim(Y_\varepsilon : \Gamma) = s - r$.

Questions. Let $L \subset (\mathbb{R}^s)^D$ be a smooth submanifold of codimension r containing the origin $0 \in (\mathbb{R}^s)^D$ and denote by L_0 the tangent space $T_0(L)$ of L. What property of L_0 ensures that L is Γ-transversal near the origin and thus every $L_\Omega \subset (\mathbb{R}^s)^\Omega$ is smooth of codimension $r|\Omega \div D|$ near the origin, i.e. when intersected with B_ε^Ω, where $B_\varepsilon \subset \mathbb{R}^s$ is a ε-ball with $\varepsilon > 0$ *independent* of Ω? Moreover, we want $\dim(Y(L) \cap B_\varepsilon^\Gamma) : \Gamma = s - r$ under a suitable assumption on L_0. (Apparently, what we need is some kind of *uniform* Γ-transversality of L_0 meaning, for example, that the corresponding map $\mathbb{R}^r(\Gamma) \to \mathbb{R}^s(\Gamma)$ is not only injective, but is left invertible in the ℓ_1-topology.) Also, we wish a more general result of this type applicable to an arbitrary (non-Γ-fixed) point $y_0 \in Y$ (rather than $y = 0$), where the corresponding tangent space $T_{y_0}(Y)$ is not Γ-invariant.

2.6. Global lower bounds on $\dim(Y(L) : \Gamma)$ for non-linear laws $L \subset \underline{X}^D$. Although we have no general result at the moment for "local" mean dimension it is possible to obtain some lower bounds for "global" infinite dimensional varieties.

An appealing example is where $\underline{X}$ is the complex projective space and L is a complex algebraic subvariety, i.e. $L \subset (\mathbb{C}P^m)^d$, $d = \operatorname{card} D$. We shall show for such L that the *corresponding subspace $Y \subset (\mathbb{C}P^m)^\Gamma$ of L-legal functions $\Gamma \to \mathbb{C}P^m$* (see 2.6.F and 2.7) *has the expected mean dimension*

$$\dim(Y : \Gamma) = 2m - \operatorname{codim}_{\mathbb{R}} L\,, \tag{$*$}$$

provided L is generic in a suitable sense (see 2.7).

In fact the upper bound on $\dim(Y : \Gamma)$ follows from that for the legal dimension

$$\operatorname{legdim} Y \leq \dim \underline{X} - \operatorname{codim} L \tag{$*$}_\leq$$

which holds true in all categories whenever one has a reasonable notion of genericity for L (i.e. when L appears as a member of a sufficiently ample family of subvarieties, see 2.7). On the other hand, the lower bound

$$\dim(Y : \Gamma) \geq \dim \underline{X} - \operatorname{codim} L \tag{$*$}_\geq$$

does not need genericity but rather homological non-degeneracy of L (see below) which, in the case of $\underline{X} = \mathbb{C}P^n$, is satisfied by *all* algebraic subvarieties $L \subset (\mathbb{C}P^m)^d$. In fact this non-degeneracy is satisfied for many (e.g. sufficiently mobile) $L \subset \underline{X}^d$, where $\underline{X}$ is any complex projective variety but (see 2.7) it is unclear if $(*)_{\geq}$ holds true for all algebraic $L \subset \underline{X}^d$.

2.6.A. Homological lower bound on the mean dimension. Let $\underline{X}$ be a compact finite dimensional locally contractible metric space, take a Cartesian power $\underline{X}^M = \underbrace{\underline{X} \times \underline{X} \ldots \times \underline{X}}_{M}$ with the sup-product metric and consider a λ-Lipschitz map from a compact metric space into $\underline{X}^M$, say $\alpha : Y \to \underline{X}^M$. We denote by N the maximal integer such that the induced *homomorphism* $H^N_{\check{\mathrm{C}}\mathrm{ech}}(\underline{X}^M) \to H^N_{\check{\mathrm{C}}\mathrm{ech}}(Y_0)$ *does not vanish.*

2.6.A′. Topological Lemma. *There exists a positive $\varepsilon = \varepsilon(\underline{X}) > 0$, such that the ε'-dimension of Y for $\varepsilon' = \varepsilon/2$ satisfies*

$$\mathrm{Widim}_{\varepsilon'}(Y) \geq N\,,$$

where this ε does not depend on M, Y, α and N.

Proof. If Y admits an ε-embedding to a polyhedron P then the cylinder of this map, say $Z \supset Y$ admits a metric extending this of Y, such that $\mathrm{dist}(z, Y) \leq \varepsilon/2$ for all $z \in Z$ (compare 2.5.C). Next we consider the M projections of $\underline{X}^M$ to $\underline{X}$, compose them with α and observe that the resulting maps $Y \to \underline{X}$ extend to continuous maps $P \to \underline{X}$ for $\varepsilon' \leq \varepsilon_0 = \varepsilon_0(\underline{X}) > 0$. In fact $\underline{X}$ embeds into some Euclidean space, say $\underline{X} \subset \mathbb{R}^n$, where we may assume our original metric in $\underline{X}$ is induced by this embedding. Since our map from Y to $\underline{X} \subset \mathbb{R}^n$ is λ-Lipschitz, it extends to a $\lambda\delta n$-Lipschitz map from $Z \supset Y$ to $\mathbb{R}^n \supset \underline{X}$. Now, as all of Z is $\varepsilon'/2$-close to Y, for $\varepsilon' = \varepsilon/2$, our Z lands δ-close to $\underline{X} \subset \mathbb{R}^n$ for $\delta \leq \frac{\varepsilon}{2}\delta n$. But the δ-neighbourhood of $\underline{X}$ in $\mathbb{R}^n$, for small $\delta \leq \delta_0(\underline{X} \subset \mathbb{R}^n) > 0$, retracts to $\underline{X}$, since $\underline{X}$ is locally contractible. This gives us the desired map $Z \to \underline{X}$ extending $Y \to X$ and all these M maps together extend the original embedding $Y \subset \underline{X}^M$ to a continuous map $Z \to \underline{X}^M$.

Finally, if $\dim P \leq N-1$, the above extension allows a homotopy of the embedding $Y \subset \underline{X}^M$ to a map which factors through a $(N-1)$-dimensional polyhedron. This makes the induced homomorphism on H^N zero. Q.E.D.

2.6.B. Homological dimension. Given $Y \subset \underline{X}^\Gamma$ we define its *projective homological dimension* relative to a given exhaustion Ω_i of Γ by projecting Y to $\underline{X}^{\Omega_i}$, denoting by N_i the maximal dimension where the corresponding cohomological map $H^*_{\text{Čech}}(\underline{X}^{\Omega_i}) \to H^*_{\text{Čech}}(Y)$ does not vanish and then by setting

$$\operatorname{prohomdim}(Y : \{\Omega_i\}) = \liminf_{i\to\infty} N_i / \operatorname{card}\Omega_i\,.$$

Next define a similar stable legal dimension for $Y = Y(L) \subset \underline{X}^\Gamma$ coming from $L \subset \underline{X}^D$ by

$$\operatorname{stlehodim}(Y : \{\Omega_i\}) = \liminf_{i\to\infty} \lim_{j\to\infty} N_{ij} / \operatorname{card}\Omega_i$$

where N_{ij} denotes the maximal dimension of non-vanishing of the cohomology homomorphism corresponding to the projection $L_{ij} : L_{\Omega_j} \to L_{\Omega_i}$, $j \geq i$.

It follows from the continuity of the Čech cohomology under the projective limits that

$$\operatorname{stlehodim} = \operatorname{prohomdim}$$

in the category of compact metric spaces. This combines with the above topological lemma and leads to the following

2.6.B′. Practical lower bound on the mean dimension. *If X is as in 2.6.A, then $Y = Y(L) \subset \underline{X}^\Gamma$ satisfies*

$$\dim(Y : \{\Omega_i\}) \geq \operatorname{stlehodim}(Y : \{\Omega_i\})$$

for all compact laws $L \subset \underline{X}^D$.

Proof. All one has to add to the above discussion is the following obvious comparison between the sup-product metric in $\underline{X}^\Omega$ and $|x - x'|$ *in* $\underline{X}^\Gamma$: *the projection* $(\underline{X}^\Gamma, |x - x'|_\Omega)$ *to* $(\underline{X}^\Omega$, sup-metric) *is 1-Lipschitz.* Q.E.D.

Remark. The dimension stlehodim (despite the ugly notation) is a computable quantity and so the above lower bound on the mean dimension is practically useful.

2.6.C. Evaluation of stlehodim in manifolds. Given a subspace B in a compact space A we denote by $\Lambda(B) \subset H^*(A)$ the part of the cohomology of A which can be represented by Čech cocycles supported arbitrarily near B, where the cohomology is taken with coefficients in a fixed field $\mathbb{K}$. Notice that $\lambda \in H^*(A)$ belongs to $\Lambda(B)$ iff its restriction to $A \backslash B$ vanishes. This is obvious. Furthermore, if A is a *closed manifold*, then $\Lambda(B)$ equals the Poincaré dual of the image of $H_*(B)$ in $H_*(A)$. This is a (small) part of the standard "Poincaré duality package" which is attached to all *homology manifolds* and also applies (with some precaution) to general *Poincaré duality spaces*. Here is another obvious property of $\Lambda(B)$,

($\star$) *if some* $\alpha \in H^*(A)$ *restricts to a trivial class on* $B \subset A$, *then* $\alpha \smile \lambda = 0$ *for all* $\lambda \in \Lambda(B)$.

This will be used below in the following way. Define corank λ, $\lambda \in H^\ell(A)$, as the maximal k, such that $\lambda \smile \alpha \neq 0$ for some $\alpha \in H^k(A)$. Notice that if A is an n-dimensional manifold (or a general Poincaré duality space), then corank $\lambda = n - \ell$ for all non-zero λ in $H^\ell(A)$. More generally, given a map $A \to C$, define $\mathrm{corank}_C \lambda$ by means of those $\alpha \in H^k(A)$ which come from $H^k(C)$. Clearly, $\mathrm{corank}_C \leq \mathrm{corank}_A = \mathrm{corank}$.

Next, set

$$\mathrm{corank}_C\, B \overset{def}{=} \mathrm{corank}_C\, \Lambda(B) \overset{def}{=} \sup_{\lambda \in \Lambda(B)} \mathrm{corank}_C\, \lambda$$

for a given $B \subset A$. Clearly, ($\star$) implies that *the latter corank bounds from below the maximal dimension k where the homomorphism $H^k(C) \to H^k(B)$ (induced by $B \subset A \to C$) does not vanish.*

Finally we observe that

$$\Lambda(B_1 \cap B_2) \supset \Lambda(B_1) \smile \Lambda(B_2) \qquad (\cap)$$

for all pairs of compact subsets B_1 and B_2 in A.

Now we return to our power space $\underline{X}^\Gamma$ and $Y = Y(L) \subset \underline{X}^\Gamma$ for a law $L \subset \underline{X}^D$, $D \subset \Gamma$. Recall that

$$Y(L) = \bigcap_{\gamma \in \Gamma} \gamma Y_L\,, \qquad (\cap_\Gamma)$$

for $Y_L \subset \underline{X}^\Gamma$ being the pull-back of L under the projection $\underline{X}^\Gamma \to \underline{X}^D$. We denote by $\Lambda^* \subset H^*(\underline{X}^\Gamma)$ the pull-back of $\Lambda(L) \subset \underline{X}^D$ under this projection and we want to apply ($\cap$) to the *infinite* intersection ($\cap_\Gamma$).

Definition of $H^\times$. *Given a commutative (or, skewcommutative) algebra H we denote by $H^\times$ the set of formal finite and infinite products,*

$$H^\times = \{h^\times = \prod_i h_i\},\ h_i \in H\,,$$

where i may run over an arbitrary index set I. We say (and this is all we care about) that some such $h^\times \in H^\times$ does not vanish, *written $h^\times \neq 0$, if $\prod\limits_{i\in J} h_i \neq 0$ for* all *finite subsets $J \subset I$.*

We shall apply the above convention to $H = H^*(\underline{X}^\Gamma)$ and denote the corresponding $H^\times$ by $H^\times(\underline{X}^\Gamma)$. Here the most interesting infinite products are of the form

$$h^\times = \bigcup_{\gamma\in\Gamma} \gamma h$$

for some $h \in H^*(\underline{X}^\Gamma)$ and we want to decide when such an $h^\times$ does not vanish.

More specifically, we define $\Lambda^\times(Y) \subset H^\times(\underline{X}^\Gamma)$ as the set of the products $\smile_{\gamma\in\Gamma} \gamma\lambda_\gamma$ for all assignments $\gamma \mapsto \lambda_\gamma \in \Lambda^*$ and we introduce the following

Definition. *Given a finite subset $\Omega \subset \Gamma$, consider an α in the image of $H^N(\underline{X}^\Omega)$ in $H^N(\underline{X}^\Gamma)$ (for the projection $X^\Gamma \to X^\Omega$) and $\lambda^\times \in \Lambda^\times(Y)$, such that $\alpha \smile \lambda^\times \neq 0$ and let N be the largest integer where such α and $\lambda^\times$ exist. Then set*

$$\operatorname{corank}(Y : \Omega) \stackrel{def}{=} \operatorname{corank}(\Lambda^\times(Y) : \Omega) \stackrel{def}{=} N/\operatorname{card}\Omega\,.$$

2.6.C$'$. Proposition. *The above corank bounds from below the stable legal homological dimension of $Y = Y(L)$ for compact laws $L \subset \underline{X}^D$,*

$$\operatorname{stlehodim}(Y : \{\Omega_i\} \geq \liminf_{i\to\infty} \operatorname{corank}(Y : \Omega_i)\,.$$

The proof is clear with the preceding discussion. Also, the following corollary is now obvious.

2.6.C$''$. *If $\underline{X}$ is a closed manifold (or a general Poincaré duality space) and $\underline{\lambda} \in \Lambda(L) \cap H^k(\underline{X}^D)$ is a class such that its lift λ to $H^*(\underline{X}^\Gamma)$ satisfies $\smile_{\gamma\in\Gamma} \gamma\lambda \neq 0$, then* $\operatorname{stlehodim}(Y : \Gamma) \geq k$ *and consequently* $\dim(Y : \Gamma) \geq k$.

2.6.D. Example: Untangled laws and monomials. Suppose $L^0 \subset \underline{X}^D$ is given by d *untangled* (systems of) equations in the (groups of) variables $\underline{x}_i$, $i = 1, \ldots, d = \operatorname{card} D$, namely by $f_i^0(\underline{x}_i) = 0$, $i = 1, \ldots, d$. In other words, L^0 equals the intersection of d-subsets L_i^0 coming from some $\underline{L}_i^0 \subset \underline{X}$ via the d projections $\underline{X}^D \to \underline{X}$ (where each $\underline{L}_i^0 \subset \underline{X}$ may be given by the equation $f^0(\underline{x}_i) = 0$). Then take some $\underline{\lambda}_i \in \Lambda(\underline{L}_i^0)$ and observe that their tensor product (monomial) $\underline{\lambda}_1 \otimes \underline{\lambda}_2 \otimes \ldots \otimes \underline{\lambda}_d$ is contained in $\Lambda(L^0)$.

Denote by $\underline{\lambda}$ the cup-product of $\underline{\lambda}_1, \ldots, \underline{\lambda}_d$ in $H^*(\underline{X})$ and suppose there exists $\underline{\alpha} \in H^k(\underline{X})$ such that $\underline{\lambda} \smile \underline{\alpha} \neq 0$, i.e. $\operatorname{corank} \underline{\lambda} \geq k$. For example, if $\underline{X}$ is a closed manifold (or a general Poincaré duality space) of dimension s and $\underline{\lambda} \in H^r(\underline{X})$, then there always exists such an $\underline{\alpha} \in H^k(\underline{X})$ for $k = s - r$.

Next, we consider the γ-translates of the monomial $\underline{\lambda}_1 \otimes \ldots \otimes \underline{\lambda}_d$ for all $\gamma \in \Gamma$ and formally cup-multiply them over Γ. The resulting Γ-*monomial* clearly equals the tensor product of Γ copies (translates) of $\underline{\lambda}$, one $\underline{\lambda}$ assigned to each $\gamma \in \Gamma$. Denote this Γ-product by $\underline{\lambda}^\Gamma$ and observe that, formally,

$$\underline{\lambda}^\Gamma \smile \underline{\alpha}^\Gamma = (\underline{\lambda} \smile \underline{\alpha})^\Gamma \neq 0$$

for the above $\underline{\alpha} \in H^{\times}(\underline{X})$.

2.6.D$'$. Corollary. *Let $L \subset \underline{X}^d$ be homologous to L^0, and so $\Lambda(L)$ contains the above monomial $\underline{\lambda}_1 \otimes \underline{\lambda}_2 \otimes \ldots \otimes \underline{\lambda}_d$. Then*

$$\dim(Y(L) : \Gamma) \geq k\,.$$

In particular, if $\underline{X}$ is an s-dimensional manifold and L is homologous to intersection of d cycles coming from some cycles $\underline{L}_i$ in $\underline{X}$ (via the d projections $\underline{X}^D \to X$, $d = \operatorname{card} D$), where $\sum_{i=1}^{d} \operatorname{codim} \underline{L}_i = r$ and the homology class represented by their intersection in $H^{s-r}(\underline{X})$ does not vanish, then

$$\dim(Y(L) : \Gamma) \geq s - r\,.$$

2.6.D$''$. Remarks. (a) This corollary is most powerful if applied to the cohomology with finite (e.g. $\mathbb{Z}/2\mathbb{Z}$) coefficients where the monomial condition is not so restrictive. Thus starting with a monomial μ_0 in $H^*(\underline{X}^D; \mathbb{Z})$ non-divisible by an integer p, one gets nonmonomial classes of the form $\mu_0 + p\mu'$ where the corollary may apply.

(b) If we work with real coefficients, then the non-vanishing of an (infinite) integer monomial μ_0 obviously yields this for $\mu_0 + \Theta\mu'$ for an integer μ' and all transcendental $\Theta \in \mathbb{R}$. Unfortunately it is not useful as the cohomology $H^*(L) \subset H^*(\underline{X}^D)$ lives over $\mathbb{Z}$ but it suggests that non-vanishing of products of the form $\smile_{\gamma\in\Gamma} \gamma\lambda$ and issuing lower bound on the mean dimension are generic phenomena. This is also confirmed by the examples we study below.

2.6.E. Nonvanishing products over uncoiled groups Γ. Let $\underline{H}$ be a (skew)commutative algebra with unit e.g. $\underline{H} = H^*(\underline{X}; K))$ and $H = \underline{H}^{\otimes \Gamma}$ (i.e. H equals the tensor product of Γ copies of $\underline{H}$, say of $\underline{H}_\gamma = H$, where the basic example is $H = H^*(\underline{X}^\Gamma))$.

Non-vanishing problem. Given an $h \in H$. Decide when the formal product $\prod_{\gamma \in \Gamma} \gamma h$ does not vanish, where we use the obvious action of Γ on H.

The simplest case, and the only one we address here, is where h is "linear", i.e. $h = \sum_{\gamma \in D} h_\gamma$ for $h_\gamma \in H_\gamma$, where $D \subset \Gamma$ is a finite subset and where all h_γ are assumed $\neq 0$. (This is somewhat opposite to the monomial case, $h = \bigotimes_{\gamma \in D} h_\gamma$ we studied earlier.)

2.6.E$'$. Proposition. *Let $h = \sum_{\delta \in D} h_\delta$. If Γ is D-uncoiled (e.g. uncoiled, see 2.2.C) then $\prod_{\gamma \in \Gamma} \gamma h \neq 0$.*

Corollary. *Let $\underline{X}$ be a closed $(r-1)$-connected manifold (i.e. its homotopy groups up to $\pi_{r-1}(\underline{X})$ vanish) and $L \subset \underline{X}^D$ be a cycle of codimension non-homologous to zero. Then*

$$\dim(Y(L) : \Gamma) \geq \dim \underline{X} - r$$

for all uncoiled amenable groups Γ, e.g. for all nilpotent and polycyclic groups without torsion.

Proof. We must show that $\pi = \prod_{\gamma \in \Omega} \gamma h \neq 0$ for all finite subsets $\Omega \subset \Gamma$. We proceed by induction on $\operatorname{card} \Omega$. Choose $\omega_0 \in \Omega$ and $\delta_0 \in D$ such that $\omega_0 \delta_0 = \gamma_0$ has a unique solution, let $\Omega_- = \Omega \backslash \{\omega_0\}$ and assume that $\pi_- = \prod_{\gamma \in \Omega_-} \gamma h \neq 0$. Then, our full Ω-product $\pi = \prod_{\gamma \in \Omega} \gamma h$ equals the product of π_- by the ω_0-translate $h_\bullet$ of $h = h_{\delta_0} + \sum_{\delta \neq \delta_0} h_\delta$. This translate can be written as

$$h_\bullet = \omega_0 h = h_{\gamma_0} + \sum_{\gamma \neq \gamma_0} h_\gamma\,.$$

Also, observe that no monomial in π_- includes a factor coming from H_{γ_0}, since $\Omega_- D$ does not contain γ_0. Thus $\pi = \pi_- h_\bullet = \pi_- \otimes h_{\gamma_0} + \varepsilon$, where no ε-term includes h_{γ_0} as a factor. Hence, no cancellation is possible and $\pi \neq 0$. Q.E.D.

2.6.E″. Remark on the Kaplansky problem. This refers to the following question. Let Γ have no torsion. Can then the group ring $K(\Gamma)$ have zero divisors? The above generalizes the standard argument showing there is no zero divisor in $K(\Gamma)$ if Γ is uncoiled (see §2.2 and [Pass]).

2.6.F. Positivity and non-cancellation in complex manifolds. Suppose we have an *ordered* (graded skewcommutative) algebra H where the order is given, by definition, by a subset $H^+ \subset H$ consisting of what we call *positive* elements, such that H^+ is closed under addition and multiplication in H and $H^+ \cap -H^+ = \{0\}$.

Example. Let H be the real cohomology algebra of $\mathbb{C}P^m$, i.e. the algebra of polynomials in a variable t truncated by the relation $t^m = 0$. Then, non-vanishing polynomials with *positive* coefficients define an order in the above sense. Notice that the *integral* positive elements in this

$$H = H^*(\mathbb{C}P^m, \mathbb{R}) \supset H^*(\mathbb{C}P^m; \mathbb{Z})$$

are exactly the Poincaré duals of fundamental classes of *complex subvarieties* in $\mathbb{C}P^m$.

An order on H induces a natural order on every tensorial power $\mathcal{H}$ of H where $\mathcal{H}^+$ is defined as the set of sums of tensor products of positive elements in H. For example, if $H = H^*(\mathbb{C}P^m)$ then its tensor power $H^{\otimes d}$ consists of truncated polynomials in d variables with the obvious notion of positivity. It is not hard to show that the Poincaré duals of complex subvarieties in $(\mathbb{C}P^m)^d$ are positive in this sense.

Now, if we look at $\mathcal{H} = H^{\otimes\Gamma}$ for a group Γ and take some positive element $h \neq 0$ there, (i.e. $h \in \mathcal{H}^+ \backslash \{0\}$ for $\mathcal{H}^+$ defined with some order in H given by $H^+ \subset H$), then the formal infinite product $\prod\limits_{\gamma\in\Gamma} \gamma h$ does not vanish. This applies, for example, to the classes in $H^*(CP^m)^D$, $D \subset \Gamma$, dual to complex subvarieties in $(\mathbb{C}P^m)^D = (\mathbb{C}P^m)^d$, $d = \operatorname{card} D$, and lead to the following

Corollary. *Let $L \subset (\mathbb{C}P^m)^D$ be a cycle of codimension r homologous to a complex algebraic one. Then*

$$\dim(Y(L) : \Gamma) \geq 2m - r$$

for all amenable groups Γ.

Standard order on $H^*(\underline{X};\mathbb{R})$. The space of real exterior forms on $\mathbb{C}^n$ has a natural (minimal in some sense) order where *positive* $2k$-forms are defined as positive combinations of pull-backs of the standard (positive!) volume from an $\mathbb{C}^k$ under non-singular $\mathbb{C}$-linear maps $\mathbb{C}^n \to \mathbb{C}^k$. (This is the only $GL_n\mathbb{C}$-invariant order on $\Lambda^2_{\mathbb{R}}(\mathbb{C}^n)$ but it seems unclear what are other orders on $\Lambda^{2k}_{\mathbb{R}}(\mathbb{C}^n)$ for $k \geq 2$.) Observe that our positive form lies in the subspace of $\Lambda^{2k}_{\mathbb{R}}$ consisting of the form invariant under the action $z \mapsto \sqrt{-1}\,z$ on $\mathbb{C}^n$, where they constitute a convex cone with non-empty interior. Forms in the interior are then called *strictly positive.*

Next, given a complex manifold $\underline{X}$, a class $h \in H^*(X;\mathbb{R})$ is called (strictly) positive if it can be represented by a form which is (strictly) positive on the tangent spaces $T_x(\underline{X})$ for $\underline{x}$ ranging over an open dense subset in $\underline{X}$. Clearly, this is a bona fide order on $H^*(\underline{X})$ in our sense.

Classical example. Embed $\underline{X}$ into some $\mathbb{C}P^N$ and intersect it with a generic hyperplane. Then the Poincar dual of this intersection is strictly positive in H^2 ($\underline{\mathrm{X}}$) assuming $\dim X > 0$.

This shows that positive elements always exist. Moreover, the Hodge theory says that every complex cycle can be "moved" to the dual of a positive cocycle. Namely, let $H_{\mathbb{C}} \subset H^*(\underline{X};\mathbb{R})$ be the span of the Poincaré duals of the fundamental classes of complex subvarieties in $\underline{X}$.

Theorem. (See [Gri-Ha].) *If $\underline{X}$ is a complex projective manifold, then the strictly positive elements constitute a cone with non-empty interior in $H_{\mathbb{C}}$. Thus, for every $h \in H_{\mathbb{C}}$, there exists a strictly positive $h^+ \in H^*$, such that $h + h^+$ is strictly positive.*

Remark. Our interest in positivity is motivated by the non-vanishing problem for products $\pi = \prod_{\gamma\in\Gamma} \gamma h$ for some $h \in H^*(\underline{X}^\Gamma) = (H^*(\underline{X}))^{\otimes\Gamma}$ which eventually come from $\underline{h} \in H^*(\underline{X}^D)$, $D \subset \Gamma$. We know that $\pi \neq 0$ if $\underline{h}$ is positive for the order relation in $H^*(\underline{X}^D) = (H^*(\underline{X}))^{\otimes d}$, $d = \operatorname{card} D$, induced by the above order in $H^*(\underline{X})$ associated to the complex structure in $\underline{X}$. But the order in $H^*(\underline{X}^D)$ coming *directly* from the complex structure in $\underline{X}^D$ usually has more positive elements than those coming from $H^*(\underline{X})$ (as some algebraic cycles in $\underline{X}^d$ do not come from products of such cycles in $\underline{X}$'s. For example, graphs of "interesting" automorphisms of $\underline{X}$ give us such cycles in $\underline{X} \times \underline{X}$).

Questions. Let $\underline{h}$ be positive with respect to the complex structure in $\underline{X}^D$. Does then π non-vanish? Let $L \subset \underline{X}^D$ be a complex subvariety of real codimension r. Does the mean dimension of $Y = Y(L)$ satisfy

$$\dim(Y : \Gamma) \geq \dim_{\mathbb{R}} \underline{X} - r\,?$$

What can be said about "positivity" of the cohomology classes in $\underline{X}$ and in $\underline{X}^D$ which are positive on all algebraic cycles?

2.6.F$'$. Representing infinite products by measures. It is hard to make sense of an infinite product $\pi = \prod\limits_{\gamma \in \Gamma} \gamma h$ for general $h \in H$, but if h is positive for a suitable order on the algebra H this can be done.

Example. Let $\underline{X} = \mathbb{C}P^\infty$ and so $H = H^*(\underline{X}^\Gamma, \mathbb{R})$ equals the algebra of polynomials in the variables x_γ, $\gamma \in \Gamma$, with the natural action of Γ, and with the standard notion of positivity. If Γ is finite, then monomials are marked by functions $\Gamma \to \mathbb{Z}_+$ indicating the degree of the letter x_γ in a given monomial. Thus each real polynomial becomes a function $p : \mathbb{Z}_+^\Gamma \to \mathbb{R}$ telling the values of coefficients of a polynomial at all monomials.

Next, look from this angle at the product over an infinite group Γ,

$$h \mapsto \pi = \pi_\Gamma = \prod_{\gamma \in \Gamma} \gamma h\,,$$

where $h \in H^*(\underline{X}^\Gamma)$, $\underline{X} = \mathbb{C}P^\infty$ is induced from $\underline{h} \in H^*(\underline{X}^D)$ as earlier. The set of monomials in the polynomials γh, $\gamma \in \Gamma$, is given by the double power set $(\mathbb{Z}_+^D)^\Gamma$, that is mapped by the above product over Γ to $\mathbb{Z}_+^\Gamma$, denoted

$$\sqcup : (\mathbb{Z}_+^D)^\Gamma \to \mathbb{Z}_+^\Gamma\,.$$

This map sends each Γ-family of monomials $\{m_\gamma \in \mathbb{Z}_+^D,\ \gamma \in \Gamma\}$ to the product $\prod\limits_\Gamma \gamma m_\gamma \in \mathbb{Z}_+^\Gamma$. Next, suppose we have functions $p_\gamma : \mathbb{Z}_+^D \to \mathbb{R}$, $\gamma \in D$, representing polynomials in x_γ, $\gamma \in D$ and we want to multiply them over Γ. To do this we limit ourselves to *positive* functions on the (countable!) set $\mathbb{Z}_+^D$ which are viewed as *measures* on $\mathbb{Z}_+^D$. Now we can multiply the measures p_γ, where the result, denoted $p_\Gamma^\times = \underset{\Gamma}{\times}\, p_\gamma$, is a measure on $(\mathbb{Z}_+^D)^\Gamma$. Of course, this measure looks rather unruly unless all p_γ are probability measures. And if p_γ have finite total masses they can be normalized to have mass one.

Finally, we push forward the product measure $p_\Gamma^\times$ to $(\mathbb{Z}_+)^\Gamma$ via our map $\sqcup : (\mathbb{Z}_+^D)^\Gamma \to \mathbb{Z}_+^\Gamma$ and declare this to be our infinite product over Γ. Notice, that for $p_\gamma = \gamma p_0$ the resulting measure is Γ-invariant. Also notice that for $\underline{X} = \mathbb{C}P^m$ with $m < \infty$, we deal with smaller spaces, namely, the *finite* set $\{0, \dots, m-1\}^D$ (instead of $\mathbb{Z}_+^D$) and the Cantor set $\{0, \dots, m-1\}^\Gamma$ (instead of $\mathbb{Z}_+^\Gamma$).

Summing up, we see that *the "fundamental cohomology class" of the infinite intersection*

$$\bigcap_{\gamma \in \Gamma} \gamma Y_L, \quad L \subset (\mathbb{C}P^m)^D, \ D \subset \Gamma,$$

where L is a complex algebraic subvariety and $Y_L \subset (\mathbb{C}P^m)^\Gamma$ is the full pull-back of L under the projection $\underline{X}^\Gamma \to \underline{X}^D$, *is representable by a Γ-invariant probability measure on the (Cantor) set of maps* $\Gamma \to \{0, \dots, m-1\}$, (where the "probability" property is achieved with an obvious normalization).

The above generalizes to arbitrary ordered real algebras H, where we have to deal with vector valued measures. The reader may enjoy persuing this more closely.

Questions. Is there a deeper relation between the algebra-geometric idea of positivity on Γ-varieties (such as $\underline{X}^\Gamma$) and (Γ-invariant) measures on associated compact Γ-spaces ? What is the nature of the space of proalgebraic cycles in such varieties as $\underline{X}^\Gamma$ and $Y(L) \subset \underline{X}^L$ where we keep track of the moduli of the cycles as well as of their "homology classes" expressed by measures on $\mathbb{Z}_+^\Gamma$? Is there a formalism of this kind associated to the Von Neumann algebra of Γ?

2.7. Generic laws $L \subset \underline{X}^D$ and upper bounds on $\dim(Y(L) : \Gamma)$. We want to extend the results of 2.2 to non-linear laws $L \subset \underline{X}^D$ and show that *generically* the mean dimension of $Y = Y(L)$ is bounded by what one may expect,

$$\dim(Y : \Gamma) \leq \dim \underline{X} - \operatorname{codim} L\,.$$

Intuitively, we think that the γ-translates of the equations defining L remain essentially independent for generic laws $L \subset \underline{X}^D$.

2.7.A. Monomial laws. Let L be the product of subvarieties $L_\delta \subset \underline{X}_\delta = \underline{X}$, $\delta \in D$, where we think of $\underline{X}^D$ as the Cartesian product of $\underline{X}_\delta = \underline{X}$, over $\delta \in D$. If $L_\delta = X_\delta$ for all but a single δ, e.g. if $\operatorname{codim} L = 1$, then, clearly, the translates of L by $\gamma \in \Gamma$ are mutually transversal and thus the legal dimension of $Y(L)$ is bounded by $\dim \underline{X} - \operatorname{codim} L$. But this may fail in general. Take, for instance, $\Gamma = \mathbb{Z}/2\mathbb{Z}$ and $L = L_1 \times L_2$

where $L_1 = L_2 \subset \underline{X}$. Then $Y(L) = L$ and $\operatorname{codim} Y = \operatorname{codim} L$ instead of the expected value $\operatorname{codim} Y = 2 \operatorname{codim} L$. However, the order of things is recovered if L_1 is transversal to L_2 in $\underline{X}$. Then, clearly, $L_1 \times L_2$ is transversal to $L_2 \times L_1$ and so the resulting $Y(L)$, being the (transversal!) intersection, $(L_1 \times L_2) \cap (L_2 \times L_1) = (L_1 \times L_2)^2$, has right codimension ($= 2 \operatorname{codim} L$).

The above reasoning applies to all groups Γ, where mutual transversality of all $L_i \subset \underline{X}$ (trivially, compare 2.2) implies that the legal codimension of $Y(L)$ is $\geq \operatorname{codim} L$.

2.7.B. Polynomial laws. Let L be a union of finitely many monomial laws, $L = \bigcup\limits_{i=1}^{k} L_i$. Here a simple example is where $\Gamma = \mathbb{Z}$, and L defined in $\mathbb{R}^D$ for $D = \{0, 1\}$ by the equation $x_0 x_1 = 0$. A sequence $\{x_i \in \mathbb{R}\}_{i \in \mathbb{Z}}$ belongs to $Y = Y(L)$ if and only if $x_i x_i = 0$ for all $i \in \mathbb{Z}$, i.e. out of two consecutive x's one must be zero. Thus $\dim(Y : \mathbb{Z}) = 1/2$ rather than zero. Yet, if we perturbe the equation to $x_0(x_1 - \varepsilon) = 0$ for $\varepsilon \neq 0$, then every sequence $\{x_i\} \in Y(L)$ looks like $\dots 0, 0, 0, x, \varepsilon, \varepsilon, \varepsilon, \dots$, where x is a free variable and thus $\dim(Y(L)) = 0$. This trivially generalizes to all Γ and polynomial laws $L \subset \underline{X}^D$, where it yields the expected bound on the dimension of $Y(L) \subset \underline{X}^\Gamma$ for *generic* polynomial laws (where all factors of all irreducible components are mutually transveral in $\underline{X}$ or at least meet across subvarieties of proper dimensions).

2.7.C. Polynomial reduction of algebraic laws. Suppose we have an algebraic subvariety $L = L_0 \subset \underline{X}^D$ which is included in algebraic family, say $L_\varepsilon \subset \underline{X}^D$, $\varepsilon \in \mathcal{E}$, such that some limit $L_{\varepsilon \to \infty}$ becomes polynomial in the above sense, i.e. becomes the union of monomial (i.e. product) varieties with factors in $\underline{X}$, where all these factors are mutually *dimensionally transversal*, i.e. all intersections $L_i \cap L_j \cap L_k$ etc. have $\operatorname{codim} L_1 \cap L_j \cap L_k \leq \operatorname{codim} L_i + \operatorname{codim} L_j + \operatorname{codim} L_k$. Then, if we work in the category of projective varieties over an algebraically closed field, we come to the following

Conclusion.

$$\operatorname{legdim}(Y(L_\varepsilon) : \Gamma) \leq \dim \underline{X} - \operatorname{codim} L_\varepsilon \tag{$\star$}$$

for generic $\varepsilon \in \mathcal{E}$, where "generic" means away from a countable union of proper subvarieties in $\mathcal{E}$.

Indeed, the dimension of our intersection is semicontinuous in $\varepsilon \in \mathcal{E}$ and if it is small for some (possibly asymptotic) value of ε, then it is generically small.

2.7.D. Examples of "polynomial" reduction of algebraic cycles. The above reduction works very well if $\underline{X}$ is a projective variety homogeneous under an action of a linear reductive group A. For example, $\underline{X}$ may be a manifold of flags in $\mathbb{C}^n$ (e.g. the Grassmann manifold $\mathrm{Gr}_{n-k}(\mathbb{C}^n)$) acted upon by $SL_n\mathbb{C}^n$. Then, the Cartesian power of the group, A^d, acts on $\underline{X}^d$ and every algebraic cycle $L \subset \underline{X}^d$ is included into the family $L_\varepsilon = a_\varepsilon L$, $a_\varepsilon \in A^d$.

Lemma. *There is a degeneration (reduction) of L to a "prodynamical cycle" within this family.*

Proof. A generic transformation $a \in A$ has *isolated* fixed points in $\underline{X}$ as follows from Thom's transversality theorem (yielding this property for homogeneous spaces of all connected Lie groups). It follows in the reductive case that there is a multiplicative 1-parameter subgroup in A, say $\mathbb{C}^\times \subset A$ (we work over $\mathbb{C}$ here, to be specific) which acts on $\underline{X}$ with *isolated* fixed points. Then, by the complex Morse theory, such an action must necessarily have a repulsive fixed point, say $x_+ \in \underline{X}$, such that the eigenvalues of $2 \in \mathbb{C}^\times$ acting on the tangent space $T_{x_+}(\underline{X})$ have $|\text{eigenvalues}| > 1$. (This was explained to me by Iiosik Bernstein.)

Now let us apply such an action to one component $\underline{X}$ of $\underline{X}^d$ and see what it does to L. For example, let $\underline{X} = P^1$, $d = 2$, and L is the diagonal. Our action of $\mathbb{C}^\times$ on P^1 has two fixed points, x_+ and x_- and aL obviously converges to $(P^1 \times x_-) \bigcup (x_+ \times P^L) \subset P^1 \times P^1$, for $z \to \infty$, $a \in \mathbb{C}^\times$. The same eventually happens to every $L \subset \underline{X}^d$ where we must apply expanding action along various $\underline{X}$-factors of $\underline{X}^d$ at some points in L. (To see it clearly, we must order all cycles $\Sigma n_i C_i$ in $\underline{X}^d$ as follows. First we use $\sum\limits_i n_i$, i.e. the number of irreducible components counted with multiplicities, which *increase* in the course of reduction and which is obviously bounded. Then, we use the dimensions of projections of L_ε to the subproducts $\underbrace{\underline{X} \times \ldots \times \underline{X}}_{d-\delta}$ which may only *decrease* in the course of reduction.

When we arrive at a cycle with a maximal number of components where each of them has minimal dimensions of projections, then this cycle is "polynomial".)

Now we can use the above $(\star)$ and conclude to the inequality

$$\operatorname{legdim} Y(aL) \le \dim \underline{X} - \operatorname{codim} L \qquad (\star_\bullet)$$

for a generic perturbation of $L \subset \underline{X}^D = \underline{X}^d$, $d = \operatorname{card} D$ by $a \in A^d$.

2.7.D′. Real case. The above argument does not work directly over $\mathbb{R}$. For example, the North Pole - South Pole action of $\mathbb{R}^\times$ on S^n may collapse all of L to a single point with all information irrevocably lost. However, we may pass to the complexification $\underline{X}(\mathbb{C}) = A(\mathbb{C})/A_0(\mathbb{C})$ where A_0 is the isotropy subgroup of some $x_0 \in \underline{X}$ and if $\underline{X}(\mathbb{C})$ is projective, then our conclusion (including $(\star_\bullet)$) applies to $\underline{X} = \underline{X}(\mathbb{R})$ acted upon by $A = A(\mathbb{R})$. For example, this works for the above S^n acted upon by $SO(n,1)$ as the corresponding subgroup $A_0(\mathbb{C})$ is parabolic in this case. But if you take S^n with the $SO(n+1)$-action the complexification trick does not work, but our conclusion may hold true all the same.

Question. Which (homogeneous) spaces $\underline{X}$ acted by A satisfy $(\star_\bullet)$? What about $\mathbb{R}^n$ acted upon by parallel translations and similarity transformations?

2.7.D″. Analytic and smooth cases. Since the required genericity of L is essentially an algebraic condition, one expects $(\star_\bullet)$ to be valid for complex (and real) analytic subvarieties $L \subset \underline{X}^D$ (which may be non-compact and/or have boundaries). In fact, the required transversality (expressing genericity) concerns the behavior of $L \subset \underline{X}^D$ at several points $x_1, \dots, x_N \in L$ where the lifts of L to $\underline{X}^\Omega$ meet. If we could deform the germs at L by A^d independently at these points, we could easily arrive at $(\star_\bullet)$ in the analytic category. In fact, such independence is achieved in the *real* analytic category if we use the group A of *all* real analytic transformations of $\underline{X}$, where $\underline{X}$ is an arbitrary real analytic manifold, and if we work over $\mathbb{C}$, we may admit L's $\subset \underline{X}^D$ which are images of holomorphic maps $f : \tilde{L} \to \underline{X}^D$ where $\tilde{L}$ is a *Stein* manifold. Then, by allowing $L_\varepsilon = f_\varepsilon(\tilde{L})$ for *all* holomorphic deformations of f, we again recapture $(\star_\bullet)$, at least in the case of a homogeneous $\underline{X}$, by a rather standard argument. Yet, I could not rigorously prove $(\star_\bullet)$ as it stands for complex analytic $L \subset \underline{X}^D$.

Finally, the above should work in the smooth category with $A = \mathrm{Diff}\,\underline{X}$ where one, probably, needs some equisingularity lemma in the spirit of Thom (compare 1.3.2.(E_1) in $[\mathrm{Gro}]_{\mathrm{PDR}}$) but I did not check the details (appearing rather straightforward to a casual eye).

2.7.E. Algebraic laws $L \subset \underline{X}^D$ for non-homogeneous $\underline{X}$. Start with a projective embedding $\underline{X} \subset P^M = \mathbb{C}P^M$ and then embed $\underline{X}^d \to P^N$ for $N = (M+1)d - 1$ in the usual way. (For example, if $d = 2$, a pair $(x_0, x_1, \dots, x_M)$, $(y_0, y_1, \dots, y_M)$ goes to $((z_{00} = x_0y_0, \dots, z_{ij} = x_iy_j, \dots, z_N = x_My_M))$. We look at the family $L_S \subset \underline{X}^d$ obtained by intersecting $\underline{X}^d \subset P^N$ with a linear (i.e. projective) subspace $S \subset P^N$ of a given codimension ℓ. Among these L_S there exist *"most degenerate"* ones which are polynomial in our sense and satisfy the transversality assumptions of 2.7.A. These come by intersecting $\underline{X}^d \subset P^N$ with "tensor products" of subspaces in P^M. (For example, the hyperplane $z_{ij} = 0$ in P^M for $N = 2(M+1)-1$ intersect $\underline{X}^2 \subset P^N$ across the union $(\underline{X}_i \times \underline{X}) \cup (\underline{X} \times \underline{X}_j)$ where $\underline{X}_i$ denotes the intersection of $\underline{X}$ with the hyperplane $x_i = 0$ in P^M.) Therefore, *generic* $L = L_S \subset \underline{X}^D = \underline{X}^d$, $d = \operatorname{card} D$, *give rise to* $Y = Y(L)$ *with the expected legal and mean dimensions,*

$$\dim(Y : \Gamma) = \operatorname{legdim}(Y : \Gamma) = \dim \underline{X} - \operatorname{codim} L\,. \qquad (\star\star)$$

In fact the upper bound on legdim follows by the above reduction argument while the lower bound depends on the homological positivity argument in 2.6.

Question. Does this conclusion (or at least the upper bound on legdim $(Y : \Gamma)$) remain valid for *all* projective embeddings $\underline{X}^d \subset P^N$?

§ 3. Harmonic maps and related spaces.

We prove here the results stated in 0.6. concerning the mean dimension of spaces of harmonic maps and of solutions of more general elliptic P.D.E. We start with a recollection of the standard properties of linear P.D.E.

3.1. Cauchy-Gårding inequality. Consider a homogeneous, uniformly elliptic system of linear P.D.E. imposed on $\mathbb{R}^s$-valued functions x in the standard unit ball $B \subset \mathbb{R}^n$, say $\mathcal{E}x = 0$. If the coefficients of the equations are smooth, then the classical regularity theorem ensures the smoothness of x. Moreover, all derivatives of x at the origin $0 \in B$ are bounded in terms of the sup-norm of x on B. Here we are mainly concerned with the first derivative (differential) Dx where the Cauchy-Gårding inequality reads

$$\|Dx(0)\| \leq C \sup_{v \in B} \|x(v)\| \tag{$*$}$$

for some constant $C = C(\mathcal{E})$.

Next suppose $\mathcal{E}$ is defined over all $\mathbb{R}^n$, where it is assumed uniformely elliptic and with all coefficients and their derivatives bounded. Then we apply $(*)$ to each unit ball $B = B(v, 1) \subset \mathbb{R}^n$ and obtain a bound on $\|Dx\|$ everywhere on $\mathbb{R}^n$,

$$\sup_{v \in \mathbb{R}^n} \|Dx(v)\| \leq C \sup_{v \in \mathbb{R}^n} \|x(v)\| \,. \tag{$**$}$$

3.1.A. Vanishing corollary. *Let x be a bounded solution x of the system $\mathcal{E}x = 0$. If x vanishes on an ε-net $\Sigma \subset \mathbb{R}^n$ with $\varepsilon < C^{-1}$, then $x = 0$.*

Proof. If $x|\Sigma = 0$, then, obviously, $\|x(v)\| \leq \varepsilon \sup_{v \in \mathbb{R}^n} \|Dx(v)\|$ for all $v \in V$. This and $(**)$ imply that

$$\sup_{v \in \mathbb{R}^n} \|x(v)\| \leq C^{-1} \varepsilon \sup_{v \in \mathbb{R}^n} \|x(v)\|$$

and so $\|x(v)|$ must vanish if $C_\varepsilon^{-1} < 1$. Q.E.D.

3.1.A′. Denote by $X^{\mathcal{E}} = X^{\mathcal{E}}_{L_\infty}$ the space of *bounded* solutions x of the system $\mathcal{E}x = 0$ and restate the above vanishing result as the following

Embedding property. *The restriction map* $R_\Sigma : X^{\mathcal{E}} \to \ell_\infty(\Sigma; \mathbb{R}^s) \subset (\mathbb{R}^s)^\Sigma$ *is one-to-one.*

In fact the above argument implies that R_Σ is a topological embedding (i.e. $R_\Sigma^{-1} : R_\Sigma(X^{\mathcal{E}}) \to X^{\mathcal{E}}$ is a *bounded* operator for the uniform topologies) and that the intersection of $R_\Sigma(X^{\mathcal{E}})$ with the unit ball in $\ell_\infty(\Gamma; \mathbb{R}^s)$ is weakly closed in $\ell_\infty(\Gamma; \mathbb{R}^s)$.

3.1.A″. Estimate on the mean dimension of $X^{\mathcal{E}}$. Take concentric i-balls $B(i) \subset \mathbb{R}^s$ of radii $i = 1, 2, \dots$, and let $X^{\mathcal{E}}(i) \subset X^{\mathcal{E}}$ consist of maps $\mathbb{R}^n \to B(i)$ satisfying $\mathcal{E}$. Clearly, all spaces $X^{\mathcal{E}}(i)$ are mutually isomorphic via the maps $x \mapsto ij^{-1}x$ sending $X^{\mathcal{E}}(j) \to X^{\mathcal{E}}(i)$, and the union of $X^{\mathcal{E}}(i)$ equals $X^{\mathcal{E}}$. Furthermore, these $X^{\mathcal{E}}(i)$ are *compact* spaces and one may speak of their mean dimensions for actions of lattices Γ on $\mathbb{R}^n$ compatible with $\mathcal{E}$. Thus we set

$$\dim(X^{\mathcal{E}} : \Gamma) \underset{\text{def}}{=} \dim(X^{\mathcal{E}}(i) : \Gamma) \,,$$

where the latter dimension does not depend on i.

Finiteness of $\dim(X^{\mathcal{E}} : \Gamma)$. Let $\mathcal{E}$ be invariant under a lattice Γ acting on $\mathbb{R}^n$. Then

$$\dim(X^{\mathcal{E}} : \Gamma) \leq \mathrm{const}_{\mathcal{E}} \operatorname{vol} \mathbb{R}^n/\Gamma \,.$$

Proof. Use a Γ-invariant net $\Sigma \subset \mathbb{R}^n$ and observe that the above embedding becomes equivariant and sends $X^{\mathcal{E}}$ to $(\mathbb{R}^{ds})^\Gamma$ where d denotes the number of elements from Σ contained in a fundamental domain of Γ. Thus

$$\dim(X^{\mathcal{E}} : \Gamma) \leq ds \,.$$

Q.E.D.

3.2. Linear P.D.E. on Riemannian manifolds. Let V be a complete Riemannian manifold and consider an elliptic operator $\mathcal{E}$ in some vector bundle over V. If the "coefficients" of $\mathcal{E}$ and its "ellipticity" are uniformely controlled by the Riemannian metric, then $(**)$ generalizes to V, provided the curvature tensor of V is C^1-bounded (probably C^0 suffices) on V. (Notice, that we do *not* need a lower bound on the injectivity radius of V, but we have it anyway in our applications where $(V, \mathcal{E})$ is invariant under a cocompact group Γ.)

3.2.A. The basic examples of such $\mathcal{E}$ are as follows:

(A.) The ordinary Laplace operator on V.

(B.) The Hodge Laplace operator on differential forms.

(C.) Various Dirac operators (where one adds sometimes the spin conditions on V).

(D.) The $\overline{\partial}$-operator, in the case where V is Hermitian.

(E.) All of the above twisted with an auxiliary vector bundle E over V, with a Euclidean connection.

Here the inequality $(*)$ applies to the coordinate charts in V (or in the unit balls $B_v(1)$ in the tangent spaces $T_v(V)$, $v \in V$ mapped to V by the exponential maps) and shows that

$$\sup_{v\in V} \|Dx(v)\| \le C \sup_{v\in V} \|x(v)\|\,, \qquad (**)_V$$

where the constant C depends only on the curvature of V, i.e. on $\sup\limits_{v\in V} \|K_v(V)\|$, and on the curvature of the implied vector bundle. In particular, we always have $(**)_V$ with some $C < \infty$ if (V, E) is acted upon by a cocompact (isometry) group. Then we have the vanishing corollary and embedding property provided (V, E) has bounded curvature. Furthermore, if (V, E) is invariant under a cocompact amenable group Γ, then, clearly,

$$\dim(X^{\mathcal{E}} : \Gamma) \le \mathrm{const}_{V,\mathcal{E}}\, \mathrm{vol}(V/\Gamma)\,.$$

3.2.B. Remarks and generalizations. (a) The vanishing corollary trivially extends to manifolds with unbounded curvature if the density ε of a net Δ is allowed to depend on v. Essentially, we need $\varepsilon(v) \le \mathrm{const}_n \|K(v)\|^{-\frac{1}{2}}$, where K incorporates the curvatures of V and E and their first derivatives if so needed. Similarly, one may admit unbounded section x with $\varepsilon(v) \approx (\sup\limits_{v\in B_v} \|x(v)\|)^{-1}$ for $B_v \subset V$ being the unit ball around $v \in V$.

In fact, one expects here a more generous density bound on Σ in the spirit of the first main theorem of the Nevanlinna theory.

(b) The above have an obvious version in a general setting where V is an arbitrary metric space and $X^{\mathcal{E}}$ is replaced by a subspace Y in the space of bounded maps $x : V \to \mathbb{R}^s$. All one needs is uniform compactness of the restriction operators from Y to functions on the balls $B(v, 1) \subset V$, for all

$v \in V$. Actually, one needs even less: if X is a linear space of bounded functions $x : V \to \mathbb{R}^s$ where all $x \in X$ with $\sup\limits_{v\in V} \|x(v)\| \leq 1$ are uniformly continuous with a given modulus of continuity then $\dim(X : \Gamma) < \infty$.

Example. Let $X : X(\lambda)$ be a linear space of functions on a Riemannian manifold V where each $x \in Y$ satisfies $\sup\limits_{v\in V} \|Dx(v)\| \leq \lambda \sup\limits_{v\in V} \|x(v)\|$ for a given constant λ. Then $\dim(X : \Gamma) < \infty$ and it may be interesting to find more specific bounds on this dimension in terms of λ and the geometry of V.

(c) The situation similar to the above example arises in the L_2-framework, where one studies the L_2-spaces $X^{\mathcal{E}}(\lambda) \subset L_2(V, \mathbb{R}^s)$ (or sections $X \to E$, in general) corresponding to the spectrum of $\mathcal{E}$ inside the λ-disk in the complex plane. Here one knows that the Von Neumann dimension $\dim_{\ell_2}(X^{\mathcal{E}}(\lambda) : \Gamma) < \infty$ for all Γ (cocompact on V) and $\lambda < \infty$. There are several candidates for the ℓ_∞-counterpart of this space. For example, one may take the weak closure of the above $X^{\mathcal{E}}(\lambda)$ in $L_\infty(\underline{X}; \mathbb{R}^s)$. Or one may look at some Γ-invariant space $Y(\lambda)$ of bounded functions, such that $\mathcal{E}(Y_\lambda) \subset Y_\lambda$ and $\sup\limits_{v\in V} \|\mathcal{E}(y)\| \leq \lambda \sup\limits_{v\in V} \|y\|$ for all $y \in Y$. One wonders whether $\dim(Y(\lambda) : \Gamma) < \infty$ for such spaces $Y(\lambda)$.

(d) Let $\mathcal{E}$ be the ordinary Laplace operator Δ on functions $V \to X$. Then one has the following geometric bound on the Von Neumann dimension of the space $X^\Delta(\lambda)$ of L_2-functions belonging to the spectrum of Δ below λ. *Suppose the Ricci curvature of V is bounded from below by -1 and let $N(\varepsilon)$ denote the minimal number of ε-balls needed to cover of ε-balls needed to cover the quotient space V/Γ. Then*

$$C_1 N(\lambda^{-\frac{1}{2}}) \leq \dim_{\ell_2}\big(X^\Delta(\lambda) : \Gamma\big) \leq C_2 N(\lambda^{-\frac{1}{2}})$$

where the positive constants C_1 and C_2 depend only on $n = \dim V$.

This easily follows from the Paul Levy isoperimetric inequality (see Ap. C in [G-L-P]). Notice in this regard that for *connected* V/Γ the bound Ricci ≥ -1 implies, by Bishop inequality, the following bound on $N(\varepsilon)$ in terms of the diameter of V/Γ,

$$N(\varepsilon) \leq \max\big(1, \varepsilon^{-n} \exp(n \operatorname{Diam} V/\Gamma)\big)$$

where V/Γ is assumed *connected* and thus

$$\dim_{\ell_2}(X^\Delta(\lambda) : \Gamma) \leq \mathrm{const}'_n \max(1, \lambda^{\frac{n}{2}} \exp(n \operatorname{Diam} V/\Gamma)\big) \qquad (\star)$$

for connected V/Γ (see [G-L-P]).

Notice that the above inequality is very far from being sharp for *infinite* groups Γ, where the following is well known.

(i) Every L_2-harmonic function on V vanishes (as is true for all connected complete non-compact manifolds V), by a standard "integration by parts" argument,

(ii)

$$\dim_{L_2}(X^\Delta(\lambda):\Gamma)\to 0 \text{ for } \lambda\to 0\,,$$

where the rate of convergence depends on Γ. For example $\dim_{\ell_2}(X^\Delta(\lambda):\Gamma)$ vanishes for small $\lambda \leq \lambda(V) > 0$, if and only if the group Γ is *non-amenable*.

Question. What are the L_∞-counterparts of the above properties? For example, does the mean dimension of the space of bounded harmonic functions vanish for all amenable groups Γ? (It is clear that $\dim^-$ introduced in 2.1.B does vanish.)

(e) Let $E\to V$ be a Γ-equivariant Euclidean vector bundle of rank s and let Δ_E be the (Bochner) Laplace operator on sections V. Then the function $\varphi_E(\lambda)=\dim_{L_2}(X^{\Delta_E}(\lambda):\Gamma)$ is related to the above $\varphi(\lambda)=\dim_{L_2}(X^D(\lambda):\Gamma)$ by the following classical

Kato Inequality.

$$\int_0^\infty e^{-\lambda\delta}\varphi_E'(\lambda)d\lambda \leq s\int_0^\infty e^{-\lambda\delta}\varphi'(\lambda)d\lambda$$

for all $\delta\geq 0$, where, observe, the derivatives φ_E' and φ' are positive (measures) since our functions are monotone increasing.

Corollary.

$$\varphi_E(\lambda)\leq se^\lambda\varphi(\lambda)\,. \tag{+}$$

(f) If $\mathcal{E}$ is a "geometric" selfadjoint operator of second order in E then it is related to Δ_E by a *Boehner formula* $\mathcal{E}=\Delta_E+B_{\mathcal{E}}$, where $B_{\mathcal{E}}$ is a symmetric endomorphism of the bundle E. Then one can bound the spectral function of $\mathcal{E}$ by that of Δ_E and the spectrum of $B_{\mathcal{E}}$. Namely, if all eigenvalues of $B_{\mathcal{E}}$ in all fibers of E are bounded from below by $-\rho$ then, clearly,

$$\varphi_{\mathcal{E}}(\lambda)\leq\varphi_E(\lambda+\rho)\leq se^{\lambda+\rho}\varphi(\lambda+\rho)\,. \tag{**}$$

For example, if $\mathcal{E} = \Delta_k$ is the Hodge-Laplace operator on k-forms (where $s = \binom{n}{k}$, $n = \dim V$), then the above $B_{\mathcal{E}}$ is minorized by the so-called *curvature operator* $R = R(V)$ and then $(\star)$ and $(**)$ give us a spectral bound on Δ_k in terms of R (which includes Ricci) and $\operatorname{diam} V/\Gamma$. This applies, in particular, to the L_2-*Betti number* $b_k(V : \Gamma)$, that is the Von Neumann dimension of the space of harmonic L_2-forms on V of rank k,

$$b_k(V : \Gamma) \leq \binom{n}{k} \exp(n^v \operatorname{Diam} V/\Gamma) \qquad (\circ)$$

provided $\rho(R) \geq -1$. (This was pointed out by Gallot and Meyer for $\Gamma = \{e\}$ in [Gal-Mey].)

Questions. (a) Can one improve over the e^λ-factor in $(+)$? (Here one may be willing or unwilling to bring the curvature of E into play.) Can one bound the mean dimension of the space of *bounded* harmonic k-form in the spirit of $(\circ)$?

Notice that a bound similar $(\circ)$ (but with a poorer dependence on n) holds true under less restrictive assumption of the *sectional curvatures* of V (rather than $R(V)$) being bounded from below by -1. This is shown in $[\mathrm{Gro}]_{\mathrm{CDB}}$ for $\Gamma = \{e\}$ but the argument equally applies to all Γ. Furthermore, that argument applies to the homology $H_k(V; K)$ for an arbitrary field K and yields a bound on $\operatorname{prodim}(H_k(V; K) : \Gamma)$ for amenable groups Γ.

(b) What is the relation between $\operatorname{prodim}(H_k(V; \mathbb{R}) : \Gamma)$ and the mean dimension of the space of bounded harmonic k-forms on V? (If one had a full-fledged Hodge theory for *bounded* forms one could immediately claim the equality of the two dimensions.)

3.2.B$'$. Harmonic functions and the maximum principle. Let $\mathcal{E}$ satisfy the maximum principle, e.g. $\mathcal{E}$ equals the ordinary Laplacian Δ on functions $V \to \mathbb{R}$. We claim that

if a bounded solution x of $\mathcal{E}$ vanishes on some net $\Sigma \subset V$ (i.e. *an ε-net with some $\varepsilon < \infty$*),*then $x = 0$. Consequently*

$$\dim(X^{\mathcal{E}} : \Gamma) = 0$$

for every amenable group cocompactly acting on $(V, \mathcal{E})$.

Proof. Let a bounded solution x of $\mathcal{E}$ vanish on some net Σ and take a sequence of points $v_i \in V$, $i = 1, \ldots$, such that $\|x(v_i)\| \to a = \sup\limits_{v \in V} \|x(v)\|$ for $i \to \infty$. If V is cocompactly acted by Γ, we translate all v_i by suitable $\gamma_i \in \Gamma$

to a fixed compact subset $V_0 \subset V$ and then (after taking a subsequence if necessary) pass to the limit $x_\infty = \lim_{i\to\infty} \gamma_i x$. This x_∞ vanishes on some (non-empty!) net, say $\Sigma_\infty \subset V$, and $\|x_\infty\|$ achieves its maximum at some point $v_0 \in V_0$. Hence $x_\infty(v) = x_\infty(v_0)$ for all $v \in V$ and since $x_\infty|\Sigma_\infty = 0$ this x_∞ vanishes everywhere. This yields the vanishing of x as $\sup\|x\| = \|x(v_0)\|$.

Next, forget about Γ and just suppose $(V, \mathcal{E})$ has locally bounded geometry. Then, instead of translating V, we move ourselves to the points v_i and pass to the (pointed Hausdorff) limit manifold $V_\infty = \lim_{i\to\infty}(V, v_i)$ with the limit operator $\mathcal{E}_\infty$ on V_∞. Then the maximum principle applies to x_∞ on V_∞ and the proof follows.

Example. If $\mathcal{E} = \Delta$ and we deal with harmonic functions, then the "bounded local geometry" refers to a bound on the curvature and the lower bound on the injectivity radius. In fact, the above argument can be easily carried through with the assumption $|K(V)| \le$ const alone, without any bound on the injectivity radius. (Probably, one needs even less, something like $K(V) \ge -$const or Ricci$(V) \ge -$const.)

Remarks. (a) The above argument, does not use the linearity of $\mathcal{E}$ and applies to all equations satisfying the maximum principle or the convex hull property. (This includes harmonic and minimal maps into Riemannian manifolds without focal points.) On the other hand, when we want to evaluate the dimension $\dim(X^{\mathcal{E}} : \Gamma)$ we compare two solutions and the linearity is used in an essential way.

(b) **Quantitative maximum principle.** The maximum principle can be expressed as follows.

If the value $\|x(v_0)\|$ is close to $\sup\|x(v)\|$, then the ratio $x(v)/x(v_0)$ is almost constant on a large ball around v_0.

More precisely,

let $\|x(v_0)\| \ge (1-\varepsilon)\|x(v)\|$, for all v in the R-ball $B(v_0, R) \subset V$ around v_0. Then $\|x(v)\| \ge (1-\delta)\|x(v_0)\|$ for all $v \in B(v, r)$, where δ and r depend on R, ε (as well as on $(V, \mathcal{E})$, but not on v_0) and $\delta \to 0$, $r \to \infty$ for $\varepsilon \to 0$ and $R \to \infty$.

Notice, that this quantitative maximal principle is equivalent to the previously used one as an obvious limit argument shows. Also observe that the quantification, i.e. the dependence of δ and r on ε and R, can be made explicit and rather precise. For example, one can use in the case of harmonic functions and maps, the mean value theorem expressing $x(v_0)$ by a weighted average of $x(v)$ on the R-ball. (Ultimately, one may appeal to the Harnack inequality.)

(b′) Notice, that the function $x(v)$ in question need be only defined on the ball $B(v_0, R)$, not on all of V. Also the almost constancy conclusion remains valid if the equation $\mathcal{E}(x) = 0$ is satisfied only approximately,

$$\|\mathcal{E}(x)\| \leq \varepsilon \|x\|,$$

where the norm in question is the sup-norm on $B(v_0, R)$ and where we assume that our x satisfies the Cauchy-Gårding inequality with the constant C independent of the above ε.

3.3. Equations where $\dim(X^{\mathcal{E}} : \Gamma) > 0$**.** The L_2-index theorem provides many instances where $\dim_{L_2}(X^{\mathcal{E}} : \Gamma)$ does not vanish but it is unclear if this implies non-vanishing of the mean dimension. On the other hand, the presence of a non-zero L_1-solution of the equation $\mathcal{E}x = 0$ (trivially) yields sufficiently many *bounded* solutions to ensure non-vanishing of the mean dimension $\dim(X_{\mathcal{E}} : \Gamma)$. An obvious way to go from L_2 to L_1 is by taking "squares" of x's (compare ...), but this is usually incompatible with the equation $\mathcal{E}x = 0$. A happy exception is the Cauchy Riemann $\overline{\partial}$ operator as the square of a holomorphic function is holomorphic. More generally, if V is a complex manifold and $E \to V$ is a holomorphic vector bundle, then one can take, for instance, the symmetric square of E, denoted E^2, and observe that the symmetric square of a holomorphic section is holomorphic. Thus

$$\dim_{L_2}(X^{\overline{\partial}}_{L_2} : \Gamma) > 0 \Rightarrow \dim_{L_\infty}(X^{\overline{\partial}\otimes 2}_{L_\infty} : \Gamma) > 0$$

(but it is unclear if $\dim_{L_\infty}(X^{\overline{\partial}\otimes 2}_{L_\infty} : \Gamma) \geq \dim_{L_2}(X^{\overline{\partial}}_{L_2} : \Gamma)$).

3.3.A. Examples. (a) Let $V = \mathbb{C}^m$ and $E_\lambda \to \mathbb{C}^n$ be a line *Hermitian holomorphic bundle*, i.e. with a given fiberwise norm) where the curvature equals $\lambda\, dz\, d\overline{z}$ on $\mathbb{C}^m$ for λ real (where $dz\, d\overline{z}$ is the standard Hermitian form on $\mathbb{C}^m$). This E can be identified with the trivial bundle $\mathbb{C}^n \times \mathbb{C} \to \mathbb{C}^n$, such that the norm of the unit section $x(v)$ equals $\exp -\lambda\|v\|^2$. If $\lambda > 0$, the unit function $x_1 : \mathbb{C}^n \to 1 \in \mathbb{C}$ becomes a holomorphic section on $\mathbb{C}^n$ which decays as $\exp -\lambda\|v\|^2$ and so is summable with all degrees. It easily follows, that the space of bounded holomorphic sections of E_λ has mean dimension equal $c_n\lambda^n$ for some constant $c_n > 0$. (Here we refer to the mean dimension with respect to some amenable exhaustian of $\mathbb{C}^n$. If E_λ is equivariant with respect to some Lattice $\Gamma \approx \mathbb{Z}^{2n}$ acting on $\mathbb{C}^n$, then this space, say X_λ, has $\dim(X_\lambda : \Gamma) = c_n\lambda^n \operatorname{vol}(\mathbb{C}^n/\Gamma)$.)

(b) Let $E \to V$ be a line bundle equivariant for some cocompact group Γ acting on V.

If the curvature form ω of E is everywhere greater than the curvature κ of the canonical bundle, i.e. $w - \kappa$ is positive definite on V, then the L_2-Euler characteristic of (the sheaf of sections of) E equals the L_2-dimension of $H^0(V, E)$, i.e. the space of holomorphic L_2-sections $V \to E$. This is the standard corollary of the vanishing theorems. On the other hand, the Euler characteristic is given by a certain characteristic class which is a topological invariant of (E, V) and which is of the order $c_1^n(E)$ for bundles E with large c_1^m, $n = \dim_{\mathbb{C}} V$. Therefore, if $w > 0$, this class for E^i is about i^n for large i and so a sufficiently high power E^i admits a non-zero holomorphic section, provided E is a *positive* line bundle, i.e. its curvature from w is positive definite.

Remark. Notice that the above can deliver sections for a given E, without taking powers, provided $w - \kappa > 0$ and $\chi(V, E) > 0$. But if we allow E^i, there is no need to appeal to the L_2-index theorem. In fact a simple application of the L_2-estimate for the $\overline{\partial}$-operator (which is essentially based on the Fredholm alternative, a baby version of the index theorem) yields lots of L_2-sections of E^i *without* any Γ-action at all.

3.3.B. Recollection on L_2-estimates. Let $V = (V, g)$ be a complete Kähler manifold and $E \to V$ a Hermitian line bundle such that $w - \kappa \geq \lambda g$ where, as above, $w = w(E)$ denotes the curvature of E, $\kappa = \kappa(V)$ stands for the curvature of the canonical line bundle of V and $\lambda > 0$ is some real number. *Then, for every smooth E-valued $(0,1)$-form z with $\overline{\partial} z = 0$, there exists a smooth section $y : V \to E$, such that*

$$\overline{\partial} y = z \text{ and } \|y\|_{L_2} \leq \operatorname{const} \lambda^{-1} \|z\|_{L_2}, \tag{$*$}$$

where "const" is universal.

This is a by now standard interpretation of the $\overline{\partial}$-estimates (see [Nap] and references therein).

We shall apply $(*)$ in order to approximate a given smooth section $x_0 : V \to E$ by a holomorphic one as follows. Consider $z = \overline{\partial} x_0$, solve $\overline{\partial} y = z$ and take $x = x_0 - y$. This x is clearly holomorphic, $\overline{\partial} x = \overline{\partial} x_0 - \overline{\partial} y = 0$ and

$$\|x - x_0\|_{L_2} \leq \operatorname{const} \lambda^{-1} \|\overline{\partial} x_0\|_{L_2}. \tag{$**$}$$

This x is close to x_0 if $\|\overline{\partial} x_0\|_{L_2}$ is small and/or λ is large. In what follows we shall be dealing with a manifold V with *bounded* curvature and high power E^i of a *positive* bundle E. Thus we assume $\lambda \geq \text{const}$ and $(**)$ becomes

$$\|x - x_0\|_{L_2} \leq \|\overline{\partial} x_0\|_{L_2} \, . \tag{$***$}$$

For example, if we want to have a *non-zero* holomorphic section x of E, all we need is an x_0, such that $\|\overline{\partial} x_0\|_{\ell_2} < \|x_0\|_{\ell_2}$.

3.3.C. Lemma. (See [Tian].) *Let E be a positive line bundle on V, and $v_0 \in V$ a given point. Then there exists a sequence of smooth sections x_i of E^i with the following properties.*
(1) *All x_i are supported in a given (small) ball $B(v_0, \rho) \subset V$.*
(2) *All x_i are holomorphic in a smaller concentric ball $B(v_0, \rho_0) \subset B(v_0, \rho)$.*
(3) $\|x_i(v_0)\|_{E^i} = 1$ *and* $\|x_i(v)\|_{E^i} < 1$ *for* $v \neq v_0$.
(4) $\|x_i\|_{L_2} \geq \text{const}\, i^{-n}$ *for some "const" independent of* i.
(5) *The pointwise norm of $\overline{\partial} x_i$ exponentially decays for $i \to \infty$, $\|\overline{\partial} x_i(v)\|_{E^i} \leq \alpha^{-i}$ for some $\alpha > 1$ and all $v \in V$.*

Proof. Since E is positive, there obviously exists a local holomorphic section x_0 near v_0 with $\|x_0(v_0)\|_E = 1$ and $\|x_0(v)\| < 1$ for $v \neq v_0$. (Actually such an x_0 exists on a rather large neighbourhood of x_0, but this is irrelevant at the moment.) We smoothly extend this x_0 to a smooth section $x_1 : V \to E$ with a support near v_0 and still having $x_1(v) < 1$ for all $v \neq v_1$ and finally take $x_i = x_1^i$. This x_i is $\geq \frac{1}{2}$ in the ball of radius $\approx \frac{1}{\sqrt{i}}$ since $\|x_0(v)\| \geq \text{const}(\text{dist}(v, v_0))^{\frac{1}{2}}$ for v close to v_0 and so its L_2-norm is at least $\text{const}\, i^{-n}$. On the other hand, $\overline{\partial}(x_1(v))$ is different from zero away from v_0 where $x_1(v) \leq 1 - \varepsilon$ and so $\|\overline{\partial} x_i(v)\|$ is bounded by $\text{const}\, i(1-\varepsilon)^{i-1}$ as required by (5). Q.E.D.

3.3.D. Remarks. (a) This construction of approximately holomorphic sections of "sufficiently positive" bundles was explained to me by Simon Donaldson about 5 years ago who used this idea for producing symplectic hypersurfaces.

(b) The above remains true if instead of the powers E^i we take an arbitrary sequence of line bundles $E_i \to V$, such that the curvatures $w_i = w(E_i)$ grow, roughly, as iw_0 for a fixed positive form w_0.

(c) Notice, we did not use the full positivity of E, but rather positivity at the point v_0 in question.

3.3.E. Corollary: Existence of holomorphic L_2- and L_1-sections.

Let $V = (V, g)$ be a complete Hermitian manifold as earlier and $E \to V$ an Hermitian line bundle such that $w - \kappa \geq \lambda g$ with $\lambda > 0$ and such tha' w is positive at some point $v_0 \in V$. Then some power E^i admits a non-zero holomorphic L_2-section. Also E^i admit non-zero holomorphic L_1-sections for all sufficiently large i.

Proof. The existence of an L_2-section is immediate from the preceding discussion and to turn L_1 we split $E^i = E^{i_1} \otimes E^{i_2}$ with large i_1 and i_2 and observe that the products of two L_2-sections is L_1. Q.E.D.

Remarks. (a) The L_2-claim remains valid for every line bundle E_i having the same positivity as E^i. Moreover, the holomorphic sections obtained by the above argument have a controlled decay at infinity. Indeed, let x_0 be a continuous section with compact support and h be the L_2-nearest holomorphic section, i.e. the normal projection of x_0 to the space of holomorphic L_2-sections. Then $y_0 = x_0 - h$ is holomorphic outside some ball, say $B(v_0, r) \subset V$, and it is *orthogonal* to all holomorphic L_2-sections. Now, take the function $\varphi : V \to \mathbb{R}_+$ which equals 1 outside a large concentric R-ball $B(v_0, R) \supset B(v_0, r)$, which vanishes on $B(v_0, r)$ and which equals $1 - (R - \mathrm{dist}(v, v_0))/(R - r)$ for all v in the annulas between the two balls. Consider the section $y_1 = \varphi y_0$ and observe that

$$\text{(i)} \qquad \langle y_1, y_0 \rangle \overset{def}{=} \int_V y_1(v) y_0(v) dv \geq \int_{C(R)} |y_0(v)|^2 dv$$

where $C(R) \subset V$ denotes the complement $V \backslash B(v_0, R)$, and

$$\text{(ii)} \qquad \|\overline{\partial} y_1\|_{L_2} \leq (R - r)^{-1} \|y_0\|_{L_2} ,$$

since $|\overline{\partial}(\varphi y_0)| = |\overline{\varphi} y_0| \leq (R - r)^{-1} |y_0|$. If our bundle is sufficiently positive, we can approximate y_1 by a holomorphic L_2-section y, such that

$$\|y - y_1\|_{L_2} \leq \mathrm{const}\, \|\overline{\partial} y_1\|_{L_2} \leq \mathrm{const} (R - r)^{-1} \|y_0\|_{L_2} .$$

It follows,

$$0 = \langle y, y_0 \rangle \geq \int_{C(R)} |y_0(v)|^2 dv - \mathrm{const}(R - r)^{-1} \|y_0\|_{L_2} ,$$

and so

$$\int_{C(R)} |y_0(v)|^2 dv \leq \mathrm{const}\, R^{-1} |y_0|_{L_2}$$

for large R and $C(R) = V \backslash B(R)$.

Finally, as h equals y_0 outside $B(v_0, R)$, our h also has its L_2-norm decaying with the rate $R^{-\frac{1}{2}}$ at infinity.

(b) Instead of the L_2-nearest h one could take the L_p-nearest one, which is unique (if it exists) for all p (including $p = 1$, where the strict convexity is due to holomorphicity). It seems not hard to show that the L_p-norm of this h has a similar decay over $C(R)$ for $R \to \infty$.

(b′) Let us indicate the proof of the decay property for (as well as the existence of) holomorphic L_p-sections in the case of locally bounded geometry. First we pass to a large *odd* power E^i of E where one has many holomorphic L_1-sections (of the form $\sum\limits_j x_j y_j$ for holomorphic L_2-sections x_j of E^{i_1} and y_j of E^{i_2} with $i_1 + i_2 = i$, compare 3.3.E). Such E^i admits $n + 1 = \dim_{\mathbb{C}} V + 1$ bounded holomorphic sections $x_0, x_1, \dots, x_n$ that are uniformly transversal to the zero $0 = V \subset E^i$ and such that their zeros $x_k^{-1}(0) \subset V$, $k = 0, 1, \dots, n$, are simultaneously uniformly transversal (see 4.3). Denote by $\widetilde{V} \to V$ the canonical ramified cover of order 2^n with the ramification locus $\Sigma = \bigcup\limits_k x_k^{-1}(0)$, observe that $\widetilde{V}$ is non-singular and that the lifted bundle $\widetilde{E} \to \widetilde{V}$ admits a square root, since $\widetilde{E}^i$ does and i is odd. Now holomorphic L_2-sections $\overline{X}$ of such square root, say $\overline{E} \to \widetilde{V}$, can be multiplicatively pushed forward to holomorphic L_1-sections x of E for $x(v) = \overline{x}(\tilde{v}_1) \otimes \overline{x}(\tilde{v}_2) \dots \otimes \overline{x}(\tilde{v}_{2^n})$ for the pullbacks $\tilde{v}_1, \dots, \tilde{v}_{2^n}$ of v and so the L_1-properties of E reduce to the L_2-theory of $\overline{E}$. Notice, that the curvature of $\overline{E}$ (as well as that of $\widetilde{E}$) vanishes along $\widetilde{\Sigma}$, but only in transversal directions, and so the metric on $\overline{E}$ can be perturbed to a one with sufficiently positive curvature, provided we had enough positivity in E to ensure that $E|\Sigma$ is more positive than the canonical bundles of the submanifolds $\overline{x}_k^{-1}(0)$ and their intersections (compare 4.3).

Then we get lots of L_1 (and hence L_p, $p \geq 1$) of holomorphic section of E with controlled L_1-decay at infinity.

3.3.E′. *Let (V, E) be acted upon by an amenable Lie group Γ with V/Γ compact. Then the space of bounded holomorphic sections of E^i for large i has positive mean dimension.*

Indeed, the presence of a single non-zero L_1-section suffices as was mentioned earlier. (See 3.3.G′ for a sharper result.)

3.3.F. Gårding inequality in E^i. In order to see how the Gårding constant for holomorphic sections $V \to E^i$ depends on i, we scale the underlying manifold (V, g) by $g \mapsto ig$. Then the curvature iw of E^i scales to w and so we have a uniform (independent of i) Gårding inequality in E^i over (V, ig). Then, coming back to g, we conclude that

The sup-norm of a holomorphic section $x : V \to E^i$ in the ε-ball $E = B(v, \varepsilon) \subset V$ bounds the differential of x, by

$$\|Dx(v)\|_{E^i} \leq C_v \varepsilon^{-1} \sup_{v \in B} \|x(v)\|_{E^i} \qquad (+)$$

for every $\varepsilon \leq i^{-1}$.

3.3.F$'$. Corollary. *The sup-norm is bounded by the L_2-norm,*

$$\|x(v)\|_{E^i} \leq C'_v \varepsilon^{-n} \left(\int_B \|x(v)\|^2 dv \right)^{\frac{1}{2}} \leq C'_v \varepsilon^{-n} \|x\|_{L_2} \qquad (++)$$

for $\varepsilon \leq i^{-1}$.

3.3.F$''$. Remarks. (a) Notice that (+) and (++) are local properties where the holomorphicity of x is only required on the ball B. Thus we can apply (++) to the solutions y_1 of the $\overline{\partial}$-problem $\overline{\partial} y_i = \overline{\partial} x_i$ satisfying the basic L_2-estimate $(*)$ from 3.3.B. These y_i are holomorphic (as well as x_i) in a small (but fixed!) ball $B(v_0, \delta) \subset V$ and then (++) applies to smaller ε-balls $B(v, \varepsilon) \subset B(v_0, \delta)$. It follows, that the holomorphic sections $x'_i = x_i - y_i$ converge to x_i *uniformly* (and exponentially fast for $i \to \infty$) on every concentric ball $B(v_0, \delta' < \delta)$. In fact, such convergence takes place also on larger balls, where $\overline{\partial} x_i \neq 0$ anymore, since the Gårding inequality remains valid for non-homogeneous situation, but we do not need this for our purposes.

(b) The constants C_v and C'_v depend on local geometry of V and E near v. Actually C_v can be bounded in terms of the curvatures of V and E while C'_v also depends on the injectivity radius of V. (In general, ε^{-n} in (++) must be replaced by $(\operatorname{Vol} B(v, e'))^{-\frac{1}{2}}$ for some ε' somewhat smaller than ε.) In particular, C_v and C'_v are bounded if V and E have bounded local geometry, e.g. if there is a cocompact isometry group Γ acting on V and on E.

3.3.G. Interpolation theorem. Let V anxd E have bounded local geometry and thus the constants C_v and C'_v are bounded on V, and let $\Sigma \subset V$ be a δ-separated subset, i.e. $\operatorname{dist}(\sigma_1, \sigma_2) \geq \delta$ for all $\sigma_1 \neq \sigma_2$ in

Σ. *Then, for every $i > \mathrm{const}_{V,E} \max(1, \delta^{-2})$ and every bounded section y of $E^i|\Sigma$, there exists a bounded holomorphic section $x : V \to E^i$, such that $x|\Sigma = y$.*

Proof. First we observe that by scaling the metric g of V, by $g \mapsto \delta^{-2} g$, we make a δ-separated set 1-separated. This also normalizes the curvature of E^i with $i \approx \delta^{-2}$ to the unit size and explaines (actually proves) the dependence of i on δ.

Now we prove the theorem for $\delta = 1$ by summing up L_1- sections of E^i. It (obviously) suffices for our purpose to have holomorphic L_1-sections $x_\sigma^\bullet : V \to E^i$, for all $\sigma \in \Sigma$ and a given $i \geq \mathrm{const}_{V,E}$, such that

(a) $\|x_\sigma^\bullet(\sigma)\|_{E^i} \geq 1/2$;

(b) the sum of the norms $\|x_\sigma^\bullet(\sigma')\|_{E^i}$ over all $\sigma' \in \Sigma$ is small, say ≤ 0.1. We recall that L_1-sections are obtained as products of L_2 sections and so we need L_2-sections, say x'_σ, satisfying (a), where (b) is replaced by a similar bound on the sums of $\|x_\sigma(\sigma')\|_{E^i}^2$. Such an x_σ is constructed by first using 3.3.C at $v_0 = \sigma$ with $\rho \leq 0.1$ and then by approximating the resulting almost holomorphic section, call it now x_σ^0, by a holomorphic one, that is our x_σ. The bounded geometry assumption makes the estimates in 3.3.C independent of σ and then 3.3.F′ applied to ε-balls around all $\sigma' \neq \sigma$ in Σ yield the required bound on the sum of $\|x_\sigma(\sigma')\|_{E^i}^2$, provided i is sufficiently large. Q.E.D.

3.3.G′. Corollary. *If (V, E) is acted upon by an amenable group Γ with compact quotient, then the mean dimension of the space of bounded holomorphic sections of E^i is about i^n, $n = \dim_{\mathbb{C}} V$.*

3.3.G″. Remark. There is a distinguished holomorphic L_2-section of E^i taking a given value $e \in E_u^i$ at a given point $u \in V$, namely the one which has the minimal L_2-norm. This section, call it $h_e(v)$, $v \in V$, controllably decays at infinity in the sense that the integrals of $\|h_e(v)\|^2$ over the complements $C(R) \subset V$ of the large R-balls $B(R, u) \subset V$ around u satisfy

$$\int_{C(R)} \|h_e(v)\|^2 dv \leq \mathrm{const}\, R^{-1} .$$

This follows from Remark (a) in 3.3.E and the Gårding inequality.

3.3.H. Interpolation with jets and transversality theorem. One can easily interpolate not only the values on Σ but also a given number r of derivatives at all $\sigma \in \Sigma$. This is done again by first constructing approximately holomorphic sections and then making them holomorphic by small perturbations, where "small" refers to the C^r-topology as is allowed by 3.3.F (which needs an obvious generalization in the case $r \geq 2$).

Let us spell out how the approximate sections come about. Start with x_0 near v_0 as in the proof 3.3.C and let Φ be a finite collection of holomorphic functions φ defined on V near v_0, such that *the r-jets of the functions $\varphi \in \Phi$ at v_0 linearly span the full space of r-jets (as do the set of monomials of degrees $\leq r$ in local coordinates.*

Now we take some sufficiently small positive ε and let $x_\varphi = (1 + \varepsilon\varphi)x_0$. Since ε is small, all x_φ satisfy $\|x_\varphi(v)\|_E \leq 1$ on the *boundary* of some small ball $B(v_0, \rho_0) \subset V$ and so can be smoothly extended with this property to V. The totality of these extended x_φ represent all r-jets at v_0. This property passes to the corresponding sections x^i_φ of E^i and then further to the *holomorphic* sections approximating x^i_φ. This is straightforward and left to the reader (who is refered to [Tian] for further results and applications).

3.3.H′. Take a subset S in the jet bundle $J^r(E^i)$ over V and let us try to move a given holomorphic section $V \to E^i$ *away from* S. This presupposes some metric on $J^r(E^i)$ and "away from S" means that the r-jet $V \to J^r(E^i)$ does not intersect an ε-neighbourhood of S for some $\varepsilon > 0$. In what follows, we assume that V and E have bounded local geometry and observe that then $J^r(E^i)$ also admits a Hermitian structure of bounded local geometry compatible with this in V. We choose and fix such structure in each $J^r(E^i)$.

We say that S is *uniformly k-dimensional*, if for each unit ball $B \subset J^r(E^i)$ and every $\delta > 0$, the intersection $S \cap B$ can be covered by at most $C\delta^{-k}$ balls of radius δ for some constant $C = C(S)$.

Uniform transversality theorem. *Let E be positive and S uniformly k-dimensional for $k < \dim V$. Then there exists $i_0 = i_0(V, E, r)$, such that for each $i \geq i_0$ every bounded section $V \to E$ can be moved away from S by an arbitrarily small (in the uniform topology) perturbation.*

Proof. The required perturbation exists over each ρ-ball in V for a fixed small $\rho > 0$ as follows from the above and the standard transversality argument. Furthermore, this argument applies to a union of such balls, say to $U = \cup B\,\mu = 1, 2, \ldots,$ if these balls are situated sufficiently far apart in V. Finally, we cover V by finitely many U's of the above kind, $V =$

$U_1 \cup U_2 \cup \ldots \cup U_N$, and apply the first perturbation over U_1, then the second, much smaller one over U_2 and so on. This "much smaller" guarantees we do not each step what we gained at the previous one and so the N'th perturbation gives us a section $x : V \to E^i$ with the jet $J^r(x) : V \to J^r(E^i)$ missing S, i.e. "moves the original section away from S". Q.E.D.

3.3.J. Further applications, generalizations and open questions. (a) As we have mentioned several times earlier, the L_2-part of our discussion applies to (non-power!) line bundles E_i with curvature $\approx i\omega$, but to go to L_1 (and thus L_∞) we need such an E_i to be tensor product of two bundles with this kind of curvature. Such decomposition is possible, for example, if $H^2(V;\mathbb{Z}) = 0$ (but the interpolation theorem, probably, remains true in all cases, compare 3.3.D(b$'$)).

(b) The full L_2-story extends to suitably positive vector bundles E of higher rank. But our "squaring argument" needs passing to tensorial powers of E. Here again, it would be nice to prove an L_1-version of the $\overline{\partial}$-estimate and this looks easy.

(c) The proof of 3.3.G yields on interpolation results for holomorphic L_p-sections of E^i for all $p \geq 1$.

(d) The classical correspondence between divisors and line bundles extends to the framework of bounded geometry. This allows, in particular, construction of many bounded sections $V \to E^i$ vanishing on a hypersurface $W \subset V$ with (sufficiently) bounded geometry.

(e) Here are several problems which seem to be solvable in the present framework:

(1) Extension of bounded (and L_p) holomorphic sections of E^i from a submanifold $\Sigma \subset V$of (sufficiently, depending on i) bounded local geometry (where the case $\dim \Sigma = 0$ is covered by the interpolation theorem).

(2) Decomposition of bounded holomorphic sections of E^i into convergent sums of L_1-sections.

(3) Construction of bounded sections of affine subbundles of sufficiently positive vector bundles. For example, solution of the equation

$$\sum_{r=1}^{s} c_r \otimes x_r = a$$

for given bounded sections c_r of E^j and of E^i with the unknown x_r bounded sections of E^i. Similarly, one is interested in the equation

$$\sum_r x_r \otimes y_r = a$$

where a is an L_2-section and the solution (x_r, y_r) must be L_1.

(f) **Kodaira embedding theorem.** The uniform transversality theorem trivially implies that the canonical map Θ from V to the projectivized space of holomorphic L_2-sections $V \to E^i$ is a holomorphic embedding for $i \geq i_0$. (Recall, that Θ is defined by sending each $v \in V$ to the space of holomorphic sections of E^i vanishing at v.) Actually, Θ is easily seen to be *locally bi-Lipschitz*, i.e. there exists a constant $C > 0$ such that

$$C^{-1}\operatorname{dist}(\Theta(v_1),\ \Theta(v_2)) \leq \operatorname{dist}(v_1, v_2) \leq C\operatorname{dist}(\Theta(v_1),\ \Theta(v_2))$$

for all pairs of disjoint points v_1 and v_2 in V satisfying $\operatorname{dist}(v_1, v_2) \leq 1$.

If V is compact, then the receiving projective space is finite dimensional and it is infinite dimensional otherwise. In the latter case we clearly have

$$\operatorname{dist}(\Theta(v_1),\ \Theta(v_2)) \to \pi/2 \text{ for } \operatorname{dist}(v_1, v_2) \to \infty\,.$$

There is (apparently) no good finite dimensional reduction of this map but nice maps $V \to \mathbb{C}P^N$ are available for all $N \geq \dim V$ in the L_∞-framework (see ...).

(g) Many naturally arising line bundles, e.g. those associated to divisors in V (say with uniformly bounded volumes in the unit balls in V, compare ....) have singular curvatures and it would be useful to extend our upper and lower bounds on the spaces of holomorphic sections to such bundles.

(h) Let $\mathcal{E}$ be some Dirac operator twisted with a Euclidean vector bundle E on V. When can one guarantee the existence of many L_p-solutions to the equation $\mathcal{E}_E x = 0$ (where the cases $p = 1$ and $p = \infty$ are especially interesting in the present context)? Here one exercises a good control over L_2-sections especially for the tensorial powers $E^{\otimes i}$ in terms of the index of the twisted operator $\mathcal{E}_{E^{\otimes i}}$ but it is unclear when there are non-trivial L_p-sections x of $E^{\otimes i}$ satisfying the equation $\mathcal{E}_{E^{\otimes i}}(x) = 0$. Similar question arises for the Hodge-Laplace operator acting on $\Lambda^*(V)$ where non-zero harmonic L_1-form may (?) appear in the presence of a non-trivial cup-product.

3.4. Non-linear equations. Let V be, as earlier, a complete Riemannian manifold and $\underline{X}$ be a compact Riemannian manifold. We are interested in smooth maps $x : V \to \underline{X}$ satisfying some elliptic system $\mathcal{E}$ of partial differential equations, where basic examples are:

(i) harmonic maps;

(ii) holomorphic maps, where the Riemannian metrics in V and $\underline{X}$ are assumed Hermitian;

(iii) maps $x : V \to \underline{X}$ whose graphs $G_x \subset V \times \underline{X}$ are minimal subvarieties.

The essential features of our equations we shall need later on are as follows.

(a) **Regularity and compactness.** Every C^1-map $x : V \to \underline{X}$ satisfying $\mathcal{E}$ is in fact C^∞-smooth. Moreover, all higher derivatives of x are bounded in terms of the first derivatives, i.e.

$$\|D^i x\| \le C_i(\|Dx\|) \tag{+}$$

for some bounded functions $C_i = C_i(V, \underline{X}, \mathcal{E})$, where $\| \quad \|$ denotes the sup-norm on functions on V, i.e. $\|D^i x\| \overset{def}{=} \sup\limits_{v \in V} \|D^i x(v)\|$. It follows, that the space of our maps x with $\|Dx\| \le \text{const}$ is *compact* for the uniform convergence on compact subsets in V.

(b) **Non-linear Cauchy-Gårding inequality.** Let x_1, $x_2 : V \to \underline{X}$ be smooth maps, where $x_1(v)$ can be joined by a *unique* minimizing with $x_2(v)$ geodesic in $\underline{X}$ for all $v \in V$. Then we can compare the differentials

$$Dx_1(v) : T_v(V) \to T_{x_1(v)}(\underline{X})$$

and

$$Dx_2(v) : T_v(V) \to V_{x_2(v)}(X)$$

using the parallel transport in $\underline{X}$ along the geodesic $[x_1(v),\ x_2(v)] \subset \underline{X}$ and take the difference $Dx_1(v) - Dx_2(v)$. Thus we can speak of the C^1-distance $\|Dx_1(v) - Dx_2(v)\|$ and set

$$\|Dx_1 - Dx_2\| \overset{def}{=} \sup_{v \in V} \|Dx_1(v) - Dx_2(v)\| \,.$$

Notice that this C^1-distance is well defined if x_1 and x_2 are C^0-close, i.e.

$$\|x_1 - x_2\| \underset{def}{=} \sup_{v \in V} \operatorname{dist}_{\underline{X}}(x_1(v),\ x_2(v)) \le \varepsilon_0 < \operatorname{InjRad} \underline{X} \,,$$

where, observe, the injectivity radius of $\underline{X}$ is positive as we assume $\underline{X}$ is compact. (Notice that one could equivalently define a C^1-distance with a given covering of $\underline{X}$ by coordinate charts where it is possible to speak of $x_1 - x_2$ locable in every chart.) Now we can state our inequality.

If x_1 and x_2 have bounded differentials and $\|x_1 - x_2\| \le \varepsilon_0$ for the above ε_0, then

$$\|Dx_1 - Dx_2\| \le C\|x_1 - x_2\| \tag{$*$}$$

for some constant

$$C = C(V, \underline{X},\ \mathcal{E}, \|Dx_1\|, \|Dx_2\|, \varepsilon_0).$$

About the proof of (a) and (b) for our examples. The property (a) is well-known for the classes of maps indicated in the above (i) - (iii) where it is derived from the corresponding elliptic regularity for non-homogeneous linear equations via the standard implicit function argument. The sufficient condition on V and $\underline{X}$ is a uniform C^1-bound on their curvatures. Then (b) follows by the trivial interpolation property of smooth maps,

$$\begin{aligned}\|Dx_1 - Dx_2\| &\le C\|x_1 - x_2\| \\ &\text{for } C = C(V, \underline{X}, \|D^2x\|,\ \|D^2x\|)\,.\end{aligned}$$

3.4.A. Embedding property. Let V and $\underline{X}$ be as earlier where we assume $\|K(V)\| \le \text{const} < \infty$. Consider the space X_c of maps $x : V \to \underline{X}$ satisfying one of the elliptic conditions (i), (ii) or (iii) and having $\|Dx\| \le c$ for a given $c > 0$. *Then there exists $\varepsilon > 0$ depending on $V, \underline{X}, \mathcal{E}$ and e, such that the restriction map from X_c to $\underline{X}^\Sigma$ for an arbitrary ε-net $\Sigma \subset V$ is an embedding.*

This follows from the Cauchy-Garding inequality by the same (obvious) argument we used in the linear case. Also, we have as a corollary, the bound

$$\dim(X_c : \Gamma) < \infty\,,$$

whenever V is isometrically and co-compactly acted upon by an amenable group Γ (which must preserve the implied complex structure in the case (iii)).

3.4.A$'$. Dependence of C and ε on $c = \sup\|Dx\|$ and the proof of 0.6.A. Harmonic and holomorphic maps are invariant under the scaling: if $x : V \to \underline{X}$ is a harmonic (holomorphic) map then it remains such if we replace V by λV and $\underline{X}$ by $\mu\underline{X}$, where the notation "λV" refers to multiplying the metric in V by a constant $\lambda > 0$ and $\mu\underline{X}$ has similar meaning. Also observe that the (ellipticity) constant C in $(*)$ can be assumed independent of λ and μ in-so-far as these λ and μ are ≥ 1, since such scaling diminishes the curvature. On the other hand, when we scale the metrics, the norms of the differentials of the maps $x : V \to \underline{X}$ scale by the rule,

$$\|Dx\|_{\lambda,\mu} = \lambda^{-1}\mu\|Dx\|\,,$$

where $\|Dx\|_{\lambda,\mu}$ is the norm measured with respect to metrics in λV and $\mu\underline{X}$. It follows, that the constant C in $(*)$ is bounded by $c\,\mathrm{const}(V, \underline{X}, \mathcal{E}, \varepsilon_0)$ if $c = \sup\|Dx\|$ is ≥ 1. This is seen by taking $\lambda = c$ and $\mu = 1$. Consequently, the above ε is bounded from below by δc^{-1}, $\delta > 0$, and so we obtain the bound of the mean dimension of X_c for large c by bc^n as was stated in $(\star)_\infty$ of 0.6.A.

Next let us see what happens if $c = \sup\|D(x)\|$ is small. Such a map x sends large R-balls in V to small ones, of radii cR in $\underline{X}$, and if we scale these small balls to the unit size by passing to $\mu\underline{X}$ with $\mu = (cR)^{-1}$ we get maps from $B(R) \subset V$ to almost Euclidean unit balls, where we assume that c is much smaller than R^{-1}. Thus we can think of the harmonic equation for map $B(R) \to \mu\underline{X}$ on each $B(R)$ as a small perturbation of the ordinary Laplace equation for maps $B(R) \to \mathbb{R}^N$, $N = \dim\underline{X}$. Namely, if x_1 and x_2 are two harmonic maps from $B(R)$ to a unit ball in $\mu\underline{X}$, then the difference $x_1 - x_2$ is approximately harmonic in the Euclidean sense, where the difference is taken in the Euclidean geometry approximating the Riemannian one in $\mu\underline{X}$. Now (b) and (b′) from 3.2.B′ imply the following

Approximate maximum principle. *Let V and $\underline{X}$ have bounded local geometries and let x_1, $x_2 : V \to \underline{X}$ be non-equal harmonic maps with $\|Dx_i\| \leq c$, $i = 1, 2$, and with $\|x_1 - x_2\| \leq \varepsilon$. Then there is a ball $B(v_0, R) \subset V$ where $x_1(v_0) \neq x_2(v_0)$ and the ratio $\|x_1(v) - x_2(v)\|/\|x_1(v_0) - x_2(v_0)\|$ is almost constant on $B(v_0, R)$, where $R \to \infty$ for c, $\varepsilon \to 0$ and where "almost" means up to a $(1 + \delta)$-factor where $\delta \to 0$ with c, $\varepsilon \to 0$.*

This trivially implies $(\star)_\circ$ in 0.6.A exactly as in the linear case considered in 3.2.B′.

Remarks. (a) We treated above only harmonic maps, but the same argument applies to the pseudo-holomorphic maps between almost complex manifolds (where it somewhat simplifies in the honestly holomorphic case).

(b) It is not hard to quantify the above and give a specific bound on $\dim(X_c : \Gamma)$ for harmonic maps and small c in terms of c, the *upper* bound on the sectional curvature of $\underline{X}$ and the rate of decay of the heat kernel in V.

3.4.B. Additional remarks and generalizations. (a) One can allow a non-compact target manifold $\underline{X}$, provided it has a uniformly bounded local geometry, i.e. $|K(\underline{X})| \leq \text{const} < \infty$ and $\operatorname{InjRad} \underline{X} \geq \varepsilon > 0$ (where only the upper bound $R(\underline{X}) \leq \text{const}$ is essential for harmonic maps). Furthermore, one may start with a general fibration $Z \to V$ (instead of the trivial one $\underline{X} \times V \to V$) and extend the discussion to sections $V \to Z$ satisfying our kind system of PDE. For example, one has $\dim(X_c : \Gamma) < \infty$ for holomorphic sections of suitable holomorphic bundles over V, e.g. those associated to the tangent bundle.

(b) If one deals with higher order elliptic systems one may need a bound on $\|D^i x\|$ for $i > 1$ to achieve the full regularity and compactness, where X_c is defined by the condition $\|D^i(x)\| \leq c$ for some sufficiently large i.

(c) It is interesting to have a possibly precise bound on $\dim(X_c : \Gamma)$ depending on specific properties of the manifolds V and $\underline{X}$.

Here is a result by A. Eremenko (see [Ere]), where $V = \mathbb{C}$, $\underline{X} = \mathbb{C}P^m$ and the maps we are concerned with are holomorphic ones.

The restriction map $x \mapsto x \mid \Delta$, sending $X_c \to (\mathbb{C}P^m)^\Delta$, is an embedding, provided $\Delta \subset \mathbb{C}$ is ε-dense for $\varepsilon < c^{-1}\sqrt{\pi/4}$. Furthermore,

$$\dim(X_c : \mathbb{C}) \leq 2mC^2/\pi\,;$$

(d) More general (but less precise) results are available for harmonic maps, where the elliptic estimates are controlled by the lower bound on Ricci curvature of V and the upper bound on the sectional curvature of $\underline{X}$ as (apparently) follows from the Yau gradient estimate (compare (b) in 3.4.A′).

(e) Our embedding result states, in effect, that two *distinct* harmonic (line) maps x and x' with bounded differentials can not coincide on a sufficiently dense subset Σ in the manifold V where the maps are defined. Much more is known for holomorphic maps, where the first main theorem of the Nevanlinna theory provides a bound on the density of Σ in terms of the growth of $\|Dx\|$ and $\|Dx'\|$ on V. This leads to the following

General problem. Consider harmonic maps x from V to $\underline{X}$ or more general maps satisfying some (linear or nonlinear) system of elliptic P.D.E. Take two non-negative functions $\sigma(v)$ and $\delta(v)$ en V and decide whether there exist two *distinct* maps x and x' from our class, such that

$$\max(\|Dx(v)\|, \|Dx'(v)\|) \leq \sigma'(v)$$

and

$$\operatorname{dist}(x(v), x'(v)) \leq \sigma(v)$$

for all $v \in V$.

Here again, one expects the bound on a suitable density of the zero set of $\delta(v)$ in terms of the asymptotic growth of $\sigma(v)$ for $v \to \delta$. More generally, one wishes to show, that if $\delta(v)$ is small on a rather dense set, then it is also small on a much larger set, provided we have some bound on $\sigma(v)$. For example, a holomorphic function x with many zeros in a ρ-disk and with a bound on $\|x(v)\|$ in the consecutive 2ρ-disk is much smaller on the ρ-disk than was suggested by the original bound on $\|x(v)\|$. Another general phenomenon of this kind is the unique continuation property for elliptic P.D.E. but all this seems far away from a desirable solution of the above problem.

3.4.C. Residual dimension for spaces of holomorphic maps. This refers to the dimension of the space of holomorphic maps $x : V/\Gamma_i \to \underline{X}$ for subgroups $\Gamma_i \subset \Gamma$ of finite index. The above argument shows that the dimension of the space of such maps x satisfying$\|Dx\| \leq c$ is bounded by $A\, c^n |\Gamma/\Gamma_i|$ for some constant $A = A(V, \Gamma, \underline{X})$ and $n = \dim_{\mathbb{R}} V$. In fact this remains valid for all our harmonic-like maps while for holomorphic maps there is a better estimate due to the following elementary (and well known)

3.4.C′. Proposition. *Let $\underline{X}$ be a complex projective variety, W be a compact connected complex manifold and let $x_0 : W \to \underline{X}$ be a holomorphic map. Then the dimension of the connected component X_0 of x_0 in the space of holomorphic maps $W \to \underline{X}$ is bounded by the volume of the image of x_0 and the maximal number $\nu = \nu(x_0)$ of irreducible components of the fibers of x_0 as follows*

$$\dim X_0 \leq A\, \nu \operatorname{Vol}_{2k} x_0(W)$$

for $k = \dim_{\mathbb{C}} x_0(W)$, $\nu = \sup_{\underline{x} \in \underline{X}} \operatorname{card} \operatorname{conn}(x_0^{-1}(\underline{x}))$, and some constant $A = A(\underline{X})$.

Proof. The dimension $\dim X_0$ is bounded by the dimension h of the space H_x of holomorphic sections of the induced bundle $x^*(T(\underline{X})) \to W$ for a generic $x \in X_0$, as these H_x make up the tangent bundle of X_0 on the non-singular locus of X_0, which is known to be a complex variety in its own right. (Here we used smoothness of $\underline{X}$ but this can be always achieved by embedding $\underline{X}$ into a smooth variety, e.g. into a projective space.) Next we observed that $T(\underline{X})$ can be embedded into a sum of several very ample line bundles over $\underline{X}$ (this is true and obvious for all vector bundles L over *projective* varieties) and the matter reduces to evaluation of the dimension

$\ell = \dim H_0(x^*(L))$. Such an L embeds $\underline{X}$ to some projective space $\mathbb{C}P^N$ and so we may think of $x^*(L)$ as the restriction of the bundle $O(1)$ to our W, now mapped to $\mathbb{C}P^N$ by composing $x : W \to \underline{X}$ and the embedding $\underline{X} \to \mathbb{C}P^N$. Notice that the product $\nu(\times) \operatorname{Vol} \times (W)$ is invariant under deformations of maps and so all we need is to estimate ℓ for a map $y_0 : W \to \mathbb{C}P^N$ in terms of $\nu = \nu(y_0) \operatorname{Vol}_{2k} y_0(W)$. We do this by induction on k as follows. Intersect $y_0(W)$ with a generic hyperplane P and observe that our number $\ell = \ell_k$ is bounded by $\ell_{k-1} + \ell'$ where ℓ_{k-1} comes from $P \cap y_0(W)$ and ℓ' is the dimension of the space of sections of $O(1)$ on W which vanish on $P \cap y_0(W)$. This space easily identifies with the space of sections of a trivial line bundle over $y_0(W)$ and so $\ell' = 1$. Thus everything reduces ℓ_0 where our variety consists of at most $\nu \operatorname{Vol} y_0(W)$ points counted with multiplicity (for the usual in $\mathbb{C}P^N$, where the volume of each subvariety equals its degree). Thus finally

$$\ell \leq \nu \operatorname{Vol}_{z_k} y_0(W) + k\,.$$

Q.E.D.

(Notice, this is sharp for the linear embeddings $W = \mathbb{C}P^k \to \mathbb{C}P^N$.)

Remark. Probably the conclusion remains true for all complex (not necessarily algebraic) $\underline{X}$ and, possibly, for more general harmonic (like) maps.

3.5. Lower bounds on the mean dimension for spaces of holomorphic maps. If $\underline{X}$ is a compact Riemannian manifold and V is complete, then, typically, the space of harmonic maps $x : V \to \underline{X}$ with $\|Dx\| \leq \text{const}$ looks zero dimensional (probably, uncountable for many generic classes of metrics on V), but I am not aware of any published result of this kind. On the other hand there are certain remarkable exceptions, such as Kähler manifolds that sustain lots of holomorphic maps and these are necessarily harmonic.

3.5.A. Example: maps $\mathbb{C} \to S^2$. These are just meromorphic functions $x = x(v)$, $v \in \mathbb{C}$, which can be constructed in abundance with bounded spherical derivatives as follows. Take a discrete subset $\Sigma \subset \mathbb{C}$ and consider meromorphic functions $\varphi_\sigma : \mathbb{C} \to \mathbb{C} \cup \infty = S^2 = \mathbb{C}P^1$ of the form $\varphi_\sigma(v) = c_\sigma(v - \sigma)^{-k}$. If the sum of these over all $\sigma \in \Sigma$ converges, we get a meromorphic function $X : \mathbb{C} \to \mathbb{C} \cup \infty = S^2 = \mathbb{C}P^1$ where one can easily control the differential dx. For example, if Σ is separated, i.e. $\|\sigma_1 - \sigma_2\| \geq \delta > 0$

for all $\sigma_1 \neq \sigma_2$ in Σ and the coefficients c_σ are bounded, then this sum obviously converges and gives us an $x : \mathbb{C} \to S^2$ with $\sup_{v \in \mathbb{C}} \|dx\| < \infty$, provided $k \geq 3$. Moreover, by varying c_σ, one can easily make such an f with prescribed values on a sufficiently rare net $\Sigma' \subset \mathbb{C}^n$ lying away from Σ. This shows, that the space X_c of holomorphic (and thus harmonic) maps $x : \mathbb{C} \to S^2$ with $\|dx\| \leq c$ has

$$\dim(X_c : \mathbb{C}) = \kappa c^2 \text{ for some } \kappa > 0 .$$

Consequently, *if a complex analytic manifold $\underline{X}$ contains a rational curve then the space of holomorphic maps $x : \mathbb{C} \to \underline{X}$ with $\|dx\| \leq c$ has positive mean dimension for all $c > 0$.*

Remark. By varying σ_i and/or rotating the sphere S^2, one can easily make an $x : \mathbb{C} \to S^2$ with $\|dx\| \leq c$ and prescribed values on a given, sufficiently sparce (depending on $c > 0$) net $\Sigma' \subset \mathbb{C}$. (See [Ere] for a finer construction of such interpolating maps $\mathbb{C} \to \mathbb{C}P^n$.)

3.6. L_2-technique for maps $V \to \mathbb{C}P^N$. Let $V = (V, g)$ be a Hermitian manifold with locally bounded geometry and $E \to V$ a strictly positive line bundle, i.e. with the curvature form w satisfying $w \geq \lambda g$.

3.6.A. Embedding Theorem. *There exists a holomorphic uniformly locally bi-Lipschitz map $x : V \to \mathbb{C}P^N$ for some $N = N(V, E)$.*

Proof. First, for each point $v \in V$, we can construct $n + 1$ L_2-sections $x_j : E^i$ for some i and $j = 0, \ldots, n = \dim V$ such that the map $V \to \mathbb{C}P^n$ defined by these sections embeds some ball $B(v, \rho)$ to $\mathbb{C}P^n$. Furthermore, by squaring x_j, we can make the sections L_1 (see 2.1.B$'$). Then, we take such sections at each point of a sufficiently rare net $\Sigma \subset V$ and by summing them up (compare 3.3.G), obtain a map $V \to \mathbb{C}P^n$ that embed the ρ-neighbourhood $U_\rho(\Sigma) \subset V$ into $\mathbb{C}P^n$ in a bi-Lipschitz manner. Finally we cover all of V by ρ-neighbourhoods of several such nets, $V = \bigcup_\nu U_\rho(\Sigma_\nu)$, $\nu = 1, \ldots, N_0$, and then the resulting map $x : V \to \mathbb{C}P^{nN_0 + N_0 - 1}$ is clearly seen to be locally bi-Lipschitz. Q.E.D.

Remark. If V is compact, the above amounts to the classical Kodaira theorem, where one can, moreover, project V from $\mathbb{C}P^N$ to $\mathbb{C}P^{2n+1}$ and then further to $\mathbb{C}P^n$ if one is not concerned so much with embeddings. But if V is non-compact, the image of V in $\mathbb{C}P^N$ may be, a priori, dense and then there is no holomorphic *Lipschitz* projection to $\mathbb{C}P^{N-1}$. However, such projection can be obtained with the uniform transversality theorem as will become clear later on.

3.6.B. Now, given a suitable holomorphic Lipschitz (i.e. with bounded differential) map $x_0 : V \to \mathbb{C}P^N$ we want to generate a larger space of such maps. To do this we take the pull-back $E \to V$ of the $O(1)$-bundle over $\mathbb{C}P^N$ and use bounded sections of E^i for this purpose. So we need E to be rather positive which is ensured by the following condition generalizing the "locally bi-Lipschitz" property.

Uniform non-degeneracy. Let $x : V \to \mathbb{C}P^N$ be a holomorphic Lipschitz map. Since V has bounded geometry, we have a local coordinate system with "bounded distortion" at each point $v \in V$ and so by looking at x and on all small balls in V we obtain a precompact family of holomorphic maps from the unit ball $B \subset \mathbb{C}^n$ to $\mathbb{C}P^N$, call them $x_v : B \to \mathbb{C}P^N$. We say that x is *uniformly non-degenerate* if every map $y : B \to \mathbb{C}P^N$ belonging to the closure of the family $\{x_v\}$, $v \in V$, (with the uniform topology) is finite to one. For example, if V is compact, then this equivalent to x itself being finite to one.

Now, it is essentially standard that if $x : V \to \mathbb{C}P^N$ is uniformly non-degenerate holomorphic Lipchitz map, then then *induced Hermitian structure in $E = x^*(O(1))$ admits a small perturbation making the curvature of E strictly positive.* (Such a perturbation can be achieved, for example, along the stratification of the locus where the differential Dx is non-injective.)

3.6.B′. Projective Interpolation theorem. *Suppose V admits a uniformly non-degenerate holomorphic Lipschitz map x_0 to $\mathbb{C}P^N$. Then for every δ-separated subset $\Sigma \subset V$ there exists a holomorphic map $x : V \to \mathbb{C}P^N$ with $\|Dx\| \leq \mathrm{const}(1 + \delta^{-1})$ taking given values at all points $\sigma \in \Sigma$, where* $\mathrm{const} = \mathrm{const}(V, x_0)$.

Proof. The line bundle $\underline{E} = O(1)$ over $\mathbb{C}P^N$ admits many (meromorphic) maps into $\mathbb{C}P^N$ different from the original projection. To see one, observe that each point (vector) in $\underline{E}$ is given by a pair (ℓ, φ) where $\ell \subset \mathbb{C}^{N+1}$ is a line and $\varphi : \ell \to \mathbb{C}$ a linear form.

Now, with a given vector $z_1 \in \mathbb{C}^{N+1}$, we associate the map $p_{z_1} : \underline{E} \to \mathbb{C}P^N$ where the line $\ell' = p_{z_1}(\ell) \in \mathbb{C}P^N$ is spanned by the vector $z_1 + \varphi^{-1}(1) \in \mathbb{C}^{N+1}$. Notice, that this p_{z_1} has poles, but it is regular in some (Zariski) neighbourhood of the zero section $\mathbb{C}P^N \subset L = x_0^*(\underline{E})$ of $\underline{E}$ to V, and observe that each holomorphic section $y_1 : V \to E$ with a sufficiently small sup-norm gives us a map of V to $\mathbb{C}P^N$, that is the composed map $p_{z_1} \circ y_1$, denoted $x_1 : V \to \mathbb{C}P^N$. Furthermore, if $z_1 \neq 0$, then the map p_{z_1} is injective on each fiber of L near zero, and so we obtain an *embedding* from the space of small sections $V \to E$ to the space of maps $V \to \mathbb{C}P^N$ close to x_0. (Consequently, the mean dimension of the space of maps $x : V \to \mathbb{C}P^N$ with $\|Dx\| \leq \text{const}$ is bounded from below by that for the space of bounded sections $V \to E$.) Then one can similarly deform x_1 using some p_{z_2} and y_2 and so on. Thus the proof would be concluded if we had the interpolation property in the bundle E.

We cannot guarantee that E itself has sufficiently many sections, but some power E^i is good for this purpose. To go from E to E^i, we consider a selfmapping ψ of $\mathbb{C}P^N$ given by polynomials of degree i such that ψ can be found with $\|D\,\psi\|$ about $\sqrt{i}$ (modeled on the standard map of $\mathbb{C}^N/\mathbb{Z}^{2N}$) and it pulls back $\underline{E}$ to $\underline{E}^i$. We compose ψ with our $x_0 : V \to \mathbb{C}P^N$ and thus promote E to E^i over V, as $(x_0 \circ \psi)^*(\underline{E}) = E^i$. Now we have as many sections as we need and the proof trivially follows from 3.2.6.

Corollary to the proof. (Compare 0.6.B.) *If V is acted upon by an amenable group Γ with a projective algebraic quotient V/Γ, then the space X_c of holomorphic maps $x : V \to \mathbb{C}P^N$ with $\|Dx\| \leq c$ satisfies*

$$\dim(X_c : \Gamma) \geq b' c^{\dim_{\mathbb{R}} V}$$

for all sufficiently large c and some $b' > 0$.

3.6.B$''$. Projective transversality theorem. *Let $x_0 : V \to \mathbb{C}P^N$ be as above and consider a subset S in the space of r-jets holomorphic maps $V \to \mathbb{C}P^N$. We want to move x_0 away from S, i.e., to have the r-jet of the moved section to lie ε-far from S for some $\varepsilon > 0$. Again, we cannot freely manipulate x_0 itself, but we can work with $x_i = \psi \circ x$ for the above map $\psi : \mathbb{C}P^N \to \mathbb{C}P^N$, where the above argument combines with the uniform transversality theorem and leads to the following conclusion,*

If the uniform dimension (see [Gro]$_{\text{PCMD}}$*) of S is strictly less than $\dim_{\mathbb{R}} V = \lambda \dim_{\mathbb{C}} V$, then there exists a holomorphic $\mathbb{R}$ uniformly non-degenerate Lipschitz map $x_i' : V \to \mathbb{C}P^N$ which is uniformly transversal to S.*

Corollaries. (a) *If* $N > \dim_{\mathbb{C}} V$, *then one can make the* x_i' *miss a small ball in* $\mathbb{C}N^N$. *This allows projections from* $\mathbb{C}P^N$ *to* $\mathbb{C}P^{N-1}$ *and eventually to* $\mathbb{C}P^n$. *Thus we obtain a holomorphic uniformly non-degenerate map* $V \to \mathbb{C}P^n$, $n = \dim V$.

(b) If $N \geq 2n-1$, *we can produce uniform immersions* $V \to \mathbb{C}P^N$, *i.e. uniformly locally bi-Lipschitz maps.*

3.6.C. Remarks and questions. (a) If $V = \mathbb{C}^n$ or if V admits a non-constant holomorphic map $V \to \mathbb{C}^n$ with bounded differential, then there are lots of holomorphic maps $x : V \to \mathbb{C}P^n$ with $\|Dx\| \leq c$ for arbitrarily small $c > 0$. On the other hand, for some V, every map $x : V \to \mathbb{C}P^n$ with sufficiently small $\|Dx\|$ is necessarily constant. This is the case, for example, for infinite cyclic coverings of compact manifolds as well as for more general V which are 1-*dimensional at infinity* in the sense of $[\mathrm{Gro}]_{\mathrm{PCAD}}$. Can one classify manifolds with this properly? Similarly, assuming V is acted upon by an amenable group Γ, what is the mean dimension of the space X_c of holomorphic maps $V \to \mathbb{C}P^N$ for small $c < 0$? Now, in general, does the dimension $\dim(X_c : \Gamma)$ depend on c, especially for $c \to 0$?

(b) What is the relation of $\dim(X_c : \Gamma)$ and the corresponding residual dimension for residually finite groups Γ ? In particular, when can a holomorphic map $x : V \to \mathbb{C}P^N$ with $\|Dx\| \leq c$ be approximated by Γ_i-periodic maps x_i with $\|D x_i\| \leq c_1$ where $\Gamma_i \subset \Gamma$ is some sequence of subgroups of finite index and c_j is independent of i ? Closely related to this is the Runge approximation problem where we look for an approximate extension of holomorphic Lipschitz maps from smaller domains in V to larger ones. Finally, one asks when holomorphic Lipschitz maps to $\mathbb{C}P^N$ extend from subvarieties $W \subset V$ with bounded local geometry to all of V.

(c) **Foliations.** Consider a manifold $\mathcal{U}$ (or a general locally compact space for this matter) foliated into complete Hermitian manifolds V and let $\mathcal{E} \to \mathcal{U}$ be a complex line bundle holomorphic along the leaves. For each point $u \in \mathcal{U}$ we take the universal covering $\widetilde{V}_u$ of the leaf $V_u \subset \mathcal{U}$ passing through u, thought of as the space of the homotopy classes of loops in $V_u \subset \mathcal{U}$ based at u so that u *canonically* lifts to $\widetilde{V}_u$ and is denoted $\tilde{u} \in \widetilde{V}_v$. Let $\widetilde{H}_u$ be the space of holomorphic L_2-sections of the bundle $\widetilde{\mathcal{E}}_u \to \widetilde{V}_u$ induced from $\mathcal{E}$ and take a vector e in the (1-dimensional) fiber $E_u \subset \mathcal{E}$ at u identified with the corresponding fiber $E_{\tilde{u}}$ of $\widetilde{\mathcal{E}}_u$. If the evaluation map $\widetilde{H}_u \to \widetilde{E}_{\tilde{u}} = E_u$ is surjective, then there exists a unique section $\tilde{h} = \tilde{h}_e : \widetilde{V}_u \to \widetilde{\mathcal{E}}_u$ having $\tilde{h}(u) = e$ and minimizing the L_2-norm $\|\tilde{h}\|_{L_2}$. Now, suppose $\mathcal{E}$ is positive along the leaves, where this positivity is uniform on $\mathcal{U}$, and also assume that all leaves have uniformly bounded local geometry. Then the above surjectivity condition is satisfied for all $e \in \mathcal{E}_u$ and all u, if not for $\mathcal{E}$ itself, then, at least, for some tensorial power $\mathcal{E}^i$ of $\mathcal{E}$. Thus every section $\varphi : u \mapsto e(u) \in \mathcal{E}^i_u$ of $\mathcal{E}^i$ gives rise to a family of holomorphic L_2-sections $\tilde{h}_u \overset{def}{=} \tilde{h}_{\varphi(u)} : \widetilde{V}_u \to \widetilde{\mathcal{E}}^i_u$, that is a section, call it $\widetilde{\varphi}$, of $\mathcal{E}^i$ lifted to the graph $\widetilde{V}$ of our foliation defined as the space of pairs $(u, \tilde{v})$, for $u \in U$ and $\tilde{v} \in \widetilde{V}_u$. According to 3.3.G$''$, each holomorphic constituent $\tilde{h}_u$ of $\widetilde{\varphi}$ has a controlled decay on the leaf $\widetilde{V}_u$ with the decay estimate independent of u. Moreover, the L_2-estimate (see $(*)$ in 3.3.B) implies that the sections $\tilde{h}_u$ are L_2-continuous in u, provided φ is continuous. In fact, if u and u' are near by points in $\mathcal{U}$ and $e \in \mathcal{E}^i_u$ and $e \in \mathcal{E}'_{u'}$ are close vectors, then the leaves $\widetilde{V}_u$ and $\widetilde{V}_{u'}$ are close on large balls $\widetilde{B}_u \subset \widetilde{V}_u$ and $\widetilde{B}_{u'} \subset \widetilde{V}_{u'}$. Then the holomorphic section $\tilde{h}_e$, on $\widetilde{B}_{u'}$ can be moved to an almost holomorphic section $\tilde{h}'$ on B_u which lies close to $\tilde{h}_{e'}$ on $\widetilde{B}_{u'}$ and has $\tilde{h}'(u) = e$. This $\tilde{h}'$ can be made zero outside $\widetilde{B}_u$ by applying an obvious cut-off argument and then we observe that the section $\tilde{h}'' = \frac{1}{2}(\tilde{h}_e + \tilde{h}')$ is also almost holomorphic and thus can be turned holomorphic as we did before using the L_2-estimate. The resulting holomorphic $\tilde{h}^\bullet$ is L_2-close to $\tilde{h}''$ and may be assumed to have $\tilde{h}^\bullet(u) = e$. On the other hand, if $\tilde{h}_{e'}$ were far from $\tilde{h}_e$, then $\tilde{h}'$ is far from $\tilde{h}_e$ as well, and then their mean would have significantly smaller L_2-norm than $\tilde{h}_e$ (as we could assume $\|\tilde{h}_{e'}\|_{L_2} \leq \|\tilde{h}_e\|_{L_2}$ by interchanging u and u' otherwise) which contradicts to minimality of the norm $\|\tilde{h}_e\|_{L_2}$. Next, we square each $\tilde{h}_u$ thus making it L_1 and then push down the resulting section $\widetilde{\varphi}^2$ from $\widetilde{V}$ to a section $\overline{\varphi}^2(v)$ of $\mathcal{E}^{2i} \to \mathcal{U}$ by integrating $\widetilde{\varphi}^2(u, \tilde{v})$ with respect to u over the leaf $V_v = V_u$ and summing up over all $\tilde{v}$ in $\widetilde{V}_u$ over v, where we need φ to be bounded (as well as continuous) on $\mathcal{U}$. In particular, if $\mathcal{U}$ is compact, we obtain, by varying φ, lots of continuous sections of $\mathcal{E}^{2i} \to \mathcal{U}$ holomorphic along the leaves and conclude that $\mathcal{U}$ *admits a continuous leafwise holomorphic and leafwise locally bi-Lipschitz map to some* $\mathbb{C}P^N$.

Notice, that such foliations exist, for example, on locally homogeneous spaces $\mathcal{U}$ of the form $K\backslash G/\Gamma$ where G is a semi-simple group without compact quotients, $K \subset G$ a (non-maximal!) compact subgroup and $\Gamma \subset G$ is a cocompact lattice.

(d) **Singular spaces V.** Probably, our results extend to singular spaces V with an obvious extension of the idea of bounded local geometry. For example, one can easily handle submanifolds V of a manifold W with bounded geometry, such that $\mathrm{Vol}_{2n}(V \cap B) \leq$ const for all unit balls B in W. In general, one needs a suitable version of $\overline{\partial}$-technique where a natural idea is to embed V into a nonsingular manifold. Alternatively, one may resolve the singularity of V and adjust the $\overline{\partial}$-lemma to sections constant (vanishing) on the pull-back of the singular locus. Alternatively, one may try L_2-techniques on the Čech resolutions of the relevant sheaves.

(e) As we mentioned earlier, the space of harmonic maps between generic Riemannian manifolds seems rather small but there are some exceptional cases besides the Kähler manifolds. For example, one may look from this angle at harmonic maps $\mathbb{R}^n \to S^N$ for all $n \geq 2$. One can also consider *n-harmonic maps* which locally minimize the energy $= \int \|Dx\|^p dv$ with $p = n$ which bubble very much like ordinary harmonic maps for $n = 2$. Here one should probably replace the uniform metric for $\int \|Dx\|^p$ by "the energy metric" and study maps $x : V \to \underline{X}$ locally minimizing $\int \|Dx\|^p dv$ and having this integral uniformly bounded over the unit balls in V. This is similar to bounding holomorphic maps $x : V \to \underline{X}$ by their "local degrees", i.e. by the volumes of their graphs within unit balls in $V \times \underline{X}$, where one can use estimates from 4.1 or, alternatively, the (first main theorem of the) Nevanlinna theory which, when it applies, gives better bounds on the mean dimensions of these spaces than 4.1, (as was pointed out to me by Alex Eremenko). Notice that in all these cases the spaces of maps with bounded local energy (or degree) are not compact and one should compactify them by allowing suitable "singular maps" best represented by certain subsets in $V \times X$ appearing as limits of graphs of the maps in question.

(f) It is worthwhile to recall at this stage that the mean dimension of a space of maps $V \to X$ appears as a limit of the ε-dimensions of spaces of maps of relatively compact domains $\Omega \subset V$. A more general class of geometric problems can be formulated for an arbitrary V, allowed to be non-complete and/or to have a boundary and for a relatively compact Ω in the interior of V. Here we take some space X_c of our (harmonic like) maps $x : V \to \underline{X}$ with a bound like $\|Dx\|_! < c$, or a similar bound on the (local or global) energy of x. Then we restrict the maps $x \in X_c$ to Ω and evaluate the ε-dimension $\mathrm{Widim}_\varepsilon$ of the resulting space $X_c|\Omega$ of maps $\Omega \to \underline{X}$ with respect to some metric in this space, e.g. the uniform metric or some energy metric. What we want to know is the asymptotic behavior of the resulting $\mathrm{Widim}_\varepsilon(X_c|\Omega)$ for growing V and Ω, where Ω remains much smaller than V. Here it is equally interesting to evaluate the minimal number N_ε of the ε-balls needed to cover $X_c|\Omega$, where the expected growth is roughly $\varepsilon^{-\mathrm{Vol}\,\Omega}$.

3.6.D. About fusion. If $\dim_{\mathbb{C}} V = 1$, then there are nonlinear techniques for producing holomorphic maps $V \to X$ where X is an *almost complex manifold* (with possibly a *non-integrable* structure) which contains "sufficiently many" rational curves. Here a given holomorphic map $x_0 : V \to x$ can be modified by "fusing" it with rational curves at the points σ of some discrete subset $\Sigma \subset V$. Recall that the analytic model for "fusion" of two curves c_1 and c_2 in X given by the equations $f_1(x) = 0$ and $f_2(x) = 0$ is the curve $C = C_\varepsilon$ given by the equation $f_1 f_2 = \varepsilon$. This C_ε for small $\varepsilon \neq 0$ looks like the connected sum of C_1 and C_2 at their intersection points.

Example. Let $X = \mathbb{C}P^N$ with an almost complex structure tamed by the standard symplectic from w on $\mathbb{C}P^N$. One can easily construct, by fusing together infinitely many rational curves, *a holomorphic Lipschitz map $x : \mathbb{C} \to \mathbb{C}P^N$ with assigned values on a given separated subset $\Sigma \subset \mathbb{C}$.* Probably, there is a similar interpolation result for all Riemann surfaces with bounded geometry. Also one may try maps into more general spaces X, e.g. into *rationally connected* algebraic manifolds X.

§ 4. Spaces of subvarieties.

Let W be a Hermitian manifold and consider the space of complex analytic subvarieties $M \subset W$ of given dimension n. All ? possible topology in $\mathcal{M}$ comes from the Hausdorff convergence on compact subsets in W. We shall use below a slightly different topology induced on $\mathcal{M}$ from the space of currents on W. Namely, for every collection Ω of continuous forms w on W of degree $2n$ with compact support, we set

$$|M_1 - M_2|_\Omega = \sup_{w \in \Omega} \left| \int_{M_1} w - \int_{M_2} w \right| .$$

Then $M_i \to M$ signifies that $|M - M_i|_\Omega \to 0$ for every finite collection Ω. Notice that the limits of M_i in this topology may acquire multiplicity. For example, the graph $M_\lambda \subset \mathbb{C}^2$ of the function $z \to \lambda z^2$, converges, for $\lambda \to \infty$, to the vertical line in $\mathbb{C}^2$ with multiplicity two (while the Hausdorff limit gives us this line without multiplicity. Our objective is the space $\widetilde{\mathcal{M}}_d$ consisting of the subvarieties $M \subset W$ of "local degree" bounded by d. This means that

$$\mathrm{Vol}_{2n}(M \cap B) \leq \alpha_W d$$

for all unit balls $B \subset W$ and a suitable normalization constant $\alpha_W > 0$ which for $M = \mathbb{C}^N$ should be chosen equal the volume of the unit Euclidean $2n$-ball. If W has bounded geometry, then our study can be reduced to that in $W = \mathbb{C}^N$ where the relevant properties of $M \in \widetilde{\mathcal{M}}_d$ become more transparent.

4.1. Normalization and Cauchy inequality. A complex analytic subvariety $M \subset \mathbb{C}^N$ can be locally represented as the graph of multi-valued holomorphic map $\mathbb{C}^n \to \mathbb{C}^{N-n}$. Namely, for each point $v \in M$, there exists a linear projection $p : \mathbb{C}^N \to \mathbb{C}^n$ so that p is finite-to-one on M. Then a germ of M at v becomes a ramified cover of a small ball $B = B(p(v), \varepsilon) \subset \mathbb{C}^n$ where it is represented by the graph of a d_v-valued map from B to $\mathbb{C}^{N-n}$. Such a map can be viewed as a singled valued holomorphic map from B to the d_v-, the symmetric power of $\mathbb{C}^{N-n}$, say $\mu : B \to S_{d_v}\mathbb{C}^{N-n}$ and by Cauchy inequality we can bound the differential of μ in a smaller ball, say in $B' = B(p(v), \varepsilon/2)$ by something like $\varepsilon/2$. (Notice that the variety $S_{d_v}\mathbb{C}^{N-n}$ is singular but it embeds into a smooth one and so one can speak of norms of derivatives of maps into it.) Thus our objective is a lower bound on ε in terms of the $2n$-volume of M.

4.1.A. Controlled normalization. We want to locate m-dimensional polydisks $D^m \subset \mathbb{C}^N$ (which are more suitable for us than $2m$-balls) for $m = N - \dim M$, such that their intersections with M are stable under ε-perturbations. Thus we say that D^m is *ε-transversal to M* if its boundary ∂D^m is ε-far from M,

$$\operatorname{dist}(\partial D^m, M) \geq \varepsilon\,.$$

Here every D^m lies in some m-dimensional affine subspace $L \subset \mathbb{C}^N$. Observe that this $D^m = (D(r))^m$ contained in the ball $B(R) \subset \mathbb{C}^m$ of radius $R = \sqrt{n}r$ and we call this R *the radius of* D^m.

Lemma. *Let M be a complex analytic subvariety in $\mathbb{C}^N$ of dimension n. Then for every $\rho > 0$ there exists a polydisk D^m in $\mathbb{C}^N$ with the following properties.*
(1) The center of D^m is located at the origine of $\mathbb{C}^N$.
(2) The radius R of D^m lies in the interval $\rho/2 \leq R \leq \rho$.
(3) D^m is ε-transversal to M, where

$$\varepsilon \geq \operatorname{const}_N \rho^{2n+1} d^{-1}(\rho) \qquad (+)$$

for $d = d(\rho)$ denoting the $2n$-volume of the intersection of M with the ρ-ball in $\mathbb{C}^N$ around the origin.

Remark. One can show, a posteriori, that such D^m exist for *all* R in the interval $\frac{1}{2}\rho \leq R \leq \frac{2}{3}\rho$.

Proof. An obvious integral geometric (or, alternatively, transversality) argument shows that almost all polycylinders D^m centered at $0 \in \mathbb{C}^N$ have empty intersection $\partial D^m \cap M$. Here one may invoke the compactness property of analytic sets with bounded volume and thus obtain a definite lower bound on $\operatorname{dist}(\partial D^m, M)$ for some R. Next, as we want a quantitative result, we recall the relevant property of M behind the compactness property which reads

$$\operatorname{Vol}_{2n}(M \cap B(v, \delta)) \geq \alpha_n \delta^{2n} \qquad (*)$$

for all $v \in M$, all balls $B(v, \delta) \subset \mathbb{C}^N$ at v and α_n equal the volume of unit Euclidean $2n$-ball. Then we take some D^m of radius R centered at $O \in \mathbb{C}^N$ and intersect it with the ε-neighbourhood M_ε of M. We measure this intersection by the minimal number $N = N(D^m, \varepsilon)$ of ε-balls needed to cover the part of this intersection lying in the "band" $D^m \backslash \frac{1}{2} D^m = D^m(\rho) - D^m(\rho/2)$. If $N \leq 0.1\,\rho/\varepsilon$, then, clearly, there is an R between $\rho/2$ and ρ such that the boundary of $D^m(R)$ misses M_ε and $(+)$ follows with this ε.

Now, assume $N \geq 0.1\,\rho/\varepsilon$ for all $D^m(R)$ and bound the volume of $M \cap B(\rho)$ from below $(*)$ as follows. First, imagine we are allowed the parallel translations of D^m by distance ρ. Then we get about $N(\rho/\varepsilon)^{2n}$ points in M with mutual distances $\geq \varepsilon/2$ and thus the volume of M covered by these translated is bounded from below roughly by $N\alpha_n(\varepsilon/2)^{2n}(\rho/\varepsilon)^{2n} = N\alpha_n\rho^{2n}$. It follows, N is bounded approximately by $\mathrm{Vol}(M \cap B(\rho)/\rho^{2n}$, hence $0.1\,\rho/\varepsilon \leq C_N \,\mathrm{Vol}(M \cap B(\rho))/\rho^{2n}$ and $(+)$ follows.

Finally, instead of translating D^m (which we are not allowed to do as the center of D^m is fixed) we rotate it around some $(m-1)$-plane L in $\mathbb{C}^m \supset D^m$. We choose this $L \subset \mathbb{C}^m$ so that the significant part of the intersection $M_\varepsilon \cap (D^m \backslash \frac{1}{2}D^m)$ lies roughly ρ-far from L, i.e. the covering number for $M_\varepsilon \cap (D^m \backslash \frac{1}{2}D^m \backslash L_{\rho'})$ is $\geq \beta_N \rho/\varepsilon$ for some $\beta_N > 0$ and $\rho' \geq \beta_N \rho$. Granted such L, the rotation of D^m gives us essentially the same volume as the above translation. Finally, to see that such L exists, we apply the same reasoning, but now we rotate L in $\mathbb{C}^m$ around some $(m-2)$-plane $L' \subset L$. Then L' is located with rotation of L' around L'' and so on down to a rotating line in $\mathbb{C}$? Q.E.D.

4.1.B. Local representation of M by multivalued function. Consider the normal ε-tube around our D^m that is $D^m \times B_0^\perp(\varepsilon) \subset \mathbb{C}^{N=m+n}$ where $B_0^\perp(\varepsilon)$ is the n-dimensional ε-ball in $\mathbb{C}^N$ normal to D^m and let M_0 denote the intersection of M with this tube. Clearly, the projection of M_0 to $B_0^\perp(\varepsilon)$ is a proper map of multiplicity $d_0 = d_0(\varepsilon) \leq \mathrm{const}_n(\mathrm{Vol}\, M \cap B(\rho))/\varepsilon^{2n}$. Thus M_0 is represented by the graph of d_0-valued map over $B_0^\perp(\varepsilon)$, say $\varphi_0 : B_0^\perp(\varepsilon) \to S_{d_0}D^m$. Then we consider such tubes centered at all points in M and cover M by a minimal number of these. Here we set $\rho = 1$ and denote by d the supremum of the volumes of intersections of the unit balls in $\mathbb{C}^N$ with M. Then we see with $(*)$ that there is a covering of M by ε-tubes, where the number of such tubes meeting each unit ball in $\mathbb{C}^N$ is bounded by $\mathrm{const}_N\, d\varepsilon^{-2n} \leq \mathrm{const}'_N\, d^{2n+1}$, where we use $(+)$ in the form $\varepsilon \geq \mathrm{const}_N\, d^{-1}$ (and where we exercise the usual freedom with the notation "const"$_N$). Finally we observe with $(+)$ that $d_0 \leq \mathrm{const}_N\, d^{2n}$ and so $\dim S_{d_0}D^m \leq \mathrm{const}_N\, md^{2n} \leq \mathrm{const}'_N\, d^{2n}$. Thus the total number of "parameters per unit volume" defining M is bounded by $\mathrm{const}_N\, d^{4n+1}$. This makes plausible that the mean dimension of the space $\widetilde{\mathcal{M}}_d$ of n-dimensional complex subvarieties $M \subset \mathbb{C}^N$ with the local degrees bounded by d satisfies the inequality

$$\dim(\widetilde{\mathcal{M}}_d : \mathbb{C}^N) \leq \mathrm{const}_N\, d^{4n+1}\,. \qquad (\star)$$

Actually, the natural conjecture (justified later on) reads

$$\dim(\widetilde{\mathcal{M}}_d : \mathbb{C}^N) \leq \mathrm{const}_N\, d^{n+1}\,, \tag{$\star\star$}$$

but we are not able to prove even the weaker inequality $(\star)$.

Here are two difficulties.

1. The above heuristic argument only applies to subvarieties close to a given one and we lack a good localization theorem saying that "the local mean dimension equals the global one". Thus we have to vary the tubes covering M which unpleasantly enlarges the exponent $4n+1$ to something of order N^2.

2. As we change a covering of M by ε-tubes, we change our representation of M by a collection of maps (this already happens near a fixed M as we appeal to Cauchy inequality). This introduces an ambiguity in our choice of a metric in M of order d^d (which probably could be greatly reduced) and this makes our exponent (even in the local case) comparably large.

Remark. One can improve the covering argument of M by ε-tubes (using tubes of variable size at different points in M) but I doubt you can bring the exponent down to $n+1$ this way. (The "difficult" M's are those having large intersections with small balls, e.g. having conical singularities of degrees $\approx d$.) On the other hand, the above 1 and 2 are purely technical problems and should be eventually resolved.

4.1.C. Parametrizations of $S_d\mathbb{C}^m$. The symmetric powers of $\mathbb{C}$ are non-singular. In fact, $S_d\mathbb{C}$ can be identified with $\mathbb{C}$ in several ways. For example, given a symmetric configuration of complex numbers $v_1, \dots, v_d$ one can associate to them the polynomial $p(z) = \prod_{i=1}^{d}(z - v_i)$ and then one uses the metric on $S_d\mathbb{C}$ corresponding to the sup-norm of functions on the disk $D(2) \subset \mathbb{C}$ of radius 2. Another useful representation of $S_d\mathbb{C}$ is by means of the symmetric functions,

$$s_1 = \sum_{i=1}^{d} m_i,\ s_2 = \sum_{i=1}^{d} m_i^2, \dots,\ s_d = \sum_{i=1}^{d} m_i^d\,,$$

where the corresponding metric is the sup-norm in the $(s_1, \dots, s_d)$-space. The two norms are bi-Lipschitz equivalent, at least in the region corresponding to m_i's with $|m_i| \leq 1$ where the Lipschitz constant can be trivially bounded by something like d^d. Indeed, going from s_i to polynomials amounts to expressing the elementary symmetric functions as polynomials in s_i. Conversely, one reconstructs s_i out of $p(z)$ by taking Cauchy integral of $z^i p'(z)/p(z)$ over the circle of radius 2 since $z^i p'(z)/p(z)$ has simple poles at m_i with residues m_i^d.

Next we observe that the natural map $S_d\mathbb{C}^m \to (S_d\mathbb{C})^m$ is finite-to-one, we take, additionally, the composition of this projection with a generic linear map of $\mathbb{C}^m$. Then the resulting map $S_d\mathbb{C}^m \to (S_d\mathbb{C})^m \times (S_d\mathbb{C})^m = (S_d\mathbb{C})^{2m}$ becomes one-to-one.

4.1.D. Embedding of $\mathcal{M}_d$ to a power space. We want to construct a sufficiently large set of m-disks in $\mathbb{C}^{N=m+n}$, so that each $M \subset \widetilde{\mathcal{M}}_d$ will be uniquely determined by intersections with these disks. (We shall eventually disregard the disks which are not ε-transversal to M.) Recall, that every m-disk in $\mathbb{C}^N$ is of the form $g\,D^m(1)$ for the standard $D(1) \subset \mathbb{C}$ and some isometry $g : \mathbb{C}^N \to \mathbb{C}^N$. Thus we can mark the disks in our set by g's. Here are our requirements on these disks and g's.

A. The set of g's is ε_0-dense in the group $\mathrm{Isom}_{\mathbb{C}}\,\mathbb{C}^N$ for the standard metric, where ε_0 should be quite small, say $\varepsilon_0 \leq \mathrm{const}_N^{-1}(d+2)^{(d+2)^N}$ for $\mathrm{const}_N = N^{N^2}$.

B. If some m-disk D is in the set, then there is a δ-dense set of rotations of this $D = D^m$ in the m-plane L spanned by D. That is g's are δ-dense in the subgroup ($\approx \mathcal{U}(m)$) of unitary transformations of L fixing the center of D. Here δ is independent of d, say $\delta = N^{-N}$. (Notice that the dependence of our constant on N is a matter of convention as they could be absorbed by the definition of the "standard" metric in $\mathrm{Isom}\,\mathbb{C}^N$. Also observe, that the only role of this condition is to take care of non-injectivity of the map $S_d\mathbb{C}^m \to (S_d\mathbb{C})^m$.)

C. With every disk D in our family, there are "sufficiently many" disks, say D_i obtained from D by parallel translation in the directions normal to D. Namely the projection of this D_i to the normal $\mathbb{C}^m$ is ε_0-dense, for the above ε_0 in some ball of radius 10 in $\mathbb{C}^m$. (This is a purely technical condition. It is not truly needed but it simplifies what follows.)

Clearly, there exists a system of disks with above properties, such that the number of these per unit volume in $\mathbb{C}^N$, i.e. meeting each unit ball (or cube) in $\mathbb{C}^N$ does not exceed $\mathrm{const}_N\,\varepsilon_0^{-K}$ for

$$K = \dim \mathrm{Isom}\,\mathbb{C}^N + n = N(2N+3) + n \leq 3N^2\,.$$

4.1.D$'$. Main Lemma. *Let $\mathcal{D}$ be a collection of disks satisfying A, B, C and let M_1 and M_2 be n-dimensional subvarieties in $\mathbb{C}^N$ from the class $\widetilde{\mathcal{M}}_d$. Suppose, that every disk $\mathcal{D} \in \mathcal{D}\,\varepsilon$-transversal to both M_1 and M_2 with the* *above ε, satisfies*

$$D \cap M_1 = D \cap M_2 .$$

Then $M_1 = M_2$.

Proof. Since $M_1 \cup M_2 \subset \mathcal{M}_{2d}$ and our ε' is so small, we can cover $\mathbb{C}^N$ by ε-tubes $g(D^m \times B(\varepsilon))$ for $D = D(1) \subset \mathbb{C}$, $B(\varepsilon) \subset \mathbb{C}^n$ and ε much larger than ε, say $\varepsilon = \mathrm{const}_N\, d^{-1}$ (see (+)) such that all disks $g\,D^m$ are in our collection and, moreover, 2ε-transversal to both M_1 and M_2. Thus, M_1 and M_2 are represented by a collection of $S_{d_g}\mathbb{C}^m$- valued maps φ_g on the corresponding ε'-balls $g(B(\varepsilon)) \subset \mathbb{C}^N$, where d_g are bounded by $\mathrm{const}'_N\, d/(\varepsilon)^{2n} \leq N^{2N} d^{2n+1}$. The intersection condition, with B, says that these functions are equal on ε'-dense subsets in these balls. Now, let δ denote the supremum of the distances between these functions over all our balls. The argument as in §3 appealing to the Cauchy inequality makes this distance $\lambda\delta$-small on concentric balls of radii, say $0.9\,\varepsilon$ with very small λ, something of order $\frac{1}{2}d^{-d}$. Thus δ-distance for one covering implies $\frac{1}{2}d^{-d}\delta$-distance for another covering which then yield $\frac{1}{2}\delta$-distance for the original covering by the discussion in 5.1.C. It follows $\delta = 0$ and the proof follows. Q.E.D.

Remark. We did not try to be sharp in the above estimates but used notations clarifying relative roles of n, N and d. Besides, there are little details to fill in, like requiring covering by $0.8\,\varepsilon'$-tubes (rather than the ε'-tubes) etc.

4.1.D$''$. We want to interprete 4.1.D$'$ as an embedding result and thus bound the mean dimension of $\widetilde{\mathcal{M}}_d$. Denote by $\Delta = \Delta(D^n)$ the union of the cones of $S_i D^m$, $i = 1, \ldots, d_0$ joint at the vertex

$$\Delta = \bigvee_{i=0}^{d_0} \mathrm{cone}(S_i D^m) ,$$

where d_0 is the smallest integer $\geq N^{2N} d^{2n+1}$, and let $\Delta_g = \Delta(g\,D^m)$. Now, for every collections of disks $g\,D^m$, g running over some subset $g \in \mathrm{Isom}\,\mathbb{C}^N$, we map $\mathcal{M}_d$ to the Cartesian product $\underset{g \in g}{\times}\ \Delta_g$ as follows. If $g\,D^m$ is 2ε-transversal to M, then the g's component of our map sends M to $M \cap gD^m$. If gD^m is not ε-transversal, we go to the joint vertex of the cones and we interplate between the two maps in some standard way. Now Main Lemma shows that this map is an embedding.

Corollary. *The mean dimension of $\widetilde{\mathcal{M}}_d$ is bounded by*

$$\dim(\widetilde{\mathcal{M}}_d : \mathbb{C}^N) \leq \mathrm{const}(N, d) \tag{++}$$

$\mathrm{const}(N,d) \leq \mathrm{const}_N \, \varepsilon_0^{-K} \dim \Delta$.

Proof. All we need, is our collection of disks being $\mathbb{Z}^{2N}$-equivariant. Then we found the mean dimension relative to $\mathbb{Z}^{2N}$ that equals that for $\mathbb{C}^N$. Q.E.D.

Remark. Our bound on K, ε and $d_0 = \dim \Delta$ are pretty awful. Better leave it just as $\mathrm{const}(N, d)$.

4.1.D″. The proof of the upper bound in 0.6.D. The above proof of (++) is essentially local in nature and trivially generalizes to subvarieties in all Hermitian manifolds with bounded local geometry. This gives us the desired (horrible but effective) upper bound in 0.6.D. The lower bound will be proven later on.

Remarks and open questions. (a) As we mentioned earlier, the constant in (++) should be bounded by $\mathrm{const}_N \, d^{2n}$, where it will be interesting to explicitely compute const_N.

(b) The above argument can be, probably, extended to 2-dimensional minimal subvarieties in Riemannian manifolds and also to pseudo-holomorphic (1-dimensional) subvarieties in almost complex manifolds W (where the easiest case if of $\dim_{\mathbb{R}} W = 4$ as we have at our disposal pseudo-holomorphic curves ε-transversal to our $M \subset W$). On the other hand, the situation seems more difficult for higher dimensional minimal subvarieties. In fact, it seems unknown if the space of n-dimensional minimal subvarieties of volume $\leq d < \infty$ in a *compact* Riemannian manifold W has finite topological dimension. (On the other hand, *generic* W's contain few minimal subvarieties and so, typically, their mean dimension should be zero for infinite groups Γ.)

(c) Clearly $\widetilde{\mathcal{M}}_d$ is empty for $d \leq d_0 = d_0(W)$, where the critical d_0 equals 1 for $W = \mathbb{C}^N$. It is not hard to see that the mean dimension is continuous at this critical value in the case of $\mathbb{C}^N$,

$$\dim(\widetilde{\mathcal{M}}_d : \mathbb{C}^N) \to 0 \text{ for } d \to 1\,,$$

and, probably, something similar holds true for all W. For example, if $|K(W)|$ is small and $\operatorname{InjRad} W \geq 1$, then the critical $d_0(W)$ is close to one and the space $\mathcal{M}_{1+\varepsilon}$ is small for small ε. In particular, if W is cocompactly acted by a discrete amenable group Γ, then

$$\dim(\mathcal{M}_{1+\varepsilon} : \Gamma) \to 0$$

for $\varepsilon \to 0$ and $|K(W)| \to 0$, as a simple argument shows. (In fact, when d is close to 1, our M's are uniformly non-singular and everything trivially reduces to linear P.D.E. Actually, this equally applies to general minimal subvarieties with $d \leq 1 + \varepsilon$, where the uniform non-singularity follows from Allard's theorem. On the other hand, we do not know how to bound the mean dimension of spaces of minimal varieties with $d \gg 1$.)

4.2. Residual dimension of $\widetilde{\mathcal{M}}_d$ and related questions. Let W be acted upon by a discrete group Γ with projective algebraic quotient W/Γ, let $\Gamma_i \subset \Gamma$ be a sequence of subgroups of finite index. The above discussion applies to submanifolds in W/Γ_i and shows, in particular, that the residual dimension of $\widetilde{\mathcal{M}}_d$ is bounded by $\operatorname{const}(W, d, \Gamma)$. We observe that Γ_i-invariant submanifolds $\in \widetilde{\mathcal{M}}_d$ descend to subvarieties in W/Γ_i of volumes $\leq \operatorname{const} d|\Gamma/\Gamma_i|$ and pose the following

Problem. Given a sequence of numbers δ_i, evaluate the dimensions of the spaces $\mathcal{M}^n_{\delta_i}(W/\Gamma_i)$ of n-dimensional subvarieties in W/Γ_i of volume $\leq \delta_i$.

Here we are interested in the *asymptotic* behavior of these dimensions for "interesting" sequences of subgroups Γ_i, where, specifically, we want to know the answer for $\delta_i = |\Gamma/\Gamma_i|^\alpha$ for a fixed α. We start with the following simple

Observation. *Let $W_\bullet$ be a compact N-dimensional manifold that admits a holomorphic finite-to-one map $\varphi : W_\bullet \to \mathbb{C}P^N$, such that the Kähler class of $\mathbb{C}P^n$ goes to a multiple of the Kähler class of $W_\bullet$ say to $\lambda[w(W_\bullet)] \in H^2(W_\bullet; \mathbb{R})$,* then

$$\dim \mathcal{M}_{\delta'}(\mathbb{C}P^N) \leq \dim \mathcal{M}_\delta(W_\bullet) \leq \dim \mathcal{M}_{\delta''}\mathbb{C}P^N$$

for $\delta' = \lambda^n(\deg\varphi)\delta$ and$\delta'' = \lambda^n\delta$ where $\deg\varphi$ denotes the toplogical degree of φ.

In fact, $\operatorname{Vol}\varphi^{-1}(\underline{M}) = \lambda^{2n}(\deg\varphi)\operatorname{vol} M$ for all $\underline{M} \subset \mathbb{C}P^n$, which yields the lower bound on $\dim \mathcal{M}_\delta(W)$, while $\operatorname{Vol}\varphi(M) = \lambda^n \operatorname{Vol} M$ for all $M \subset W_\bullet$ which gives us the upper bound.

Remarks. (a) This observation applies, strictly speaking, only to those $W_i = W/\Gamma_i$ where Γ_i acts *freely* on W in order to have W_i non-singular. But everything (and obviously) equally works in the singular case.

(b) The above inequalities are most efficient for small $\deg\varphi$ but for our $W_i = W/\Gamma_i$we only guarantee maps $\varphi_i : W_i \to \mathbb{C}P^N$ with $\deg\varphi_i = \mathrm{const}\,|\Gamma/\Gamma_i|$ and one cannot do better in most (?) cases, e.g. for the groups Γ satisfying Kazhdan's property T (see [Gro]$_{\mathrm{MIKM}}$). On the other hand, there are cases where $\deg\varphi_i \leq$ const independently of i, e.g. for coverings of an Abelian variety.

Now we recall the standard bounds for $\dim\mathcal{M}^n_\delta(\mathbb{C}P^N)$.

4.2.A. Lemma. *The space* $\mathcal{M}^n_{\delta_1}(\mathbb{C}P^N)$ *of irreducible n-dimensional subvarieties in* $\mathbb{C}P^N$ *of degree δ satisfies*

$$\frac{(\delta+1)(\delta+2)\dots(\delta+n+1)}{(n+1)!} - 1 \leq \dim\mathcal{M}^n_\delta(\mathbb{C}P^N)$$
$$\leq (N-n)\left(\frac{(\delta+1)(\delta+2)\dots(\delta+n+1)}{(n+1)!} - 1\right) \leq \mathrm{const}_N\,\delta^{n+1}\,.$$

Proof. If $N-n=1$ then M's are given by homogeneous polynomials of degree δ and so

$$\dim\mathcal{M}^n_\delta(\mathbb{C}P^N) = \frac{(\delta+1)\dots(\delta+n)}{n!}\,.$$

Next, if $N-n\geq 2$, we project M to $N-n$ $(n+1)$-planes in general position in $\mathbb{C}P^N$ and observe that M appears as an irreducible component of the intersection of the pull-backs of these components. Q.E.D.

Example: Abelian varieties. Let Γ be a Lattice acting on $\mathbb{C}^N$ with projective algebraic quotient $W_\bullet = \mathbb{C}^N/\Gamma$, e.g. $\Gamma = i\,\mathbb{Z}^{2N}$, $i=1,2,\dots$. Then there is our map $\varphi : W_\bullet \to \mathbb{C}P^N$ with $\deg\varphi \leq \mathrm{const} < \infty$ independently of Γ with $\lambda = (\mathrm{Vol}\,W_\bullet)^{-\frac{1}{N}}$ (where the Kähler metric in $\mathbb{C}P^N$ is normalized to have $\mathrm{Vol}\,\mathbb{C}P^N = 1$). Then $\dim\mathcal{M}^n_\delta(W_\bullet)$ is approximately (i.e. up to a multiplicative constant) equal to

$$(\delta\lambda^n)^{n+1} = \delta^{n+1}(\mathrm{Vol}\,W_\bullet)^{-\frac{n(n+1)}{N}}\,.$$

Thus, if we set $d = \delta/\mathrm{Vol}\,W_\bullet$, we get

$$\dim\mathcal{M}^n_\delta(W_\bullet) \sim d^{n+1}(\mathrm{Vol}\,W_\bullet)^{\frac{(n+1)(N-n)}{N}}\,. \qquad (*)$$

If $N = n + 1$, this becomes

$$\dim \mathcal{M}_d^n(W_\bullet) \sim d^{n+1} \operatorname{Vol} W_\bullet$$

and gives us the following bound on the residual dimension of the space $\widetilde{\mathcal{M}}_d = \widetilde{\mathcal{M}}_d(\mathbb{C}^N)$ (of n-dimensional subvarieties M with $\operatorname{Vol} M \cap B(1) \leq d$ for all unit balls $B(1)$),

$$\operatorname{resdim} \widetilde{\mathcal{M}}_d \leq \operatorname{const} d^{n+1} .$$

This improves our earlier bound (with a poor dependence on d) and suggests that the mean dimension of $\widetilde{\mathcal{M}}_d$ must be asymptotic to d^{n+1}. Here is a more general

Conjecture. Let W be a Hermitian manifold of bounded local geometry and $\{B_i\}_{i\in I}$ be a collection of balls of radii $r_i \leq 1$, such that the concentric balls of radii $r_i/2$ cover W. Consider the space $\mathcal{M}$ of n-dimensional subvarieties $M \subset W$, such that $\operatorname{Vol}(M \cap B_i) \leq d_i r_i^n$ for all i and given $d_i \geq 0$. Then

$$\dim \mathcal{M} \leq \operatorname{const} \sum_{i\in I} d_i^{n+1} ,$$

where the constant depends only on $N = \dim W$ and the implied bound on the local geometry of W.

The above conjecture truly makes sense only for compact W, where in general one should use a suitable "dimension per unit volume" in W. For example, if W is cocompactly acted upon by an amenable group Γ and the system $\{B_i\}$ is Γ-invariant, then the mean dimension $\dim(\mathcal{M} : \Gamma)$ should be bounded by $\operatorname{const} \sum_i d_i^{n+1}$ for i running over a fundamental domain $J \subset I$, i.e. a subset such that $\Gamma J = I$.

Remark. For certain manifolds W, e.g. for $W = \mathbb{R}^N$, it is interesting to look at M's defined with systems of balls B_i where r_i are unbounded, say for concentric balls of radii $i \to \infty$ in $\mathbb{C}^N$. Then one may try to evaluate some "asymptotic dimension" of $\mathcal{M}$ in the spirit of the Nevanlinna theory. For example, let $\mathcal{M}_\varphi$ denote the space of n-dimensional subvarieties M in $\mathbb{C}^N$, such that $\operatorname{vol}(M \cap B(R)) \leq \varphi(R)$ for a given function $\varphi(R)$ and all concentric R-balls $B(R) \subset \mathbb{C}^N$ around the origin. Denote by $\mathcal{M}_\varphi(R)$ the space of subvarieties in $B(R)$ of the form $M \cap B(R)$ for all $M \in \mathcal{M}_\varphi$. Then one may ask what is the asymptotic behavior of $\dim_\varepsilon \mathcal{M}_\varphi(R)$ for $R \to \infty$ (and eventually for $\varepsilon \to 0$) with respect to the Hausdorff metric in $\mathcal{M}_\varphi(R)$.

A particularly interesting case is $\varphi(R) = CR^p$ for some $p > n$. (If $p = n$, M is necessarily algebraic and so $\dim_\varepsilon \mathcal{M}_\varphi(R)$ is uniformly bounded.)

Now, let us look at the above asymptotic relation $(*)$ for $\operatorname{codim} M \geq 2$, i.e. for $N - n \geq 2$. Here the exponent $\frac{(n+1)(N-n)}{2}$ is strictly greater than 1, and so $(*)$ yields no bound at all on the residual (as well as on mean) dimension of $\widetilde{\mathcal{M}}_d$. However, this does not contradict $(++)$ from 4.1.D$''$ but rather shows that majority of subvarieties $M \subset W$ are highly non-uniformly distributed in W for $N - n \geq 2$ and so $(++)$ does not apply. This suggests the following

Algebraic questions. Consider the space $\mathcal{M}^n_\delta(\mathbb{C}P^N)$ of algebraic subvarieties in $\mathbb{C}P^N$ of dimension n and degree δ. How many irreducible components of $\mathcal{M}^n_\delta(\mathbb{C}P^N)$ lie in the interval $[\delta^{\alpha_1},\ \delta^{\alpha_2}]$ for given $0 < \alpha_1 < \alpha_2 \leq n+1$? Here we are most interested in the asymptotic behavior of this number for $\delta \to \infty$, where a good answer is plausible for large α_1, e.g. $\alpha_1 > n$.

To get some perspective look at the space $\mathcal{M} \subset \mathcal{M}^n_\delta(\mathbb{C}P^N)$ of complete intersections M of hypersurfaces of degrees $\delta_1 \geq \delta_2 \geq \ldots \geq \delta_{N-n}$. Its dimension is easy to evaluate by looking at the normal bundle of M or by rescaling $\mathbb{C}P^N$ by δ_1 (which makes the volume of hypersurfaces of degree δ_1 equal that of the rescaled $\mathbb{C}P^N$ and then applying $(++)$ to the rescaled picture). Thus one easily shows that $\dim \mathcal{M} \sim \delta\delta_1^n$ and so each $M \in \mathcal{M}$ is contained in a hypersurface of degree, δ_{N-n} which is roughly bounded by $\left(\delta^{\frac{n+1}{n}}/d^{\frac{1}{n}}\right)^{\frac{1}{N-n-1}}$ for $d = \dim \mathcal{M}$ (since $\delta = \delta_1\delta_2 \ldots \delta_{N-n}$). This suggests that for *every* irreducible variety $\mathcal{M} \subset \mathcal{M}^n_\rho(\mathbb{C}P^N)$ of (large) dimension D, each $M \in \mathcal{M}$ is contained in a hypersurface of degree $\leq \delta'$, where δ' can be (reasonably) evaluated in terms of D and δ. For example, if $D \geq \varepsilon\delta^{n+1}$, then one expects $\delta' \leq \varepsilon' = \varepsilon'(\varepsilon, N)$, and if $D \geq \delta^{n+1-\alpha}$ for a small $\alpha > 0$, then $\delta' < d$ for $\delta \geq \delta_0 = \delta_0(\alpha, N)$. (Notice that holomorphic maps $\mathbb{C}P^n \to \mathbb{C}P^N$ with images of degree δ make a variety of dimension about δ^n whose generic members do not, apparently, lie in hypersurfaces of degrees $< \delta$ and so "small" should be at least "smaller than one".)

There is another idea also expressing non-uniform distribution of subvarieties of codimension ≥ 2 in $\mathbb{C}P^N$. For example, one may seek a non-trivial upper bound on the dimension of the subspace $\mathcal{N}^n_\delta \subset (\mathbb{C}P^N)^k$ consisting of those k-tuples of points which lie on *some* subvariety $M \subset \mathbb{C}P^N$ of dimension n and degree δ. For example, if $k \geq \varepsilon\delta^{n+1}$, then, probably, $\operatorname{codim} \mathcal{N}^n_\delta > 0$ for all $\delta \geq \delta_0 = \delta_0(\varepsilon, N)$. In fact our inequality $(++)$ suggests, that no configuration of points $(x_1, \ldots, x_k) \in \mathcal{N}^n_\delta$ can be uniformly dense in $\mathbb{C}P^N$, i.e. the ρ-neighbourhood of $\{x_1, \ldots, x_k\} \subset \mathbb{C}P^N$ must have small measure for ρ not much exceeding $(\varepsilon\delta^{n+1})^{-N}$ and $\delta \geq \delta_0(\varepsilon, N)$.

4.3. Construction of subvarieties in W. Let W admit a positive line bundle E of locally bounded geometry. Then W admits a holomorphic uniformly non degenerate Lipschitz map x to $\mathbb{C}P^N$, $N = \dim W$. The pull-backs of subvarieties in $\mathbb{C}P^N$ are, clearly, in our class $\widetilde{\mathcal{M}}_{d\to\infty}$ and by varying x one sees that $\widetilde{\mathcal{M}}_d$ has positive mean dimension. Actually, by a direct application of the uniform transversality theorem one obtains bounded holomorphic sections $x : W \to E^i$ which are uniformly transversal to the zero section in the obvious sense. The zero set $x^{-1}(0) \subset W$ of such an X is a manifold with bounded local geometry of dimension equal $\dim W - 1$ and so one obtains by induction such submanifolds of all codimensions. This combines with an obvious scaling argument and shows, in particular, that in the presence of cocompact amenable action *the mean dimension of the space $\widetilde{\mathcal{M}}_d$ of n-dimensional submanifolds $M \subset W$ with the bound* $\operatorname{Vol} B \cap M \leq d$ *for all unit balls $B \subset W$ satisfies*

$$\dim(\widetilde{\mathcal{M}}_d : \Gamma) \geq \operatorname{const} d^{n+1}$$

for all $n \leq N$, some $\operatorname{const} = \operatorname{const}(W, \Gamma) > 0$ *and all sufficiently large d.*

Remarks and final questions. Since every complex subvariety is minimal, one sees with the above theorem, for example, that the space $\mathcal{M}_d$ of $2m$-dimensional minimal subvarieties $M \subset \mathbb{R}^N$ with the volume bound by $\operatorname{Vol} M \cap B \leq d$ has $\dim(\mathcal{M}_d : \mathbb{R}^N) > 0$ for all $N \geq 2m + 2$ and $d > \operatorname{Vol} B^{2m}$. But it is unclear if this dimension is positive for minimal surfaces in $\mathbb{R}^3$ (where one can use the Weirstrass representation to generate minimal surfaces).

Another situation where one may expect positive mean dimension is that of pseudo-holomorphic subvarieties $M \subset W$ with $\dim_{\mathbb{R}} M = 2$, but here one needs a different technique for producing sufficiently many of them in suitable almost complex manifolds W.

Finally, we mention special Lagrangian submanifolds and related classes of complex submanifolds, e.g. $M \subset \mathbb{C}^N$ isotopic relative to a given (symmetric or anti-symmetric)bi-linear form on $\mathbb{C}^N$. Unfortunately, the lack of examples precludes us from asking meaningful questions.

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