ENDOMORPHISMS OF SYMBOLIC ALGEBRAIC VARIETIES

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§ 1. Ax' surjunctivity theorem.

1.A. Strict embeddings and surjunctivity. A map between sets is called a *strict embedding*, denoted $f : X \underset{\neq}{\subset} Y$, if it is one-to-one but *not* onto. Then, following Gottschalk, (see [Gott, 1972]) a map $f : X \to Y$ is called *surjunctive* if it is *not* a strict embedding. In other words f is surjunctive iff it is either surjective or non-injective.

1.B. Theorem [Ax, 1968]. *Every regular selfmapping of a complex algebraic variety X is surjunctive.*

In other words *no* X admits a strict embedding $X \underset{\neq}{\subset} X$. Or, put it yet another way, "one-to-one" implies "onto" for every regular map $f : X \to X$.

If $X = \mathbb{C}^n$ this specializes to the following earlier result by Bialynicki-Barula and Rosenlicht, (see [BB-R, 1962]).

1.B′. *Every complex polynomial self-mapping of $\mathbb{C}^n$ is surjunctive.*

Repeat, this signifies that *no* strict polynomial embedding $\mathbb{C}^n \underset{\neq}{\subset} \mathbb{C}^n$ is possible, i.e. every injective polynomial map $\mathbb{C}^n \to \mathbb{C}^n$ is surjective.

§ 2. Proalgebraic spaces and their endomorphisms.

Recall that a *complex proalgebraic space* means a projective limit X of a projective system of complex algebraic varieties X_i where i runs over a directed set I. The simplest and most important case is where $I = \mathbb{N} = \{1, 2, \ldots\}$. Here our projective system can be given by a string of $\mathbb{C}$-varieties X_i and connecting regular maps, also called *projections*, $X_1 \underset{\pi_2}{\leftarrow} X_2 \underset{\pi_3}{\leftarrow} \ldots \underset{\pi_i}{\leftarrow} X_i \underset{\pi_{i+1}}{\leftarrow} \ldots$. Then the projective limit $X = \underset{\leftarrow}{\lim} X_i$ consists of the sequences $x_i \in X_i$ such that $\pi_i(x_i) = x_{i-1}$ for all $i = 2, 3, \ldots$. For example, the complex Euclidean spaces $\mathbb{C}^i$, $i = 1, 2, \ldots$, form such a projective system for the natural projections $\pi_{i+1} : \mathbb{C}^{i+1} \to \mathbb{C}^i$ and the limit, denoted $\mathbb{C}^\infty$, consists of all infinite sequences $(z_1, z_2, \ldots, z_i \ldots)$ for $z_i \in \mathbb{C}$.

A *proregular map* $f = f_\infty$ between two proalgebraic spaces $X = X_\infty = \underset{\leftarrow}{\lim} X_i$ and $Y = Y_\infty = \underset{\leftarrow}{\lim} Y_j$ is given, by definition, by an order preserving map between the underlying directed sets, $I \to J$, denoted $i \mapsto j = j(i)$ and by regular maps $f_i : X_i \to Y_j$ such that all diagrams commute

$$\begin{array}{ccc} X_{i_1} & \longrightarrow & Y_{j_1} \\ {\scriptstyle \pi_{i_1,i_2}} \downarrow & & \downarrow {\scriptstyle \pi_{j_1,j_2}} \\ X_{i_2} & \longrightarrow & Y_{j_2} \end{array}$$

where π_{i_1,i_2} and π_{j_1,j_2} are the connecting maps of our projective systems defined for all $i_1 \geq j_2$ and $j_1 \geq j_2$. (In the above case of $I = \mathbb{N}$ we have $\pi_{i_1,i_2} = \pi_{i_2} \circ \pi_{i_2-1} \circ \ldots \circ \pi_{i_1}$).

2.A. Example : polynomial mappings $\mathbb{C}^\infty \to \mathbb{C}^\infty$. A function $p : \mathbb{C}^\infty \to \mathbb{C}$ is called a *polynomial* if it is a polynomial depending on finitely many complex variables, say $z_{i_1}, z_{i_2}, \ldots, z_{i_k} \in \mathbb{C}$ among the infinity of $z_1, z_2 \ldots$ making up $\mathbb{C}^\infty$. In other words p is obtained by composing the projection of $\mathbb{C}^\infty$ to some $\mathbb{C}^k$ with an ordinary complex polynomial $\mathbb{C}^k \to \mathbb{C}$. Then a *polynomial map* f of $\mathbb{C}^\infty$ into itself, $f : z = (z_1, z_2, \ldots z_i \ldots) \mapsto z' = (z'_1, z'_2, \ldots z'_i \ldots)$ is defined by $z'_i = p_i(z)$ for some polynomials p_i on $\mathbb{C}^\infty$. It is easy to see that these polynomial maps $\mathbb{C}^\infty \to \mathbb{C}^\infty$ are the same as proregular self mapping (endomorphisms) of $\mathbb{C}^\infty$ thought of as a proalgebraic space.

2.B. Our central problem. *Find simple conditions which ensure surjunctivity of a proregular endomorphism of a proalgebraic space.*

First, let us make clear that some conditions are unavoidable.

2.B$'$. Counterexamples. (a) The complement $X = \mathbb{C} \backslash \{0, 1, 2, \ldots\}$ is proalgebraic being the decreasing intersection of quasi-affine varieties $X_i = \mathbb{C} \backslash \{0, 1 \ldots i\}$. And the (obviously proregular) map $z \mapsto z - 1$ is one-to-one but *not* onto.

(b) The polynomial map $f : \mathbb{C}^\infty \to \mathbb{C}^\infty$ for $f : \{z_1, z_2, \ldots\} \mapsto \{0, z_1, z_2, \ldots\}$ is also one-to-one but not onto.

The above may look discouraging. Yet we shall see that surjunctivity is rather typical for maps commuting with a sufficiently large automorphism group (or pseudo-group) acting on X. Here is our basic example.

2.C. Endomorphisms of $\underline{X}^\infty = \underline{X}^\Delta$. Let $\underline{X}$ be an arbitrary (finite dimensional !) complex algebraic variety and Δ be a countable set. Then the infinite Cartesian power $X = \underline{X}^\Delta$, i.e. the space of all $\underline{X}$-valued functions on Δ, comes along with a natural proalgebraic structure. One can see this, for example, by enumerating Δ, i.e. by bijecting $\mathbb{N} = \{1, 2, \ldots\} \leftrightarrow \Delta$ and thus identifying $\underline{X}^\Delta$ with

$$\underline{X}^\infty = \underline{X}^\mathbb{N} = \{x_1, x_2, \ldots, x_i, \ldots\}_{x_i \in \underline{X}} = \lim_{\leftarrow} \underline{X}^i = \underline{X}^{\{1,2,\ldots i\}}.$$

Or, more invariantly, one can use the directed set I of all finite subsets $\Omega \subset \Delta$ (I can not bring myself to denote them $i \subset \Delta$, I rather have $\Omega \in I$). Here one has the projective system $\{\underline{X}^\Omega, \Omega \in I\}$ with the projections $X^\Omega \to X^\Sigma$ for all $\Omega \supset \Sigma$ corresponding to restrictions of functions $x = x(\omega) \in \underline{X}^\Omega$ from Ω to Σ. One can see that the projective limit of $\underline{X}^\Omega$ is isomorphic to the above $X^\infty = \lim\limits_{\leftarrow} \underline{X}^i$ (where, recall, the Cartesian powers X^i form a projective system with the projections $\underline{X} \leftarrow \underline{X}^2 \leftarrow \ldots \leftarrow \underline{X}^i \leftarrow \ldots$) in the category of proalgebraic spaces and proregular maps. In fact, take an exhaustion of Δ by finite subsets $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_i \subset \ldots, \bigcup\limits_i \Omega_i = \Delta$. Then the projective limit $\lim\limits_{\leftarrow} \underline{X}^{\Omega_i}$ obviously equals the projective limit $\lim\limits_{\leftarrow} \underline{X}^\Omega$ over all Ω since the subsets Ω_i *exhausting* Δ are *cofinal* in the set of all finite Ω's in Δ. On the other hand the spaces $\underline{X}^{\Omega_i} = \underline{X}^{N_i}$, $N_i = \operatorname{card} \Omega_i$, make a cofinal subsystem in $\{X^i\}_{i=1,2,\ldots}$ and so $\lim\limits_{\leftarrow} \underline{X}^{\Omega_i} = \lim\limits_{\leftarrow} \underline{X}^i$.

Now, suppose we have a group Γ acting on Δ and thus on $X = \underline{X}^\Delta$ and let $f : X \to X$ be a proregular Γ-equivariant map.

2.C$'$. Theorem. *If Γ is a locally compact (e.g. discrete) amenable (e.g. solvable) group and the action of Γ on Δ has finitely many orbits (e.g. transitive), then f is surjunctive, i.e. one-to-one $\Rightarrow$ onto for these maps f.*

Remark. In many examples the isotropy subgroups $\Gamma_\delta \subset \Gamma$, $\delta \in \Delta$, are compact and then Δ admits a structure of a locally finite graph invariant under Γ. Yet we do not have to make this assumption as the general case of the theorem trivially reduces to the one where Γ_δ are compact.

We prove the theorem in 5.M$''$ exploiting Ax' idea of reduction surjunctivity from the algebraic category to finite sets $\underline{X}$.

2.C$''$ Question. Are Γ-equivariant proregulator selfmappings of $\underline{X}^\Gamma$ are surjunctive for all, not necessarily amenable, groups Γ acting on Δ with finitely many orbits ?

One knows, this is true for *initially subamenable* (e.g. residually amenable) groups (see 4.G) but the question remains open even in the more traditional framework of *symbolic dynamics*, i.e. for *finite* sets $\underline{X}$, where the problem was raised by Gottschalk in [Gott] and persued in [La].

2.C$'''$ Generalizations. Theorem 2.C$'$ extends to some endomorphisms of proalgebraic spaces where the global symmetry group is replaced by partial symmetries. The spaces we consider in this paper arise starting from $\underline{X}^\Delta$ where Δ is a countable connected graph with *bounded valency*, i.e. having at most $d < \infty$ edges at each vertex. Such a Δ has many partial symmetries γ i.e. graph isomorphisms $\gamma : D \leftrightarrow D'$ between finite subgraphs D and D' in Δ and we require our f to be compatible with the transformations $\underline{X}^D \leftrightarrow \underline{X}^{D'}$ induced by some of these γ. Then we consider a certain space X° of "orbits" of these transformations $X^D \leftrightarrow X^{D'}$ where we have a natural map $f^\circ : X^\circ \to X^\circ$. Here we are able to prove surjunctivity of f° under suitable amenability assumptions on Δ (see 7.G). In fact, we prove in § 7 a more general theorem applicable to proregular endomorphisms of *orbit completions* X° of certain (sufficiently soft or "stable") proalgebraic subvarieties in $\underline{X}^\Delta$.

2.D. Examples. Let Δ be a graph with exactly d edges issuing from each vertex $\delta \in \Delta$ and let, moreover, we are given an ordering of these edges at each $\delta \in \Delta$. Thus the vertices adjacent to δ can be enumerated by d indices, call them $\delta_i = \delta_i(\delta) \in \Delta$, $i = 1, \ldots, d$, for each $\delta \in \Delta$. (If we allow several edges between pairs of vertices, we may have $\delta_i = \delta_j$ for $i \neq j$ and loops at δ make $\delta_i = \delta$ for some i). Next, let $p : \underbrace{\underline{X} \times \underline{X} \times \ldots \times \underline{X}}_{d} \to \underline{Y}$ be a $\underline{Y}$ valued function in d-variables $x_i \in \underline{X}$, $i = 1, \ldots, d$, and let us construct a map $f = f_p : \underline{X}^\Delta \to \underline{Y}^\Delta$ as follows. Think of each $x \in \underline{X}^\Delta$ as an $\underline{X}$-valued function on Δ and send x to the function y on Δ where the value of y at each $\delta \in \Delta$ is determined by the values of x at the adjacent vertices $\delta_i = \delta_i(\delta) \in \Delta$ according to the rule

$$y(\delta) = p(x(\delta_1), x(\delta_2), \ldots, x(\delta_d)). \qquad (*)$$

For instance, one might have $\underline{X} = \underline{Y} = \mathbb{C}$ and f defined with $p = x_1 + x_2 \ldots + x_d$ by

$$y(\delta) = \sum_{i=1}^{d} x(\delta_i), \text{ for } \delta_i = \delta_i(\delta).$$

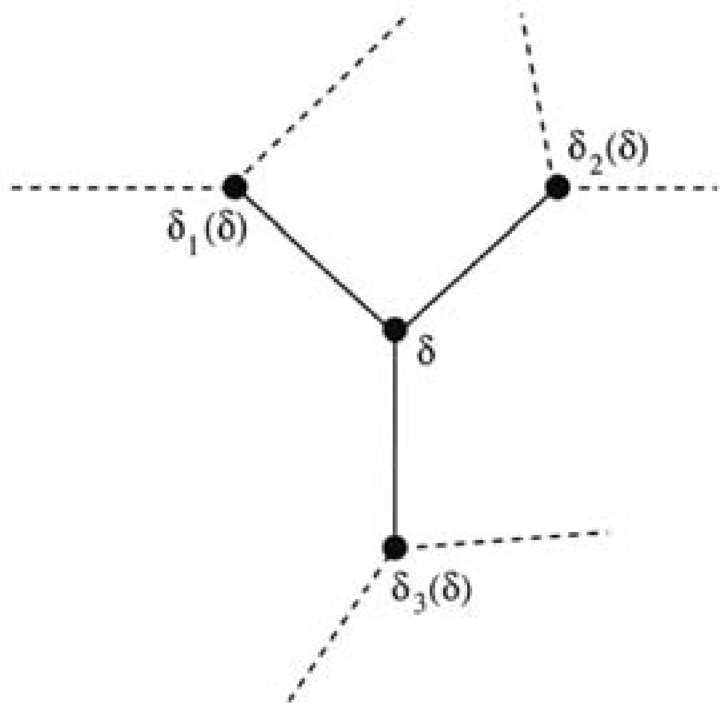

Fig. 1

In general, if $\underline{X}$ is an algebraic variety and $p : \underline{X}^d \to \underline{Y}$ is regular, then the resulting $f : \underline{X}^\Delta \to \underline{Y}^\Delta$ is clearly proregular.

The construction $p \leadsto f_p$ can be thought of as a kind of a "symbolic dynamic functor" in two variables p and Δ, where $p \in \mathrm{Maps}(f : \underline{X}^d \to \underline{Y})$ and Δ is a *d-regular* graph (i.e.

with d edges at each vertex) with *local order* (i.e. with orderings of the edges at each vertex). The range of this "functor" is our $f_p = f_p^\Delta : \underline{X}^\Delta \to \underline{Y} \in \mathrm{Maps}(\underline{X}^\Delta \to \underline{Y}^\Delta)$.

We are mostly concerned in this paper with the case where $\underline{X} = \underline{Y}$ and so f maps $\underline{X}^\Delta$ into itself. We seek assumptions on Δ and on p which make f surjunctive. There is one case where everything is clear, namely for $d = 1$ and $y(\delta) = p(x(\delta))$. Here the graph structure is not relevant at all and the map f just repeats p infinitely many times. Thus the surjunctivity of f trivially reduces to that for p itself. But, in general, the combinatorics of Δ enters in a subtle way and we do not actually know how to handle f itself rather than its orbit completion mentioned above. Yet no completion problem arises for homogeneous graphs as is explained below.

2D$'$. Cayley graphs. Let Γ be a group with a distinguished set of generators, say $D = \{\gamma_1, \ldots, \gamma_d\}$. Then *the Cayley graph* Δ of (Γ, D), by definition, has Γ for the vertex set, where γ and γ' are joined by an edge whenever $\gamma^{-1}\gamma' \in D$. The group Γ acts by graph automorphisms on Δ via the left translations and this action induces an obvious (shift) action of Γ on $X = \underline{X}^d = \underline{X}^\Gamma$. (In fact Δ may have more global and/or partial symmetries but we do not need them at the present moment). The map $f = f_p : \underline{X}^\Gamma \to \underline{X}^\Gamma$ defined with (*) clearly commutes with the action of Γ (as it commutes with all graph isomorphisms) and so we arrive at an instance of a Γ-invariant proregular selfmapping of $\underline{X}^\Gamma$. In fact, it is easy to show that every proregular Γ-equivariant map f from $\underline{X}^\Gamma$ to $\underline{Y}^\Gamma$ arises in this manner with some $D \subset \Gamma$ and a regular map p from $\underline{X}^D = \{\text{maps } D \to \underline{X}\}$ to $\underline{Y}$. Actually this $f = f_p : \underline{X}^\Gamma \to \underline{Y}^\Gamma$ can be described without any reference to the Cayley graph as follows. The value of $y = f(x)$ at a given $\gamma \in \Gamma$ is obtained by first restricting $x = x(\gamma)$ to the γ-translate of D, denoted $\gamma D \subset \Gamma$ and then by evaluating p on this restriction $(x|\gamma D) \in \underline{X}^{\gamma D} = \underline{X}^D$, where $X^{\gamma D}$ is identified with X^D via the correspondence $\gamma : \delta \leftrightarrow \gamma\delta$. That is

$$y(\gamma) = p(x(\gamma\delta)_{\delta\in\Delta}). \qquad (**)$$

So our f appears here as an *algbraic difference operator* acting on functions $x : \Gamma \to \underline{X}$. (For instance, if $\Gamma = \mathbb{Z}$, this can be written in the traditional form $f : x(k) \mapsto y(k),\ k \in \mathbb{Z}$, for

$$y(k) = p(x(k-m), \ldots, x(k), \ldots, x(k+n))$$

for some m and n, such that $m + n + 1 = d$).

Clearly, the above f is the special case of the maps considered in 2.C. and so theorem 2.C$'$. applies. This yields the surjunctivity of $f = f_p$ for every amenable (e.g. solvable) group Γ.

Remark. What makes the picture attractive in our eyes, is the definition of a transcendental object, the map $f : \underline{X}^\Gamma \to \underline{Y}^\Gamma$, via a single regular map between complex algebraic varieties, our $p : \underline{X}^d \to \underline{Y}$, which we regard as an "elementary" or "finitary" object. And it is amazing, how logically convoluted our surjunctivity becomes when translated back to the (elementary) first order language ! (see 7.K$''$). Actually, this translation and keeping track of a multitude of quantifiers needed for our proof, constitute an essential part (painful but apparently unavoidable) of the present paper.

2.D$''$. Example of an orbit completion. Let p_0 and p_1 be two different regular maps $\underline{X}^D \to \underline{Y}$ and let i be a 0,1-function on Γ, i.e. a map $i : \Gamma \to \{0,1\}$. Define $f = f_i : \underline{X}^\Delta \to \underline{Y}^\Delta$ by

$$f : x(\gamma) \mapsto y(\gamma) = p_{i(\gamma)}(x(\gamma\delta)_{\delta\in\Delta}). \qquad (**)'$$

In other words, the value $y(\gamma)$ is computed either with p_0 or with p_1 depending on the value 0 or 1 of i at this $\gamma \in \Gamma$.

The function i may be thought of as a member of the space $\{0,1\}^\Gamma$ where the group Γ acts by the obvious (shift) transformations. If we replace i by γi for some $\gamma \in \Gamma$, we get an essentially same map f. However, if we take the full orbit $\Gamma i \in \{0,1\}^\Gamma$ and consider some j from the *closure* I of Γi in the *product topology* of $\{0,1\}^\Gamma$, then this map $j : \Gamma \to \{0,1\}$ may look rather different from all γi and $f_j : \underline{X}^\Delta \to \underline{Y}^\Delta$ may be quite dissimilar to all $f_{\gamma i}$. For example, if $\Gamma = \mathbb{Z}$ and $i(\gamma) = 0$ for $\gamma \le 0$ and $i(\gamma) = 1$ for $\gamma > 0$, then among j's one finds $j(\gamma) \equiv 0$ and $j(\gamma) \equiv 1$.

Now we let $X^\circ = X \times I$ and $Y^\circ : Y \times I$ where f° maps X° to Y° by $(x,i) \mapsto (f_i(x), i)$. This is an instance of our orbit completion (see 7.C$'$ for the general case) where we claim that *f° is surjunctive for $\underline{X} = \underline{Y}$ if the group Γ is amenable and the function $i : \Gamma \to \{0,1\}$ is quasihomogeneous* in the following sense.

Denote by $\Gamma' = \Gamma'(i,D)$ for some $D \subset \Gamma$ the set of those $\gamma \in \Gamma$ for which the restriction of i on $\gamma D \subset \Gamma$ equals $i|D$ where we identify D and γD by $\delta \leftrightarrow \gamma\delta$ for all $\delta \in D$. We say that Γ' is cofinite in Γ (or makes a *net* in Γ) if there exists a *finite* subset $D' \subset \Gamma$, such that $\Gamma' D' = \Gamma$, i.e. finitely many right translates of Γ' cover all Γ (where the action we used was the left action of Γ on Γ). Finally we call i *quasihomogeneous* if the above $\Gamma' = \Gamma'(i,D)$ is cofinite for all finite subsets D in Γ.

Notice that X° and Y° are (naturally) *proalgebraic* and f° is a *proregular Γ-equivariant* map for the diagonal action of Γ on $X = X \times I$ and on $Y^\circ = Y \times I$. In general, however the spaces like X° are neither proalgebraic nor do they possess global symmetries while the role of I is played by some space of marked graphs Δ (Compare § 6).

Remarks. ***(a) Symbolic algebraic geometry.*** It seems to me that infinite dimensional spaces such as Γ-equivariant (pro)algebraic subvarieties in $\underline{X}^\Gamma$ and Γ-equivariant pro-regular mapping between these provide a meaningful meeting point between algebraic geometry and symbolic dynamics. Our Ax-type theorem illuminates a tiny region as the two domains come into contact but the entire field remains in the dark. (See $[\mathrm{Gro}]_{\mathrm{TIDS}}$ for a different view on the symbolic algebraic geometry).

(b) Algebraic varieties associated to graphs. Here is an example of how the "symbolic" idea leads to an attractively explicit class of algebraic manifolds. Start with an *algebraic graph* A, i.e. a subvariety in a Cartesian square of an algebraic variety, say $A = (\underline{X} \times \underline{X},\ \underline{Y} \subset \underline{X} \times \underline{X})$. Here we think of $\underline{X}$ as the set of vertices of A and $\underline{Y}$ plays the role of the set of edges. Notice that these edges are *directed* as $\underline{X} \times \underline{X}$ consists of ordered pairs (x, x'). (If we want to mimic an ordinary undirected graph rather than a *digraph*, we should take $\underline{Y}$ invariant under the canonical involution on $\underline{X} \times \underline{X}$. On the other hand, if we care for multiple edges and loops, we may take a non-injective morphism $\underline{Y} \to \underline{X} \times \underline{X}$).

Now, for every abstract digraph $\Delta = (\Delta, E \subset \Delta \times \Delta)$ (where we use the same notation for the vertex set of Δ and Δ itself), we consider the space A^Δ of the maps $\Delta \to \underline{X}$ sending E to $\underline{Y}$. (For general digraphs $\underline{Y} \to \underline{X} \times \underline{X}$ and $E \to \Delta \times \Delta$ one should use pairs of compatible maps $\Delta \to \underline{X}$ and $E \to \underline{Y}$). For instance if $\underline{Y} \subset \underline{X} \times \underline{X}$ is given by an equation $f(x, x') = 0$, then $A^\Delta \subset \underline{X}^\Delta$ consists of the strings of variables $x_\delta \in \underline{X}$ indexed by the vertices $\delta \in \Delta$ such that $f(x_\delta, x_{\delta'}) = 0$ whenever δ and δ' are adjacent in Δ, i.e. $(\delta, \delta') \in E \subset \Delta \times \Delta$.

Question. Suppose we know everything about the varieties $\underline{X}$ and $\underline{Y}$. What can we say about A^Δ for a given *finite* graph Δ? Specifically, when is A^Δ non-singular and what are its Betti numbers ?

Example. Assume $\underline{X}$ is defined over $\mathbb{C}$ and $\underline{X} \times \underline{X}$ is embdded to some $\mathbb{C}P^N$, intersect $\underline{X} \times \underline{X}$ with a hyperplane $H \subset \mathbb{C}P^N$ and take $\underline{Y} = \underline{Y}_H = (\underline{X} \times \underline{X}) \cap H$.

Now we specify the above question to the present case where we take $\underline{Y} = \underline{Y}_H$ for a *generic* (possibly depending on Δ) hyperplane H. In fact, the problem becomes much easier (and still interesting) if we take *different generic* H's for different edges of Δ. To make it simple we express $\underline{Y}$ by the equation $f_H(x, x') = 0$ and then, instead of sticking to $A^\Delta = A^\Delta(H)$ given by the *identical* equations $f_H(x_\delta, x_{\delta'}) = 0$, $(\delta, \delta') \in E \subset \Delta \times \Delta$, we allow H to depend on the edges of Δ. Namely, we take some collection of hyperplanes $\mathcal{H} = \{H_e \subset \mathbb{C}P^N\}$ indexed by the edges $e = (\delta, \delta') \in E \subset \Delta \times \Delta$ and define $A^\Delta(\mathcal{H}) \subset \underline{X}^\Delta$ by the equations $f_{H_e}(x_\delta, x_{\delta'})$ written down at all edges $e = (\delta, \delta')$ of Δ. Here the genericity has more power as we may perturb H_e's independently of each other and it is easy to see that the resulting variety $A^\Delta(\mathcal{H})$ is non-singular for non-singular $\underline{X}$ and generic $\mathcal{H} = \{H_e\}$. Thus the topology of $A^\Delta(\mathcal{H})$ does not depend on $\mathcal{H}$ for generic $\mathcal{H}$ and one is challenged to figure out what this topology actually is. (See pp 210-214 in [Gro]$_{\mathrm{PDR}}$ for specific examples and geometric applications).

Generalizations. If one starts with a $\underline{Y} \subset \underbrace{\underline{X} \times \underline{X} \times \ldots \times \underline{X}}_{d}$ for $d \geq 2$, then the corresponding space A^Δ makes sense for every *d-hypergraph*. Similarly one may use the language of *algebraic simplicial spaces* (cf. dimensions $d > 1$) but all that, albeit useful linguistically, does not enrich the class of varieties A^Δ.

The structure of the paper. The next § 3 is devoted to an elementary algebra geometric discussion around the Ax theorem, where we sketch, in particular, the topological proof due to Bialynicki-Barula and Rosenlicht as well as the generalization of that by Borel. Then in § 4 we prove basic properties of proalgebraic varieties we need in future. Also we explain how surjunctivity fares when we go from one group to a "nearby" group and prove surjunctivity for *initially subfinite* groups (see 4.G$''$). The following § 5 starts with a brief introduction to the first order theories. We explain the ideas of distance and approximation for models and, in particular, bring forth the *extended Lefschetz principle* (see 5.E$'$) constituting the major idea of Ax' argument based on an approximation of $\mathbb{C}$ by the fields $\overline{\mathbb{F}}_p = \bigcup_\nu \mathbb{F}_{p^\nu}$. Then we explain how one should reformulate the notions of injectivity and surjectivity in the proalgebraic category in order to make surjunctivity amenable to the extended Lefschetz principle. This reformulation is used in the case of Γ-equivariant proregular maps $f : \underline{X}^\Gamma \to \underline{X}^\Gamma$ where the surjunctivity is reduced to that of shift endomorphisms $S^\Gamma \to S^\Gamma$ where S is a finite set. Thus we prove surjunctivity for amenable groups Γ where the corresponding feature for shift endomorphisms is rather obvious (and well known).

Our main constructions are exposed in § 7 with § 6 presenting basic graph theoretic terminology in the spirit of the above 2.C''' and 2.D. We explain at the begining of § 7 how partial symmetries of graphs Δ act on proalgebraic varieties X associated to Δ, this is what we call *holonomy* H, and then we introduce the *holonomy* (orbit) *completion* X° of X (see 7.C) generalizing 2.D''. Then we isolate the essential properties of Δ and H needed for surjunctivity of $f^\circ : X^\circ \to X^\circ$. (These may look rather heavy and arbitrary at the first sight but I think that most of them will prove relevant in further development of "symbolic algebraic geometry"). We formulate our main surjunctivity theorem in 7.G' and prove it in the following sections by essentially repeating the steps used in § 5 : first translating everything to the first order language, then applying the extended Lefschetz principle and finally using a counting argument borrowed from the topological entropy. All this is, essentially, a routine; yet I could not find a two page argument taking care of all details of the picture. On the other hand, I did not attempt to state and prove everything in the maximal generality. But I tried to indicate different possible directions and perspectives around the Ax theorem without tying them all up by a unifying formalism (that would make the article twice shorter but, in my view, unreadable). My primary goal was to initiate a meaningful conversation between the three well established domains : model theory, algebraic geometry and symbolic dynamics.

Finally, we explain in § 8 the "Garden of Eden" surjunctivity theorem originated in 1963 in the theory of cellular automata.

§ 3. More about Ax' theorem.

3.A. Let us explain in simple terms why no complex algebraic variety X admits a strict embedding into itself. First we recall the standard

Open embedding Lemma (see 3.19 in [Har]) *Let $f : X \to Y$ be an injective regular map between equidimensional complex algebraic varieties. Then the image $f(X) \subset Y$ is Zarisky open, i.e. $f(X) = Y \backslash A$ for an algebraic subvariety $A \subset Y$.*

3.B. Now apply this to an injective polynomial map $f : \mathbb{C}^n \to \mathbb{C}^n$ and show, following [BB-R, 1962], that the image $U : f(\mathbb{C}^n) \subset \mathbb{C}^n$ can not be homeomorphic (not even homotopy equivalent) to $\mathbb{C}^n$ unless $A = \mathbb{C}^n \backslash U$ is empty. Indeed, take a non-singular point $a \in A$ at which $\dim_a A = m \geq 0$ and let $S_\epsilon^{2(n-m)-1}$ be a small ϵ-sphere in the normal space $N_a(A)$. This sphere is non-trivially linked with A and hence is non-homologous to zero in $U = \mathbb{C}^n \backslash A$. Thus $H_{2(n-m)-1}(U) \neq 0$. Q.E.D.

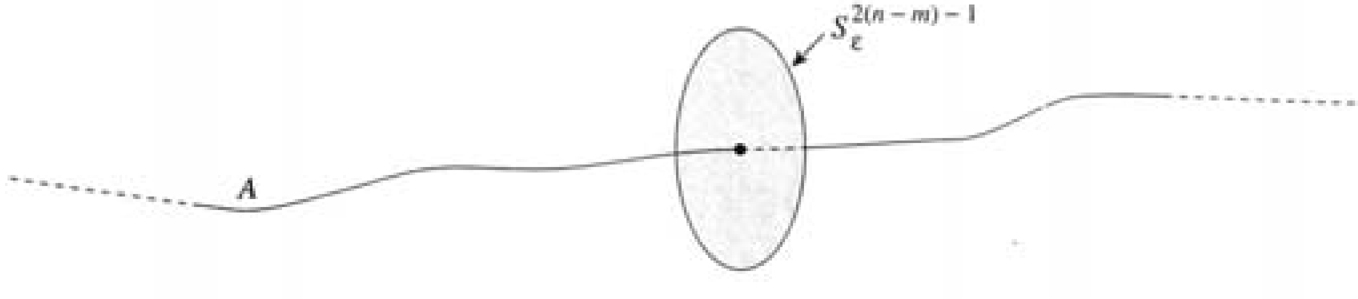

Fig. 2

3.C. The Ax theorem (see [Ax, 1968]) claims, in general, that

$$\text{one-to-one} \Longrightarrow \text{onto}$$

for all regular selfmappings $f : X \to X$ where X is an arbitrary complex algebraic variety (or more generally the set of K-points of a variety over K where K is an algebraic closed field). This is rather obvious if X is a projective or more generally, complete (and thus compact) variety, as no proper subset of such X is homeomorphic (not even homotopy equivalent) to X. On the other hand the case of a Zariski open subset $X \subset \mathbb{C}^n$ is already interesting. It is not totally obvious that a rational map $f : \mathbb{C}^n \to \mathbb{C}^n$ which is regular on this X and *injects* X into itself must send X onto all of X. Yet, following Borel (see [Bor, 1969]) one can show that X can not be homotopy equivalent to $X \backslash A$ for a non-empty Zariski closed (or even arbitrarily *constructible*) subset $A \subset X$. In fact the above homological argument gains in efficiency when applied to an iterate $f^{(i)} = \underbrace{f \circ f \circ \ldots \circ f}_{i}$ for large i. Here $U_i = f^{(i)}(X) \subset X$ is obtained by removing i disjoint homeomorphic copies of A from X,

$$U_i = X \backslash (A \cup f(A) \cup f^{(2)}(A) \cup \cdots).$$

Since X has finite topological type, the major contribution to the homology $H_{2k-1}(U_i)$ for $k = \operatorname{codim}_{\mathbb{C}} A$ comes from i small spherical $(2k-1)$-cycles that are linked to $A, f(A), \ldots, f^i(A)$. Thus

$$\operatorname{rank} H_{2k-1}(U_i) \geq i - \operatorname{const}(X)$$

with $\operatorname{const}(X) \leq \operatorname{rank} H_*(X)$. Now, clearly, U_i is *not* homotopy equivalent to X for $i > 2 \operatorname{const} X$ which trivially implies that $X = U_0$ can not be homotopy equivalent to $U = U_1$ (as $U_j \sim U_{j+1}$ for $j = 0, \ldots, i-1$ would make $U_0 \sim U_i$).

It is not hard, following Borel, to extend this idea, homology + iteration, to all complex algebraic varieties and prove Ax's "one -to-one" implies "onto" theorems in full generality. Then, one can invoke *Lefschetz' principle* and derive Ax' theorem for all algebraically closed fields of zero characteristic (see [Bor, 1969]).

3.D. The original proof by Ax is of more formal (model theoretic) nature where surjunctivity of regular selfmappings is derived from that for selfmapping of finite sets via a suitable modulo p reduction (see 5.F). We extend this powerful idea to the infinite dimensional context where the modulo p reduction lands in the category of symbolic dynamical systems replacing the finite sets of Ax' argument (see 5.M). Now, not all "symbolic endomorphisms" are surjunctive, but whenever they are such, so are also the corresponding proalgebraic ones (see 5.M''). Thus Ax' idea gains an extra edge in the infinite dimensional case which hardly can be matched by a topological argument in the spirit of 3.B. and 3.C. (Eventually, the topological proof will be rendered infinite dimensional as well, I believe.)

3.E. The converse to Ax' theorem fails to be true : the map $x \mapsto x^2$ is onto but not one-to-one and $(x_1, x_2) \mapsto ((1+x_1)^2, x_1x_2)$ is not even finite-to-one.

Also, this theorem *does not* extend to general complex analytic maps $\mathbb{C}^n \to \mathbb{C}^n$ for $n \geq 2$ as manifested by the famous Fatou example. Yet it may be true for special classes of such maps $f: \mathbb{C}^n \to \mathbb{C}^n$, e.g. for those where the differential $Df(x)$ (and/or its inverse D^{-1}) grows slowly (strongly subexponentially for $x \to \infty$).

3.F. The first (as far as I checked the references) one-to-one $\Rightarrow$ onto result was proven for *real* polynomial maps $\mathbb{R}^2 \to \mathbb{R}^2$ in 1960 by D.J. Newman. It was extended to real polynomial maps $\mathbb{R}^n \to \mathbb{R}^n$ by Bialynicki-Barula and Rosenlicht and then to regular maps of *normal* (e.g. non-singular) real algebraic varieties by Borel who was inspired by the Ax' theorem (see [New], [BB-R], [Ax], [Bor]). Notice, that Ax' method applies to certain non-closed fields, (see 5.F', but not to $\mathbb{R}$). The main point in the real case is to observe that a regular map $f: X \to Y$ between smooth equidimensional $\mathbb{R}$-varieties has a well defined degree mod 2 over a Zariski open subset in Y. It follows, $Y \backslash f(X)$ is Zariski closed in Y for injective maps f and then the mod 2 homology + iteration do the job for maps $f: X \to X$ (The mod 2-degree idea was explained to me by Slava Charlamov).

3.G. Constructible sets and maps. A subset A in an algebraic variety X is called *locally closed* if it is a difference $\overline{A} \backslash \Sigma$ where $\overline{A}$ and Σ are subvarieties in X, i.e. Zariski closed subsets. Notice, that this representation can be made canonical with $\overline{A}$ being the Zariski closure of A and $\Sigma = \overline{A} \backslash A$. Next, a subset $A \subset X$ is called *constructible* if it is a finite union of locally closed subsets. Notice, that this makes sense for a variety X defined over any field K if the words "set", "subset" etc. refer to the set $X(K)$ of K-points of X. (We dealt before exclusively with $K = \mathbb{C}$ and did not distinguish notationally X and $X(\mathbb{C})$).

It is rather obvious that *a pull-back of a constructible subset under a regular map is constructible*. What is deeper is the following classical

CIT : Constructible image theorem. *If the field K is algebraically closed then the image of a constructible subset $A \subset X(K)$ under a regular map $f: X \to Y$ is constructible in $Y(K)$* (see 3.19 in [Har]).

Basic example. Let $A \subset K^{n+1}$ be an affine subvariety. Then the projection of A to K^n is constructible, provided K is an *algebrically closed* field.

Now we can define *constructible maps* $f : X(K) \to Y(K)$ as those where the graphs $\Gamma_f \subset X(K) \times Y(K)$ are constructible subsets. In particular, one may speak of *constructibly isomorphic* spaces $X(K)$ and $Y(K)$. These mean X and Y can be decomposed into mutually biregularly isomorphic constructible pieces. Then one easily sees that every variety $X(K)$ is constructively isomorphic to a constructible subset in some affine space K^N.

3.G$'$. Ax' theorem for constructible maps. *Let K be an algebraically closed field and $A \subset X(K)$ be a constructible subset in an algebraic variety X over K. Then every constructible selfmapping of A is surjunctive.*

The proof is identical to that of Ax' theorem (see 5.F$''$). What is amusing is a combination of this theorem with the following

3.G$''$. Conservation property. Consider a rational selfmapping of X, i.e. a regular map $f : U \to X$ where U is a Zariski open subset in X, i.e. $U = X \setminus \bigcup_{i=1}^{N} Y_i$ where Y_i are *irreducible* subvarieties in X and where X itself is assumed irreducible.

CP. *If f is one-to-one and $\dim Y_i \leq d$, $i = 1, \ldots, N$ then the complement of the image, say $Y' = X \setminus f(U)$ can be covered by subvarieties Y'_i, $i = 0, 1, \ldots, N$, where Y'_i are irreducible of dimension $\leq d$ and Y'_0 is a union of subvarieties of dimension $\leq d - 1$.*

In fact, Ax' argument reduces this to the case of finite fields (see 5.F) where one applies the Lang-Weil theorem claiming that the number of F-points in an irreducible variety of dimension d is approximately $(\operatorname{card} F)^d$ for most finite fields F (see App. C in [Har]).

Remark. This "conservation law" for the number of complementary components can be alternatively obtained by the topological reasoning of Borel but this requires a little effort.

3.G$'''$. One may also try to count the complementary components of different dimensions. This is possible, for example if $Y = X \setminus U$ and $Y' = X \setminus f(U)$ have isomorphic d-dimensional parts and then one can count the $(d-1)$-dimensional ones. Technically speaking, one should work in the constructible category and observe that CP makes sense and remains valid for constructible maps $f : U \to X$.

Questions. (a) What is the maximal set of numerical invariants of varieties in a given birational equivalence class reflecting the order relation inherent in the above version of Ax' theorem ? (The varieties obtained by blow-ups at different non-singular rational points are apparently numerically undistinguishable from one another as was pointed out to me by Fedia Bogomolov).

(b) Let two algebraic varieties X and Y admit embeddings to a third one, say $X \hookrightarrow Z$ and $Y \hookrightarrow Z$, such that the complements $Z \setminus X$ and $Z \setminus Y$ are biregular isomorphic. How far are X and Y from being birationally equivalent ? For example, can two non-isomorphic curves in $\mathbb{C}^2$ (or in P^2) have isomorphic complements ?

3.H. On the order relation $X \underset{\neq}{\subset} Y$. Ax' method allows one to operate with algebraic varieties over $\mathbb{C}$ as if they were natural numbers with the order relation $m < n$ corresponding to the strict embedding relation $X \underset{\neq}{\subset} Y$. In particular, one can take minima, maxima, minmax etc for families of these varieties as follows.

Let $\pi : X \to B$ be a morphism of varieties over $\mathbb{C}$. Then there exists a "maximal" fiber $X_{\max}$, i.e. $\pi^{-1}(b_0) \subset X$ for some $b_0 \in B$, such that it admits no strict embedding into another fiber $\pi^{-1}(b)$ for all $b \in B$. Similarly, there exists a "minimal" fiber $X_{\min}$ such that no $\pi^{-1}(b)$ strictly embeds into $X_{\min}$.

One can bring the two statements together by considering two parametric families of varieties, say $X_{b,c}$. Here one proves the existence of a "minmax" point $(b_0, c_0) \in B \times C$. This means X_{b_0,c_0} is "maximal" in c and $\forall b \exists c$, s.t. X_{b_0,c_0} receives *no* strict embedding from X_{bc}. Similarly one states and prove the existence of maxminmax etc (see 5.H).

3.I. Ax theorem for subconstructible spaces. Given the notion of a constructible subset in K^n, one goes on building new spaces as follows.

Take a constructible subset $A \subset K^n$. Next, take a constructible subset $R \subset A \times A \subset K^{2n}$ which is an *equivalence* relation, i.e. symmetric and transitive. Define B as A/R and call these B *subconstructible spaces*, where (A, R) is called a *presentation* of B. A subset $B_3 \subset B_1 \times B_2$ is called *subconstructible* if it lifts to a constructible subset in the corresponding product $A_1 \times A_2 \subset K^{2n}$. Then define subconstructible morphisms $B_1 \to B_2$ as maps with subconstructible graphes.

3.I′. Theorem. *Let U_1 and U_2 be subconstructibly isomorphic subspaces in some subconstructible space B over an algbraically closed field K. Then their complements have equal dimension, say δ, and equal numbers of irreducible components of dimension δ. In particular every subconstructible selfmapping of B is surjunctive.*

About the proof. First we observe that the notion of dimension and of irreducible components obviously extend to our category and so the statement of the theorem makes sense. Then we reduce the general case to that of the fields $K = \overline{\mathbb{F}}_p$ (see 5.E′). Finally, we consider the minimal field $\mathbb{F}_q \supset \overline{F}_p$ for some $q = p^\nu$ so that the coefficients of polynomials involved in the definition of B, U_1, U_2 and the isomorphism $f : U_1 \leftrightarrow U_2$ (Recall that everything was defined in terms of algebraic subsets in $\mathbb{C}^n$ (or $\mathbb{C}^{2n}$) appearing as zero sets of some polynomials). Then the Galois group of $\overline{\mathbb{F}}_p = \overline{\mathbb{F}}_q$ over $\overline{\mathbb{F}}_q$, generated by the Frobenius automorphisms $x \mapsto x^q$, acts on B and this action preserves U_1, U_2 and commutes with f. Thus everything follows from the fact that B equals an increasing union of the periodic orbits of the Galois-Frobenius action (compare 5.F).

Remark. The above theorem adds nothing new compared to 3C′-G‴ as every B is subconstructibly isomorphic to a constructible subset in $\mathbb{C}^n$. Yet the logic of the above definitions suggests something new for dim $= \infty$, namely the notions of *subproconstructible spaces and morphisms* (see 4.F‴ and 7.P).

3.J. The surjunctivity can be extended from maps $X(\mathbb{C}) \to X(\mathbb{C})$ to regular maps $X(\mathbb{C}) \to Y(\mathbb{C})$ where $X(\mathbb{C})$ and $Y(\mathbb{C})$ are conjugate by some Galois automorphisms of $\mathbb{C}$. In fact if $X(\mathbb{C}) \subsetneq g(X)(\mathbb{C})$ for some variety X over $\mathbb{C}$ and $g \in \text{Galois}(\mathbb{C}/\mathbb{Q})$ then Borel's argument delivers strict embeddings $X(\mathbb{C}) \subsetneq g^k(X)(\mathbb{C})$ with the "homological" size of the complement about k. Thus every regular map $f : X(\mathbb{C}) \to g(X)(\mathbb{C})$ is surjunctive.

Remark. This extends with little effort to the (sub)constructible category. In fact a model theoretic reduction to finite fields is also possible here. This is rather obvious for X and Y defined over $\overline{\mathbb{Q}}$ and the general case can be derived, I believe, along similar lines.

§ 4 Approximation, surjunctivity and symbolic dynamics.

One can sometimes prove surjunctivity of a selfmapping $f : X \to X$ by suitably approximating it by a family of maps $f_\nu : X_\nu \to X_\nu$ which for some reason are known to be surjunctive. For example, these X_ν may be f-invariant subsets in X. If their union equals X then the surjunctivity of all $f_\nu : f|X_\nu$ obviously implies surjunctivity of f.

4.A. Example : Surjunctivity over $\overline{\mathbb{F}}_p$. Recall that the field $\overline{\mathbb{F}}_p$, i.e. the algebraic closure of the prime field $\mathbb{F}_p$ (consisting of residues modulo a prime number p) equals the increasing union of finite fields $\overline{\mathbb{F}}_p = \bigcup_{\nu=1}^{\infty} \mathbb{F}_{p^\nu}$. Then, for every regular selfmapping f of a variety X over $\overline{\mathbb{F}}_p$, we observe that both X and f are defined over some $\mathbb{F}_{p^{\nu_0}}$ and so the map $f : X(\overline{\mathbb{F}}_p) \to X(\overline{\mathbb{F}}_p)$ sends the $\mathbb{F}_{p^\nu}$-points into themselves for all $\nu \geq \nu_0$. Thus the maps $f_\nu : X(\mathbb{F}_{p^\nu}) \to X(\mathbb{F}_{p^\nu})$, $\nu \geq \nu_0$, approximate f in the above sense as the union of $\mathbb{F}_{p^\nu}$-points over $\nu \geq \nu_0$ gives us exactly all $\overline{\mathbb{F}}_p$-points of X, i.e. $X(\overline{\mathbb{F}}_p) = \bigcup_{\nu \geq \nu_0} X(\mathbb{F}_{p^\nu})$. But the maps $f_\nu = f|X(\mathbb{F}_{p^\nu})$ are surjunctive as the sets $X(\mathbb{F}_{p^\nu})$ are finite and so the surjunctivity of f follows.

Remark. Notice that onto $\Rightarrow$ one-to-one over each $\mathbb{F}_{p^\nu}$ but this does not pass to the union $\overline{\mathbb{F}}_p = \bigcup_\nu \mathbb{F}_{p^\nu}$, unlike our surjunctivity implication "one-to-one $\Rightarrow$ onto".

4.B. Now, suppose, X is given a topology and the union of $X_\nu \subset X$ is dense in X. Then we can derive surjunctivity of f from that of $f_\nu = f|X_\nu$ provided we can prove that the image $f(X) \subset X$ is necessarily closed and so "dense image" $\Rightarrow$ "onto".

4.B′. Example. Let X be a *compact* topological space and $f : X \to X$ a continuous selfmapping. Suppose f admits a family $X_\nu \subset X$, $\nu = 1, 2, \ldots$, of finite f-invariant subsets whose union is dense in X. Then, clearly, f is surjunctive.

4.B″. Shift endomorphisms. Let S be a finite set (alphabet) and $X = S^{\mathbb{Z}}$, where the infinite power space

$$S^{\mathbb{Z}} = \ldots \times S \times S \times \ldots \times S \times S \times \ldots$$

is given the product topology (with respect to which $S^{\mathbb{Z}}$ is homeomorphic to the Cantor set). The natural action of $\mathbb{Z}$ on this X is called *"shift"* or the full $\mathbb{Z}$*-shift on the alphabet* S. One may think of elements in $S^{\mathbb{Z}}$ of doubly infinite words with letters from S. Then the generator $1 \in \mathbb{Z}$ shifts each letter to its neighbour's place to the right.

$$\begin{matrix} \ldots aabbababcacdab \ldots \\ \ldots aabbababcacdab \ldots \end{matrix} \downarrow \text{shift.}$$

Here finite orbits of a group action are called *periodic orbits* and their (finite!) cardinalities are called *periods*. For example orbits of period 5 for the shift look like this

$$\ldots abbdcabbdcabbdc \ldots$$

Obviously, there are only finitely many points of a given period, namely $p^{\operatorname{card} S}$ points of period p. Also it is clear that periodic points are dense in X. Indeed every word $x \in X$ can be approximated by p-periodic words x_p for $p = 1, 2 \ldots$. To do this we just periodically repeat the *initial block* of x. Namely think of x as an S-valued function, $x = x(i) \in S$, $i \in \mathbb{Z}$, and define $x_{p_q}(i)$ by two conditions,

(1) $x_{p_q}(i) = x(i)$ for i in the segment $-q, \ldots, 0, \ldots, p - q$,

(2) $x_{p_q}(i + p) = x_{p_q}(i)$, $i \in \mathbb{Z}$.

These x_{p_q} are clearly p-periodic and if $q, p - q \to \infty$ then $x_{p_q} \to x$ in our product topology.

Clearly, every *shift endomorphism*, i.e. a continuous $\mathbb{Z}$-equivariant mapping $f : X \to X$, sends each subset $X_p \subset X$ of p-periodic points into itself and *so f is always surjunctive.*

Notice, that there are lots and lots of shift endomorphisms. Indeed take an arbitrary map $\varphi : S^{m+n+1} \to S$ and then define $f : x \mapsto y$ by

$$y(i) = \varphi(x(i - m), x(i - m + 1), \ldots, x(i), \ldots, x(i + n))$$

(as we did in 2.C‴ for regular maps p). Clearly this f is continuous and commutes with the shift. Conversely, one can easily show (compare (∗) below) that every shift endomorphism comes this way. On the other hand one has no clear picture yet of all shift *automorphisms* (see [He]).

4.C. "Varieties", "regular maps" and the prodiscrete topology. We want to generalize the above to infinite sets S, equipped with extra structures, e.g. to algebraic varieties over $\mathbb{C}$. So we start with some subcategory of the category of sets, where the objects are called *"varieties"* and morphisms called *"regular"* maps. The we take projective limits of our *"varieties"* and call these "provarieties" and their morphisms, i.e. projective limits of "regular" maps defined as in § 2, are called *"proregular"* maps. In what follows, we stick to projective limits over countable directed systems admitting a cofinal subsystem isomorphic to $\mathbb{N} = \{1, 2, \ldots\}$ and so everything reduces to projective limits of sequences,

$$X = \varprojlim X_i \text{ for } X_1 \leftarrow X_2 \leftarrow \ldots \leftarrow X_i \leftarrow \ldots,$$

as in § 2. (But in our applications, we shall use the directed systems over finite subsets of a countable set, compare § 7).

Prodiscrete metric $|x-x'|$ on X. We define a metric on each X_i, denoted $|x_i - x_i'|$ by $|x_i - x_i'| = 2^{-i}$ if $x_i \neq x_i'$ and $|x_i - x_i'| = 0$ otherwise. Then we set $|x-x'| = \sum_{i=1}^{\infty} |x_i - x_i'|$ for sequences $x_i, x_i' \in X_i$ representing $x, x' \in X$. The topology corresponding to $|x-x'|$ is called *prodiscrete*. Here convergence $x_\nu \underset{\nu\to\infty}{\to} x$ in X for $x = (x_i)$ and $x_\nu = (x_i)_\nu$ amounts to stabilization :

$$(x_i)_\nu = x_i \text{ for } \nu \geq \nu_0 = \nu_0(x, i).$$

Clearly *the metric space* $(X, |x-x'|)$ *is complete.* Furthermore, *every "proregular" map is uniformly procontinuous* i.e. uniformly continuous relative the metric $|x-x'|$.

And in the full category of sets (where all maps are allowed) the converse is true : every uniformly procontinuous map is "proregular". In particular, every uniformly procontinuous Γ-equivariant map $f : S^\Gamma \to T^\Gamma$, where Γ is a countable group and S and T are arbitrary sets, is given by a map $\varphi : S^D \to T$ for some finite subset $D \subset \Gamma$, according to the recipe described in 2.D′, i.e. by the formula

$$f(x)(\gamma) = \varphi(x(\gamma\delta)_{\delta\in D}) \qquad (*)$$

where recall $x \in S^\Gamma$ and $y = f(x) \in T^\Gamma$ are viewed as S and T-valued functions on Γ and D is identified with γD via the translation $\delta \to \gamma\delta$, $\delta \in D$.

4.C′ "Subalgebraic" sets and their intersections. A subset $A \subset X$ is called "algebraic" if it is a finite intersection of pull-backs of points of "regular" maps from X to some "varieties". We also agree that $A = X$ is "algebraic". Then we define "subalgebraic" subsets as images of "algebraic" ones under "regular" maps.

Definition of "SA"IP. We say that our category has *"subalgebraic" intersection property* ("SA"IP) if every countable decreasing family of *non-empty* "subalgebraic" subsets in each X has *non-empty* intersections.

Examples. (a) "SA"IP is obviously satisfied in the category of finite sets.

(b) "SA"IP is satisfid in the category of compact topological spaces and continuous maps. In fact every (possibly uncountable) decreasing family of non-empty compact sets has a non-empty intersection.

(c) Finite dimensional vector spaces over an arbitrary field and the affine maps satisfy "SA"IP. This is clear.

4.C″. SAIP *for uncountable algebraically closed fields.* Every subalgebraic subset in an algebraic variety over an algebraically closed field is constructible and then SAIP follows from the well known *countable intersection property,*

CIP. *The intersection of a decreasing (possibly uncountable) family of constructible subset in the set of K-points of an algebraic variety X over K is non-empty, provided K is an uncountable algebraically closed field.*

This is obvious for an irreducible one-dimensional X, where the constructible subsets are all of the form X minus a finite subset. These can not have an empty intersection as X is uncountable. Then the case of $\dim X \geq 1$ follows by an easy induction argument.

4.C‴ Remark about $K = \mathbb{R}$ and ${}^*\mathbb{R}$. This CIP also holds for the real numbers, (as well as for every real closed uncountable field), but it does not apply to subalgebraic subsets. However, (as was pointed out to me by Udi Hrushovski) CIP does hold for semialgebraic (and thus for subalgebraic) subsets over $K = {}^*\mathbb{R}$, i.e. for the non-standard model ${}^*\mathbb{R}$ of real numbers obtained as the ultraproduct of countably many copies of $\mathbb{R}$ over a non-principle ultrafilter (see Ch. 3 in [HML]). For example, the subsets $A_n \subset {}^*\mathbb{R}$ defined by the inequalities $A_n = \{x \in {}^*\mathbb{R} \mid x \geq n\}$ have a *non-empty* intersection $\bigcap_{n=1}^{\infty} A_n$ which consists of all "infinitely large" real numbers. In fact, every (countable) first order theory admits a *saturated* model where CIP is valid for all subsets definable in this theory (see Ch. 3 in [HML]).

4.D. Closed image property (ClImP) for "proalgebraic" maps. *If a category of "varieties" and "regular" maps has the "subalgebraic" intersection property then the image of an arbitrary "proregular" map $f : X \to Y$ for $X = \lim_{i\to\infty} X_i$, $X = \lim_{j\to\infty} Y_i'$ is closed in Y for the prodiscrete topology.*

This is well known with the proof obtained by the standard compactness argument. To see it in a nutshell we start with the following non-emptiness property.

4.D′. *If $(X_i, i \in I)$ is a countable projective system in a "SA"IP-category where all X_i are non-empty, then the projective limit $X = \lim_{\leftarrow} X_i$ is also non-empty.*

Proof. Let $X_i^j \subset X_i$ denote the image $\pi_{ji}(X_j) \subset X_i$ and $X_i^\infty = \bigcap_{j=i}^{\infty} X_i^j$. These subsets are non-empty by "SA"IP and we claim that $\pi_{i+1,i}$ sends X_{i+1}^∞ *onto* X_i^∞ for all i. In fact, $\pi_{i+1,i}^{-1}(x) \cap X_{i+1}^j$ is non-empty for all $x \in X_i^\infty$ and $j \geq i+1$ as $x \in X_i^\infty \subset X_i^j$. Hence, the intersection $\pi_{i+1,i}^{-1}(x) \cap X_{i+1}^\infty$ is also non-empty by "SA"IP and so x comes from some $x' \in X_{i+1}^\infty$. Finally the above "onto" makes the projection $X \to X_i$ also onto and so $X = \lim_{\leftarrow} X_i$ is non-empty.

Proof of 4.D. Take $y \in Y$ and let $y_j = \pi_{\infty,j} \in Y_j$ for the projection $\pi_{\infty,j} : Y = Y_\infty \to Y_j$. Denote by $X_i(y) \subset X$ the pullback of y_j under $f_i : X_i \to Y_j$ and observe that

$$y \in \text{ Closure } f(X) \Longleftrightarrow X_i(y) \cap \pi_{\infty,i}(X) \neq \emptyset.$$

Hence, $X_i(y)$ is non-empty and since $\pi_{i+1,i}$ (obviously) sends $X_{i+1}(y)$ to $X_i(y)$, the spaces $X_i(y)$ make a projective system, such that $X(y) = \lim_{\leftarrow} X_i(y) = f^{-1}(y)$.

Now 4.D$'$ applies to $\{X_i(y)\}$ and shows that $f^{-1}(y)$ is non-empty. Q.E.D.

Remark. Observe that "SA"IP was used in X_i's and not in Y_j's. This is similar to what happens to maps between topological spaces : to have $f(X)$ closed in Y one needs compactness of X but not of Y.

4.E. Surjunctivity of shift endomorphisms over residually finite groups. A countable group Γ is called residually finite if one of the following five equivalent conditions is satisfied.

(1) There exists a family of subgroups of finite index $\Gamma_p \subset \Gamma$ such that $\cap\Gamma_p = \{\mathrm{id}\}$.

(2) There exists an embedding of Γ into a *profinite* group, i.e. to a projective limit of finite groups.

(3) There is a continuous faithful action of Γ on a compact space such that the periodic (i.e. finite) orbits are dense.

(4) The periodic orbits for the natural (shift) action of Γ on $\{0,1\}^\Gamma$ are dense.

(5) The periodic orbits in S^Γ are dense in the prodiscrete topology in S^Γ (which is the same as the infinite Cartesian product topology) for every set S.

The equivalence of these properties is obvious and well known. The most common is (1) and the one we need is (5). The implications (1) $\Rightarrow$ (5) follows by the argument we used in 4.B$''$ to prove the density of periodic points in $S^{\mathbb{Z}}$. In the present case we take suitable fundamental domains $\Delta_p \subset \Gamma$ of the subgroups $\Gamma_p \subset \Gamma$ and extend the restrictions $x|\Delta_p$ to all of Γ by Γ_p-periodic functions $x = x(\gamma)$.

4.E$'$. Let our category of "varieties" and "regular" maps admit finite Cartesian products and define the infinite products as the projective limits of finite ones. We assume our category is surjunctive, i.e. all "regular" selfmappings of "varieties" are surjunctive, and that it satisfies the "subalgebraic" intersection property. For example, this can be the category of the complex algebraic varieties. Now ClImP and the general discussion in 4.B. trivially imply the following

Residually finite Surjunctivity Theorem. *Every "proregular" Γ-equivariant self-mapping $f : \underline{X}^\Gamma \to \underline{X}^\Gamma$ is surjunctive for all "varieties" $\underline{X}$ in our category and all residually finite groups Γ.*

The only point in the proof which still needs an explanation is a clarification of the structure of the sets of periodic points. Here for every $\Gamma_p \subset \Gamma$, we have the fixed point set for Γ_p, denoted $X_p \subset X$, whose points are often called Γ_p-*periodic*. This X_p can be identified with the set of maps of the coset Γ/Γ_p into $\underline{X}$, or equivalently with the set of $\underline{X}$-valued functions on a fundamental domain $\Delta_p \subset \Gamma$. In any case X_p equals a finite Cartesian power of $\underline{X}$, i.e $X_p = \underline{X}^{\Delta}p = \underline{X}^q$ for $q = \operatorname{card}\Delta_p = \operatorname{card}\Gamma/\Gamma_p$ and our f is surjunctive on X_p.

4.E$''$. Remarks and open questions. (a) There are many examples of residually finite groups to which the above applies, e.g. all finitely generated subgroups in Lie groups are residually finite.

(b) If the conclusion of the theorem holds true for an incresing family of group Γ_ν then it obviously holds for $\Gamma = \cup\Gamma_\nu$. Thus our theorem extends to *locally residually finite groups*, i.e. those where every finitely generated subgroup is residually finite. For example all countable subgroups in Lie groups are ℓ.r.f.

(c) The above surjunctivity theorem is well known for *finite* sets $\underline{X}$ and, probably, due to Gottschalk (compare [Hed]) who introduced the word "surjunctive" and raised the following

Gottschalk problem [Gott, 1972]. Is every shift endomorphism of S^Γ surjunctive for all finite sets S ?

This question generalizes to every surjunctive category with finite Cartesian products. But the positive answer, probably, needs something like "SA"IP even for such groups as $\mathbb{Z}$.

(d) The category of real algebraic varieties is surjunctive but has no "SA"IP. So we do not know how to prove (or disprove) surjunctivity of $\mathbb{R}$-endomorphisms of $\underline{X}$ for residually finite groups Γ. Yet we gain "SA"IP if we pass to the field $^*\mathbb{R}$ of nonstandard real numbers where "SA"IP holds true and surjunctivity follows.

4.E″. Generalizations. Suppose we have a family of *normal* subgroups $\Gamma_p \subset \Gamma$, of possibly infinite index, such that $\bigcap\limits_p \Gamma_p = \{\mathrm{id}\}$. Then we have by the above argument,

The Subgroup approximation theorem. *If all quotient groups $\Gamma'_p = \Gamma/\Gamma_p$ are surjunctive relative to a category of "varieties" with "SA"IP, then Γ is also surjunctive.*

Here a group Γ is called *surjunctive* (compare [Gott]) if every Γ-equivariant "proregular" selfmapping $f : \underline{X}^\Gamma \to \underline{X}^\Gamma$ is surjunctive for all "varieties" $\underline{X}$ in our category. (We tacitly assume here that finite Cartesian products are defined in our category).

Another possible generalization concerns more general "proalgebraic varieties" (which are not necessarily Cartesian powers) with Γ-actions. The above argument applies whenever Γ_p-periodic points are dense in X and there are many interesting Γ-actions, e.g. on "subprovarieties" in $\underline{X}^\Gamma$ where periodic points are dense (compare 4.F‴ and 7.P).

4.F. Diagonal intersection property and uniform injectivity. An injective map between metric spaces, say $f : X \to Y$ is called *uniformly injective* if the inverse map f^{-1} from $f(X) \subset Y$ to X is uniformly continuous for the metrics $\mathrm{dist}_Y \mid f(X)$ and dist_X on X. In other words for each $\epsilon > 0\ \exists\ \delta > 0$, such that $\mathrm{dist}_X(x, x') \geq \epsilon \Rightarrow \mathrm{dist}_Y(f(x), f(x')) \geq \delta$.

We are interested in the case of "proregular" maps f between projective limits $X = \varprojlim X_i$ and $X' = \varprojlim Y_j$ where our $f : X \to Y$ is given by "regular" maps $f_i : X_i \to Y_j\ j = j(i)$. What we want to show is that in certain categories the injective "proregular" maps are uniformly injective.

Definition. A subset $A \subset X \times X$ is called *subdiagonal* if it is obtained by pulling back the diagonal by the square of a map f' in our category, say by $f' \times f' : X' \times X' \to Y \times Y$, and then by pushing this pullback forward to X by $\pi' \times \pi' : X' \times X' \to X \times X$ for some $\pi' : X' \to X$. We call $B \subset X \times X$ *codiagonal* if there exists $\pi'' : X \to X''$, such that B equals the pullback of the complement to the diagonal in $X'' \times X''$ under $\pi'' \times \pi''$. Finally $C \subset X \times X$ is called $\Box$-"algebraic" if it is the pull-back of a point under the square $X \times X \to X^\bullet \times X^\bullet$ of some map $X \to X^\bullet$.

Diagonal intersection property (DIP). This says that for every codiagonal $B \subset X \times X$ and an arbitrary decreasing sequence A_i of subdiagonal subsets in $X \times X$, such that $A_i \cap B$ is nonempty for every $i = 1, 2,$ the infinite intersection $\bigcap_{i=1}^{\infty} A_i \cap B$ is also non-empty. Furthermore, if $C \subset X \times X$ is $\Box$-"algebraic" such that $A_i \cap B \cap C$ is non-empty for all i, then $\bigcap_{i=1}^{\infty} A_i \cap B \cap C$ is also non-empty.

4.F$'$. Clearly, algebraic varieties over an uncountable algebraically closed field have DIP since all these B and A_i's are constructible (see 3.G and 4.C$'$). Also the category of linear spaces and affine maps over an arbitrary field has DIP. But compact spaces do not have DIP (yet they have "SA"IP).

4.F$''$. Proposition. *In a category with* DIP *injectivity implies uniform injectivity of "proregular" maps.*

In particular, this is true for proregular maps of proalgebraic spaces over $\mathbb{C}$.

Proof. If the projective limit $f = \lim\limits_{\leftarrow} f_i : X \to Y$ for $f_i : X_i \to Y_j$ is *not* uniformly injective for our prodiscrete metric, then there exists i_0, such that $\forall i \geq i_0 \; \exists \; x, x' \in X_i$ such that

$$\pi_{i,i_0}(x) \neq \pi_{i,i_0}(x') \qquad (i)$$

$$f_i(x) = f_i(x'). \qquad (ii).$$

We observe that the pairs of points $(x, x') \in X_i \times X_i$ satisfying (i) equals the pullback of the complement of the diagonal in X_{i_0} while the pairs satifying (ii) make the pullbacks of the diagonal in $Y_j \times Y_j$. Thus the projection B_{i_1} of the first set to $X_{i_1} \times X_{i_1}$ for some $i \geq i_1 \geq i_0$, is a codiagonal in $X_{i_1} \times X_{i_1}$ independent of i (as it equals $(\pi_{i_1,i_0} \times \pi_{i_1,i_0})^{-1}((X_{i_0} \times X_{i_0})\backslash\text{diagonal})$) while the projections of the subsets defined by (ii) make a decreasing sequence $A_{i_1,i} \subset X_{i_1} \times X_{i_1}$ of subdiagonal subsets, such that $B_{i_2} \cap A_{i_1,i} \neq \emptyset$ for all $i = 1 \ldots$. Then the infinite intersection $B_{i_1} \cap A_{i_1,\infty}$ for $A_{i_1,\infty} = \bigcap_{i=i_1}^{\infty} A_{i_1,i}$ is also non-empty and these intersections make a projective system for the projections

$$\pi_{i_2,i_1} \mid B_{i_2} \cap A_{i_1,\infty} \to B_{i_1} \cap A_{i_1,\infty},$$

where each point in the projective limit, denoted $(x, x') \in B_\infty \cap A_{\infty,\infty}$, satisfies $x \neq x'$ and $f(x) = f(x')$ by the very construction of our sets A_i and B_i. Then we show that this projective limit is non-empty by intersecting $B_{i_2} \cap A_{i_2,i}$ with $C_{i_2} = (\pi_{i_2,i_1} \times \pi_{i_2,i_1})^{-1}(z)$ for $z \in B_{i_j} \cap A_{i_1,\infty}$ and showing (compare 4.D) that the resulting intersection remains non-empty as we send $i \to \infty$ and pass to $B_{i_2} \cap A_{i_1,\infty} \cap C_{i_2}$. Thus the assumption of non-injectivity led us to non-injectivity of f. Q.E.D.

The main moral of the above story is that *proregular maps of proalgebraic varieties over $\mathbb{C}$ behave similar to (even slightly better than) continuous maps between compact spaces: The images are closed and inverses of injective maps are uniformly continuous.*

In fact, this remains valid if we replace $\mathbb{C}$ by any (countably) saturated model of a given first order theory (see Ch. 3 in [HML]), e.g. by the field $^*\mathbb{R}$ of non-standard real numbers.

This is well known to model theorists in a slightly different language as was pointed out to me by Udi Hrushovski.

Allowing Galois action. One can enlarge the category of regular maps acting on K-points of algebraic varieties by composing them with Galois K-automorphisms preserving these varieties. For example one can compose polynomial maps $f : \mathbb{C}^m \to \mathbb{C}^n$ with the actions of automorphisms g_1 and g_2 in Gal $\mathbb{C}$ (diagonally) acting on $\mathbb{C}^m$ and on $\mathbb{C}^n$, i.e. taking $g_2 f g_1 : \mathbb{C}^m \to \mathbb{C}^n$. Such composed maps still have the same essential features as the regular ones : *they preserve constructible subsets under taking pullbacks and images. Thus the above density and the uniform injectivity property remain valid in this extended category.* In fact one can generalize further by applying different Galois automorphisms to different components of our manifolds. Moreover, one can allow (non-regular) constructible mappings with different Galois maps on different pieces of finite constructible decomposition of the varieties in question; all that matters is the preservation of constructible subsets under taking images and pullbacks.

4.F‴.On subproalgebraic spaces. One may try to bring together the categories of compact spaces and continuous maps and of proalgebraic spaces and proregular maps. The relevant objects are defined as quotients of proalgebraic spaces by equivalence relations $Y = X/R$ where $R \subset X \times X$ must be a proalgebraic (or proconstructible) subspace in $X \times X$ (compare 3.I). Probably these quotients have the same basic properties as proalgebraic spaces but I do not see at the moment interesting examples where R sufficiently mixes together the (prodiscrete) topology and the algebraic structure on X. In fact it is not so easy to construct by hand "proalgebraic" *equivalence* relations $R \subset X \times X$ as the transitivity of R is hard to satisfy. (The reader may ponder on how the plain quotients X/R with profinite X and R give rise to the immensely rich realm of compact spaces). On the other hand quotients are easy to come by for profinite dimensional vector spaces.

4.G. Initial approximation of groups. Let Δ be a countable set. Then each exhaustion of Δ by finite subsets, say $\Omega_1 \subset \Omega_2 \subset \ldots \subset \Omega_i \subset \ldots$ with $\cup \Omega_i = \Delta$ gives us a (prodiscrete) metric on the set of subsets in Δ viewed as the space of $\{0,1\}$-valued functions on Δ. In fact, such exhaustion identifies $\{0,1\}^\Delta$ with the projective limit $X = \varprojlim X_i$ for $X_i = \{0,1\}^{\Omega_i}$ and then we use our metric $|x-y| = \sum_i |x_i - y_i|$ for $|x_i - y_i| = 2^{-i}, 0$ as in 4.C. Furthermore, the quotient sets of Δ are determined by equivalence relations $R \subset \Delta \times \Delta$

and so each exhaustion of $\Delta \times \Delta$, where we use one of the forms $\{\Omega_i \times \Omega_i\}$ gives us a metric in the set of quotients of Δ.

We are going to use this metric on the set of factor groups of a given countable free group with a given exhaustion. In fact, we shall truly need this only for finitely generated groups Γ with a given finite generating set, thought of as a (possibly non-injective) map of a given finite set into Γ, say $a : D \to \Gamma$. Then the free group F_D spanned by D is exhausted by sets of D-words of length $\le i$ that are $\Omega_i = (D \cup D^{-1}) \cup (D \cup D^{-1})^2 \cup \ldots \cup (D \cup D^{-1})^i$ and so all groups Γ with generators *marked* by D, (i.e. with maps ($D \to \Gamma$ generating Γ), appear as quotients of F_D. Thus we have our metric, denoted $|\Gamma - \Gamma'|$ on the set of these *marked* groups.

Initially subfinite and initially subamenable groups. Say that a finitely generates group Γ admits *an initial approximation* by groups from a given family, say $\{\Gamma_\alpha\}_{\alpha \in \mathcal{A}}$, if for every finite generating set $D \subset \Gamma$ there exists a sequence of groups $\Gamma_{\alpha_i} \in \{\Gamma_\alpha\}$ and maps $a_i : D \to \Gamma_{\alpha_i}$, such that the a_i-marked groups Γ_{α_i} converge to Γ in the above metric.

The idea is, that these Γ_{α_i} are "initially close" to Γ, as $|\Gamma - \Gamma_{\alpha_i}| \le 2^{-i}$ signifies that one can not tell Γ from Γ_{α_i} by looking at the elements representable by D-words of length $\le i$.

Next, a (possibly infinitely generated) Γ is called *initially subfinite* if every finitely generated subgroup $\Gamma' \subset \Gamma$ admits an initial approximation by finite groups. (Such groups were introduced in [Ve-Go] under the name of "locally embeddable into finite groups"). Similarly, we define *intially subamenable* groups, (where the definition of amenability for groups is explained in 6.E).

Remark and example. Obviously, every residually finite group is initially subfinite. And it is easy to show (see [Ve-Go]) that every *finitely presented* initially subfinite group is residually finite. And *the group Γ of permutations of $\mathbb{Z}$ generated by $\mathbb{Z}$-translations and the permutations with finite supports is finitely generated and initially subamenable without being residually finite* (see [Ve-Go]).

Surjunctivity and initial approximation. Let us consider again some catgory of "varieties" and "regular" maps satisfying "SA"IP (see 4.C′) and DIP (see 4.F).

4.G′. Initial approximation theorem for groups. *If a finitely generated group Γ admits an initial approximation by a family of groups Γ_α which are surjunctive relative to our category then Γ is also surjunctive.*

4.G″. Corollary. *If Γ is initially subfinite, then every Γ-equivariant complex proalgebraic endomorphism $\underline{X}^\Gamma \to \underline{X}^\Gamma$ is surjunctive (where, recall $\underline{X}$ stands for an arbitrary complex algebraic variety).* This follows from 4.C″ and 4.F″ (compare 4.E).

Remark on $^*\mathbb{R}$. It is unclear if the above remains true for proalgebraic endomorphisms over $\mathbb{R}$ but everything works fine if we replace $\mathbb{R}$ by the field $^*\mathbb{R}$ of non-standard real numbers as follows from 4.C‴. and 3.F. *Thus* $^*\mathbb{R}$*-endomorphisms of* $\underline{X}^\Gamma$ *are surjunctive for all normal* $^*\mathbb{R}$*-varieties* $\underline{X}$ *and all initially subfinite groups* Γ.

Remark on locally surjunctive groups. It is obvious that the union of an increasing family of surjunctive groups is surjunctive. Therefore, if every finitely generated subgroup in a countable group Γ is surjunctive then so is Γ. This somewhat enlarges the scope of applications of the above theorems.

Question. Are $^*\mathbb{R}$-endomorphisms surjunctive for amenable (and thus for initially subamenable) groups Γ ? We know this is true for $\mathbb{C}$-endomorphisms as it follows from our surjunctivity theorem 2.C′.

We start the proof of 4.G′. with a dynamical interpretation of our convergence $\Gamma_\alpha \to \Gamma$. We consider the shift space with an arbitrary "alphabet" S over the free group $F = F_D$ generated by $D \subset F_D$ and observe that the shifts over all quotient groups Γ of F_D imbeds into this shift, $S^\Gamma \subset S^F$ consists of the functions $F \to S$ *invariant* under the action of the normal divisor $R_\Gamma \subset F$ defining Γ by $\Gamma = F/R_\Gamma$.

4.G‴. Convergence criterion. *Convergence of factor groups* $\Gamma_\alpha =$ *of* F *to* Γ *is equivalent to the Hausdorff convergence of the subsets* $S^{\Gamma_\alpha} \subset S^\Gamma$ *to* $S^\Gamma \subset S^F$ *for every non-empty set* S.

Recall, that the space S^F, being a projective limit, carries a prodiscrete metric, defined via a given exhaustion Ω_i of $F = F_D$, where we may use e.g.

$$\Omega_i = D \cup D^{-1} \cup (D \cup D^{-1})^2 \cup \ldots \cup (D \cup D^{-1})^i.$$

Then one may use the *Hausdorff metric* in the set of subsets of S^Γ. Recall, that the Hausdorff convergence $S^{\Gamma_\alpha} \underset{\alpha\to\infty}{\to} S^\Gamma$ signifies that

(1) S^{Γ_α} is contained in an ϵ-neighbourhood of S^Γ for $\epsilon \underset{\alpha\to\infty}{\to} 0$.

(2) S^{Γ_α} becomes arbitrarily dense near S^Γ for $\alpha \to \infty$. That is the ϵ-neighbourhood of S^{Γ_α} contains S^Γ for all $\alpha \geq \alpha_0 = \alpha_0(\epsilon)$.

Next, we exhaust each factorgroup Γ by projections of given $\Omega_i \subset F$ to Γ, denoted $\Omega_i(\Gamma) \subset \Gamma$. We observe that each $S^{\Omega_i(\Gamma)}$ embeds to S^Γ by extending function $\Omega_i \to S$ by constants outside Ω_i and each $S^{\Omega_i(\Gamma)} \subset S^\Gamma$ is ϵ-dense in S^Γ for $\epsilon = 2^{-i}$. Next if $|\Gamma - \Gamma'| \leq 2^{-i}$, then $\Omega_i(\Gamma)$ is identical to $\Omega_i(\Gamma')$ and so $S^{\Omega_i(\Gamma)} = S^{\Omega_i(\Gamma')}$. This implies the Hausdorff convergence $S^{\Gamma_\alpha} \to S^\Gamma$ for $\Gamma \to \Gamma_\alpha$. And the converse (for non-empty S) is equally clear (and, actually, unneeded for our theorem).

4.H. Expansiveness. An action of a group Γ on a metric space X is called ϵ*-expansive* if

$$\forall x \neq y \in X \ \exists \ \gamma \in \Gamma, \text{ s.t. } \operatorname{dist}(\gamma(x), \gamma(y)) \geq \epsilon.$$

In other words, the distance between every two different orbits is at least ϵ.

Now we look from this angle on the shift action of Γ on S^Γ with our prodiscrete metric defined with some exhaustion of Γ on Ω_i and observe that

4.H′. *The shift action is ϵ-expansive for $\epsilon = 1$.*

This is obvious and, of course, well known.

Finally, we call an action of X on Γ *expansive* if it is ϵ-expansive for some $\epsilon > 0$.

4.H″. Injectivity lemma. *Let Γ expansively act on X and let $f : X \to X$ be a Γ-equivariant uniformly continuous map which is uniformly injective* (see 4.F) *on some Γ-invariant subset $X_0 \subset X$. Let X_α be a sequence of Γ-invariant subsets coming close to X_0, i.e. X_α is contained in the δ-neighbourhood of X_0 where $\delta = \delta(a) \to 0$ for $\alpha \to \infty$. Then f is injective on X_α for all sufficiently large α.*

Proof. In order to prove that $f(x_\alpha) \neq f(y_\alpha)$ for $x_\alpha \neq y_\alpha \in X_\alpha$ it suffices to check this for $x'_\alpha = \gamma(\alpha_j)$ and $y'_\alpha = \gamma(y_j)$ and suitable $\gamma \in \Gamma$, since f is Γ-equivariant. We chose γ so that $\text{dist}(\gamma(x_\alpha), \gamma(y_\alpha))$ becomes $\geq \epsilon$ for the "expansive constant" ϵ and we take α so large that x'_α and y'_α lie δ-close to X_0 for a small enough δ. Then we take points x'_0 and y'_0 in Y_0 which are δ-close to x'_α and y'_α so that $\text{dist}(x'_0, y'_0) \geq \epsilon/2$, as we could assume $\delta \leq \epsilon/4$. Now, by the uniform injectivity of f on X_0, we have $\text{dist}(f(x'_0), f(y'_0)) \geq \delta'$ for some $\delta' = \delta'(\epsilon) > 0$ and by the uniform coninuity of f we have

$$dist(f(x'_\alpha), f(y'_\alpha)) \geq \delta' - \rho(\delta)$$

for $\rho(\delta) \to 0$ as $\delta \to 0$. Thus $f(\alpha'_\alpha) \neq f(y'_\alpha)$ if δ is small enough which is achieved with large α. Q.E.D.

4.I. Proof of 4.G′. Our map $f : S^\Gamma \to S^\Gamma$, whose surjunctivity is in question, is given by a map $\varphi : S^D \to S$ as in 4.C. Then the same φ defines an F-equivariant selfmapping of S^F for the free group $F = F_D$, denoted $\tilde{f}$, which maps the subset $S^{\Gamma'} \subset S^F$ into itself for each factor group Γ' of F. Now, if $\Gamma_\alpha \to \Gamma$, we have the Hausdorff convergence $S^{\Gamma_\alpha} \to S^\Gamma$ and then the injectivity of $f = \tilde{f}|S^\Gamma$ yields injectivity of $\tilde{f}|S^{\Gamma_\alpha}$ for large α as f is uniformly injective by 4.F″. Then, by our assumption, f sends each S^{Γ_α} *onto* itself and then the Hausdorff convergence $S^{\Gamma_\alpha} \to S^\Gamma$ and continuity of f imply that the map $f : S^\Gamma \to S^\Gamma$ has dense image and surjectivity of f follows by 4.D.

4.J. Further applications and non-applications. Theorem 4.G′ applies, besides the category of regular maps of complex algebraic varieties, to finite dimensional vector spaces (over our arbitrary field) and affine maps between these. On the other hand, there are surjunctive categories which have "SA"IP but not DIP. For example the category of closed topological manifolds (or, more generally, of pseudomanifolds, which includes real analytic spaces) is surjunctive : *no closed manifold $\underline{X}$ can be strictly embedded into itself.* These manifolds, being compact, also satisfy "SA"IP and so residually finite groups Γ are surjunctive relative to this category, i.e. every procontinuous Γ-equivariant map

$f : X^\Gamma \to \underline{X}^\Gamma$ is surjunctive and similar results hold true for *continuous* maps with respect to the *ordinary product topology* in X^Γ (see [Gro]$_{\text{TIDS}}$).

4.K. DIP and "SA"IP for projective limits. If some category (of "varieties" and "regular maps") satisfies "SA"IP then clearly, the same remains true for the corresponding category built of "proalgebraic" spaces and "proregular" maps in-so-far as our projective limits are taken over countable directed sets. The same applies to "SA"IP where one should track down the definitions slightly more carefully (hoping they were set up right). Here is a kind of a corollary.

Let Γ_1 and Γ_2 be groups which are surjunctive relative to every surjunctive category satisfying "SA"IP *and/or* DIP. *Then $\Gamma = \Gamma_1 \times \Gamma_2$ is also surjunctive relative to this class of categories.*

Unfortunately, I see no meaningful application. Yet one may try to generalize this to (more) general group extensions $1 \to \Gamma_j \to \Gamma \to \Gamma_2 \to 1$ where such a result may be useful.

§ 5 Reduction modulo a prime in finite and infinite dimension.

We start with a brief recollection of the language of the first order theories.

5.A. A first order structure. Such a structure on a set A is given, by definition, by a sequence of *relations* $R_1, R_2, \ldots$ in variables $a, b, c \ldots \in A$, say R_i is a relation in n_i variables $a_1, a_2, \ldots, a_{n_i} \in A$. Here "relation" means either a subset in A^{n_i} or, better, a $\{0,1\}$-function on A^{n_i} where $R(a_1, \ldots, a_{n_i}) = 1$ is interpreted as "R is satisfied by $a_1, \ldots, a_{n_i}$" or as "being true" while $R = 0$, means "not satisfied" or "false".

5.B. Example of groups. The basic (algebraic) relations of the group theory are $R_1(a,b,c) = \{ab = c\}$ and $R_2(ab) = \{a = b^{-1}\}$. Thus the relation $R_1 : A^3 \to \{0,1\}$ define the multiplication table on A. Then one can derive more complicated relations using conjunction and disjunction. For example, $\{abc = d\}$ is expressible as $\{(ab = e_1) \wedge (e_2 c = d) \wedge (e_1 = e_2)\}$. Thus one arrives at all algebraic relations like $\{(a^2 b^3 c^{-1} d^{-3} a^5 = bcd) \wedge \cdots\}$.

All of the above are called *quantifier free* relations. Now, bringing in quantifiers, we may have something like $R = \{\forall x \exists y, s.t.\ x = y^{-1}\}$. We think of this as a relation of zero number of variables, and it happens to be true for all groups A. But the relation $\{\forall x \exists y, s.t.\ y^2 = x\}$ is true for $A = \mathbb{R}$ but not for $A = \mathbb{Z}$.

To see the formalism clearer one needs a more complicated example, such as $R = R(a,b) = \{\forall x \exists y, s.t.\ x^2 a = b y^2\}$. So this $R \subset A \times A$ consists of the pairs (a,b), where for every $x \in A$ one can find $y \in A$, such that $x^2 a = b y^2$.

Recall, that the variables $a, b, c, \ldots$ here are called *free* and those attached to quantifiers, $x, y, \ldots$ are called *bound*. Notice, that given any relation $R(a, b, c, d, e, \ldots)$ (where we only indicate free variables) we can form new relations bounding (some of) these variables, e.g. by taking $R_\bullet(d, e, \ldots) = \forall a\ \exists b\ \forall c\ R(a, b, c, d, e, \ldots)$. (This is analogous to integration where, for example, $\int(ax + by + cx^2 + dy^3 + ex + \cdots)\mathbf{d}x\mathbf{d}y$ has x and y as "bound" variables and $a, b, c, d, e, \ldots$ are "free". Then one can "bound" a, b, c by further integrating over $\mathbf{d}a\mathbf{d}b\mathbf{d}c$. Similarly, one "bounds" variables in analysis by taking max and min over them, as in $\max_x \min_y (ax^2 + by + cx^2 \ldots)$.

5.C. Theories and models. A *first order theory* is determined by some basic relations which are given some names. For example "the product relation" is the name for what we have in groups. Another name for this is $\{ab = c\}$, where a, b and c here are not interpreted yet as elements of any set, but rather as letters similar to $p, r, o \ldots$ in "product". In general, we just name the basic relations by some letters $R_1, R_2, \ldots$ etc. In our examples we have finitely many of these basic relations, but we can generate further relations using conjunction and disjunction as earlier. Furthermore, we can add quantifiers and have the full set of relations syntactically generated by R_1, R_2 etc. such as

$$\{\exists x \forall y, s.t.\ R_1(x, y, a, b, c) \vee R_2(x, d)\}.$$

Next a *model of a theory* given by relations $R_1, R_2 \ldots$ is a set A with some relations in the earlier sense bearing the names of R_1, R_2 etc. For example, a *model of the semigroup theory*, or just a *semigroup* is a set A with a ternary relation called "product relation" : $A^3 \to \{0, 1\}$ or $\{ab = c\} \subset A^3$ or anything you want, say just plain R, or R_{product}. What is important is the possibility to compare this product for different sets A and eventually to define various kinds of approximation of one model of a given theory by a family of such models.

5.D. Initial approximation revisited. Let $R_1, R_2, \ldots$ be a sequence of relations in a given theory $\mathcal{F}$ which include the basic relations, take a model A of $\mathcal{T}$ and a finite subset $D = \{a, \ldots, a_n\} \subset A$. Now, given two such models with D's, say $A = (A, D)$ and $A' = (A', D')$ for $D' = \{a'_1, \ldots, a'_n\}$, we look at the maximal i, such that the relations $R_1, R_2, \ldots, R_i$ hold true in A with respect to the variables $a_i \in D \subset A$, if and only if the corresponding relations hold true for a'_i. Then we set

$$\text{"dist"}(A, A') = 2^{-i}$$

for this maximal i. It is not truly a distance, since "dist" $= 0$ does not necessarily imply $A = A'$ but it certainly looks very much as a distance in all other respects.

5.D$'$. Example : Convergence for groups. If $\mathcal{F}$ is the group theory and $R_1, R_2 \ldots$ is the sequence of *all quantifier free* relations, then the above distance is essentially the same we had in 4.G, where "essentially" means the two metrics lead to the same notion of approximation.

Remark. One can refine the above metric by using an exhaustion of A by subsets D_j and defining "dist$''(A, A')$ with respect to such exhaustions as 2^{-m} for the maximal $m \underset{\text{def}}{=} \min(i, j)$, such that the first i relations hold true for the variables from the first j subsets $D_1, \ldots, D_j$ in A and $D'_j, \ldots, D'_j$ in A' correspondingly.

5.E. Syntactic distance. Two models of a theory are called *(elementary) equivalent* if they have identical sets of true formulas without free variables.

Example : Lefschetz principle. *Every two algebraically closed fields of the same characteristic are equivalent.*

In other words, every algebraically geometric statement which is true in one field K is true in all of them.

Warning. Be careful, that your statement is expressible in the language of the first order theory of fields. For example the statement : "if a function $f : K \times K \to K$ is a polynomial in each variable then it is a polynomial" is true (and rather easy to prove) for $K = \mathbb{C}$ but not for $K = \overline{\mathbb{Q}}$. What is wrong here is speaking of "arbitrary functions on $K \times K$" which are not expressible in the first order language of fields.

Next, for a given theory $\mathcal{F}$, we enumerate all relations without free quantifiers and define dist(A, A') between the equivalence classes A and A' of models of our theories. That is

$$\text{dist}(A, A') = \text{"dist"}(\tilde{A}, \tilde{A}')$$

for some representatives $\tilde{A}$ and $\tilde{A}'$ of A and A' (where the implied set D is empty).

5.E$'$. Extended Lefschetz principle. *Let K_ν be a sequence of equivalence classes of algebraically closed fields of finite characteristics converging to infinity. Then* dist$(K, K_\nu) \to 0$ *for every algebraically closed field K of zero characteristic.*

This is standard (see [HML], Ch. 1, Proposition 2.8, for instance) and rather obvious. The idea is that every finite sequence of relations $R_1, R_2, \ldots, R_i$ contains $\{\forall x,\ px = 0\}$ only for finitely many primes $p = p_i$ serving as characteristics of K_i.

5.E$''$. Remark. We formulated the extended Lefschetz principle in "geometric" terms. Algebraically speaking, it says that

if a relation R without free variables holds true for a sequence of algebraically closed fields with characteristics $\to \infty$ then it holds true for all algebraically closed fields of characteristic zero. Conversely, the validity of an R for a single algebraically closed field of characteristic zero, implies that for all algebraically closed fields of characteristics $\geq p_0 = p_0(R)$.

5.F. Proof of Ax' theorem. Start with polynomial maps $f : \mathbb{C}^n \to \mathbb{C}^n$. Clearly, the statements

$$R_{\text{sur}}(f) = \text{"}f \text{ is onto" } \textit{and}$$

$$R_{\text{inj}}(f) = \text{"}f \text{ one-to-one"}$$

are expressible in the first order language with the coefficients of f serving as free variables and the variables $x \in \mathbb{C}^n$ being bound variables. For example for $f = ax^2 + bx + c$ being onto is expressible by $\{\forall y \exists x, s.t. ax^2 + bx + c = y\}$. Then, the surjunctivity relation for f, i.e. $R_\bullet(f) = R_{sur}(f) \vee R^\perp_{inj}(f)$, where $R^\perp$ means negation of R is also of the same nature. On the other hand, the desired relation $\forall f R_\bullet f$ is not of the first order as we are not allowed to quantify over polynomials, only over variables. But every f of a given degree $\leq d$ is given by a string of variables that are the coefficients of f and so $\forall f R_\bullet(f)$ is expressible by a *sequence* of first order formulas without free variables, namely $\forall f_d R_\bullet(f_d)$ for f_d referring to polynomials of degree $\leq d$.

Now the general case of the Ax theorem reduces to the case of the fields $\overline{\mathbb{F}}_p = \cup F_{p^i}$ where the surjunctivity is obvious (compare 4.A).

5.F′. Remark on the role of finite fields. Not every R which holds for all finite fields is necessarily held for $\mathbb{C}$. (For example onto implies one-to-one for finite fields but not for all infinite fields K.) However, this is true for all K which can be approximated by $\mathbb{F}_{p^i}$ for our metric dist. These are the fields which are equivalent (e.g. isomorphic) to ***ultraproducts of finite fields*** (see $[\mathrm{Ax}]_1$ and [HML], Ch 3).

What is special about the surjunctivity relation $\{\forall f R_\bullet(f)\}$ is the stability of this under increasing union of fields. Thus all fields which are elementary equivalent to unions of ultraproducts of finite fields enjoy the surjunctivity of polynomial endomorphisms.

5.F″. Ax' theorem for constructible maps. We have proved surjunctivity so far only for polynomial maps $K^n \to K^n$ for an algebraically closed field. But the same argument applies to all constructible selfmaps f of constructible subsets $X \subset K^N$. Actually, every such X and the map $f : X \to X$ are defined by a finite set of first order formulas and so the extended Lefschetz principle reduces everything to $\overline{\mathbb{F}}_p$. Then a simple argument shows that f is given by a collection of rational maps, each mapping some stratum of X to X, and so our f maps $X(\mathbb{F}_{p^i})$ into itself for all sufficiently large i. Hence f is surjunctive over $X(\mathbb{F}_{p^i})$ and the extended Lefschetz principle applies. Notice, that this version of Ax' theorem yields all surjunctivity statements over $\mathbb{C}$ claimed in §3 for regular, constructible and subconstructible maps.

5.G. Surjunctivity for families of maps (compare $[\mathrm{Ax}]_2$). Let X_b be a constructible family of constructible sets, i.e. $X_b = \pi^{-1}(b) \in X$ for a constructible map $\pi : X \to B$. Then we say that a map $f : X \to X$ sending each fiber X_b to $X_{b'}$, for $b' = b'(b) \in B$, is *one fiber surjunctive* if $\exists b \in B$, such that the map $f : X_b \to X_{b'}$ is surjunctive.

If the underlying field is algebraically closed, then every constructible f is one fiber surjunctive.

This is obvious by the above proof of Ax' theorem.

5.H. Maximal and minimal fibers. We want to show, that the above family X_b contains a maximal fiber $X_{\max} = X_{b_0}$ which admits no strict embedding f into any other fiber X_b and similarly, we look for a minimal fiber $X_{\min}$ receiving no strict embeddings from X_b, $\forall b \in B$ (see 3.H). But now watch out : the relation $\{\exists b_0 \forall f \ldots\}$ has non-first order "$\forall f$" under the existence quantifier and it cannot (at least not obviously so) be reduced to first order formulas of the field theory. But if we limit f to maps defined by polynomials of degree $\leq d$, or, equivalently, defined by first order formulas of size $\leq d$, we have a bona fide first order relation $\{\exists b_0 \forall f_d \ldots\}$ to which the basic principle applies. Furthermore, our one fiber surjunctivity relation is stable under union of fields, at least in the case we need, i.e. for the field $K = \overline{\mathbb{F}}_p = \underset{i}{\cup} \mathbb{F}_{p^i}$. To see this consider the set F_d of all strict embeddings between the fibers $X_b \subset X$ defined via polynomials of degrees $\leq d$. This F_d appears as a constructible set in some $\overline{\mathbb{F}}_p^N$ and we have the tautological map defined on $X' \times F_d$ for $X' = \pi^{-1}(B')$, where $B' \subset B$ corresponds to all non-maximal fibers, by $\tau : (x, f) \mapsto f(x) \in X \times F_d$. Now non-existence of $X_{\max}$ amounts to $B' = B$ which gives us an endomorphism of $X \times F_d$ which is *not* one fiber surjunctive. Similarly, non-existence of the minimal fiber leads to the same conclusion for the action of τ on the set $\tau^{-1}(X'')$ where $X'' = \pi^{-1}(B'')$ for $B'' \subset B$ consisting of all non-minimal fibers. Thus we prove the validity of

$$\{\exists b_{\max} \forall f_d \ldots\} \text{ and } \{\exists b_{\min} \forall f_d \ldots\}$$

for all algebraically closed fields K. What remains, is to satisfy these relations for infinitely many d's which amounts to taking intersections of subsets of b's satisfying $\{b \in B | \forall f_d \ldots\}$ for all d. But we know, that intersections of non-empty chains of constructible subsets are non-empty for *uncountable algebraically closed fields* and thus we prove the existence of X_{max} and X_{min} for such fields. Similarly, we prove, for example, the existence of $\max_c \min_b X_{bc}$ claimed in 3.H, i.e. the relations $\{\exists b_0, c_0, \forall b, \forall f : X_{bc_0} \to X_{b_0 c_0}$ is surjunctive and $\forall c \exists b$ such that $\forall f : X_{b_0 c_0} \to X_{bc}$ is surjunctive $\}$ for all uncountable algebraically closed fields (I did not look for counterexamples for countable fields). Then one proceeds with minmaxmin, maxminmaxmin, etc. In all cases, one first limits oneself to the sets F_d of strict embeddings between the fibers of degrees $\leq d$ and shows that these can not be too large over our parameter space $B \times C \times D \times \ldots$, where the bound comes from what we have for families of finite sets. In fact the underlying idea comes from varieties X defined over $\overline{\mathbb{Q}}$. These can be reduced $\bmod p$ for all sufficiently large $p \geq p_0(X)$. Then we can assign the numbers $c_{p,i}(X) = \operatorname{card} X(\mathbb{F}_{p^i})$ for all large p and i measuring the size of X in some sense. For example, if $X \underset{\neq}{\subset} X'$ then $c_{p_i}(X) < c_{p_i}(X')$ for all sufficiently large p and i.

Sobering remark. All this may look quite interesting but one should keep in mind that

(1) The existence of a strict embedding $X \underset{\neq}{\subset} X'$ is a strong condition. No surprise there are so few of them !

(2) *Constructible order relations* on algebraic varieties are rare beasts with a rather primitive anatomy (see Observation below) allowing the reduction of all minmax properties back to the Ax theorem.

Observation. Consider an order relation on B, denoted $a \succ b$, such that the set of pairs satisfying $a \succ b$, denoted $\{\succ\} \subset B \times B$ is constructible in $B \times B$. Denote by $B_i \subset B$ the subset of those $b \in B$ for which there exists a chain $a_1 \succ a_2 \succ \ldots \succ a_i \succ b$. Clearly, this is a constructible subset in B (as we assume here the underlying field is algebraically closed.)

If i is sufficiently large, depending on the relation, then there exists a Zariski dense open subset $U \subset B_i$ and a constructible equivalence relation, say $\{\sim\} \subset U \times U$ which equals to our order on U, i.e. $\{\sim\} = \{\succ\} \cap U \times U \subset B_i \times B_i$.

Proof. There exists, by the Hilbert theorem, an i, such that $B_{i+1} \subset B_i$ is Zariski dense in B_i. This is our i with $U \subset B_i$ being the maximal open subset such that $U \times U$ is contained in the intersection of $\{\succ\}$ and $\{\prec\}$ in $B_i \times B_i$, where $\{\prec\}$ is obtained from $\{\succ\}$ by the involution $(a,b) \to (b,a)$ of $B_i \times B_i$).

Corollary. *If the relation admits arbitrary long chains $a_1 \succ a_2 \succ \ldots \succ a_i$ then there exists a point $b \in B$, such that $b \succ b$.*

Remark. The above can be applied, besides the relation $a \underset{d}{\succ} b$ given by (the existence of) strict embeddings $X_a \to X_b$ of degree $\leq d$, to surjective non-injective maps $X_a \to X_b$ or to equidimensional maps $X_a \to X_b$ of topological degree ≥ 2, or to surjective maps from subsets, $X_a \underset{\neq}{\supset} Y_a \to X_b$, etc., where one should bound the algebraic degrees of our maps by d in order to keep the order relation $a \underset{d}{\succ} b$ constructible. Eventually $d \to \infty$ and we arrive at $b \in B$ with $b \succ b$ by intersecting countably many constructible sets as we did earlier.

5.I. Initialization of surjectivity for proregular maps. Given a proregular map $f : X \to Y = \underset{\leftarrow}{\lim} \, (f_i : X_i \to Y_j)$, $i, j \in \mathbb{N}$, we want to express the onto property, as well as one-to-one and eventually surjunctivity, by a sequence of first order relations. Here is such a sequence of relations.

Initial surjectivity. A proregular map f, or rather the projective system of maps defining f, is called (i, j^+)-*surjective* if the image of $f_i : X_i \to Y_j$ contains the image of the projection $\pi_{j^+,j} : Y_{j^+} \to Y_j$ where j^+ is assumed $\geq j = j(i)$. Then f is called *initially surjective* if $\forall i \exists j^+$, s.t. f is (i, j^+)-surjective.

Notice that the above makes sense for a general "proregular" map. The particular structure of X_i, Y_j and f_i is irrelevant at the moment. Also observe the following trivial implication

$$\textit{surjectivity} \implies \textit{initial surjectivity}$$

if the projections $Y_j \to Y_{j-1}$ are onto for all j. Moreover

$$\textit{density of } f(X) \subset Y \Rightarrow \textit{initial surjectivity} \qquad (\star)$$

provided the projective system X_j is image stable according to the following definition.

Image stability. Call a projective system $\{X_i\}$ *$(i_0, i_0+\ell)$-image stable* if $\pi_{i_0+k,i_0}(X_{i_0+k}) = \pi_{i_0+\ell,i_0}(X_{i_0+\ell}) \subset X_i$ for all $k \geq \ell$. Then call $\{X_i\}$ *image stable* if $\forall i_0 \exists \ell$ s.t. it is $(i_0, i_0+\ell)$-image stable.

The above $(\star)$ obviously holds in every category of "proregular" maps with "density" referring to the prodiscrete topology in Y. What is slightly less trivial is *the validity of $(\star)$ for proalgebraic varieties over uncountable algebraically closed field (and saturated models, in general) without the image stability assumption.*

Indeed, if $f_i(X_i) \subset Y_j$ contains the intersection of the images, $\bigcap_{j' \geq j} \pi_{j',j}(Y_{j'}) \subset Y_j$, then $\exists j_0'$, s.t. $f(X_i) \supset \pi_{j_0',i'}(X'_{j_0'})$. This follows from the countable intersection property for constructible sets (see 4.C$''$).

Also observe the opposite implication.

$$\text{initial surjectivity} \Rightarrow \text{surjectivity}, \qquad (\star')$$

provided the category in question satisfies "SA" IP.

In fact, this follows by our proof of 4.D.

The spaces $X_\bullet, X_+$ and a topological interpretation of initial surjectivity. Denote by $X_\bullet$ the disjoint union of the spaces X_i with the metric $|x-x'|$ defined as follows. Add a dummy element, say 0, to each X_i and embed the extended X_i to their projective limit in the obvious way,

$$x_i \mapsto (x_i, 0, 0, \ldots) \in X \sqcup 0 = \lim_{\leftarrow} (X_i \sqcup 0).$$

Then the prodiscrete metric on $X \sqcup 0$ restricts to all $X_i \subset X \sqcup 0$ and is denoted by $|x-x'|$ as earlier. Clearly, $X \subset X \sqcup 0$ equals the set of the limits of all convergent sequences $\{x_i \in X_i\}$. Alternatively one could introduce $X_+ =$ (metric completion of $X_\bullet$) for the metric $|x-x'|$ on $X_\bullet$ and then *define* the projective limit $X = X_\infty = \lim_{\leftarrow} X_i$ as the complement $X_+ \setminus X_\bullet$.

Now we observe that

(a) X_{i+1} is contained in the ϵ-neighbourhood of $X_i \subset X_\bullet$ for $\epsilon = 2^{-i}$.

(b) the map $f_\bullet : X_\bullet \to Y_\bullet$ associated to a projective system of maps $\{f_i : X_i \to Y_j\}$ is uniformly continuous for our metrics on $X_\bullet$ and on $Y_\bullet$.

(c) a projective system of maps $\{f_i : X_i \to Y_j\}$ is initially surjective iff for every $\epsilon > 0$ there exists j, such that Y_j is contained in the ϵ-neighbourhood of $f(X_\bullet) \subset Y_\bullet$

(d) a projective system $\{X_i\}$ is image stable iff $X \subset X_+$ equals *the Hausdorff limit* of the subsets $X_i \subset X_\bullet \subset X_+$ for $i \to \infty$, which amounts in the present case to the following property : for every $\epsilon > 0$ $\exists i$, such that X_i is contained in the ϵ-neighbourhood of X_+.

5.I′. Reduction of surjectivity modulo a prime $p \to \infty$. Suppose our X, Y and f are defined over $\mathbb{Z}$ so that we may speak of reduction modulo p and define $f(\mathbb{F}_{p^\nu}) : X(\mathbb{F}_{p^\nu}) \to Y'(\mathbb{F}_{p^\nu})$ for all p and ν. In fact we shall only need $p \geq p_0$ and $\nu \geq \nu_0$ and so all we say will equally apply to provarieties and maps defined over a number field.

If there exists a sequence of finite fields $K_\nu = \mathbb{F}_{p_\nu^\nu}$ for infinitely many positive integers $\nu \in \{\nu\} = \{\nu_1 < \nu_2 < \ldots\}$ and $p_\nu \to \infty$ for $\nu \to \infty$, such that the map $f(K_\nu)$ are surjective, the $f(\mathbb{C})$ is also surjective, provided the projective systems $\{Y_j(K_\nu)\}$ are uniformly image stable, i.e. $\forall_{j_0} \exists_\ell$ s.t. $\{Y_j(K_\nu)\}$ are $(j_0, j_0 + \ell)$ *image stable for all ν.*

Proof. The surjectivity and the uniform image stability obviously imply *uniform initial surjectivity* of the projective system of maps defining $f(K_\nu)$, i.e. $\forall i \exists j^+$ such that $f(K_\nu)$ is (i, j^+)-surjective for all v. This implies, that $f(\overline{\mathbb{F}}_{p_\nu^\nu})$ is also (i, j^+)-surjective, at least for all large ν by the following

Trivial Lemma. *Let $f : A \to B$ and $\pi : B^+ \to B$ be morphisms of varieties over $\mathbb{Z}$ (or over any number field for this matter). Then for all sufficiently large p and ν, the inclusion of the images $f(A(\mathbb{F}_{p^\nu}) \supset \pi(B^+(\mathbb{F}_{p^\nu}))$ implies that $f(A(\overline{\mathbb{F}}_p)) \supset \pi(B^+((\overline{\mathbb{F}}_p))$.*

Now $f(\mathbb{C})$ is also (i, j^+)-surjective by the extended Lefchetz Principle (see ...) and $(\star')$ applies. Q.E.D.

Counterexample. One can not go by without the uniform image stability condition. Indeed look at the map $z \mapsto \pi_i(z) = z^i(z^i - 1)$. This is onto over $K = \mathbb{C}$ but if K is a finite field with q elements, $q = (\text{prime})^\nu$, and $i = q - 1$, then π_i maps all of K to $0 \in K$. Thus the projective limit of the sequence $K \underset{\pi_1}{\leftarrow} K \underset{\pi_2}{\leftarrow} \ldots \underset{\pi_i}{\leftarrow} K \underset{\pi_{i+1}}{\leftarrow} \ldots$ equals $\{0\}$. It follows that the obvious map of $X = \{0\}$, viewed as the projective limit of the sequence $\{0\} \leftarrow \{0\} \leftarrow \ldots$, to $Y = \lim(K, \pi_i)$ is surjective over every finite field K but it is not surjective over $K = \mathbb{C}$.

One can make the above even more convincing with $\xi_j : K^2 \to K$ defined by $\xi_j : (z_1, z_2) \mapsto \pi_j(z_1) + (j+1)g_j(z_1, z_2)$ where g_j is a generic polynomial of degree $2j$ without constant term with integer coefficients. This ξ_i is *surjective* over $\mathbb{C}$ and has (unlike ψ_j) *irreducible fibers*, while viewed over the finite field K with q-elements it maps K^2 to $\{0\}$ for $j = q - 1$. Now we take $\varphi_1 = \xi_1 : K^2 \to K$, then $\varphi_2 = \xi_2 \times \xi_2 : K^4 \to K^2$, next $\varphi_3 = (\xi_3 \times \xi_3) \times (\xi_3 \times \xi_3) : K^8 \to K^4$ and so on. All these maps are surjective for $K = \mathbb{C}$ with irreducible fibers, while the projective limit of this system $Y_j = K^{2^j}$ *over each finite field* equals $\{0\}$. Thus the obvious map $f : \{0\} \to \underset{\leftarrow}{\lim}\, Y_j$ is surjective over all finite fields but not over $\mathbb{C}$.

5.J. Uniform irreducibility of fibers. The image stability over K boils down to showing that certan regular maps, say $Y'_{j+1} \to Y'_j$ are all *onto* over K, i.e. have non-empty fibers over all K-points in Y'_j for $j = 1, 2, \ldots$. If K is a finite field, then non-emptiness of a variety (fiber) A can be derived from non-emptiness and irreducibility of A over $\overline{K}$ by the corollary due to Lang of the celebrated theorem of A. Weil. On the other hand, if some

$\mathbb{Z}$-variety is irreducible over $\mathbb{C}$, then it is also irreducible over all fields of sufficiently large characteristic $\kappa = \kappa(A)$. What, in general, is missing, is a bound on this κ independent of j and for this we need some uniformity of the $\mathbb{C}$-irreducibility with respect to j. Actually, when we turn to our applications (see § 7) we shall be able to work with only finitely many j's at a time. Mor generally we could require the maps $Y'_{j+1} \to Y'_j$, viewed as fibrations, to be induced from a single fibration, $\Pi : Z \to B$ (possibly with non-connected base) with irreducible fibers. Then this $Z \to B$ will have irreducible fibers over the fields with characteristic $\geq \kappa \geq \kappa(\Pi : Z \to B)$ and then this irreducibility will be transmitted to all $Y'_{j+1} \to Y'_j$ induced by regular maps $Y'_{j+1} \to B$.

Warning. Consider $Z \subset \mathbb{C}^3$ given by the equation $z_1 z_2 = b$ and non-equation $b \neq 0$. This obviously fibers over $B = \mathbb{C}\backslash\{0\}$ with irreducible fibers. And this irreducibility remains intact over any field of arbitrary characteristic. However, if we take some *integer point* $b_0 \in \mathbb{C}\backslash\{0\}$ then the fiber over this point, i.e. $z_1 z_2 = b_0$, becomes reducible over each field K where the characteristic divides b_0. This happens because the embedding $\{\bullet\} \mapsto b_0 \in \mathbb{C}\backslash\{0\}$ does not define any embedding $\{\bullet\} \mapsto K\backslash\{0\}$ if the characteristic of K divides b_0. Yet everything works fine for affine and/or projective varieties defined over $\mathbb{Z}$, where all $\mathbb{Z}$-morphisms can be reduced modulo each prime. For example, if we take the affine realization of $B = \mathbb{C}\backslash\{0\}$ as $B^\bullet \subset \mathbb{C}^2$ given by the equation $bc = 1$, then the (z_1, z_2)-fibers of $Z^\bullet = \{z_1 z_2 = b,\ bc = 1\}$ over $B^\bullet$ retain irrducibility over all $\mathbb{Z}$-points of Z reduced modulo any prime. In fact, this $\mathbb{Z}^\bullet$ has only 4 integer points, all with the coordinates ± 1.

Example. Consider $X \subset (\mathbb{C}^n)^\infty$ given by equations $\{g_i(x_i, x_{i+1}) = 0\}$ where each variable $x_1, x_2, \ldots, x_i, x_{i+1}, \ldots$ runs over $\mathbb{C}^n$ and g_i are polynomial (maps) with integer coefficients $\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^k$. Then the truncated subvarieties $X_i \subset (\mathbb{C}^n)^i$ form a projective system where, each projection $X_{i+1} \to X_i$ can be obviously induced from the projections of the variety

$$Z_i = \{x_i, x_{i+1} \mid g_i(x_i, x_{i+1}) = 0\} \subset \mathbb{C}^n \times \mathbb{C}^n$$

to $B = \mathbb{C}^n$, $\Pi_i : (x_i, x_{i+1}) \mapsto x_i$. Now, if these projections $Z_i \to B$ have irreducible fibers, and if the degrees and the coefficients of all polynomials p_i are bounded by a constant independent of i (e.g. all g_i are mutually equal), then any propolynomial endomorphism f of X with integer coefficients which is onto on $\mathbb{F}_{p^\nu}$-points of X, (i.e. $F(\mathbb{F}_{p^\nu}) : X(\mathbb{F}_{p^\nu}) \to X(\mathbb{F}_{p^\nu})$ is onto) for $\nu \to \infty$ and $p_\nu \to \infty$ is also onto on $\mathbb{C}$-points of X, as a simple argument using the Lang-Weil theorm shows.

Remark. We do not formulate the most general notion of uniform irreducibility. When we refer to it we mean some condition on the fibers of our maps over $\mathbb{C}$ which suffices for irreducibility over $\overline{\mathbb{F}}_p$ for large p when we invoke the extended Lefschetz principle. Recall, what we eventually need is the image stability of our projective system $\{Y_j\}$ on $\mathbb{F}_{p^\nu}$-points for large p and ν and this property is trivially satisfied for the projective system of the Cartesian powers, $Y_j = \underline{Y}^{N_j}$, regardless of irreducibility of $\underline{Y}$. So we sometimes include this case in the "uniform fiber irreducibility" category.

5.K. $\mathbb{Z}$-families over proconstructible spaces. It may happen we have to study not an individual proregular map $f : X \to Y$ but a family of those, say $f_\mu : X \to Y$ where μ runs over some proconstructible moduli space defined over $\mathbb{Z}$. In other words we have a class of maps f defined by infinitely many first order formulas of the field theory where the spaces X and Y may also be taken from some classes described by such formulas. For example, one may look at polynomial endomorphisms of X from the above example where the admissible X's are those where all g_i are mutually equal and where the admissible f's are given by (sequences of) polynomials, each having exactly 55 non-zero coefficients. The question is whether the "onto" property for a given map from this class on K_ν-points for all our K_ν implies that for $\mathbb{C}$. This can be reduced to a single map of larger spaces, say $\mathcal{F} : \mathcal{X} \to \mathcal{X}'$, where $\mathcal{X}$ and $\mathcal{X}'$ are fibered over some proconstructible moduli space of spaces and maps (compare 5.G) and $\mathcal{F}$ is fiber preserving. So all we have done for individual maps f applies to $\mathcal{F} = \{f_\mu\}$, where some caution is needed in the treatment of uniform irreducibility of fibers of the maps in the projective system defining $\mathcal{X}'$. We shall not persue this suject matter in full generality but shall treat it on the case by case basis.

5.L. Injectivity and initial injectivity. We start with a proregular map $f : X \to Y$ between proalgebraic varieties defined over $\mathbb{Z}$ and ask ourselves if injectivity of f on $\mathbb{C}$-points implies that for sufficiently many finite fields $K_\nu = \mathbb{F}_{p_\nu^\nu}$, i.e. for finite fields of arbitrarily large characteristics p_ν and degrees ν over the respective prime fields $\mathbb{F}_{p_\nu}$.

Definition. Call a projective system of maps $\{f_i : X_i \to Y_i\}$ (i_0, i_+)*-injective* for given $i_0 = 1, \dots,$ and $i_+ \geq i_0$, if $\pi_{i_+,i_0}(x) \neq \pi_{i_+,i_0}(x') \Rightarrow f_{i_+}(x) \neq f_{i_+}(x')$ for all $x, x' \in X_i$. Then call $\{f_i\}$ *initially injective* if $\forall i_0 \exists i_+$ s.t. $\{f_i\}$ is (i_0, i_+)-injective.

This is (obviously) equivalent to the *eventual uniform injectivity* of the associated map $f_\bullet : X_\bullet \to Y_\bullet$ which means that for every $\epsilon > 0$ $\exists \delta > 0$ and $i = 1, \dots,$ such that

$$|x - x'| \geq \epsilon \Rightarrow |f_\bullet(x) - f_\bullet(x')| \geq \delta$$

for all $x, x' \in X_\bullet \setminus \bigcup_{j=1}^{i-1} X_j$.

It follows that

$$\textit{uniform injectivity of } f = \lim_{\leftarrow} f_i \Rightarrow \textit{ initial injectivity of } \{f_i\} \qquad (*)$$

provided the projective system $\{X_i\}$ *is image stable* (see 5.I).

Corollary. *If our category has DIP, then*

$$\textit{injectivity of } f \Rightarrow \textit{ initial injectivity of } \{f_i\}. \qquad (**)$$

In particular $(**)$ *holds for proregular maps over uncountable algebraically closed fields.*

5.L′ When does "(i_o, i_+)-injective" imply "injective". It is obvious that initial injectivity implies injectivity. But what we need is the implication

$$(i_0, i_+)\text{-injective} \Rightarrow \text{injective} \qquad (*')$$

for fixed but possibly large i_0 and i_+. It is immediate that (i_0, i_+)-injectivity yields the following "ϵ_0-injectivity" of $f : X \to Y$,

$$|x - x'| > \epsilon_0 = 2^{-i_0} \Rightarrow f(x) \neq f(x') \text{ for all } x, x' \in X.$$

Then we recall (see 4.H) that this is sufficient for injectivity in the presence of an expansive action on X so we arrive at the following

Lemma. *If X is endowed with uniformly continuous and expansive action of a group Γ then there exists i_0, such that for every $i_+ \geq i_0$ (i_0, i_+)-injectivity implies injectivity for all Γ-equivariant uniformly continuous map $f : X \to Y$.*

This applies, in particular, to Γ-invariant "subvarieties" in $X = \underline{X}^\Gamma$ with the shift action of Γ on X.

Remark. The above remains true for certain expansive *orbit structures* on X *not* associated to groups (which are implicitly present in § 7).

5.L$''$. Injectivity reduced modulo p. Let $f : X \to Y$ be a proregular map between proalgebraic varieties, everything defined over $\mathbb{Z}$, such that f is injective on the set $X(\mathbb{C})$ of $\mathbb{C}$-points of X. Then, assuming the projective system X_i defining X is image stable, we conclude that the map f is initially injective and thus (i_0, i_+)-injective for some pairs (i_0, i_+) with $i_0 \to \infty$. Now, the (i_0, i_+)-injectivity is expressible by first order formulas and so the map f is also (i_0, i_+)-injective on $X(\overline{\mathbb{F}}_p)$ for all $p \geq p_0 = p_0(i_+)$, according to the extended Lefschetz principle. Finally, if we are in a situation where (i_0, i_+)-injectivity $\Rightarrow$ injectivity, e.g. for subshifts $X \subset \underline{X}^\Gamma$, we conclude to injectivity of f on $X(\overline{\mathbb{F}}_p)$ and thus to injectivity on $X(\mathbb{F}_{p^\nu})$ for all ν and $p \geq p_0$.

Remarks. The above trivially extends to proconstructible varieties and maps defined over $\mathbb{Z}$ or, more generally, over algebraic number fields. Also, as in the case of surjectivity, one may apply the above discussion to proalgebraic and proconstructible families of maps defined over $\mathbb{Z}$ (or more generally for classes of varieties and maps defined by first order formulas).

5.M. Reducing surjunctivity modulo p. Given an injective map $f : X \to Y$ as above, we want to establish its surjectivity on $\mathbb{C}$-points provided we know that injectivity $\Rightarrow$ surjectivity for K_ν-points for $K_\nu = \mathbb{F}_{p_\nu^\nu}$ for a sequence $\nu, p^\nu \to \infty$. This, in fact, can be done under the following three assumptions.

(1) The system X_i defining X is image stable on $\mathbb{C}$-points of X_i.

(2) There exists i_0, such that (i_0, i_+)-injectivity $\Rightarrow$ injectivity on $\overline{\mathbb{F}}_p$-points of X for each $i_+ > i_0$ and for all $p \geq p_0 = p_0(i_0)$.

(3) The system Y_j is uniformly fiber irreducible over $\mathbb{C}$.

Indeed (1) gives us the implication

$$\text{injectivity} \Rightarrow (i_0, i_+)\text{-injectivity},$$

on $\mathbb{C}$-points of X for all $i = 1, 2...$ and some $i_+ = i_+(i_0)$. Next this passes to $\overline{\mathbb{F}}_p$ by the extended Lefschetz principle and then yields $\overline{\mathbb{F}}_p$-injectivity which trivially implies $\overline{\mathbb{F}}_{p^\nu}$-injectivity for $p > p_0$. The latter injectivity implies $\overline{\mathbb{F}}_{p^\nu}$-surjectivity by our assumption, which then yields initial $\overline{\mathbb{F}}_{p^\nu}$-surjectivity due to (3) as we indicated in 5.J. Finally, this yields initial surjectivity and, hence, surjectivity on the $\mathbb{C}$-points (see 5.I$'$).

5.M$'$. Surjunctivity for families. Often we want to prove surjunctivity for a (pro)regular map f defined over $\mathbb{C}$, rather than over $\mathbb{Z}$ (or any number field for this matter). For example, Ax' theorem claims surjunctivity for *all* $\mathbb{C}$-endomorphisms of algebraic $\mathbb{C}$-varieties. And in our case we may encounter *arbitrary* Γ-equivariant $\mathbb{C}$-proregular self-mappings of $X = \underline{X}^\Gamma$. We handle this matter as earlier by including our map f, and X if necessary, into a family, say $f_b : X_b \to X'_b$, where B is a proalgebraic space defined over $\mathbb{Z}$ (or over a number field) or more geneally, a proconstructible space eventually defined by first order formulae. Thus we have the global mapping $\mathcal{F} = \{f_b\} : \mathcal{X} \to \mathcal{X}'$ for $\mathcal{X} = \bigcup_{b \in B} X_b$ and $\mathcal{X}' = \bigcup_{b \in B} X'_b$,where now all objects, $\mathcal{F}$, $\mathcal{X}$ and $\mathcal{X}'$ can be defined over $\mathbb{Z}$ (e.g. by first order formulas). however, the problem we face now has changed. Instead of the implication "one-to-one $\Rightarrow$ onto" for $\mathcal{F}$, we want to prove this for every f_b-constituent of $\mathcal{F}$ individually. This is done by restricting $\mathcal{F}$ to $\mathcal{Y} \subset \mathcal{X}$ consisting of the union of X_b over the part $A \subset B$, defined by the condition

$$\{b \in A | f_b \text{ one-to-one on } X_b(\mathbb{C})\}.$$

Now, our map $\mathcal{F}$ restricted to $\mathcal{Y}$, say $\mathcal{F} : \mathcal{Y} \to \mathcal{Y}'$ for $\mathcal{Y}' = \bigcup_{b \in A} X'_b$, *is* injective and if we can prove it is surjective, we shall arrive at the desired surjunctivity of the original $f = f_{a_0}$. This works perfectly well in the case of ordinary (finite dimensional) varieties over $\mathbb{C}$, since $A \subset B$ as well as $\mathcal{Y} \subset \mathcal{X}$ and $\mathcal{Y}' \subset \mathcal{X}'$ are defined by first order relations and so both $\mathcal{Y}$ and $\mathcal{Y}'$ are constructible spaces defined over $\mathbb{Q}$ and eventually over $\mathbb{Z}$. But, in general, for proalgebraic varieties and maps, injectivity is not expressible by first order formulas, unless we have injectivity $\leftrightarrow$ initial injectivity. And this implication is ensured, as we know (see 5.L) by the image stability of X. Thus we must chose $\mathcal{Y} \subset X$ and $\mathcal{Y}' \subset \mathcal{X}'$ more carefully, so that all $X_b, b \in A$, retain the needed properties of X, X' and our $f' : X \to X'$, namely (1) and (3) of 5.K. which are given by (infinitely many) first order formulas. And as for (2), this is of a combinatorial nature and must be required independently for all maps under scrutinity. Here is our example where everything goes smoothly along these lines.

5.M$''$. Proof of 2.C$'$ for $\Delta = \Gamma$. Let $f : \underline{X}^\Gamma \to \underline{X}^\Gamma$ be a Γ-equivariant proregular map defined over $\mathbb{C}$ and observe that the above discussion yields the following conclusion.

($\star\star$) *f is surjunctive on $\mathbb{C}$-points of our $X = \underline{X}^\Gamma$, provided every continuous Γ-equivariant selfmapping $S^\Gamma \to S^\Gamma$ is surjunctive for every finite set S.*

In fact, each f is given by a single regular map $\varphi : \underline{X}^d \to \underline{X}$ for $d = \operatorname{card} D$ and $D \subset \Gamma$ being a finite subset (see 4.C). We include $\underline{X}$ into a family $\underline{\mathcal{X}} = \{\underline{X}_b\}$, of varieties over B

where B and $\underline{X}$ live over a number field (which can be reduced further to $\mathbb{Z}$ with the first order language) and we denote by Φ_b the set of all regular maps $\varphi_b : X_b^N \to X_b$ of degree $\leq$ degree (φ). then we take the infinite Cartesian power of $\underline{X}$ over B time $\Phi = \bigcup_{b \in B} \Phi_b$, i.e. $\mathcal{X} = \bigcup_{b \in B} (\underline{X}_b^\Gamma \times \Phi_b)$. This $\mathcal{X}$ is defined over a number field (which can be reduced to $\mathbb{Z}$ in our first order language) and we have a natural proregular map $\mathcal{F} = \{f_b\} : \mathcal{X} \to \mathcal{X}$ also defined over a number field (or $\mathbb{Z}$ if one wishes) which sends each fiber $X_b^\Gamma \times \Phi_b$ into itself. By the map $(x, \varphi_b) \mapsto (f_b(x), \varphi_b)$ where f_b is built out of $\varphi_b : \underline{X}_b^d \to \underline{X}_b$ as usual (see 4.C). Now the surjunctivity discussion from 5.K′. applies and $(\star\star)$ follows.

What remains to prove 2.C′ for $\Delta = \Gamma$ is the following result from symbolic dynamics.

5.M‴. *Let S be a finite set, Γ an amenable group and $X' \underset{\neq}{\subset} S^\Gamma$ be a closed Γ-invariant subset. Then the topological entropy of the Γ-action on X' satisfies the strict inequality*

$$\operatorname{ent}(X' : \Gamma) < \operatorname{ent}(S^\Gamma : \Gamma)(= \log \operatorname{card} S).$$

This is standard for $\Gamma = \mathbb{Z}$ and well known to experts (e.g. to Benjy Weiss) for all amenable Γ (compare 8.D). In fact we shall establish this in a more general situation (without explicitly referring to the entropy) in the course of the proof of the surjunctivity theorem 7.G′ for initially amenable graphs Δ (see 7.L′).

Now surjunctivity of the shift endomorphisms is trivial as a strict embedding of S^Γ to S^Γ would land on a compact, and hence closed, subset X' in S^Γ and we would have $S^\Gamma = X' \underset{\neq}{\subset} X = S^\Gamma$ which contradicts the above strict inequality for the entropy.

Remarks. (a) Our proof of $(\star\star)$ equally applies to Δ with an action of a locally compact group with finitely many orbits as in 2.C′ and the 2.C′ follows with an obvious generalization fo 5.K‴. (See § 7 for a more general results).

(b) surjunctivity of amenable groups implies that for initially subamenable ones (see 4.G′ and 6.E) and so surjunctivity of all proregular (and also proconstructible) Γ-equivariant selfmappings $\underline{X}^\Gamma \to \underline{X}^\Gamma$ is ensured for all initially subamenable groups Γ.

§ 6 Infinite graphs : symetries, amenability, asymptotic dimension.

We collect here basic definitions concerning infinite graphs Δ. Later on we look at projective systems parametrized by Δ (or rather by finite subsets in Δ) and augment the purely graph theoretic properties of Δ by extra data coming from these projective systems.

6.A. Partial symmetries and limits of graphs. From now on Δ denotes a countable, usually infinite *connected*, graph where we do not notationally distinguish between the graph and the set of its vertices denoted $\delta \in \Delta$. This is possible, strictly speaking, only if our graph has *no loops* and an *at most one* edge between *each pair* of vertices. So, to simplify the life, we assume Δ does have these two properties (and we leave to the reader the adjustment needed for general graphs Δ with loops and multiple edges).

We define $\mathrm{dist}_\Delta = \mathrm{dist}(\delta, \delta')$ on Δ as the minimal length of a path of edges joining δ and δ' in Δ. This is an integer valued *metric* on Δ if Δ is *connected*. Otherwise Δ becomes somewhere infinite, namely it gives infinite distance between different connected components but this does not prevent one from using it as a metric on each component. Notice that dist_Δ carries the same information as the graph structure on Δ.

In general, we do not expect Δ to have non-trivial automorphisms, i.e. isometries for the metric dist_Δ, but it may have many *partial* isometries, namely bijective maps between subsets in Δ preserving the metric dist_Δ. For example let us assume that Δ has *bounded valency*, i.e. each vertex $\delta \in \Delta$ has at most d adjacent edges for some $d < \infty$ independent of δ. Then, clearly, for every $r < \infty$ there are at most *finitely many* isometry classes of r-balls in Δ where the r-hall $D = D(\delta, r) \subset \Delta$ around $\delta \in \Delta$ is defined by

$$D = \{\delta' \in \Delta | \, \mathrm{dist}(\delta, \delta') \leq r\}.$$

Thus we have infinitely many mutually isometric r-balls in each *infinite* graph Δ and we are interested in isometries between such balls, denoted $\gamma : D \leftrightarrow D'$.

Such isometries allow one to define limits $\Delta^\bullet$ of a fixed graph Δ with a given sequence of points $\delta_i \in \Delta$, $i = 1, 2, \ldots$ called *markings* in Δ, going to infinity (in Δ) for $i \to \infty$. To construct this $\Delta^\bullet = \lim_{i \to \infty} (\Delta, \delta_i)$ we need isometries between certain balls around δ_i, namely $\gamma_i : D(\delta_i, r_i) \leftrightarrow D(\delta_{i+1}, r_i)$, for some sequence $r_i \to \infty$. We may assume (by restricting some isometries to smaller concentric balls $D(\delta_i, r_i')$ if necessary) that r_i increase with i and then compose each γ_i with the inclusion $D(\delta_{i+1}, r_i) \subset D(\delta_{i+1}, r_{i+1})$. Thus we get a sequence of isometric embeddings, still denoted γ_i,

$$D(\delta_1, r_1) \overset{\gamma_1}{\hookrightarrow} D(\delta_2, r_2) \overset{\gamma_2}{\hookrightarrow} \ldots \overset{\gamma_{i-1}}{\hookrightarrow} D(\delta_i, r_i) \overset{\gamma_i}{\hookrightarrow}$$

and then the union, or rather the inductive limit of these balls, is denoted $\Delta^\bullet = \lim_{i \to \infty} (\Delta, \delta_i)$, or, more precisely, $\lim_{i \to \infty} (\Delta, \delta_i, \gamma_i)$. This limit carries the obvious graph structure or, equivalently, a metric coming from that in the balls $D(\delta_i, r_i)$. (In fact the metric space $\Delta^\bullet$ equals the Hausdorff limit of the marked metric spaces (Δ, δ_i), (see [GLP]). Notice, that if δ_i do not go to infinity, i.e. remain in a finite subset in Δ, then the graph $\Delta^\bullet$ is isomorphic to Δ, but we typically obtain uncountably many graphs $\Delta^\bullet$ corresponding to various sequences $\delta_i \to \infty$ in Δ. We are primerly interested in locally finite graphs Δ where the condition $\delta_i \to \infty$ is equivalent to $\text{dist}(\delta_0, \delta_i) \to \infty$. Observe that not every sequence δ_i gives rise to $\Delta^\bullet$ as we need the above mutually isometric balls around δ_i of radii $\to \infty$ but such sequences are abundant if Δ has bounded valency. In fact, every sequence δ_i in such a Δ obviously admit a subsequence where the required balls exist.

Observe next that every limit graph $\Delta^\bullet$ is *locally isometric* to Δ in the sense that each ball $D_\bullet$ in $\Delta^\bullet$ of radius $r < \infty$ is isometric to some r-ball in Δ. In fact $\Delta^\bullet$ is exhausted by the balls $D(\delta_\bullet, r_i) \subset \Delta^\bullet$ around the distinguished point (marking) corresponding to the sequence $\{\delta_\bullet\}$ and these balls are tautologically isometric to the balls $D(\delta_i, r_i) \subset \Delta$. Conversely, every graph Δ' locally isometric to Δ is (globally) isometric to some limit graph $\Delta^\bullet$ of Δ corresponding to certain δ_i and γ_i. This is obvious. It is equally obvious, that the local isometry is a partial order relation between graphs, written $\Delta' \prec \Delta$, but this is not, in general, an equivalence relation. In fact, there are special graphs Δ, called (dynamically) *minimal*, such that $\Delta' \prec \Delta$ implies $\Delta \prec \Delta'$ and the following fact is standard.

6.A$'$. Minimality Lemma. *Every graph Δ of bounded valency admits a minimal limit graph $\Delta^\bullet$ i.e. a minimal graph $\Delta^\bullet$ satisfying $\Delta^\bullet \prec \Delta$.*

This is proven with Zorn's lemma by a usual compactness argument. In fact, this can also be derived from the existence of a minimal leave saturated subset in a compact foliated space. (Here it is the Hausdorff moduli space of marked graphs Δ with valencies bounded by some constant, see [GLP]). Also recall the following obvious (and well known)

6.A$''$. Criterion for minimality. *A graph Δ of bounded valency is minimal if and only if for very ball D in Δ of radius $r < \infty$ there exists a number $R = R(D) < \infty$, such that every R-ball $D^+ \subset \Delta$ contains an r-ball $D' \subset D^+$ (non-concentric to D^+ in general) isometric to D* (where "only if" does not need "bounded" valency).

6.B. Additional structures on Δ, local order and distinguished isometries. Given an additional structure on Δ, e.g. a coloring of Δ into finitely many colors, we may limit our isometries to these preserving our structure and thus obtain a subset of partial isometries of Δ. A particularly important structure in this respect is *a local order* on Δ, i.e. an ordering of the edges at each vertex $\delta \in \Delta$. If the graph Δ is locally finite, this local order eliminates non-trivial isometries fixing a point in Δ. In fact, there is at most one (if any) isometry between balls $D \leftrightarrow D'$ sending the center of D to that of D' and preserving the local orders in these balls induced from a preassigned local order in Δ. On the other hand, such ordering does not drastically reduce the overall partial symmetry of Δ as every ball of finite radius admits at most *finitely many* different local orders. In particular, if Δ is a locally ordered graph of bounded valency, it still satisfies the following

Precompactness property. *Every infinite sequence of points in Δ contains a subsequence, say $\delta_i \in \Delta$, such that each ball of radius i around δ_i, i.e. $D(\delta_i, i)$ admits an order preserving isometry γ_i to the ball $D(\delta_{i+1}, i)$. In particular, these exists a limit $\Delta^\bullet$ of* *the marked locally ordered graphs* (Δ, δ_i) *and this* $\Delta^\bullet$ *carries a natural (limit) structure of marked locally ordered graph.*

In fact, there are at most finitely many isomorphism classes of ordered balls in Δ which yields the compactness property.

Similarly, the minimality lemma and the minimality criterion trivially generalize to graphs with extra structures where the relevant role of these structures is a limitation on local isometries in Δ. This can be expressed with the following

6B′ Pseudogroups Γ of isometries of graphs Δ. Such a pseudogroup Γ is defined as a distinguished subset of partial isometries of Δ, called Γ-*isometries* and denoted as earlier $\gamma : \Omega \to \Omega'$ for $\gamma \in \Gamma$ and some Ω and Ω' from Δ (depending on γ) where the following four axioms must be satisfied.

(A) Γ contains the identity map $\mathrm{Id}_\Omega : \Omega \to \Omega$ for every $\Omega \subset \Delta$.

(B) $\gamma \in \Gamma \Rightarrow \gamma^{-1} \in \Gamma$

(C) If $\gamma : \Omega \to \Omega'$ and $\gamma' : \Omega' \to \Omega''$ are in Γ then also $\gamma'\gamma : \Omega \to \Omega''$ is in Γ.

(D) For every $\gamma \in \Gamma$, $\gamma : \Omega \to \Omega'$ its restriction $\gamma : \Omega_0 \to \Omega_0' = \gamma(\Omega_0)$ is also in Γ for all $\Omega_0 \subset \Omega$.

If, for example, Γ is an isometry group acting on Δ then Γ can be restricted to all subsets $\Omega \subset \Delta$ for $\gamma : \Omega \to \Omega' = \gamma(\Omega)$ thus giving us a pseudogroup in the above sense. But usually our pseudogroups do not come from global isometries of Δ.

Γ-limits $\Delta^\bullet$ of Δ. These are defined with sequences of points $\delta_i \in \Delta$ and Γ-isometries $\gamma_i : D(\delta_i, r_i) \to D(\delta_{i+1}, r_i)$ where $r_i \to \infty$. Such a limit graph $\Delta^\bullet = \lim_{i\to\infty} (\Delta, \delta_i, \gamma_i)$ can be thought of as an increasing union (inductive limit) of the balls $D(\delta_i, r_i)$ embedded into each other by γ_i and so $\Delta^\bullet$ comes along with a marking $\delta_\bullet \in \Delta^\bullet$ corresponding to the sequence $\{\delta_i\}$ as well as embeddings $D(\delta_\bullet, r_i) \to \Delta$ sending these ball bijectively (and isometrically) onto $D(\delta_i, r_i) \subset \Delta$. This distinguishes a certain pseudogroup acting on $\Delta^\bullet$, denoted $\Gamma^\bullet$, which consists of isometries $\gamma \in \Gamma$ pulled from Δ to $\Delta^\bullet$ via the above bijections between balls. So we see that $(\Delta^\bullet, \Gamma^\bullet)$ is *locally* Γ-*isometric* to $\Delta = (\Delta, \Gamma)$ in an obvious sense.

The following definitions 6.C,D distinguish sufficiently large pseudogroups Γ which will be needed for the surjunctivity property in § 7.G.

6.C. Cofiniteness. Dealing with Γ-isometries between balls, $\gamma : D \to D'$ we shall insist on the center going to the center and call balls Γ-*isometric* if there exists such an isometry $\gamma \in \Gamma$ between them. We call a pseudogroup Γ *cofinite* on Δ if for every $r = 0, 1, \ldots,$ there are at most finitely many mutually non-Γ-isometric balls of radius r.

Example. If Γ comes from an isometry group acting on Δ then, clearly, it is cofinite if and only if the (global) action of the group has finitely many orbits in Γ.

6.C′. Compactness. If Γ is a cofinite pseudogroup on Δ then every infinite sequence of points in Δ obviously admits a subsequence, say $\delta_i \in \Delta$, such that the ball $D(\delta_i, i)$ is Γ-isometric to $D(\delta_{i+1}, i)$ for all $i = 1, 2, \ldots$.

Recall that the resulting limit graph $\Delta^\bullet = \lim_{i\to\infty} (\Delta, \delta_i, \gamma_i)$ also possesses (the limit) pseudogroup structure and it is *locally Γ-isometric to* Δ.

6.D. Dense pseudogroups Γ and quasihomogeneity. We say that two points δ and δ' in Δ are *r-equivalent* with respect to Γ if the r-balls around these points are Γ-isometric. Then (the action of) Γ is called *dense on* Δ if, for every $r = 0, 1, \ldots$, each *non-empty* r-equivalence class of points in Δ, say $\Delta' \subset \Delta$ constitutes a *net* in Δ. that is, there exists an $R = R(\Delta') < \infty$, such that Δ' meets every ball of radius R in Δ.

Notice that this condition is similar to the one used in the above minimality criterion. In fact one observes that there are some distinguished (tautological) isometries between some balls in $\Delta^\bullet$ and in Δ. Using these, one generates a new (huge) *pseudogroup* $\Gamma^\bullet$ acting on (subsets in) the *disjoint union* $\Delta^\bullet$ of all Γ-limits $\Delta^\bullet$ of Δ. Namely $\Gamma^\bullet$ is the minimal pseudogroup acting on $\Delta^\bullet$ which restricts to Γ on Δ and which contains the above mentioned distinguished isometries (that are $D(\delta_\bullet, r_i) \to D(\delta_i, r_i) \subset \Delta$ for the balls $D(\delta_\bullet, r_i) \to D(\delta_i, r_i) \subset \Delta^\bullet = \lim_{i\to\infty} (\Delta, \delta_i, \rho_i)$) for all $\Delta^\bullet$. Thus one can speak of local $\Gamma^\bullet$-isometry relation between different limits $\Delta_1^\bullet$ and $\Delta_2^\bullet$ of Δ (which are connected components of $\Delta^\bullet$) where the implied isometries from r-balls of $\Delta_1^\bullet$ to $\Delta_2^\bullet$ must belong to $\Gamma^\bullet$. Then one defines the order relation $\Delta_1^\bullet \succ \Delta_1^\bullet$ as earlier and (easily) proves the minimality criterion for Γ-graphs Δ with the cofiniteness condition staying for "bounded valency".

Finally a graph Δ is called *quasi-homogeneous* with respect to a given pseudogroup Γ if the (partial !) action of this Γ on Δ is cofinite and dense.

If Γ comes from an isometry group with finitely many orbits in Δ, then Δ is obviously quasi-homogeneous with respect to Γ, as cofiniteness implies density in this case. This is not true in general; however, we have here the following generalization of the minimality lemma.

If Δ is a locally finite graph and Γ is cofinite on Δ, then some Γ-limit $(\Delta^\bullet, \Gamma^\bullet)$ of Δ is quasi-homogeneous.

There is nothing new here to the proof compared to 6.A′ (and we do not even need this for our surjunctivity theorem. The role of this lemma is to illuminate the notion of quasihomogeneity.

6.E. Amenability, uniform amenability and initial subamenability. Define the boundary $\partial\Omega \subset \Delta$ for each $\Omega \subset \Delta$ as the set of points $\delta \in \Delta$ where the unit ball $D(\delta, 1) \subset \Delta$ intersect Ω as well as the complement $\Delta \backslash \Omega$.

An exhaustion of Δ by finite subsets $\Omega_i \subset \Delta$, $i = 1, 2, \ldots$, is called *amenable* if

$$\operatorname{card} \partial\Omega_i / \operatorname{card} \Omega_i \to 0 \text{ for } i \to \infty.$$

This means that large "domains" Ω_i in Δ have relatively small boundaries.

Then a graph Δ is called *amenable* if it admits an amenable exhaustion. Intuitively, Δ is amenable if it has "negligible boundary at infinity".

This notion applies, in particular, to Cayley graphs of finitely generated groups Γ and coincides with the traditional definition of amenability for Γ : a group Γ is called amenable if every continuous action of Γ on compact topological space admits an invariant measure. (All this is well known, see [Gree] for instance). In particular amenability of a Cayley graph Δ does not depend on the choice of generators in Γ which are involved in the definition of Δ.

6.E′. Call a graph Δ *uniformly* amenable if there exists a function $R(r,\epsilon) = R_\Delta(r,\epsilon)$, for $r = 1, 2, \dots,$ and $\epsilon > 0$, such that for each r-ball $D = D(\delta, r) \subset \Delta$ there exists a subset Ω in Δ pinched between D and the concentric R-ball $D^+ = D(\delta, R)$, i.e. $D \subset \Omega \subset D^+$, such that

$$\operatorname{card} \partial\Omega \leq \epsilon \operatorname{card} \Omega$$

Clearly, the uniform amenability implies amenability. In fact, Δ *is uniformly amenable if and only if all limit graphs* $\Delta^\bullet$ *(i.e. all connected graphs locally isometric to* Δ*) are amenable, where "if" needs the assumption of* Δ *having bounded valency.* The proof is straightforward and as far as our applications go we could postulate the uniform amenability of all $\Delta^\bullet$ to start with (see 7.G).

6.E″. A graph Δ is called *initially subamenable* if for every $r = 1, 2, \dots,$ every $\epsilon > 0$ and every finite subset $D \subset \Delta$ (where one can restrict oneself to balls $D \subset \Delta$) there exists a graph Δ' with a finite subset $\Omega'_\epsilon \subset \Delta'$ such that

(a) Δ' is r-locally isometric to Δ on Ω'_ϵ. That is all r-balls $D(\delta', r)$, $\delta' \in \Omega'_\epsilon$, are isometric to some r-balls in Δ.

(b) Ω'_ϵ contains an isometric copy of D, i.e. some $D' \subset \Omega'_\epsilon$ is isometric to D

(c) $\operatorname{card} \partial\Omega'_\epsilon / \operatorname{card} \Omega'_\epsilon \leq \epsilon$.

For example, if Δ admits a sequence of free isometric actions by groups Γ_i on Δ such that the quotients $\Delta_i = \Delta/\Gamma_i$ are amenable (e.g. finite) graphs, such that for every *finite* subset $D \subset \Delta$ there exists an i where the quotient map $\Delta \to \Delta_i$ is *injective* on D. Then, clearly, Δ is initially subamenable.

Next, we say that Δ is uniformly initially subamenable if there exists a function $R = R_\Delta(r,\epsilon)$ for $r = 1, 2, \dots$ and $\epsilon > 0$, with the following property. For every subset $D \subset \Delta$ of diameter $\leq r$ there exists a graph Δ' and a finite subset $\Omega' \subset \Delta'$ of diameter $\leq R$ satisfying the above conditions (b) and (c) and the following strengthened version of (a)

(a^+) Δ' is r-locally isometric on Ω' to the R-neighbourhood $D^{+R} \subset \Delta$ of D, i.e. each r-ball $D(\delta', r)$, $\delta' \in \Omega'$, must be isometric to some ball in D^{+R}.

Notice that (a^+) is equivalent to (a) for quasihomogeneous graphs Δ and only such graphs will appear in our applications.

6.E‴. If a graph Δ comes along with an extra structure, e.g. a local order, one modifies the above definitions in the obvious way by requiring Δ' to carry the same kind of structure and where all (local) isometries in question must preserve this structure. The most important structure for us is a distinguished pseudogroup Γ of isometries of Δ where one may think of Δ' appearing in the definition of (uniform) initial amenability as some graph glued out of r-balls in Δ by some isometries from Γ on some subsets of these balls.

Example. The Cayley graph Δ of a finitely generated group Γ is (uniformly) initially subamenable for the natural Γ-structure on Δ, iff Γ is initially subamenable. But it may (?) happen Δ is initially amenable as a graph without Γ being initially subamenable.

6.F. Asymptotic dimension growth of metric spaces. Let Δ be an arbitrary metric space (e.g. a connected graph) and define *the dimension of Δ on the scale λ*, for a real number $\lambda > 0$, denoted $\dim(\Delta|\lambda)$ as the minimal number N such that Δ can be covered by $N+1$ subsets, say $\Delta_0, \Delta_1, \ldots, \Delta_N$ where each Δ_i decomposes into the union of *uniformly bounded* subsets separated by distances $\geq \lambda$. Thus $\Delta_i = \bigcup_j B_{ij}$ for j running over some (usually infinite) index set J, where

(a) $\operatorname{diam} B_{ij} \leq \text{const} < \infty$, for all i, j, (recall, $\operatorname{diam} B \underset{\text{def}}{=} \sup_{b,b' \in B} \operatorname{dist}(b, b')$)

(b) $\operatorname{dist}(B_{ij}, B_{ij'}) \geq \lambda$ for all $i = 0, \ldots, N$ and all pairs j and $j' \neq j$, where $\operatorname{dist}(B, B') \underset{\text{def}}{=} \inf_{b,b'} \operatorname{dist}(b, b')$ for $b \in B$ and $b' \in B'$.

What we are actually interested in is the asymptotic behavior of $\dim(\Delta|\lambda)$ for $\lambda \to \infty$. This is a very robust invariant unsensitive to changing bounded pieces of Δ. For example every net $\Delta' \subset \Delta$ has the same asymptotic behaviour of $\dim(\Delta'|\lambda)$ as $\dim(\Delta|\lambda)$. In particular every net $\Delta \subset \mathbb{R}^n$ has $\dim(\Delta|\lambda) = n$ for all sufficiently large λ. The same is true for nets in the n-dimensional hyperbolic space and in every n-dimensional symmetric space of non-compact type as well (see $[\text{Gro}]_{\text{A1}}$). But in general, one has a poor idea of the growth of $\dim(\Delta|\lambda)$. (Of course, it is obvious that $\dim(\Delta|\lambda) \leq \text{const}^{\lambda}$ for graphs Δ of bounded valency but one does not know how $\dim(\Delta|\lambda)$ can grow, for example, if Δ equals the universal covering of a finite aspherical 2-dimensional polyhedron).

6.F′. On locality of $\dim(\Delta|\lambda)$. Observe that *every graph $\Delta^\bullet$ locally (i.e. r-locally for all $r > 0$) isometric to Δ has $\dim(\Delta^\bullet|\lambda) \leq \dim(\Delta|\lambda)$ for all $\lambda > 0$, provided Δ has bounded valency.*

However, this does not give us *any* estimate on $\dim(\Delta^\bullet|\lambda)$ if $\Delta^\bullet$ is r-locally isometric to Δ with a given $r \gg \lambda$. This suggests another dimension

$$\dim_{\text{loc}}(\Delta|\lambda) \underset{\text{def}}{=} \limsup_{r \to \infty} \dim(\Delta_r^\bullet|\lambda)$$

where $\Delta_r^\bullet$ runs over all graphs r-locally isometric to Δ.

It is clear that

$$\dim_{\text{loc}}(\Delta|\lambda) \geq \dim(\Delta|\lambda)$$

for all λ and in many cases the two dimensions are equal or, at least, have the same asymptotic growth for $\lambda \to \infty$. But it is unknown what is the relation between these dimensions in general.

Example. The nets Δ in $\mathbb{R}^n$ obviously have $\dim_{\mathrm{loc}}(\Delta|\lambda) = \dim(\Delta|\lambda) = n$ for all large λ. Probably, $\dim_{\mathrm{loc}}(\Delta|\lambda) = \dim(\Delta|\lambda)$ for all connected Lie groups with invariant Riemannian metrics and large λ as well as for homogeneous Riemannian manifolds in general.

Remark. The above definition of $\dim_{\mathrm{loc}}$ is not, a priori, quasi-isometry invariant for $\lambda \to \infty$. This could be remedied either by defining $\dim'_{\mathrm{loc}}(\Delta|\lambda, L, \rho) = \sup_{\Delta'} \dim_{\mathrm{loc}}(\Delta'|\lambda)$ where Δ' runs over all metric spaces admitting ρ-nets which are L-bilipschitz equivalent to certain ρ-nets in Δ. Then one could send $\lambda \to \infty$, extract some asymptotic invariant from $\dim'_{\mathrm{loc}}(\Delta|\lambda, L, \rho)$ and finally send $L, \rho \to \infty$. For instance one could take

$$\limsup_{L,\rho\to\infty} \limsup_{\lambda\to\infty} \dim'_{\mathrm{loc}}(\Delta|\lambda, L, \rho)$$

or

$$\limsup_{L,\rho\to\infty} \limsup_{\lambda\to\infty} \lambda^{\alpha} \dim'_{\mathrm{loc}}(\Delta|\lambda, L, \rho)$$

for a given $\alpha \leq 0$.

A more logical definition would be as follows. Consider all $\Delta^\bullet$ where every r-ball admits a ρ-net L-bilipschitz equivalent to a ρ-net in some r-ball in Δ. Set

$$\dim_r(\Delta|\lambda, L, \rho) = \sup_{\Delta^\bullet} \dim(\Delta|\lambda)$$

over all such $\Delta^\bullet$ and let

$$\dim^\bullet_{\mathrm{loc}}(\Delta|\lambda, L, \rho) = \limsup_{\lambda\to\infty} \dim_r(\Delta|\lambda, L, \rho).$$

Then proceed as earlier by first sending $\lambda \to \infty$ and then letting $L, \rho \to \infty$.

All these dimensions seem to coincide for simple examples but the overall picture remains unclear.

§ 7. Proalgebraic varieties over infinite graphs.

We start with an observation showing that graphs enter rather naturally once we start looking at proalgebraic varieties. For example, consider a proregular mapping between products, say $f : X \to Y$ for $X = \underset{\delta\in\Delta}{\times} \underline{X}_\delta$ and $Y = \underset{\delta\in\Delta}{\times} \underline{Y}_\delta$, where Δ is a countable set and $\underline{X}_\delta$ and $\underline{Y}_\delta$ are algebraic varieties. Such an f is given by a system of regular mappings $f_\delta : X \to \underline{Y}_\delta$ for all $\delta \in \Delta$ where each f_δ actually depends on finitely many variables among $x_\delta \in \underline{X}_\delta$, say on x_b for $b \in B = B(\delta) \subset \Delta$. Thus we have a finite subset $B = B(\delta) \subset \Delta$ assigned to each $\delta \in \Delta$ and we make a *graph* with the vertex set Δ by joining δ_1 and δ_2 with an edge iff the intersection $B(\delta_1) \cap B(\delta_2)$ is non-empty.

The structure of this graph Δ is the main combinatorial characteristic of f and we shall persue our quest for surjunctivity in the graph theoretic language of Δ.

Remark. There is a finer *directed graph structure* on Δ where δ_1 and δ_2 are joint by an arrow $\delta_1 \to \delta_2$ whenever $\delta_1 \in B(\delta_2)$. But we shall be oblivious to this in the present article.

A special case of the above graph structure on Δ arises when we have a proalgebraic subvariety Z in $\mathbb{C}^\Delta$, where δ_1 and δ_2 are joined by an edge whenever there is a polynomial, among those defining Z, which depends simultaneously on the variables x_{δ_1} and x_{δ_2}. Here every edge is labelled by (a vector of) complex numbers, the coefficients of polynomials involving x_{δ_1} and x_{δ_2}. This essentially augments the pure combinatorics of Δ and a similar labelling for the above $f : X \to Y$ is needed for answering the surjunctivity question.

7.A. Graphs, metrics and finite propagations. We consider a connected graph Δ with the metric dist $=$ dist$_\Delta$ as is measured by the length of edge paths in Δ. Typically, our Δ is an infinite locally finite graph of bounded valency (compare 2.D and 6.A).

Our main objects of study are projective systems $\{X_\Omega\}$ over finite subsets $\Omega \subset \Delta$, in particular, subsystem of products which we call *subproduct* systems. Here the ambient system of products consists of the products of the "fibers" $X_\delta = X_{\{\delta\}}$ over subsets $\Omega \subset \Delta$ where $X_\Omega \subset \underset{\delta\in\Omega}{\times} \underline{X}_\delta$ for all $\Omega \in \Delta$. Thus the subproduct property says that the map $X_\Omega \to \underset{\delta\in\Omega}{\times} X_\delta$, that is the product of the projections $X_\Omega \to X_\delta$, is injective for all $\delta \in \Omega$ and all $\Omega \subset \Delta$. And we agree in all cases that the space corresponding to *the empty set* consists of a single point, $X_\emptyset = \{\bullet\}$.

We say that a subproduct system has propagation $\leq \ell$ if the inclusion $x \in X_\Omega$ for a given $x \in \underset{\delta\in\Omega}{\times} X_\delta \supset X_\Omega$ is detectible by looking at the balls of radii $\leq \ell$ around all points in Ω. That is, $x \in X_\Omega$ if and only if the *restrictions* of x to the intersections $\Omega \cap D(\delta, \ell)$ are contained in $X_{\Omega\cap D(\delta,\ell)}$ for all $\delta \in \Omega$, where, recall, the ball $D(\delta, \ell)$ is defined as

$$D(\delta, \ell) = \{\delta' \in \Delta | \operatorname{dist}(\delta, \delta') \leq \ell\}$$

and where the word "restriction" refers to the projection $X_\Omega \to X_{\Omega\cap D(\delta,\ell)}$ as x's in the product are *viewed as functions* (or sections) $\delta \mapsto x(\delta) \in \underline{X}_\delta$ on Δ. Then "bounded propagation" means "propagation $\leq \ell$" for some $\ell < \infty$. Notice that this corresponds to the notion of "finite type" customary applied to subshift of the full shift space in the framework of the symbolic dynamics.

Similarly, a morphism between projective systems over Δ, $\{f_\Omega : X_\Omega \to Y_{\Omega^-}\}$ is called of *propagation* $\leq \ell$, if it is determined by maps f_D for the balls $D = D(\delta, \ell)$ for all $\delta \in \Delta$. More precisely (and more generally) we require that the subset $\Omega^- = \Omega^-(\Omega) \subset \Delta$ consists of all ℓ-*interior* points in Ω, where $\delta \in \Omega$ is called ℓ-*interior* if $D(\delta, \ell) \subset \Omega$. In particular, if $\{Y_\Omega\}$ is a subproduct system, then the above can be equivalently expressed by saying that the value of $y = f_\Omega(x)$ at each point δ is determined by the restriction $x|D(\delta, \ell)$, i.e. the values of x on the ball $D(\delta, \ell)$. Then we speak of bounded propagation as earlier if $\ell < \infty$ and we apply this terminology to projective limits X and maps between these, $f = \lim\limits_{\leftarrow} f_\Omega : X \to Y$.

7.A′. Basic example (compare 2.D). Let X and Y be infinite product spaces over Δ, say $X = \underset{\delta \in \Delta}{\times} X_\delta$ and $Y = \underset{\delta \in \Delta}{\times} Y_\delta$. Suppose we have maps $\varphi_\delta : \underset{\delta' \in D_\delta}{\times} X_{\delta'} \to Y_\delta$, for $D_\delta = D(\delta, \ell)$, assigned to all points $\delta \in \Delta$. This defines a map $f : X \to Y$, where $x \mapsto y$ such that the value of y at each $\delta \in \Delta$ equals $\varphi_\delta(x|D)$. Thus a collection of "finitary" objects, our maps φ_δ, defines a "transcendental" map $f : X \to Y$. Clearly, such f has bounded propagation and this is the only remnant of the finitary origin of f.

7.A″. Generalization of subproducts. Let us replace one point sets $\{\delta\}$ by the balls $D(\delta, \ell)$ of a fixed radius ℓ and consider the Cartesian product of the projection from X_Ω to $X_{\Omega(\delta)}$ for $\Omega(\delta) = \Omega \cap D(\delta, \ell)$, that is $\Pi_\Omega(\ell) : X_\Omega \to \underset{\delta \in \Omega}{\times} X_{\Omega(\delta)}$. We say that our system is a *generalized subproduct*, if there exists $\ell < \infty$, such that the maps $\Pi_\Omega(\ell)$ are injective for all $\Omega \subset \Delta$. This generalization comes naturally in the bounded propagation framework but it is not truly necessary. In fact, it can be reduced to the subproduct case by introducing an auxiliary projective system $\{X_\Omega^{+\ell} \underset{\text{def}}{=} X_{\Omega^{+\ell}}\}$, where $\Omega^{+\ell} \subset \Delta$ denotes the ℓ-*neighbourhood* of Ω, i.e. the union of the balls $B(\delta, \ell)$ around all $\delta \in \Omega$. Thus everything we shall eventually prove for subproduct systems could be extended to generalized subproducts.

7.B. Holonomy over Δ. Let $\{X_\Omega\}_{\Omega \subset \Delta}$ be a projective system over a graph Δ. A *holonomy map* h between (projective systems over) balls D and D' in Δ is given by the following data

(i) an isometry $\gamma = \gamma_h : D \to D'$ sending the center of D to that of D',

(ii) a bijective map from the projective system $\{X_\Omega\}_{\Omega \subset D}$ to $\{X_{\Omega'}\}_{\Omega' \subset D'}$. This means there are bijective maps $h_\Omega : X_\Omega \to X_{\Omega'}$ for all $\Omega \subset D$ and $\Omega' = \gamma(\Omega)$ which *commute with the restriction maps* $X_{\Omega_1} \to X_{\Omega_2}$ and $X_{\Omega'_1} \to X_{\Omega'_2}$ for all pairs Ω_1 and $\Omega_2 \subset \Omega_1$ in D (where, recall "restriction maps" refer to projections π_{Ω_1, Ω_2} constituting our projective system).

The most important holonomy map among h_Ω is h_D and we often write h instead of h_D.

A holonomy over Δ is defined as a set H of holonomy maps defined between certain (pairs of) balls D and D'. The balls admitting holonomies between them are called *equivalent*. Notice that equivalent balls are *isometric* but the converse does not have to be true. Also we do not assume the uniqueness of holonomy h between two given equivalent balls. (Actually this non-uniqueness causes some technical complications by making certain constructions non-canonical).

(Pseudogroup) Axioms for holonomy. We assume below that our holonomies satisfy the following four axioms (compare 6.B$'$).

(1) The *identity* Id_D, given by the identity map $D \to D$ and the identity map $X_\Omega \to X_\Omega$, $\Omega \subset D$, is in H for all balls $D \subset \Delta$.

(2) $h \in H \Leftrightarrow h^{-1} \in H$

(3) If h and h' are in H where h is defined between D and D' and h' is between D' and D'', then their composition $h' \circ h$ defined between D and D$''$ is also in H.

(4) If a ball D_0 is contained in D then the (obvious) restriction of each holonomy from D to D_0 belongs to the holonomy over D_0 (where D_0 is not necessarily a concentric ball).

Notice that we do not bother to define the holonomy over non-balls $\Omega \subset \Delta$ but this could be easily done by restricting those from balls $D \supset \Omega$. This being done, we obtain a pseudogroup $\Gamma = \Gamma(H)$ acting on Δ consisting of all $\gamma = \gamma_h$, $h \in H$.

7.B$'$. Holonomies commuting with endomorphisms. Suppose we have an endomorphism between two projective systems over Δ, say $\{f_\Omega : X_\Omega \to X_{\Omega_-}\}$ where we assume $\Omega_- \subset \Omega$ for all $\Omega \subset \Delta$. A holonomy in this situation refers to h's which commute with f. (Similarly one may speak of holonomies for maps between different systems or for extra structures in $\{X_\Omega\}$ not coming from maps, but we deal mostly with endomorphisms in this paper).

7.B$''$. Rigid holonomy. A holonomy is called *rigid* if there is *at most one* h covering given $\gamma : D \to D'$ where "covering" means $\gamma = \gamma_h$.

Basic example of holonomy. (Compare 2.D). Let the graph Δ have bounded valency, i.e. at most $d < \infty$ edges issuing from every vertex. Suppose we have a partition of Δ into N subsets $\Delta_1, \dots, \Delta_N$ such that the points $\delta \in \Delta_k$, $k = 1, \dots, N$ have the same valency in Δ, say d_k for all $\delta \in \Delta_k$. Furthermore, we assume that we are given a *local order* on Δ, i.e. an ordering of the edges adjacent to each vertex and thus an ordering of the neighbour points to δ, call them $\delta_c(\delta) \in D(\delta, 1)$ with $c = 0, 1, \dots, d_k$ where $\delta_0(\delta) = \delta$. Next let $\underline{X}$ be an algebraic variety and let $p_k : \underline{X}^{d_k+1} \to \underline{X}$ be some regular maps for $k = 1, \dots, N$. Then these p_k define an endomorphism f of the Cartesian power $X = \underline{X}^\Delta$ defined as follows, $f : x \mapsto y$ where the value of y at each point $\delta \in \Delta_k$, $k = 1, \dots, N$, is given by the values of x at the neighbouring points $\delta_c = \delta_c(\delta), c = 0, \dots, \delta_k$ via p_k according to the following rule (compare 2.D),

$$y(\delta) = p_k(x(\delta_0), x(\delta_1), \dots, x(\delta_{d_k})).$$

This map f clearly has bounded propagation with the implied $\ell = 1$. Then the structure of f as well as the structure of the corresponding endomorphism of the projective system of the Cartesian powers $X_\Omega \overset{\text{def}}{=} \underline{X}^\Omega$, say $\{f_\Omega : X_\Omega \to X_{\Omega^-}\}$, where Ω^- equals the 1-interior of Ω, is determined, besides the given finite set of polynomials p_k, by the pure combinatorics of the graph Δ, the partition $\Delta = \cup\Delta_k$ and the local ordering of Δ. Thus every isometry $\gamma : D \to D'$ preserving the partition and the local order on Δ lifts to a map $h : X_D \to X_{D'}$ compatible with the maps f_Ω where "lifts" mean $\gamma_h = \gamma$. In other words this h serves as a holonomy map for the projective system $\{f_\Omega : X_\Omega \to X_{\Omega^-}\}$.

Now we declare two points δ_1 and δ_2 in Δ *ℓ-equivalent* if the balls $D(\delta_1, \ell)$ and $D(\delta_2, \ell)$ are equivalent with respect to $\Gamma = \Gamma(H)$ in the sense of 7.B which amounts here to the existence of the above isometry $D(\delta_1, \ell) \to D(\delta_2, \ell)$ compatible with the partitions. Clearly there are at most finitely many ℓ-equivalence classes of points in Δ for every $\ell = 1, 2, \ldots$. Thus we have lots of holonomy for *infinite* graphs Δ as we may have sequences δ_i of mutually ℓ-equivalent points in Δ going to infinity (compare § 6).

Observe that the holonomy in this example is quite rigid : there is at most one h corresponding to an isometry $D \to D'$ and this is further enhanced by the local order in Δ as the local order preserving isometry $D \to D'$, sending the center of D to the center of D' is, obviously, unique (if existing at all), as we have already mentioned in 6.B.

7.C. Holonomy orbit completion. Let $\{X_\Omega\}$ be a projective system with a holonomy H over Δ. A sequence of balls $D_i = D(\delta_i, i) \subset \Delta$, $\delta_i \in \Delta$, $i = 1, 2, \ldots$, is called *holonomic* if, for every $i = 2, 3, \ldots$ the (concentric) ball $D_i^- = D(\delta_i, i-1)$ is $\Gamma(H)$-*equivalent* to D_{i-1}, i.e. if there exists an isometry $\gamma = \gamma_h \in \Gamma(H)$ from D_i^- to D_{i-1}. Given such a sequence we can organize the spaces $X_i^\bullet \underset{\text{def}}{=} X_{D_i}$ into a projective system by composing the restriction maps $X_{D_i} \to X_{D_i^-}$ with some holonomy maps $h_i : X_{D_i^-} \to X_{D_{i-1}} = X_{i-1}^\bullet$. Thus every choice of h_i gives us a projective system denoted $\{\pi_i^\bullet : X_i^\bullet \to X_{i-1}^\bullet\}$. For example, if we take $\delta_i = \delta_0$ independent of i. Then the balls $D(\delta, i)$ give us an exhaustion of Δ (if Δ is connected as we always assume) and if $h_i = \mathrm{Id}_{D(\delta,i)}$ for all i, the projective limit $X^\bullet = \varprojlim X_i^\bullet$ equals our original $X = \varprojlim X_\Omega$ "viewed from the point δ_0". (Shifting "the point of view" and sending it to infinity in all possible ways lead to our "orbit completion" defined below). Also nothing essentially new happens if δ_i stay in a bounded subset in Δ but everything may change if $\delta_i \to \infty$, i.e. if $\mathrm{dist}(\delta_0, \delta_i) \to \infty$ for $i \to \infty$. Here we have a sequence of balls $D_i = D(\delta_i, i) \subset \Delta$ and of embeddings $D_i \hookrightarrow D_{i+1}$ given by $\gamma_{h_{i+1}}^{-1}$. These define the limit graph $\Delta^\bullet = \varinjlim (\Delta, \delta_i, \gamma_{h_{i+1}}^{-1})$ which is essentially the union $\bigcup_{i=1}^{\infty} D_i$ (see 6.A). Since our holonomy maps $h_i : X_{D_i^-} \to X_{d_{i-1}}$ for $D_i^- = D(\delta_i, i-1)$ are, in fact, isomorphisms of the projective systems X_Ω and $X_{\Omega'}$ for all $\Omega \subset D_i^-$ and $\Omega' = \gamma_h(\Omega) \subset D_{i-1}$ we have not only the sequence $X_i^\bullet$ but the whole projective system $\{X_{\Omega^\bullet}^\bullet\}_{\Omega^\bullet \subset \Delta^\bullet}$. If the holonomy is rigid, i.e. completely determined by the metric (or combinatorial) structure in Δ, then this $\{X_{\Omega^\bullet}^\bullet\}$ emerges as the limit of $\{X_\Omega\}$ when we move our reference point $\delta_i \in \Delta$ to infinity. If the original system $\{X_\Omega\}$ is homogeneous, then the limit $\{X_{\Omega^\bullet}^\bullet\}$ is, of course, isomorphic to $\{X_\Omega\}$ and we do not get anything new. In general, $\Delta^\bullet$ is *locally isometric* to Δ, that is every ball in $\Delta^\bullet$ is isometric to some ball in Δ (but not, necessarily, vice versa (see 6.A). Yet the global geometry of $\Delta^\bullet$ may be far from that of Δ and $\{X^\bullet_{\Omega^\bullet}\}$ may be even further away from $\{X_\Omega\}$, although $\{X^\bullet_{\Omega^\bullet}\}$ is locally isomorphic to $\{X_\Omega\}$ (compare 6.A).

7.C′. Spaces X° and $X^\sqcup \subset X^\circ$. We denote by $X^\circ = X^\circ(H)$ the disjoint union of the spaces $X^\bullet = \lim\limits_{\leftarrow} X_i^\bullet$ over all holonomic sequences of balls $D_i \subset \Delta$ and all sequences of holonomies $h_i : X_{D_i^-} \to X_{D_{i-1}}$. This X°, called the *holonomy completion* of $\{X_\Omega\}$ and/or of $X = \lim\limits_{\leftarrow} X_\Omega$, can be naturally represented as a projective limit $X^\circ = \lim\limits_{\leftarrow} X_i^\circ$ where each X_i° equals the disjoint union of the spaces X_{D_i} labelled by sequences of holonomies

$$h_i : X_{D_i^-} \to X_{D_{i-1}},\ h_{i-1} : X_{D_{i-1}^-} \to X_{D_{i-2}}, \ldots, h_2 : X_{D_{2^-}} \to X_{D_1}$$

and the union is taken over all such sequences. This X° may be very big and not convernient to work with. For example, it does not carry, in general, a natural proalgebraic structure when we start with a projective system of algebraic varieties X_Ω. But it may contain subspaces which are "small" and yet "sufficiently representative".

Projective systems $\{X_i^\sqcup\}$. Suppose, for every i, we are given a collection $\mathcal{D}_i^\sqcup$ of balls D in Δ of radius i and denote by $X_i^\sqcup = X_{\mathcal{D}_i^\sqcup}^\sqcup$ the disjoint union of the spaces X_D over all $D \in \mathcal{D}_i^\sqcup$. We assume, for each ball $D \in \mathcal{D}_i^\sqcup$, there exists $D'_- \in \mathcal{D}_{i-1}^\sqcup$ such that the concentric $(i-1)$-ball $\mathcal{D}_- \subset D$ is $\Gamma(H)$-equivalent to D'_-. Then for each $D \in \mathcal{D}_i$ we compose the restriction map $X_D \to X_{D_-}$ with some holonomy $X_{D_-} \to X_{D'_-}, D'_- \in \mathcal{D}_{i-1}^\sqcup$, and thus obtain a map $X_i^\sqcup \to X_{i-1}^\sqcup$. All these make a projective system denoted $(\pi_i^\sqcup = X_i^\sqcup \to X_{i-1}^\sqcup)$ and the projective limit $X^\sqcup = \lim\limits_{\leftarrow} X_i^\sqcup$ obviously embeds into X°.

Example. If $X = \underline{X}^\Gamma$ for a group Γ, with the holonomy generated by the natural (shift) action of Γ on X, then $X^\circ = X \times \Gamma^{\mathbb{N}}$. In fact, we have a unique holonomy $h : X_D \to X_{D'}$ for every two balls D and D' of equal radii as $D' = \gamma D$ for some $\gamma \in \Gamma$. Thus every sequence of points $\gamma_i \in \Gamma$ admits a sequence of holonomies over the i-balls around γ_i and X° identifies with the disjoint union of the copies of X indexed by all sequences $\{\gamma_i\} \in \Gamma^{\mathbb{N}}$. But in the case of $X = \underline{X}^\Gamma$ one can reduce X° back to X, exactly because it consists of disjoint and mutually non-interacting copies of X. In fact, one can divide X° by the group $\Gamma^{\mathbb{N}}$ naturally acting on X° and arrive at $X = X^\circ/\Gamma^{\mathbb{N}}$.

Group H°. In general, every holonomy H gives rise to a huge group, denoted H° and acting on X°, where every $h^\circ \in H^\circ$ is, by definition, a function on the set of all balls $D \subset \Delta$ assigning to each D a holonomy map h between D and another ball $D' \subset \Delta$. Such functions $h(D)$ make a semigroup for the natural composition

$$D \underset{h(D)}{\longrightarrow} D' \underset{h'(D')}{\longrightarrow} D''$$

and the invertible elements constitute our group H°. Clearly, this H° acts on holonomic sequences of balls and then it acts on sequences of holonomies between these balls by conjugation. Thus H° acts on X° but the quotient X°/H° may be rather pathological such as $\mathbb{C}/\operatorname{Gal}_{\mathbb{C}}$, for example. So it is better to deal with the following

"Fundamental domains". A projective system $\{X_i^\sqcup\}$ is called a "fundamental domain" for H in $\{X_i^\circ\}$ if for each i-ball $D \subset \Delta$ there exists a $\Gamma(H)$-equivalent ball in the collection $\mathcal{D}_i^\sqcup$ associated to this $\{X_i^\sqcup\}$. We also express this property by saying that $X^\sqcup = \varprojlim X_i^\sqcup$ is a "fundamental domain" in X. But such an $X^\sqcup$, in general, is not a true fundamental domain for H° as the H°-orbit of $X^\sqcup$ may be smaller than all of X°. This is due to the possible non-uniqueness of a holonomy between balls. In fact, given a holonomic sequence of balls, $D_1, \dots, D_i, \dots$, one may have a priori two sequences of holonomies $h_i, h_i' : X_{D_i^-} \to X_{D_{i-1}}$, such that there is no holonomies $h_i^\circ = X_{D_i} \to X_{D_i}$ satisfying $h_i' = h_i^\circ h_i (h_i^\circ)^{-1}$ for all i. (Of course such h_i° exist for $i \le i_0$ but there may be a problem with $i \to \infty$). On the other hand, no difficulty of this kind appears if the holonomy H is rigid (as in the above case of $X = \underline{X}^\Gamma$ where X itself serves as a fundamental domain in X° and where the action of H° on X° reduces to that of $\Gamma^{\mathbb{N}}$.) Also notice that in the rigid case X° itself is of the form $X^\sqcup$ for $\mathcal{D}_i$ being the collection of *all* i-balls in Δ for $i = 0, 1, 2, \dots$, but it is not so in general.

7.C$''$. Cofiniteness and density. A holonomy H is called *cofinite* if the action of the pseudogroup $\Gamma(H)$ is cofinite on Δ in the sense of 6.C. Similarly, H is called *dense* if $\Gamma(H)$ is dense on Δ.

If the holonomy is cofinite, then the system $\{X_i^\circ\}$ admits a *fundamental domain* $\{X_i^\sqcup\}$ with *finite* sets $\mathcal{D}_i$ for all $i = 1, \dots$.

Among these $X^\sqcup$ there are minimal ones where each $\mathcal{D}_i$ contains exactly one representative in the $\Gamma(H)$-equivalence class of i-balls. Sometimes such a minimal $X^\sqcup$ is unique up to an isomorphism (given by some $h_0 \in H^\circ$). In general, however, given two such domains which are projective limits $\varprojlim X_{D_i}^\sqcup$ and $\varprojlim X_{D_i'}^\sqcup$, all we can claim is an isomorphism of these projective systems up to a given finite level i_0, but this isomorphism does not always survive when $i_0 \to \infty$ as we mentioned earlier.

What is good about these "finite" fundamental domains in any case is them being *proalgebraic* when we start with a projective system of *algebraic* varieties X_Ω.

In general, every fundamental domain $X^\sqcup$ in X° (regardless of its "finiteness") gives a good view on X° from each point $\delta \in \Delta$. In particular if $\{X_\Omega\}$ comes with an endomorphism f, then the initial injectivity of $f^\sqcup$ does not depend on a particular choice of a fundamental domain $(X^\sqcup, f^\sqcup)$ and, in fact, is equivalent to the initial injectivity of f° for the natural representation $(X^\circ, f^\circ) = \varprojlim (X_i^\circ, f_i^\circ)$ as in 7.C$'$.

7.D. Holonomy in the proalgebraic category. Let $\{X_\Omega\}_{\Omega \subset \Delta}$ be a projective system of algebraic varieties over a field K which may harbour additional structures, such as an endomorphism $\{f_\Omega\}$ of $\{X_\Omega\}$. We want to keep track of all symmetries, i.e. all "isomorphisms" between the projective systems $\{X_\Omega\}_{\Omega \subset D} \leftrightarrow \{X_\Omega'\}_{\Omega' \subset D'}$ for all balls in

D and D' in Δ of equal radii, where "isomorphisms" must be compatible with extra structures, e.g. with endomorphisms f_Ω of X_Ω. The obvious "isomorphisms" to consider are *biregular mappings over* K (compatible with $\{f_\Omega\}$ if this is required) and holonomies of this kind are called *regular*.

For example, if $\{X_\Omega\}$ is a subproduct system, say $X_\Omega \subset \underline{X}^\Omega$, $\Omega \subset \Delta$, we may start with the obvious lifts of isometries $\gamma : D \to D'$ to regular maps $h : \underline{X}^D \to \underline{X}^{D'}$ where h *identifies* the $\underline{X}$-components in $\underline{X}^D$ and in $\underline{X}^{D'}$ according to γ. Then we select those γ for which h sends X_Ω to $X_{\Omega'}$ for all $\Omega \subset D$ and $\Omega' = \gamma\Omega$, and commutes with all f_Ω.

The above holonomy can be enlarged by adding extra (non-identities) regular maps between $\underline{X} = \underline{X}_\delta$ and $\underline{X} = \underline{X}'_\delta$, which can lead to a non-rigid regular holonomy. Yet the rigidity can sometimes be recaptured by incorporating the holonomy groups into X_Ω's (see 7.0).

Galois holonomy. We want to eventually relax the notion of ℓ-equivalent points δ and δ' in Δ replacing the (essential) equality of our projective systems over balls $D = D(\delta, \ell)$ and $D' = D(\delta', \ell)$ by their *elementary equivalence* or, for $K = \mathbb{C}$, by the *Galois equivalence*. For example two cubic polynomials $p = x^2 + y^3 + ay + b$ and $p' = x^2 + y^3 + a'y + b'$ on $\mathbb{C}^2$ are *not* biregular equivalent for *generic* $a' \neq a$ and $b' \neq b$. Yet there is an automorphism of $\mathbb{C}$ sending (a, b) to (a', b'), if a' and b' as well as a and b are algebraically independent. Thus p and p' are Galois equivalent.

The Galois equivalence over $\mathbb{C}$ (or over any algebraically closed field for this matter) adequately represents the *syntactic (elementary) equivalence* of the corresponding algebra-geometric objects expressed in the first order language of the field theory. This equivalence between two objects means, that everything we may say in this language which is true for the first object is also true for the second one.

Definitions of Galois holonomy H_{Gal}. Let $\{X_\Omega\}_{\Omega\subset\Delta}$ be a projective system of K-points of algebraic varieties X_Ω over K with a system of endomorphism $f_\Omega : X_\Omega \to X_{\Omega^-}$ for all finite $\Omega \subset \Delta$ and some (possibly empty) $\Omega^- = \Omega^-(\Omega) \subset \Delta$. Then the absolute Galois group $\mathrm{Gal}\, K = \mathrm{Aut}\, K$ acts on $\{X_\Omega\}$. *A Galois holonomy map* over a pair of balls D and D' in Δ is given by the following data : a center preserving isometry $\gamma : D \to D'$ and a biregular equivalence of the projective systems $\{X_\Omega\}_{\Omega\subset\Delta}$ to $\{gX_{\Omega'}\}_{\Omega'=\gamma\Omega\subset D'}$ for some $g \in \mathrm{Gal}\, K$, where the biregular equivalence is supposed to preserve all extra structures, such as f_Ω if these are present in the picture.

Accordingly, δ_1 and δ_2 in Δ are called *Galois ℓ-equivalent*, if there is a Galois holonomy map over $D(\delta_1, \ell)$ and $D(\delta_2, \ell)$.

If K is algebraically closed, then "Galois ℓ-equivalence" can be renamed into "elementary ℓ-equivalence", but we stick to the former even though we are primerly interested in $K = \mathbb{C}$. Accordingly we use the notation H_{Gal} for the (maximal) holonomy consisting of *all* biregular maps composed with all Galois automorphisms of the field $K (= \mathbb{C})$.

7.E. Local and global stability. A projective system $\{X_\Omega\}_{\Omega\subset\Delta}$ is called (globally) ℓ-stable (compare the image stability in 5.I) if for every triple of finite subsets $\Omega_0 \subset \Omega_1 \subset \Omega_2$ in Δ the image of the projection $\pi_{2,0} : X_{\Omega_2} \to X_{\Omega_0}$ equals the image of $\pi_{1,0} : X_{\Omega_1} \to X_{\Omega_0}$, provided Ω_1 contains the ℓ-neighbourhood of Ω_0. In particular the 0-*stability* amounts to surjectivity of the projections (or restrictions) $X_\Omega \to X_{\Omega_0}$ for all finite $\Omega \subset \Delta$ and $\Omega_0 \subset \Omega \subset \Delta$. In other words every $x_0 \in X_{\Omega_0}$ can be extended to $\Omega \supset \Omega_0$. Similarly, the ℓ-stability for $\ell > 0$ says, in effect, that every $x_0 \in X_{\Omega_0}$ can be extended from the ℓ-interior $\Omega_0^{-\ell}$ of Ω_0 to every $\Omega \supset \Omega_0$, i.e. there exists $x \in X_\Omega$ such that $x|\Omega_0^{-\ell} = x_0|\Omega_0^{-\ell}$. Here , a priori, we allow only finite subsets Ω, but by repeating the process we can extend x_0 from $\Omega_0^{-\ell}$ to all of Δ, i.e. to construct $x \in X = \lim\limits_{\leftarrow} X_\Omega$ with the projection to $X_{\Omega_o^{-\ell}}$ equal to $x_0|\Omega_0^{-\ell}$.

Terminology : "stable and ℓ-stable". When we say "stable" we mean "there exists ℓ, such that $\{X_\Omega\}$ is ℓ-"stable" and similarly we understand other forms of stability considered in the following sections.

Stability and mixing. Our stability is similar to the uniform mixing property (see [Ru-We]) for topological dynamical systems, such as subshifts in $X = \underline{X}^\Gamma$ for a group Γ with the natural (shift) action on X. In fact our notion of stability (as well as its variations displayed below) extend to a more general framework of foliated spaces X, (where the leaves in our present case come with structures of graphs $\Delta^\bullet$ locally isomorphic to a given Δ).

7.E′. Localization of stability. We say that a projective system is *L-locally ℓ-stable* if the above extension of $x_0 \in X_{\Omega_0}$ from $\Omega_0^{-\ell}$ to $\Omega \supset \Omega_0$ is possible for all $\Omega \subset \Delta$ with diam $\Omega \leq L$ and all $\Omega_0 \subset \Omega$. It is obvious that

if a system $\{X_\Omega\}$ of propagation $\leq \ell_0$ is L-locally 0-stable for $L \geq 4\ell_0 + 2$ then it is (globally) ℓ-stable.

In fact, we can extend x_0 from $\Omega_0 \cap B_1$ to x_1' on B_1 for every B_1 with diam $B_1 \leq L$, and then define x_1 on $\Omega_1 = \Omega_0 \cup B_1^{-2\ell_0}$ by the conditions $x_1|\Omega_0 = x_0$ and $x_1|B_1^{-2\ell} = x_1'|B_1^{-2\ell}$. The ℓ_0- propagation bound ensures this x_1 is indeed contained in X_{Ω_1} and the inequality $L \geq 4\ell_0 + 1$ allows us to choose B_1 such that Ω_1 is strictly greater than Ω_0. For example, one could take the ball $D(\delta_1, L/2)$ for B_1 with δ outside Ω_0 and then the resulting $\Omega_1 = \Omega_0 \cup D(\delta, L/2 - 2\ell_0)$ contains δ_1.

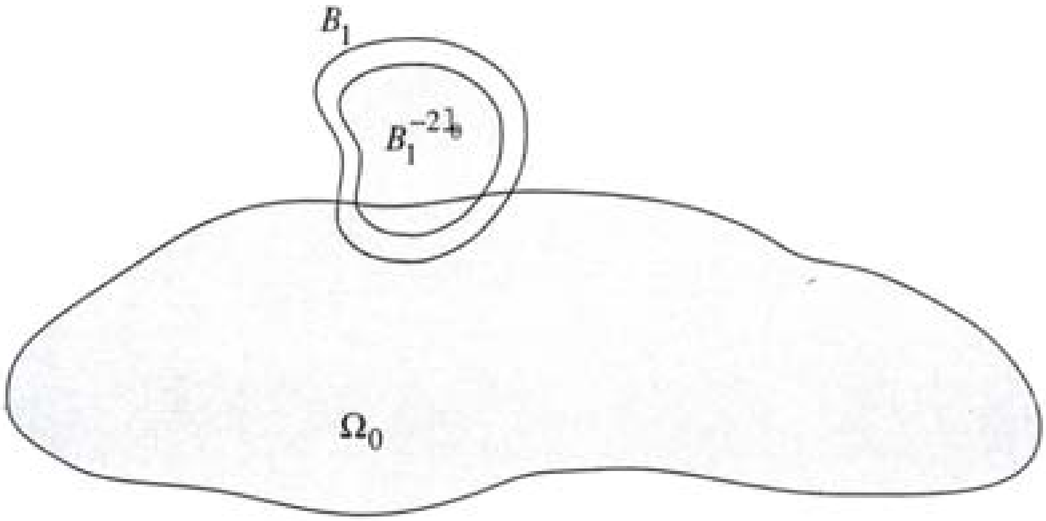

Fig. 3

Then we can add B_2 to getting $\Omega_2 \underset{\neq}{\supset} \Omega_1$ and so on unless we reach a desired $\Omega \supset \Omega_0$.

Now let us extend this argument to $\ell > 0$. Here we have an extra complication as we have to replace Ω_0 by its ℓ-interior at every stage of the extension and so as we go from Ω_0 to Ω adding $B_1, B_2, \ldots, B_k$ we reduce Ω_0 to $\Omega_0^{-k\ell}$ with k depending on (the size of) $\Omega \supset \Omega_0$. Notice, that this problem would be less severe if the subsets B_j, $j = 1, \ldots, k$, were disjoint and far from each other, farther than $2\ell_0$. In this case we could make our extension independently around each B_j thus reducing Ω_0 only to $\Omega_0^{-\ell}$. More generally, if we could obtain all of Δ and thus each $\Omega \supset \Omega_0$ by adding subsets $\Delta_0, \ldots, \Delta_N$ to Ω_0 where each Δ_i is a union of disjoint B_{ij} which are mutually far from each other, we would get an extension from $\Omega_0^{-(N+1)\ell}$ to Ω. So we invoke the idea of asymptotic dimension (see 6.F) and arrive at the following

7.E″. Localization of ℓ-stability. *Let Δ have sublinear growth of the asymptotic dimension, i.e. $\dim(\Delta|\lambda)/\lambda \to 0$ for $\lambda \to \infty$. Then for every ℓ_0 and ℓ there exist L and $L^+ = \ell^+(L, \ell, \Delta)$, such that every L-locally ℓ-stable projective system of propagation $\leq \ell_0$ is (globally) ℓ^+-stable.*

In fact, the assumption on $\dim(\Delta|\lambda)$ gives us subsets $\Delta_0, \ldots, \Delta_N$, such that their $2(\ell_0 + \ell)(N+1)$-interiors cover Δ and such that each Δ_i consists of far away B_{ij}'s where "far away" means dist $\geq 2\ell_0$. Then the above argument works and delivers $\ell^+ = \ell(N+1)$.

Density of periodic points. Suppose we have a sequence of graphs Δ^r, $r = 1, 2, \ldots$, with projective systems $\{X_\Omega^r\}_{\Omega \subset \Delta^r}$ which are r-locally isomorphic to $\{X_\Omega\}_{\Omega \subset \Delta}$ and such that the original system on the r-ball $D_r = D(\delta_0, r) \subset \Delta$ is identified with X^r on some ball $D_r^r = D(\delta_r, r) \subset \Delta^r$.

Example. Consider a group Γ_r acting on Δ and on $\{X_\Omega\}$, such that $\text{dist}_\Delta(\delta, \gamma\delta) \geq 2r$ for all $\delta \in \Delta$ and $\gamma \neq \text{id}$ in Γ_r. Then one could take $\Delta^r = \Delta/\Gamma_r$ with the (naturally defined) quotient system X^r over Δ^r satisfying the above conditions. Then the space $X^r = \varprojlim X^r_\Omega$, $\Omega \subset \Delta^r$, can be identified with the space of Γ_r-invariant (or Γ_r-periodic) points in $X = \varprojlim X_\Omega$, $\Omega \subset \Delta$.

In general, points $x \in X^r$, called *Δ^r-points in X*, can be viewed as periodic points in X, although there is no embedding $X^r \subset X$. Yet, the space $X^r_{D^r_r}$ identifies with $X^r_{D_r}$ and so it makes sense to ask whether (the union of all) Δ^r-points are *dense* in X, where we view both X and $X_{D^r_r}$ as subsets in the metric space X_+ defined in 5.I. Thus the density of Δ^r-points means that each $x \in X$ restricted to D_r can be extended from r_0-interior of $D^r_r = D_r$ to all of Δ^r where $r_0 \to 0$ with $r \to \infty$.

Clearly, the ℓ-stability of the systems $\{X^r_\Omega\}$ for some ℓ independent of r is sufficient for this density but the ℓ-stability of $\{X_\Omega\}$ on Δ does not, a priori, suffice. However, the above localization argument shows that if $\dim_{loc}(\Delta|\lambda)$ has *sublinear growth then Δ^r-points are dense in X, provided the original system $\{X_\Omega\}$ over Δ is stable.*

There is one simple case where the assumption on $\dim_{loc}$ is unneeded. One can assume instead that our X admits an *absolutely invariant* element $x_0 \in X$ which passes to $X^r_0 \subset X^r$ over all Δ^r. For example, if we deal with $\Delta^r = \Delta/\Gamma_r$ this must be an $x \in X$ fixed by all groups Γ_r. Then the ℓ-stability of X_Ω on the ball $D_r \subset \Delta$ identified with $D^r_r \subset \Delta^r$ allows an extension of each x from the ball $D^r_{r-\ell}$ to Δ^r by making it x_0 outside D^r_r.

7.E‴. Strong ℓ-stability. Let us introduce another version of the ℓ-stability allowing localization witout extra assumptions on Δ.

Definition. A projective system is called *strongly ℓ-stable* if the following condition (loc) is sufficient for extendability of x_0 from Ω_0 to Ω for all finite subset Ω_0 and $\Omega \supset \Omega_0$ in Δ.

(loc). *For every $\delta \in \Omega_0$ the restriction $x_0|\Omega_0 \cap D(\delta, \ell)$ is extendable* to $D(\delta, \ell)$.

Clearly, this strong ℓ-stability implies the ℓ-stability.

Strong localization lemma. *If a system is L-locally strongly ℓ-stable for $L \geq 2\ell + 1$ then it is strongly ℓ-stable.*

In fact, this follows by the same (trivial) argument we used to localize the 0-stability.

Corollary. *Strong stability implies density of ("periodic") Δ^r-points in the context of the previous section.*

Remark. Stability is a rather transparent property if the underlying graph is a tree, e.g. the Cayley graph of a free group, $\mathbb{Z}$ for instance. In fact stability for free groups is essentially equivalent to *the topological mixing property* which is satisfied by "generic" subproduct systems (compare [Ru-We] and see $[\text{Gro}]_{\text{TIDS}}$ for further discussion). But the picture is not so clear for general graphs (and groups) where exhibiting meaningful stable systems is not totally trivial even for (Cayley graphs of) groups like $\mathbb{Z}^n$, $n \geq 2$ (compare $[\text{Gro}]_{\text{TIDS}}$).

7.F. Vertical irreducibility. Our preoccupation with the localization of stability was motivated by necessity of expressing this property in the first order language where we can apply the extended Lefschetz principle reducing algebra geometric problems over $\mathbb{C}$ to those over $\overline{F}_p$. Then we need a further reduction, to *finite* subfields $\mathbb{F}_{p^r} \subset \overline{\mathbb{F}}_p$, and this is achieved with the following

Definition. A projective system (of K-points) of algebraic varieties, $\{X_\Omega\}_{\Omega\subset\Delta}$, over a field K is called *ℓ-vertically irreducible* if the fibers (i.e. the pull-backs of the points) of the projection (restriction) map $X_{\Omega^{+\ell}} \to X_\Omega$ are irreducible for all $\Omega \subset \Delta$ (where, recall $\Omega^{+\ell}$ stands for the ℓ-neighbourhood of Ω).

We shall see that this property over $\overline{\mathbb{F}}_p$ combines with a suitably localized ℓ-stability and yields ℓ-stability on $\mathbb{F}_{p^r}$-points of our projective system by the Weil-Lang theorem (see 5.J and 7.K$''$).

7.G. Surjunctivity theorem for quasi-homogeneous endomorphisms over amenable graphs. Let Δ be a countable connected graph and $\{f_\Omega : X_\Omega \to X_{\Omega^-}\}_{\Omega\subset\Delta}$ be a projective system of $\mathbb{C}$-varieties and regular maps. We make the following assumptions on our objects.

I. Bounded propagation. $\{X_\Omega\}$ is a subproduct system of bounded propagation and the system of maps $\{f_\Omega\}$ has also bounded propagation.

This means that everything is detectable by looking at our objects restricted to the balls in Δ of a fixed (possibly large) radius ℓ_0. Thus our "transcendental" map $f : X \to X$ is determined by Δ and the countable collection of elementary objects that are finite projective systems $\{f_\Omega : X_\Omega \to X_{\Omega^-}\}_{\Omega\subset D(\delta,\ell_0)}$ for all $\delta \in \Delta$.

II. Density of the Galois holonomy H_{Gal} on Δ. This means, that for every $\ell = 1,2\ldots$, every non-empty Galois ℓ-equivalence class in Δ is a net in Δ in the sense of 6.D.

In other words every algebraic *ℓ-property* of $\{f_\Omega\}$ observed at some point $\delta_0 \in \Delta$, i.e. a property of $\{f_\Omega\}_{\Omega\subset D(\delta,\ell_0)}$ will recur in the vicinity of each point $\delta \in \Delta$.

III$^-$. Cofiniteness of Galois holonomy H_{Gal}. This requires that for each ℓ there are at most finitely many mutually Galois (i.e. elementary) non-ℓ-equivalent points $\delta \in \Delta$.

So, eventually, f is determined by a *finite* collection of projective systems $\{f_\Omega\}$ over balls $D(\delta_i,\ell)$, $i = 1,\ldots,N$, representing all ℓ-equivalence classes. But unfortunately we do not know how to handle the "Galois ambiguity" lurking behind the definition of this ℓ-equivalence and so we shall need a stronger cofiniteness condition stated in III below.

III. Rigid cofiniteness of H on Δ. This requires the existence of a regular rigid holonomy H on $\{f_\Omega : X_\Omega \to X_\Omega\}$ which is cofinite.

If we have a rigid cofinite H, then the projective system over all of Δ can be *canonically* reconstructed from its restriction to some large L-ball $D \subset \Delta$, where the restricted $\{f_\Omega\}_{\Omega \subset D}$ is augmented by H restricted to the pairs of ℓ-balls $D(\ell)$ and $D'(\ell)$ in $D = D(L)$ (which are not necessarily concentric to $D(L)$). Now we can truly claim that our transcendental $f : X \to X$ is given by an (elementary) algebraic object, that is $(f, H)|D$. This is important in our modulo p reduction argument which applies only to the properties of *algebraic* objects expressible in the elementary language of the field theory. (Notice that the elementary property of $f|D$ which we prove, some kind of "initial surjunctivity", is obtained with an appeal to the transcendental f. Thus one can think of our approach in two complementary ways. On one hand, we establish some (surjunctivity) property of a *transcendental* object, our f, by applying the *elementary* theory of fields to the algebraic objects (i.e. $f, H|D$). On the other hand, we derive some *elementary algebraic property* of $f|D$ by *transcendental means* (which are not so transcendental after all as they reduce to counting points in a finite projective system over larger domains $\Omega \supset D$ in Δ, where we extend our system using the holonomy H on D).

IV. Stability. This means ℓ-stability for some $\ell \geq 0$ (which can be set equal to the propagation ℓ_0 of our system).

V. Vertical irreducibility. Here "vertical" reads "ℓ-vertical for some $\ell \geq 0$".

Notice that this condition fails to be true for product projective systems such as $\{X_\Omega\} = \{\underline{X}^\Omega\}$ where $\underline{X}$ is a reducible algebraic variety and where our surjunctivity theorem (stated below) holds with no problem. One could somewhat artificially bring the two cases together by allowing $\{X_\Omega = X'_\Omega \times \underline{X}^\Omega\}$ for some vertically irreducible system $\{X'_\Omega\}$ and arbitrary (possibly reducible) $\underline{X}$. In fact it is not hard to formulate a comprehensive condition of "controlled vertical reducibility" sufficient for our theorem, but we leave this to the pleasure of the reader.

VI. Amenablility. We require that *the graph Δ is initially uniformly subamenable with respect to the pseudogroup $\Gamma = \Gamma(H)$* associated to our rigid holonomy H.

Recall (see 6.E) that uniform amenability on Δ (which refers to no Γ) suffices for this. Also observe that the uniformity is not needed for the essential part of our reasoning.

Now we return to the Galois holonomy H_{Gal} that is the set of *all* Galois holonomy maps $X_D \to X_{D'}$ and let (X°, f°) denote the corresponding holonomy completion of X. Recall, that (X°, f°) appears as the disjoint union of projective systems of the form $\{X^\bullet_{\Omega^\bullet}, f_{\Omega^\bullet}\}_{\Omega^\bullet \subset \Delta^\bullet}$ where $\Delta^\bullet$ is some limit of marked graphs (Δ, δ_i) for $\delta_i \to \infty$ and the system $\{X^\bullet_{\Omega^\bullet}, f_{\Omega^\bullet}\}$ is a limit (essentially, a prodiscrete limit, where "convergence" means "stabilization", compare 4.C) of (X_Ω, f_Ω) marked by points $\delta_i \in \Delta$ going to infinity.

7.G′. Surjunctivity Theorem. *Let a projective system of complex algebraic varieties X_Ω and regular maps $f_\Omega : X_\Omega \to X_{\Omega^-}$ over a graph Δ satisfy the above conditions I-VI, that are*

I. Bounded propagation of $\{X_\Omega\}$ and $\{f_\Omega\}$.

II. Density of the Galois holonomy on Δ.

III. Cofiniteness of a regular rigid holonomy H on Δ.

IV. Stability of $\{X_\Omega\}$.

V. Vertical irreducibility.

VI. Uniform initial subamenability of Δ.

Then the map $f^\circ : X^\circ \to X^\circ$ is surjunctive for $(X^\circ, f^\circ) = (X^\circ, f^\circ)(H_{\mathrm{Gal}})$, i.e.

$$f^\circ \textit{ is injective } \Rightarrow f^\circ \textit{ is surjective.} \qquad (o)$$

Furthermore, the maps $f^\sqcup : X^\sqcup \to X^\sqcup$ are also surjunctive for all fundamental domains $X^\sqcup$ for H_{Gal} in X° (where, observe $f^\sqcup = f^\circ|X^\sqcup$),

$$f^\sqcup \textit{ is injective } \Rightarrow f^\sqcup \textit{ is surjective.} \qquad (\sqcup)$$

About the proof. We shall (essentially) reduce (o) and ($\sqcup$) to a similar property of projective systems of *finite* sets by applying a reduction modulo p argument to a suitable fundamental domain $X^\sqcup \subset X^\circ$. This reduction goes in three steps.

(1) Translation of (o) and ($\sqcup$) into the first order language of the field theory.

(2) Invoking the extended Lefschetz principle and thus reducing the $\mathbb{C}$-problem to the corresponding $\overline{\mathbb{F}}_p$-problems for all primes p.

(3) Passing from $\overline{\mathbb{F}}_p$ to finite subfields $\mathbb{F}_{p^\nu} \subset \overline{\mathbb{F}}_p$.

The step (1) is performed in 7.H-J$'''$ where the key role is played by I (i.e. bounded propagation) and III$^-$ (Galois cofiniteness which is weaker than III). These conditions say, in effect, that our projective system is determined by finite (local in Δ) data and the combinatorics of Δ. One also uses ℓ-stability at this stage, but this is a rather technical matter.

The full strength of III, i.e. confiniteness of a rigid *regular* holonomy is needed at the next stage when we apply the Lefschetz principle. One seems to need here regular rather than Galois holonomy as the latter does not belong to the first order language of the field theory.

The condition V, i.e. the vertical irreducibility, is used at the step (3) where it ensures (by the Lang-Weil theorem) a suitable stability of the projective system of $\mathbb{F}_{p^\nu}$-varieties.

Notice that neither II (Galois density) nor VI (amenability) has been used so far. The initial amenability is used to prove surjunctivity of certain endomorphisms of projective system of finite sets (that are $\overline{\mathbb{F}}_{p^\nu}$-points of our varieties) by a counting argument (with some kind of *entropy* lurking behind the scene) where a suitably adjusted II makes this counting work in the desired way. Actually one could drop II altogether with the following weakening of the conclusion.

7.G″. If f° *is injective, then there exists a (limit) projective system* $\{X^\bullet_{\Omega^\bullet}, f_{\Omega^\bullet}\}_{\Omega^\bullet\subset\Delta^\bullet}$ *locally isomorphic to the original one, where the map* $f^\bullet = \varprojlim f^\bullet_{\Omega^\bullet}$ *is surjective. This can be written as*

$$f^\circ \textit{ is injective } \Rightarrow \textit{ some } f^\bullet \textit{ is surjective.} \qquad (o \to \bullet)$$

(Here "locally" means " r-locally for *all* $r = 1, 2, \ldots,$").

Finally, we confess that our proof of 7.G′ and 7.G″ do not use the reduction modulo p all way, as we make a shortcut at some moment which limits our considerations to projective systems over some auxiliary *finite* graph Δ'' rather than the original (infinite) Δ (see 7.L). On the other hand such a full reduction is considerd for its own sake in 7.N with some extra assumptions on $\{X_\Omega\}$.

Why f° and $f^\sqcup$? It would be more pleasant to prove the surjunctivity for f itself rather than for f° and/or $f^\sqcup$. The technical reason of bringing in these maps is the need for Δ-*uniform injectivity* of f as is explained in the next section. (Probably, there are some examples where f itself is not surjunctive under our assumptions but these must be rather exceptional).

7.H. Δ-uniform injectivity. Every exhaustion $\{\Omega_i\}$ of Δ represents X as the projective limit of a sequence, namely of $X_i = X_{\Omega_i}$ and every such representation defines our prodiscrete metric $|x - x'|$ on X (see 4.C). Now, for each $\delta \in \Delta$ we exhaust Δ by i-balls $\Omega_i = D(\delta, i) \subset \Delta$ and denote by $|x - x'|_\delta$ the resulting metric in X.

If we think of $x \in X$ as (X_δ-valued) functions on Δ, then the metric $|x - x'|_\delta$ reflects our perception of (pairs of) functions viewed from the point δ. We clearly see any distinction between x and x' at the points δ in Δ near δ, but as $\delta_\bullet$ goes further away from δ the functions x and x' come closer eventually merging in our eyes for $\mathrm{dist}(\delta_\bullet, \delta) \to \infty$. (Recall, that the inequality $|x - x'|_\delta \le 2^{-i}$ is equivalent to $x|D(\delta, i) = x'|D(\delta, i)$)).

Observe, that every two metrics $|x - x'|_{\delta_1}$ and $|x - x'|_{\delta_2}$ are bi-Lipschitz equivalent. In fact, the ratio between the two is (obviously) bounded by $2^{\mathrm{dist}(\delta_1, \delta_2)}$. Thus every finite set of these metrics is as good as any single one. But the totality of these metrics for δ ranging over all of Δ carries more topological information than our individual metrics $|x - x'|_\delta$. We are especially concerned with *uniform injectivity* of our maps $f : X \to X$ distinguished according to the following quite general

Definition. Consider a space X with a family of metrics denoted $|x - x'|_\delta$, $\delta \in \Delta$, and call a selfmapping f of X Δ-*uniformly injective* with respect to this family if for every $\epsilon > 0$ there exists $\epsilon' = \epsilon'(\epsilon) > 0$, such that $|x - x'|_\delta \ge \epsilon \Rightarrow |f(x) - f(y)|_\delta \ge \epsilon'$ for all $\delta \in \Delta$ (compare 4.F).

Lemma. *If the system* $\{X_\Omega, f_\Omega\}$ *of* $\mathbb{C}$*-varieties and regular maps is Galois cofinite* (see III$^-$ in 7.G) *then injectivity of* f° *implies* Δ*-uniform injectivity of* f *as well as* Δ*-uniform injectivity of every* $f^\bullet : X^\bullet \to X^\bullet$.

Proof. Let us choose a particular fundamental domain $X^\sqcup$ for H_{Gal} with finte sets of balls $\mathcal{D}_i^\sqcup$, such that

(a) if a ball $D = D(\delta, i)$ has $\mathrm{dist}(\delta, \delta_0) \leq i + 1$ for a fixed point (marking) $\delta_0 \in \Delta$, then it is contained in $\mathcal{D}_i^{\sqcup}$.

(b) The ball D'_- we take with the above D (see 7.C′ for notations) is $D_- = D(\delta, i-1)$ and the holonomy $X_{D_-} \to X_{D_-}$ must be the identity. Thus the (vertical) arrows in the projective system $\{X_i^{\sqcup}\}$ become eventually the old restriction maps for the inclusion between concentric balls of large radii.

Since the sets $\mathcal{D}_i$ are finite, the spaces $X_i^{\sqcup}$, being finite unions of algebraic varieties, are also algebraic and the projections $X_i^{\sqcup} \to X_{i-1}^{\sqcup}$ are compositions of regular maps with Galois automorphisms. Thus the injectivity of the map $f^{\sqcup} = \lim f_i^{\sqcup}$ (which is a "part" of f°) implies its uniform injectivity for the prodiscrete metric attached to this projective system $\{X_i^{\sqcup}\}$. On the other hand, the Δ-uniform injectivity of $\{f_\Omega : X_\Omega \to X_{\Omega^-}\}$ is equivalent, essentially by definition, to uniform injectivity of yet another system, denoted $\{X_i^{\bullet}, f_i^{\bullet}\}$, where $X_i^{\bullet}$ equals the disjoint union of X_{D_i} over all i-balls $D \subset \Delta$ with the projections $X_i^{\bullet} \to X_{i-1}^{\bullet}$ corresponding to the restriction to the concentric balls and with $f_i^{\bullet}$ being made of f_{D_i} in the obvious way. Thus $X^{\bullet} = \lim\limits_{\leftarrow} X_i^{\bullet}$ equals the disjoint union of copies of X marked by the points $\delta \in \Delta$ and so this $X^{\bullet}$ maps into $X^{\sqcup}$ with this map being isometric on each copy of X in $X^{\bullet}$ and with $f^{\bullet}$ going to $f^{\sqcup}$. This is possible due to our choice of $X^{\sqcup}$ (motivated by a possibility to have such a map). It easily follows that the uniform injectivity of $f^{\sqcup}$ implies this property of $f^{\bullet}$ and consequently the Δ-uniform injectivity of f. Q.E.D.

Now we introduce a "mixed" surjunctivity property of f via the following implication

$$f \text{ is } \Delta\text{-uniformly injective} \Rightarrow f^{\circ} \text{ is surjective.} \qquad (\star)_o$$

The above lemma obviously leads to the following

Corollary. *The implication $(\star)_o$ yields the (o)-part of the surjunctivity theorem*

$$(\star)_o \Rightarrow (o).$$

(And $(\sqcup)$ of that theorem also follows from $(\star)_o$ for *stable* projective systems over Δ as we shall see below).

Notice that surjectivity of f° amounts to the surjectivity of all $f^{\bullet}$ "locally isomorphic" to f. In particular, $(\star)_o$ tells us that

$$\Delta\text{-}\textit{uniform injectivity of } f \Rightarrow \textit{surjectivity of } f \qquad (\star)_-$$

and we shall see later on that $(\star_-) \Rightarrow (\star)$ as well as $(\star)_- \Rightarrow (\sqcup)$ for stable systems $\{X_\Omega\}$. This reduces our surjunctivity theorem to proving $(\star)_-$ under the assumptions I-VI.

7.H′. Remarks on Δ-uniform continuity etc. If we reverse the inequalities in the definition of the Δ-uniform injectivity we arrive at the notion of a Δ-*uniformly continuous map*, such that

$$|x - x'|_\delta \leq \epsilon \Rightarrow |f(x) - f(x')|_\delta \leq \epsilon'$$

for some $\epsilon' = \epsilon'(\epsilon)$ independent of δ, all $\delta \in \Delta$ and all x, x' in X. Similarly one defines Δ-*uniform Lipschitz* property for f

$$|f(x) - f(x')|_\delta \leq C|x - x'|_\delta$$

for a constant C independent of $\delta \in \Delta$ and all $\delta \in \Delta$.

Observe, that if $f = \underleftarrow{\lim}\, f_\Omega$, where the system of maps $\{f_\Omega\}$ over a graph Δ has propagation bounded by ℓ, then f is Δ-uniformly Lipschitz with $C = 2^\ell$.

It is also clear that the converse is true for maps between product spaces $X = \underset{\delta\in\Delta}{\times} \underline{X}_\delta$ and $Y = \underset{\delta\in\Delta}{\times} \underline{Y}_\delta$. Namely

$$\Delta\text{-}\mathit{Lipschitz\ for\ } f \Rightarrow \mathit{bounded\ propagation\ for\ } \{f_\Omega\}.$$

Finally observe that one can reconstruct the graph structure on Δ by looking at the metrics $|x - x'|_\delta$. Namely, δ_0 and δ_1 are joined by an edge, if and only if the corresponding metrics are 2-Lipschitz equivalent

$$\tfrac{1}{2}|x - x'|_{\delta_0} \leq |x - x'|_{\delta_1} \leq 2|x - x'|_{\delta_0}.$$

If one replaces "2-Lipschitz" by "C-Lipschitz" with some $C \geq 2$ one gets another graph, say Δ_C where edges correspond to paths of length $\leq \log_2 C$ in Δ. This Δ_C is quite similar to Δ (it is *quasiisometric* to it) and so the choice of (large !) C is not so essential. (This suggests a similar construction of a graph structure on an arbitrary set Δ of metrics on a space X).

7.I. Uniform initial injectivity and the proof of the implications $(\star)_- \Rightarrow (\star)_o$, $(\star)_o \Rightarrow (\sqcup)$ and $(\star) \Rightarrow (o)+(\sqcup)$. We say that a projective system of maps $\{f_\Omega : X_\Omega \to Y_{\Omega^-}\}_{\Omega\subset\Delta}$ is (i_0, i_+)-injective at $\delta \in \Delta$, where $0 \leq i_0 \leq i_+$ if the map $f_{D_+} : X_{D_+} \to Y_{D_+^-}$, over the ball $D_+ = D(\delta, i_+)$, (where, recall $D_+^- = D(\delta, i_+ - \ell_0)$) does not identify points in X_{D_+} with non-equal projections to $X_{D(\delta,i_0)}$. That is

$$x|D(\delta, i_0) \neq x'|D(\delta, i_0) \Rightarrow f_{D_+^-}(x) \neq f_{D_+^-}(x')$$

for all x and x' in f_{D_+}.

We say that our system $\{f_\Omega\}$ is *uniformly initially injective on* Δ if $\forall i_0 \exists i_+$ s.t. it is (i_0, i_+)-injective at all $\delta \in \Delta$.

Remark. If $\{X_\Omega\}$ is a *subproduct* system then the uniform $(0, i_+)$-injectivity obviously implies $(i_0, i_0 + i_+)$-injectivity and so there is no need to look at $i_0 > 0$.

(a) Lemma. *If a system $\{f\}_\Omega$ over a graph Δ is uniformly initially injective, then the system $\{f^\circ\}$ (and thus every system $\{f_i^\sqcup\}$) is initially injective in the sense of 5.L.*

Conversely, if $\{f_i^{\sqcup}\}$ is initially injective for some fundamental domain $\{f_i^{\sqcup}\}$ in $\{X_i^{\circ}\}$, then the system $\{f_\Omega\}$ is uniformly initially injective on Δ.

This is obvious and true for all projective systems over graphs, just unwind the definitions.

Uniform image stability and the ℓ-stability. A projective system $\{X_\Omega\}_{\Omega\subset\Delta}$ is called $(i_0, i_0+\ell)$-*stable* at $\delta \in \Delta$ if the image of the restriction map (projection) $X_{D_k} \to X_{D_0}$ for $D_k = D(\delta, i_0 + k)$ and $D_0 = D(\delta, i_0)$ does not depend on k for $k \geq \ell$. This stability *on* Δ means stability at evry $\delta \in \Delta$ and *uniform image stability* signifies that $\forall i_0 \exists \ell$, s.t. the system is $(i_0, i_0 + \ell)$-stable on Δ.

It is immediate that for each $\ell = 0, 1, \dots$

$$\ell\text{-}stability \Rightarrow uniform\ image\ stability,$$

since the ℓ-stability (defined in 7.E) says that for all finite subsets $\Omega_0 \subset \Delta$ and all Ω containing the ℓ-neighbourhood $\Omega_0^{+\ell} \supset \Omega_0$, the image of the restriction map $X_\Omega \to X_{\Omega_0}$ does not depend on Ω.

On the other hand the uniform image stability perfectly matches the notion of the image stability from 5.I according to the following

(b) Obvious lemma. *If a projective system $\{X_\Omega\}_{\Omega\subset\Delta}$ is uniformly image stable then all systems $\{X_i^{\sqcup}\}$ as well as $\{X_i^{0}\}$ are image stable. Conversely, if some fundamental domain $\{X^{\sqcup}\}$ is image stable, then $\{X_\Omega\}$ is Δ-uniformly image stable.*

Now we recall that endomorphisms f of *image stable* projective systems (obviously) satisfy (compare 5.L)

$$f \text{ is uniformly injective} \Leftrightarrow f \text{ is initially injective} \qquad (+)$$

and *proregular maps over* $\mathbb{C}$ satisfy (see 5.L)

$$injectivity \Rightarrow initial\ injectivity. \qquad (*)$$

In fact, we want to apply the latter implication to a projective system $\{f_i^{\sqcup} = X_i^{\sqcup} \to X_i^{\sqcup}\}$ where $X_i^{\sqcup}$ are constructed for the holonomy H_{Gal} with *finite* collections of balls $\mathcal{D}_i^{\sqcup}$ for all i. Here every $X_i^{\sqcup}$ is a *finite* union of algebraic varieties X_D, $D \in \mathcal{D}_i^{\sqcup}$, and so it is an algebraic $\mathbb{C}$-variety itself. Yet our projections $\pi_i^{\sqcup} : X_i^{\sqcup} \to X_{i-1}^{\sqcup}$ are not $\mathbb{C}$-regular maps as they are composed with Galois automorphisms applied to the components X_D of $X_i^{\sqcup}$. However $(*)$ still holds true (see 4.F$''$) since our projections, albeit contaminated by Galois automorphisms, still preserve constructible subset under direct and inverse images. (Alternatively, one could "unwind" the Galois part in $\pi_i^{\sqcup}$, i.e. to construct a truly proalgebraic system, say $\{\tilde{f}_i^{\sqcup} : \tilde{X}_i^{\sqcup} \to \tilde{X}_i^{\sqcup}\}$ that is equivalant to $\{f_i^{\sqcup} : X_i^{\sqcup} \to X_i^{\sqcup}\}$ by bijective (Galois) maps $X_i^{\sqcup} \leftrightarrow \tilde{X}_i^{\sqcup}$ commuting with f_i's.) Thus for *finite* $\mathcal{D}_i^{\sqcup}$ we have

$$injectivity\ of\ f^{\sqcup} \Rightarrow uniform\ injectivity\ of\ f^{\sqcup}.$$

(c) Conclusion. *Let $\{X_\Omega, f_\Omega\}_{\Omega\subset\Delta}$ be a stable projective system of $\mathbb{C}$-varieties with cofinite Galois holonomy. Then injectivity of $f^\sqcup$ on some "fundamental domain" $X^\sqcup \subset X^\circ$ for H_{Gal} implies uniform initial injectivity of $\{f_\Omega\}$ on Δ,*

$$\textit{injectivity of } f^\sqcup \Rightarrow \textit{uniform initial injectivity of } \{f_\Omega\}.$$

Proof. We can assume, taking a smaller "subdomain" in $X^\sqcup$ if necessary, that $X^\sqcup$ is made out of $X_i^\sqcup$ with finite $\mathcal{D}_i^\sqcup$ and so the above implications work. Then we conclude with $(+)$ that $f^\sqcup$ is initially injective and applying (a) we arrive at the uniform initial injectivity of $\{f_\Omega\}$ on Δ. Q.E.D.

Remark. Observe that the reverse implications are obvious,

$$\textit{uniform initial injectivity of } \{f_\Omega\} \textit{on } \Delta \Rightarrow \textit{uniform injectivity of } f$$

and

$$\textit{uniform initial injectivity of } \{f_\Omega\} \textit{on } \Delta \Rightarrow \textit{uniform injectivity of } f^\circ \textit{ and of all } f^\sqcup.$$

for an arbitrary holonomy H on $\{X_\Omega, f_\Omega\}$.

Now let us modify our implication $(\star)_-$ to the following one, denoted $(\star)$ in sequel.

$$\textit{uniform initial injectivity of } \{f_\Omega\} \textit{on } \Delta \Rightarrow \textit{surjectivity of } f = \varprojlim f_\Omega. \qquad (\star)$$

We shall prove $(\star)$ in the remaining part of §7 for *all* projective systems of $\mathbb{C}$- varieties satisfying the assumptions I-VI of the surjunctivity theorem. Here we only observe that $(\star)$ in this generality yields the surjunctivity theorem,

$$(\star) \Rightarrow (o) + (\sqcup).$$

In fact, if the system $\{f_\Omega\}$ is uniformly initially injective then so are also all (limit) systems $\{f^\bullet_{\Omega^\bullet}\}_{\Omega^\bullet\subset\Delta^\bullet}$ which are locally isomorphic to $\{f_\Omega\}$. It is equally clear that the assumptions I-VI, due to their locality and uniformity, also pass from $\{f_\Omega\}$ to all $\{f^\bullet_{\Omega^\bullet}\}$ and so $(\star)$ for all systems satisfying I-VI yields $(\star^\bullet)$, i.e. the following implication follows from $(\star)$,

$$\textit{uniform initial injectivity of } \{f_\Omega\} \textit{on } \Delta \Rightarrow \textit{surjectivity of all } f^\bullet \qquad (\star^\bullet).$$

Then, obviously

$$\textit{surjectivity of all } f^\bullet \Rightarrow \textit{surjectivity of } f^\circ \Rightarrow \textit{surjectivity of all } f^\sqcup,$$

while (c) reduces the left hand side of $(\star)$ to that of $(\sqcup)$. Q.E.D.

Finally, we observe that this argument also shows that $(\star)_- \to (\star)_\circ$ and $(\star)_\circ \Rightarrow (\sqcup)$, but we do not care about this anymore as we now deal exclusively with $(\star)$.

7.J. Initialization and localization of $(\star)$. We want to reduce $(\star)$ to a property of the projective system $\{f_\Omega : X_\Omega \to X_{\Omega^-}\}_{\Omega\subset\Delta}$ expressible in the first order language of the field theory and we start by adjusting the notion of initial surjectivity from 5.I to Δ. We say that the system $\{f_\Omega\}_{\Omega\subset\Delta}$ is $(i, i+k)$-*surjective* at $\delta \in \Delta$ if the image $f_D(X_D) \subset X_{D^{-\ell_0}}$ contains the image of the restriction map (projection) $X_{D^{+k}} \to X_{D^{-\ell_0}}$, where D denotes the ball $D(\delta, i)$ in Δ, and $D^{-\ell_0} = \mathcal{D}(\delta, i-\ell_0)$, $D^{+k} = D(\delta, i+k)$ and where ℓ_0 is the propagation of $\{f_\Omega\}$. Then we introduce the following "initialization" of $(\star)$ denoted $(\star)_i = (\star)_i(\delta_0, i_+, k_1)$

$$(0, i_+)\text{-}injectivity\ of\ \{f_\Omega\} on\ \Delta \Rightarrow (i, i+k_1)\text{-}surjectivity\ at\ \delta_0. \qquad (\star)_i$$

7.J′. Lemma. *Let $\delta_0 \in \Delta$ be an arbitrary point, $i_+ = 1, 2 \ldots$, a number and $k_1 = k_1(i)$ be an arbitrary function. Then the implications $(\star)_i$ for all $i = 1, 2 \ldots$, yield $(\star)$. In fact one only needs $(\star)_i$ for $i \geq j_0$, i.e.*

$$\bigwedge_{i=j_0}^{\infty} (\star)_i \Rightarrow (\star)$$

for every $i_+ = 1, 2 \ldots$.

Proof. Uniform initial injectivity amounts to $(0, i_+)$-injectivity for some i_+ while surjectivity follows from $(i, k_1(i))$-surjectivity valid for all large i by $(\star')$ in 5.I. : Q.E.D.

7.J″. Restricting $(\star)_i$ **to a ball.** Take the R-ball $D_R = D(\delta_0, R) \subset \Delta$, let $D_R^- = D(\delta_0, R - \ell_0 - i_+)$ and consider the following implication, denoted $(\star)_i(R) = (\star)_i(R; \delta_0, i_+, k_1 = k_1(i))$,

$$(0, i_+)\text{-}injectivity\ of\ \{f_\Omega\} on\ D_R^- \Rightarrow (i, i+k_1)\text{-}surjectivity\ at\ \delta_0 \qquad (\star)_i(R)$$

where we assume that

$$R \geq R_1(i) = i + k_1(i) + \ell_0 + i_+. \qquad (*)$$

It is obvious that

$$(\star)_i(R) \Rightarrow (\star)_i$$

for every i, and every R satisfying $(*)$.

Next we modify $(\star)_i(R)$ by replacing "$(i, i+k_1)$-surjectivity at δ_0" by "$(i, i+k_1)$-surjectivity on the ball $D(\delta_1, \rho) \subset D_R$ for *some* $\delta_1 \in D_{R-2\rho} = D(\delta_0, R-2\delta)$ where "on the ball" means "at all point in this ball". We write the resulting implication as

$$(0, i_+)\text{-}injectivity\ on\ D_R^- \Rightarrow (i, i+k_1)\text{-}surjectivity\ on\ some\ D(\delta_1, \rho). \qquad (\star)_i(R, \rho)$$

7.J‴. Lemma. *For each $i = 1, 2 \ldots$, there exists $\rho_0 = \rho_0(i)$, such that for every $\rho > \rho_0$ and every $R \geq 2\rho + i + k_1(i) + \ell_0 + i_+$, the implication $(\star)_i(R, \rho)$ yields $\star_i(R)$,*

$$(\star)_i(R, \rho) \Rightarrow \star_i(R).$$

Proof. If the system $\{f_\Omega\}$ is *not* $(i, i+k_1)$-surjective at δ_0, then each ball of radius $\rho \geq \rho_0(i)$ in Δ contains a point δ where $\{f_\Omega\}$ is not $(i, i+k_1)$-surjective. This follows from the density of the Galois holonomy (see II in 7.G) since the $(i, i+k_0)$-surjectivity is (obviously) a Galois invariant property (actually it is invariant under arbitrary automorphisms of $\{f_\Omega\}$ but it seems silly to go beyond Galois when we deal with algebraic varieties).

7.K. Reduction $(\star)_i(R, \rho)$ modulo p. Observe that the implication $\star_i(R, \rho)$ is a sentence in the first order language of the field theory or at least it becomes such a sentence if we limit the degrees of all varieties X_Ω, $\Omega \in D_R$, and maps f_Ω. Since our system has propagation $\leq \ell_0$ it is sufficient to bound the degrees of all X_Ω and f_Ω for $\operatorname{diam} \Omega \leq 10\ell_0$. We denote such a bound by d_0 (which also incorporate the dimensions of the fibers X_δ for all $\delta \in D_R$). Also, we have to bound the degrees of the holonomy maps $X_D \to X_{D'}$. Since we deal with rigid holonomies, these maps are determined by their constituents $X_\delta \to X_{\delta'}$ for all $\delta \in D$ and so we only have to bound degrees of the latter maps which we do with d_0 again. So we define $\star_i(R, \rho, d_0)$ as the above implication $\star_i(R, \rho)$ limited to systems $\{X_\Omega, f_\Omega\}$ where the degrees of X_Ω, f_Ω and H are bounded in the above (ℓ_0-local) sense. This restricted $\star_i(R, \rho, d_0) = \star_i(i_+, k_1, R, \rho, d_0)$ is a bona fide first order sentence in the elementary field theory. We want to establish this $\star_i(R, \rho, d_0)$ for large R under the following assumptions.

I_R. **The propagation of $\{X_\Omega\}$ and $\{f_\Omega\}$ on D_R are bounded by ℓ_0.** This means, for our subproduct system $\{X_\Omega\}$, that $x \in X_\Omega$ for a given $\Omega \subset D_{R-\ell_0}$, iff the restriction of x to $\Omega \cap D(\delta, \ell_0)$ is contained in $X_{\Omega \cap D(\delta, \ell_0)}$ for all $\delta \in \Omega$. And similarly, the value of $f(x)$ at each $\delta \in D_{R-\ell_0}$ is determined by the values of x on the ℓ_0-ball around δ.

II_R. **No assumption on H_{Gal}.**

III_R. **Holonomy on D_R corellated with Γ.** We consider some pseudogroup of isometries acting on Δ, denote it Γ (where, eventually, $\Gamma = \Gamma(H)$ for our regular holonomy H) and we assume that the system $\{X_\Omega, f_\Omega\}_{\Omega \subset D_R}$ admits a rigid holonomy H_R with $\Gamma(H_R) = \Gamma | D_R$.

IV_R. **Stability.** We assume our system $\{X_\Omega\}_{\Omega \subset D_R}$ to be ℓ-stable for some ℓ which, to save notation, we set equal to the propagation ℓ_0.

V_R. **Vertical irreducibility.** This means ℓ-vertical irreducibility on D_R with $\ell = \ell_0$.

Now the time came to introduce the following

Technical definition. Say that a graph Δ with a distinguished pseudogroup of isometries Γ is *locally surjunctive over a* family of fields $\{K\}$ if $\forall \ell_0, i_+, i, d_0 \ \exists k_1 \forall \rho \ \exists R_1$

s.t. for $\forall R > R_1$ the assumptions I_R, III_R, IV_R and V_R on a projective system of K-points of K-varieties over an arbitrary R-ball in Δ yield the implication $\star_i(R, \rho, d_0) = \star_i(i_+, k_1, \ell_0, \rho, R, d_0)$ for all fields $K \in \{K\}$.

Notice that ℓ_0 enters into the conditions I_R, IV_R and V_R . We also agree that "irreducibility" always means "irreducibility over the algebraic closure of the field K in question. Also, it must be understood that the (ℓ_0-local) degrees of our projective system and of the holonomy are bounded by d_0 in the above sense.

Now we use the extended Lefschetz principle and reduce $\star_i(R, \rho, d_0)$ modulo p as follows.

7.K′. *If a graph Δ of bounded valency is locally surjunctive over a family $\{\overline{\mathbb{F}}_p\}$ with infinitely many primes p, then it is locally surjunctive over every algebraically closed field of characteristic 0, in particular, over $\mathbb{C}$.*

We express this schematically by

$$\star_i(R, \rho)/\{\overline{\mathbb{F}}_p\} \Rightarrow \star_i(R, \rho)/\mathbb{C} \qquad (\overline{p} \Rightarrow \mathbb{C})$$

Next, we want to go from $\overline{\mathbb{F}}_p$ to finite fields $\overline{\mathbb{F}}_{p^\nu}$ and we do it with the following

7.K″. Lemma. *If Δ is locally surjunctive over a family of finite fields $\{\mathbb{F}_{p^\nu}\}$ where every p which appears in this family comes along with arbitrary large exponents ν, (i.e. if $\mathbb{F}_{p^{\nu_0}} \in \{\mathbb{F}_{p^\nu}\}$ then also $\mathbb{F}_{p^{\nu_i}} \in \{\mathbb{F}_{p^\nu}\}$ with arbitrary large ν_i), then Δ is also locally surjunctive over $\{\overline{\mathbb{F}}_p\}$.*

Proof. Since $\overline{\mathbb{F}}_p = \bigcup_i \mathbb{F}_{p^{\nu_i}}$ for $\nu_i \to \infty$ everything is trivial except for the (strong) ℓ-stability. The problem is that a projective system defined over $\mathbb{F}_{p^{\nu_0}}$ may be ℓ-stable on the corresponding system of the $\overline{\mathbb{F}}_p$-points but not on $\mathbb{F}_{p^\nu}$-points for any (finite) $\nu > \nu_0$. Consequently a morphism may be $(i, i+k)$-surjective on the $\overline{\mathbb{F}}_p$-points without being such on $\mathbb{F}_{p^\nu}$-points. But our vertical irreducibility condition, which says, in effect, that $\overline{\mathbb{F}}_p$-varieties, which are fibers of projections (restriction maps) in our projective system, are absolutely irreducible and hence, must be non empty over $\mathbb{F}_{p^\nu}$ when they are non empty $\overline{\mathbb{F}}_p$ by the Lang-Weil theorem. A word of caution is needed here. The Lang-Weil theorem, which claims that every absolutely irreducible variety V over $\mathbb{F}_{p^\nu}$ has a $\mathbb{F}_{p^\nu}$-point needs this variety to be projective. We do not assume our varieties V are projective, but we allow to enlarge p and/or ν if necessary, and then the Lang-Weil theorem holds for all (possibly non-projective) V. This suffices for the proof of the Lemma, which can be written as

$$\star_i(R, \rho)/\{\mathbb{F}_{p^\nu}\} \Rightarrow \star_i(R, \rho)/\{\overline{\mathbb{F}}_p\}. \qquad (p \Rightarrow \overline{p})$$

Then we combine $((\overline{p} \Rightarrow \mathbb{C}))$ and $(p \Rightarrow \overline{p})$ and see that local surjunctivity of (Δ, Γ) over $\mathbb{C}$, reduces to that over finite fields, i.e.

$$\star_i(R, \rho)/\{\mathbb{F}_{p^\nu}\} \Rightarrow \star_i(R, \rho)/\mathbb{C}.$$

On the other hand we know that $\bigwedge_i \star_i(R,\rho)$ yields the surjunctivity theorem (over $\mathbb{C}$) and so all that is left is to prove $\star_i(R,\rho)$ over finite fields. In fact we are going to prove the following purely combinatorial Lemma concerning projective systems of finite sets over initially amenable finite graphs D.

7.L. Consider a locally finite graph Δ (which here may be finite as well as infinite) of bounded valency with a distinguished finite subset $D \subset \Delta$ and a pseudogroup Γ of isometries acting on Δ such that for some (large but fixed) $r > 0$ and every $\epsilon > 0$ there exists a graph D' with a finite subset $\Omega'_\epsilon \subset \Delta'$ satisfying the following three conditions (compare 6.E$''$).

(a) Ω'_ϵ is r-locally Γ-isometric to D in Δ. This means every r-ball in Δ' with the center in Ω'_ϵ is Γ-isometric to some ball in Δ with the center in D.

(b) Ω'_ϵ contains a Γ-isometric copy of D.

(c) $\operatorname{card}\partial\Omega'_\epsilon / \operatorname{card}\Omega'_\epsilon \leq \epsilon$.

Next, let $\{X_\Omega\}_{\Omega\subset\Delta}$ be a projective subproduct system of finite sets over Δ with selfmappings $\{f_\Omega : X_\Omega \to X_{\Omega^-}\}$ commuting with the projections in the system, such that the following conditions are satisfied.

I_D. The propagation of $\{X_\Omega, f_\Omega\}$ is bounded by some number $\ell_0 \leq r/4$. In particular, this means $\Omega^- = \Omega^{-\ell_0}$ for all $\Omega \subset \Delta$.

III_D. The system $\{X_\Omega, f_\Omega\}$ admits a rigid holonomy with $\Gamma(H) = \Gamma$.

IV. **Local stability.** The system $\{X_\Omega\}$ is $2r$-locally ℓ_0-stable.

Then $\{f_\Omega\}$ satisfy the following property.

7.L$'$. Local combinatorial surjunctivity. *If the system of maps $\{f_\Omega\}$ is $(0,i_+)$-injective on Δ for some $i_+ \leq r/4$, then there exists a ball $D_\rho = D(\rho,\delta) \subset \Delta$ with $\delta \in D$ and $\rho = r/4$ such that the map $f_{D_\rho} : X_{D_\rho} \to f_{D_\rho^-}$ sends X_{D_ρ} onto the image of the restriction map $X_{D_\rho} \to f_{D_\rho^-}$ for $D_\rho^- = D(\delta, \rho - \ell_0)$.*

Proof. We proceed in four steps.

Step 1. *Reducing the general case to that where Γ preserves some local ordering on Δ.* This is done by first limiting Δ to the r-neighbourhood $D^+ \subset \Delta$ (the rest of Δ takes no part in the action anyway) and then considering the disjoint union, say $\Delta^\sqcup$ of the copies of D^+ corresponding to all possible local orderings (see 6.B) of D^+. This $\Delta^\sqcup$ comes with a natural local ordering and a pseudogroup $\Gamma^\sqcup$ preserving this ordering. Namely $\Gamma^\sqcup$ consists of those $\gamma : \Omega_1 \to \Omega_2$ from Γ (where Ω_1 and Ω_2 may lie in different copies say D_1^+ and D_2^+ of D^+) which preserve the local ordering on $\Delta^\sqcup$. The system $\{X_\Omega, f_\Omega\}$ obviously extends to $\Delta^\sqcup$ and so we may assume from now on that Γ itself preserves a local order on Δ to start with. (Notice, that we needed local finiteness of Δ at this point to have D^+ finite). So we do assume that $\Delta = D^+$ is locally ordered, Γ preserves the order and Δ' is locally ordered as well (where we can use any local order on Δ' we wish, as the copies of D^+ in $\Delta^\sqcup$ give all possibilities).

Step 2. *Extension* $\{X_\Omega, f_\Omega\}$ *to* Δ'. Now, for each $\Omega' \subset \Delta'$ contained in some r-ball $D(\delta', r)$, for $\delta' \in \Omega'$, we have Γ-isometry $\gamma' : \Omega' \to \Omega \subset D(\delta, r)$ for some $\delta \in D$ and this γ', if it exists for some Ω, is unique as it preserves local order. Then we can *define* $X_{\Omega'}$ and $f_{\Omega'}$ by declaring $X_{\Omega'} = X_\Omega, f_{\Omega'} = f_\Omega$. Finally we define $X_{\Omega'}$ for all (larger) subsets Ω' in Δ' by postulating the subproduct property of the new system and the propagation $\leq \ell_0$ condition. Since ℓ_0 is significantly smaller than r, this uniquely defines our extended system. It is also clear that the extended system $\{f_{\Omega'}\}$ is $(0, i_+)$-injective since i_+ is small compared to r.

Step 4 *Proving that there exists a ball* $D'_\rho = D(\delta', r/4) \subset \Delta'$, $\delta' \in \Omega'$, *such that* $f_{D'_\rho}$ *sends* $X_{D'_\rho}$ *onto the image of the projection* (restriction) $X_{D'_\rho} \to X_{D'_{\rho - \ell_0}}$. This is the main step which is performed by a simple entropy style counting argument presented below which works if ϵ is sufficiently small.

Suppose we have some points $\delta'_1, \delta'_2, \dots, \delta'_N$ in the r-interior of Ω'_ϵ with mutual distances $\geq 2r$ such that the above onto property fails to the true for all balls $D(\delta'_i, r/4)$, $i = 1, \dots, N$. Denote by X_i the images of the projections (restrictions) $X_{D(\delta'_i, \rho)} \to X_{D(\delta'_i, \rho - \ell_0)}$, $\rho = r/4$, $i = 1, \dots, N$, and consider the projection π_i from $X = X_{\Omega'_\epsilon}$ to these X_i. The local stability of our system extended to Δ gives us the r-local ℓ_0-stability on Ω'_ϵ. this implies (by a trivial globalization argument as in 7.E′) that the product map

$$\pi_1 \times \pi_2 \times \cdots \times \pi_N : X \to X_1 \times X_2 \times \cdots \times X_N$$

is *onto*. Furthermore, for each $i = 1, \dots, N$ the fibers of the projection $\pi_i : X \to X_i$ satisfy

$$\operatorname{card} \pi_i^{-1}(x) \geq \alpha \operatorname{card} X \qquad (*)_i$$

for all $x \in X_i$ and some $\alpha = \alpha(\Delta) > 0$ independent of Δ'. In fact

$$\alpha^{-1} \leq \max_{D_r} \bigcap_{\delta \in D_r} \operatorname{card} X_\delta \qquad (+)$$

for all r-balls D_r in D or, equivalently, in Ω. In fact the stability of our system on the ball $D_i = D(\delta'_i, r/2)$ reduces $(*)_i$ to a similar lower bound on the cardinality of the "complementary space" to X_i. This space, say Y_i, is defined as the image of the restriction map $X_{\Omega'_\epsilon} \to X_{\Omega'_i}$ for $\Omega'_i = \Omega'_\epsilon \setminus D_i$. Here it is obvious that

$$\operatorname{card} Y_i \geq \alpha \operatorname{card} X$$

with the above bound on α^{-1}.

The inequality $(*)_i$ shows that missing a single point $x_i \in \pi_i(X) \subset X_i$ by the map $f_{D'_\rho}$ bounds the image of the map $f_{\Omega'_\epsilon}$ by

$$\operatorname{card} f_{\Omega'_\epsilon}(X) \leq (1 - \alpha) \operatorname{card} X.$$

We want to iterate this N times and obtain the inequality

$$\operatorname{card} f_{\Omega'_\epsilon}(X) \leq (1 - \alpha)^N \operatorname{card} X, \qquad (*)^N$$

where we need, roughly speaking, the inequalities $(*)_i$ to be independent for $i = 1, \ldots, N$. In other words, we need a quantitative version of the onto claim for the above map $\pi_1 \times \pi_2 \times \cdots \times \pi_N : X \to X_1 \times X_2 \times \cdots \times X_N$ incorporating N independent versions of $(*)_i$ as follows.

Take some $x_i \in X_i$, for all $i = 1, 2, \ldots, N$, and set $X(i) = \bigcap_{j=1}^{i} \pi_i^{-1}(x_i)$. In other words, $X(i)$ equals the pull-back of $(x_1, x_2, \ldots, x_i) \in X_1 \times X_2 \times \cdots \times X_i$ under the map $\pi_1 \times \pi_2 \times \cdots \times \pi_i : X \to X_1 \times X_2 \times \cdots \times X_i$.

Now we claim the following refinement of $(*)$

$$\operatorname{card} X(i) \geq \alpha \operatorname{card} X(i-1) \qquad (*)_i^i$$

for all $i = 2, 3, \ldots, N$, and α^{-1} bounded by $(+)$ as earlier.

Proof of $(*)_i^i$. We argue as in the proof of $(*)_i$ with the space $X(i-1)$ playing the the role of X. Namely, we denote by Π_i the restriction of π_{i-1} to $X(i-1)$ and observe that $X(i) = \Pi_i^{-1}(X_i)$. To apply our previous reasoning to $\Pi_i : X(i-1) \to X_i$ we need this $X(i-1)$ to emerge as the projective limit of a projective system over Ω'_ϵ where this system must be ℓ_0-stable on the ball $D_i = D(\delta'_i, r/2)$. We define this system $\{X_{\Omega'}(i-1)\}$, $\Omega' \subset \Omega'_\epsilon$, as a subsystem in $\{X_{\Omega'}\}$ where $x \in X_{\Omega'}$ is contained in $X_{\Omega'}(i) \subset X_{\Omega'}$ if and only if the restriction of x to $D(\delta'_j, r/4 - \ell_0)$ equals x_j for $j = 1, 2, \ldots, i-1$. Notice that this condition makes sense only if $D(\delta'_j, r/4 - \ell_0) \subset \Omega'$; otherwise, it is a vacuous (and, in particular, $X_{\Omega'}(i-1) = X_{\Omega'}$ if $D(\delta'_j, r/4 - \ell_0) \not\subset \Omega'$ for all $j = 1, 2, \ldots, i-1$) . Clearly, this projective system $\{X_{\Omega'}(i)\}$ equals to the old $\{X_{\Omega'}\}$ on D_i and so the proof of $(*)_i$ yields $(*)_i^i$ as well. Q.E.D.

Now, obviously $(*)_i^i \Rightarrow (*)^N$ exactly as we wanted it. On the other hand the $(0, i_+)$-injectivity of $\{f_\Omega\}$ implies that the cardinality of this image is bounded from below by

$$\operatorname{card} f_{\Omega'_\epsilon}(X) \geq \operatorname{card} X_{\Omega'_-}$$

where $\Omega'_- \subset \Omega'_\epsilon$ denotes here the $(\ell_0 + i_+)$-interior of Ω'_ϵ. Next, we invoke the bound on $\operatorname{card} \partial\Omega'_\epsilon$, that is

$$\operatorname{card} \partial\Omega'_\epsilon \leq \epsilon \operatorname{card} \Omega'_\epsilon$$

and the bound on $\operatorname{card} X_{\delta'}$ by a constant C depending on Δ but not on Δ'. It follows that

$$\operatorname{card} X_{\Omega'_-}(X) \geq \operatorname{card} X / C^{\epsilon \operatorname{card} \Omega'_\epsilon}$$

and so

$$(1-\alpha)^N \geq 1/C^{\epsilon \operatorname{card} \Omega'_\epsilon}.$$

But, recall, α (as well as C) is fixed, independently of Δ' while ϵ can be made arbitrarily small. Then, unless $\partial\Omega'_\epsilon$ is empty and everything is obvious, each connected component of Δ' grows in diameter as $\epsilon \to 0$ and so we can find as many disjoint balls as we want in Ω'_ϵ. Moreover, since the graphs Δ' have uniformly bounded valences, we can find in each of them $N \geq \beta \operatorname{card} \Omega'_\epsilon$ balls $D(\delta'_i, \rho)$, $i = 1, \dots, N$, where $\beta > 0$ does not depend on Δ', such that these balls lie in the r-interior of Ω'_ϵ and have $\operatorname{dist}(\delta_i, \delta_j) \geq 2r$ for $i \neq j$. Thus we arrive at the inequality

$$(1-\alpha)^{\beta \operatorname{card} \Omega'_\epsilon} \geq 1/C^{\epsilon \operatorname{card} \Omega'_\epsilon}$$

which leads to a contradiction for $\operatorname{card} \Omega'_\epsilon \to \infty$ and $\epsilon \to 0$. So we must admit the existence of a ball $D'_\rho = D(\delta', r/4)$ in $\Omega' \epsilon$ where $f_{D'_\rho}$ goes *onto* the image of the restriction map from $X_{D'_\rho}$ to $X_{D'_{\rho - t_0}}$.

Finally, we recall that our system over Δ' is locally isomorphic to the original system over Δ and so such a ball already exists in Δ.

7.M. Conclusion of the proof of the surjunctivity theorem. It is obvious that the local combinatorial surjunctivity yields $\star_i(R, \rho)$ over all finite fields and with the above $((\overline{p} \Rightarrow \mathbb{C}))$ and $(p \Rightarrow \overline{p}))$ we obtain $\star_i(R, \rho)/\mathbb{C}$ for all i. This yields the desired (o) and ($\sqcup$) in 7.G$'$, as was shown in the preceding sections. Q.E.D.

Finally we observe that initial amenability of a Cayley graph of q group Γ implies this property for Γ and so our claims on surjunctivity over Γ follows from (o) with the discussion in 6.E.

7.N. Lefschetz principle for the global surjunctivity. Our reduction of the surjunctivity over $\mathbb{C}$ to a combinatorial injectivity took place after the problem had been reduced to a first order proposition. It would be more esthetically pleasing to have such a reduction in the original setting which is transcendental but is not overburdened by nested quantifiers. Here are several such reductions. We start with the (almost) homogeneous situation where a locally compact group acts on a countable set Δ withe finitely many orbits and with compact isotropy subgroups. (One can assume here Δ is a locally finite graph and $\Gamma \subset \operatorname{isom} \Delta$). Then

(A) *If for every finite set F all continuous Γ-equivariant selfmappings of F^Δ are surjunctive, then such are also all Γ-equivariant proregular maps $f : \underline{X}^\Delta \to \underline{X}^\Delta$ where $\underline{X}$ is an arbitrary algebraic $\mathbb{C}$-variety. In fact one can claim here surjunctivity of all proconstructible selfmappings of $\underline{X}^\Delta$ for all K-constructible spaces $\underline{X}$ where K is an arbitrary uncountable algebraically closed field.*

Next we look at Γ-equivariant endomorphisms of certain Γ-*invariant* subsets $Y \subset F^\Delta$ and $X \subset \underline{X}^\Delta$. These appear as projective limits of Γ-invariant projective subsystems $\{Y_\Omega \subset F^\Omega\}$ and $\{Y_\Omega \subset \underline{X}^\Omega\}$. We observe that Δ admits a Γ-equivariant locally finite graph structure and so we can attribute such property as stability, bounded propagation etc. to these systems and thus to X and Y.

(A$_1$) *Let surjunctivity be satisfied by all stable Γ-equivariant $Y \subset F^\Delta$ of bounded propagations (i.e. for all stable subshifts of finite type) and all finite sets F. Then one also has surjunctivity for all Γ-equivariant proregular maps of stable and vertically irreducible Γ-invariant proalgebraic subvarieties of bounded propagation X in $\underline{X}^\Delta$, where $\underline{X}$ is the set* *of K-point of an arbitrary algebraic variety over K for K being an uncountable algebraically closed field and where we additionally assume that* $\dim(\Delta|\lambda)$ *grows sublinearly for* $\lambda \to \infty$, *i.e.* $\limsup_{\lambda\to\infty} \lambda^{-1}\dim(\Delta|\lambda) = 0$.

Remark. Such properties as bounded propagation, stability etc, we attribute to Y and X, refer, in fact to the projective system defining these spaces.

(A_1') *The same as above with "strongly stable" instead of "stable" and with no assumption on* $\dim(\Delta|\lambda)$ *anymore.*

Notice that in both cases (A_1) and (A_1') one may allow arbitrary proconstructible endomorphisms over uncountable algebraically closed fields. Actually, one can admit more general fields, such as increasing unions of ultraproducts of $\mathbb{F}_{p^\nu}$ with $p, \nu \to \infty$, and proregular maps. (Probably one can replace "proregular" by prodefinable in this case).

This proof of these is indicated below in 7.N′. (Notice that for all we know the combinatorial surjunctivity assumption may always be satisfied in the cases we consider and so these reductions have a purely academic interest at the present moment).

Now we generalize the above by considering a graph Δ with a given *pseudogroup* of isometries, say Γ acting on Δ. We study projective systems over Γ with rigid holonomies H, such that $\Gamma(H) = \Gamma$, and where the holonomy must be regular (or at least, constructible) when we deal with $X \subset \underline{X}^\Delta$ for algebraic varieties $\underline{X}$. Here, given an endomorphism f of X, we also need to look at all $f^\bullet : X^\bullet \to X^\bullet$ locally isomorphic to $f^\bullet$, e.g. coming from the projective systems $\{X^\bullet_{\Omega^\bullet}, f^\bullet_{\Omega^\bullet}\}_{\Omega^\bullet\subset\Delta^\bullet}$ over all graph $\Delta^\bullet$ locally Γ-isomorphic to Δ, for the above (fixed) Γ. Then we deal with the following *weak surjunctivity*

$$\text{all } f^\bullet \text{ are injective} \Rightarrow \text{some } f^\bullet \text{ is surjective.}$$

Notice that this is equivalent to the *strong surjunctivity*,

$$\text{all } f^\bullet \text{ are injective} \Rightarrow \text{all } f^\bullet \text{ are surjective}$$

in the presence of an auxiliary (possibly non-regular, e.g. Galois) holonomy which is dense on Δ (compare discussion in 7.G′ concerning II).

(B, B_1, B_1') *The above* (A), (A_1) *and* (A_1') *extend to the present case with "surjunctivity" replaced by "weak surjunctivity". And this is also true for the strong surjectivity if the holonomy H is dense as well as cofinite on Δ.*

Here again we do not know if *all* maps in question are *always* surjunctive but making counterexamples seem more feasable with non-homogeneous graphs Δ.

7.N′. About the proofs. We have seen in sections 7.J-J‴ how surjunctivity over Δ reduces to a finitely statement $\star_i(R, \rho)$ which admits a reduction to a finite field. What we need to do now is to go backward and to *derive* $\star_i(R, \rho)$ from the global surjunctivity, at least over finite fields (or in the combinatorial framework in general). This is done by unwinding our argument in 7.J-J‴ where the key step is a deriving surjectivity from initial surjectivity. This needs stability of our system which is, a priori, *not* a first order property. But r-*local* stability is eventually expressible in the first order language as well as the strong r-local stability. And we know (see 7.E-E''') that the latter is localizable while the former is localizable if $\dim(\Delta|\lambda)$ grows sublinearly. So the reduction modulo p is possible in these cases. We leave the reader at this point to the (questionable) pleasure of browsing through all our quantifiers and checking that everything works as expected.

7.O. About non-rigid holomies. Non-rigidity of a holonomy is manifested by the presence of non-trivial *holonomy groups* $H^0(\Omega)$, $\Omega \subset \Delta$, where $H^0(\Omega) \subset H$ consists of all $h : X_\Omega \to X_\Omega$ with $\gamma_h : \mathrm{Id} : \Omega \to \Omega$. These $H^0(\Omega)$ make a projective system of groups over Δ where each group $H^0(\Omega)$ acts on the space X_Ω and, moreover, on the whole projective (sub)system $\{X_{\Omega'}\}_{\Omega' \subset \Omega}$.

Examples (a). Let $X_\Omega = (\mathbb{C}^n)^\Omega$ and the holonomy is defined with all *linear* maps between the fibers $X_\delta = \mathbb{C}^n \to \mathbb{C}^n = X_\delta$ for all $\delta, \delta' \in \Delta$. Then $H^0(\Omega) = (GL_n)^\Omega$ is an algebraic group algebraically acting on the spaces $(\mathbb{C}^n)^\Omega = \mathbb{C}^{n\,\mathrm{card}\,\Omega}$.

(b) The ultimate regular holonomy in the above example is given by the group A_n of *all* biregular automorphisms of $\mathbb{C}^n$. It is not an algebraic group for $n \geq 2$.

There is an obvious way to rigidify a holonomy, just replace Ω_Ω by $\underline{X}_\Omega = X_\Omega / H^0(\Omega)$. The resulting quotient spaces are "almost algebraic" for an algebraic group $H^0(\Omega)$, namely, these are constructible spaces. So we can admit non-rigid algebraic holonomies as well as those which are contained in algebraic ones. Here one must be careful to make sure that injectivity and surjectivity of our maps do not suffer from the above factorization. For this we need all restriction maps $H^0(\Omega) \to H^0(\Omega^-)$ to be onto which can always be achieved with stablization by redefining

$$H^0_{\mathrm{new}}(\Omega) = \cap \text{ (images of the projection from } H^0(\Omega^+) \text{ for all } \Omega^+ \supset \Omega).$$

Also, if we do not want to leave the category of algebraic varieties, we can replace the "fibers" $X_\delta = X_{\{\delta\}}$ acted upon by H^0_δ by Zariski open subsets $Y_\delta \subset (X_\delta)^N$ for large N such that the diagonal action of H^0_δ on Y_δ is *free* (assuming the actions of H^0_δ on X_δ are faithful) and thus making quotients Y_δ / H^0_δ look more agreable.

7.P. Further directions. It seems that a natural framework for our surjunctivity theorem is given by the category of subproalgebraic spaces, i.e. suitable quotients of proalgebraic spaces with *approximate* actions of certain pseudogroups satisfying some "stability" (including expansiveness, a kind of uniform mixing and possibly ϵ-shadowing) and amenability (which may be unnecessary) conditions. (One may start here with *algebraic sofic systems*, i.e. images of subproduct systems with finite propagation under proregular morphisms with finite propagation).

In fact it may be worthwhile to start with reformulating our surjunctivity theorem in terms of the spaces $X = \varprojlim X_\Omega$ and X° themselves without dirctly mentioning X_Ω. All this does not seem hard to accomplish but I have not worked out satisfactory examples (and making up examples looks like a non-trivial issue) to justify the efforts needed for such generalization. But what appears to me more exciting is developing a general view (forget surjunctivity!) on the equivariant proalgbraic category, e.g. visualizing "symbolic algebraic geometry" of Γ-invariant proalgebraic subvarieties in $\underline{X}^\Gamma$ and Γ-equivariant proregular maps between these varieties.

§ 8. Appendix: Garden of Eden, entropy and surjunctivity.

Recently, Antonio Machi explained to me that the dynamical surjunctivity problem surfaced earlier in cellular automata under the name of *Garden of Eden theorem*. Here, following the idea of Moore and Myhill (but not their terminology), we say that a map f from a subproduct space $X \subset \times_{\delta\in\Delta} \underline{X}_\delta$ to some Y is *preinjective*, if $f(x) \neq f(x') \Rightarrow x \neq x'$ provided $x(\delta) = x'(\delta)$ for all but finitely many $\delta \in \Delta$ (where $x(\delta)$ stands for the projection of $x \in X \subset \underline{X}_\delta$ to $\underline{X}_\delta$). For example, if X is a linear space of functions x on Δ and f is a linear operator, then "preinjective" amounts to "injective on the subspace of functions $X_0 \subset X$ having finite support". (In the language of "Garden of Eden" one speaks of *mutually erasable patterns* x and x'. This means $x \neq x'$, yet $f(x) = f(x')$, where $x(\delta) = x'(\delta)$ for $\delta \in \Delta\backslash$ (finite subset). Then preinjectivity expresses the absence of mutually erasable patterns.)

Clearly, preinjectivity is much weaker than injectivity; yet it is often good enough to imply surjectivity. For example, let Γ be a finitely generated amenable group, $\underline{X}$ be a finite set and $f : \underline{X}^\Gamma \to \underline{X}^\Gamma$ a continuous equivariant map (which necessarily has finite propagation).

8.A. Theorem. (see [Ce-Ma-Sca]). *The map f is surjective if and only if it is preinjective.*

8.A′. Remarks. (a) This result for $\Gamma = \mathbb{Z}^n$ is due to Moore and Myhill and is called the *Garden of Eden theorem* where $x \in \underline{X}^\Gamma$ is called a *Garden of Eden configuration* if it is *not* in the image of f. Thus "surjective" acquires a nostalgic overtone: "no way to reach the Garden of Eden". (The implication

$$\text{surjective} \Rightarrow \text{preinjective}$$

was proven by Moore in [Moo] followed by the converse implication

$$\text{preinjective} \Rightarrow \text{surjective}$$

observed by Myhill in [Myh]. Notice that the latter sharpens the surjunctivity as defined by Gottschalk.)

(b) Machi and Mignosi has proven earlier (see [Ma-Mi]) these results for groups Γ of subexponential growth.

(c) The proof of the theorem in all cases depends on an entropy type computation similar to what we do in 7.L. Actually the meaning of the theorem becomes clearer if the entropy enters the statement as well as the proof so we give a definition of entropy suitable for this purpose.

8.B. Entropy. Let $X \subset \underset{\delta\in\Delta}{\times} \underline{X}_\delta$ be a subproduct space and $\Omega_i \subset \Delta$, $i = 1, 2, \ldots$, be a sequence of finite subsets. Let $\overline{X}_i = \overline{X}_{\Omega_i} \subset \underset{\delta\in\Omega_i}{\times} \underline{X}_\delta$ denote the "restriction" of X to Ω_i, i.e. the projection of X to the finite product $\underset{\delta\in\Omega_i}{\times} \underline{X}_\delta$ and set

$$\operatorname{ent}(X) = \operatorname{ent}(X : \{\Omega_i\}) = \liminf_{i\to\infty} \operatorname{card}^{-1}(\Omega_i) \log(\operatorname{card} \overline{X}_i).$$

Clearly, the entropy is monotone for inclusions between subsets, i.e. all $X' \subset X$ have

$$\operatorname{ent} X' \leq \operatorname{ent} X\,,$$

and more significant inequalities of this nature are indicated below.

8.C. Monotonicity. Let Δ be an infinite connected graph of bounded valency (as in § 6) and consider a map of *bounded propagation* between two subproduct spaces, say $f : X \to Y$ for $X \subset \underset{\delta\in\Delta}{\times} \underline{X}_\delta$ and $Y \subset \underset{\delta\in\Delta}{\times} \underline{Y}_\delta$, where "bounded propagation" means that the value $y(\delta)$ for $y = f(x)$ and a given $\delta \in \Delta$, depends only on the values of x on the ℓ-ball $D(\delta, \ell) \subset \Delta$ for some $\ell < \infty$ independent of δ.

8.C′. *If the sequence Ω_i is amenable, i.e.* $\operatorname{card} \partial\Omega_i / \operatorname{card} \Omega_i \underset{i\to\infty}{\longrightarrow} 0$, *(see 6.E), the cardinalities of $\underline{X}_\delta$ are bounded, i.e.* $\sup_{\delta\in\Delta} \operatorname{card} \underline{X}_\delta < \infty$, *and the map f is surjective, then*

$$\operatorname{ent} Y \leq \operatorname{ent} X$$

i.e. $\operatorname{ent} f(X) \leq \operatorname{ent} X$ *for all maps f.*

Proof. The cardinality of $\overline{Y}_i = \overline{Y}_{\Omega_i}$ does not exceed that of $\overline{X}_{\Omega_i^{+\ell}}$, where $\Omega_i^{+\ell}$ denotes as earlier the ℓ-neighbourhood of Ω_i, i.e. $\Omega_i^{+\ell} = \Omega_i \cup \partial_\ell\Omega_i$ where $\partial_\ell\Omega$ denotes the *ℓ-iterated boundary* of Ω, i.e. the set of the centers of the ℓ-balls which meet Ω as well as the complement of Ω. Clearly, the amenability of Ω_i and the bounds on the valency of Δ and the cardinality of $\underline{X}_\delta$ make

$$\lim_{i\to\infty} \operatorname{card} \partial_\ell\Omega_i / \operatorname{card} \Omega_i = 0\,,$$

and (A) trivially follows as the contribution from $\partial_\ell\Omega_i$ to $\log \operatorname{card} \overline{Y}_{\Omega_i}$ is bounded by $\operatorname{card} \partial_\ell\Omega_i (\log \sup \operatorname{card} \underline{X}_\delta)$ which is $O(\operatorname{card} \Omega_i)$.

Splicable spaces. Given $\Omega \subset \Delta$ and two "functions" x_0 and x_1 in $\underset{\delta\in\Delta}{\times} \underline{X}_\delta$ we define their *splice* over Ω as the function x on Δ which equals x_0 on Ω and x_1 outside Ω. We say that a subspace $X \subset \underset{\delta\in\Delta}{\times} \underline{X}_\delta$ is *ℓ-splicable* if the conditions $x_0, x_1 \in X$ and $x_0 = x_1$ on $\partial_\ell \Omega$ imply $x \in X$ for all finite subsets $\Omega \subset \Delta$.

Example. If X equals the projective limit of an ℓ-stable (see 7.E) projective system over Δ of propagation $\leq \ell$ then, clearly, X is ℓ-splicable.

8.C″. *If X is ℓ-splicable for some ℓ and $f : X \to \underset{\delta\in\Delta}{\times} \underline{Y}_\delta$ is a preinjective map of bounded propagation then $Y = f(X)$ has* $\operatorname{ent} Y = \operatorname{ent} X$.

Proof. If $\operatorname{ent} Y < \operatorname{ent} X$, then $\operatorname{card} \overline{Y}_{\Omega_i}$ is much smaller than $\operatorname{card} \overline{X}_{\Omega_i + \ell}$ for large i and so there are two functions x_0 and x_1 which are different on Ω_i but such that $f(x_0) = f(x_1)$ on Ω_i. Moreover, one can assume these x_0 and x_1 are equal on $\partial_\ell \Omega_i$ as the latter condition only has a minor effect on the cardinalities for large i. Then the splice x of x_0 and x_1 is also in X, where x and x_1 now equal at infinity and have equal images in Y. Thus making f *not* preinjective. Q.E.D.

8.D. Strict monotonicity. Now, let us express the computation at Step 4 in 7.L′ in the language of entropy. We consider infinitely many ρ-balls $D_j \subset \Delta$, $j = 1, \ldots$ which constitute a net in Δ, i.e. some R-neighbourhood of their union equals all of Δ. Then we consider a subset X' in a subproduct space $X \subset \underset{\delta\in\Delta}{\times} \underline{X}_\delta$ such that $X' \subset X$ is strictly smaller than X on every ball D_j, i.e. $\overline{X}'_{D_j} \subsetneq \overline{X}_{D_j}$.

8.D′. *If X is a stable space of bounded propagation (i.e. it equals the projective limit of a system $\{X_\Omega \subset \underset{\delta\in\Omega}{\times} \underline{X}_\delta\}$ with these properties) then*

$$\operatorname{ent} X' < \operatorname{ent} X\,,$$

where the entropy is measured with respect to a given amenable sequence $\Omega_i \subset \Delta$ and where we assume as earlier that the valency of Δ and the cardinalities of $\underline{X}_\delta$ are bounded.

Proof. We may assume (throwing away some balls if necessary) that the mutual distances between the balls are large say $\leq 10\ell$ for ℓ being the stability and the propagation constant. We take our balls within some large amenable Ω_i, say $D_{j_1}, \ldots, D_{j_N}$, $N = N_i = N(\Omega_i)$, and consider the map from $\overline{X}_i = \overline{X}_{\Omega_i}$ to the product of $\overline{X}_{j_k} = \overline{X}_{D_{j_k}}$ as in 7.L′. Here again this map

$$\pi_1 \times \pi_2 \times \ldots \pi_N : \overline{X}_i \to \overline{X}_{j_1} \times \ldots \times \overline{X}_{j_N}$$

is onto. Moreover, missing a single point in each X_j diminishes the cardinality of $\overline{X}_i$ by at least a factor $1 - \alpha$ for a fixed $\alpha > 0$. Thus

$$\operatorname{card} \overline{X}'_{\Omega_i} \leq (1 - \alpha)^{N_i} \operatorname{card} \overline{X}_{\Omega_i}\,,$$

where $\lim_{i\to\infty} \inf N_i / \operatorname{card} \Omega_i > 0$ due to the density of the balls D_j in Δ, and our claim follows.

8.E. Preinjectivity corollary. *Let $f : X \to \underset{\delta\in\Delta}{\times} \underline{Y}_\delta$ be a map of bounded propagation where the corresponding projective system of maps $\{f_\Omega : X_\Omega \to \underset{\delta\in\Omega}{\times} \underline{Y}_\delta\}$ admits a rigid dense holonomy. Then the equality $\operatorname{ent} f(X) = \operatorname{ent} X$ implies that f is preinjective.*

Proof. If f is not preinjective, there exists a system of balls D_j making a net in Δ, and of pairs $f(x_j)$ of functions x_j and $x'_j \neq x_j$ in X, such that each x'_j equals x_j outside D_j and $f(x'_j) = f'(x_j)$ for all j. Actually, the existence of a single ball follows from the definition of (non)preinjectivity and then the holonomy carries them densely spread in Δ. Then we take $X' \subset X$ consisting of those x, where no restriction of x to any D_j equals $x_j \mid D_j$ (but may be equal to x'_j). Clearly $f(X') = f(X)$ while $\operatorname{ent}(X') < \operatorname{ent}(X)$ by 8.5.A. Thus $\operatorname{ent} f(X) = \operatorname{ent} f(X') \leq \operatorname{ent}(X') < \operatorname{ent}(X)$ by 8.4.A.

8.E′. Surjectivity corollary. *Let $f : X \to Y$ be the projective limit of a system $\{f_\Omega : X_\Omega \to Y_\Omega\}$ of finite propagation admitting a dense rigid holonomy where $\{Y_\Omega\}$ is stable. Then the equality $\operatorname{ent} Y = \operatorname{ent} f(X)$ implies that f is surjective.*

Proof. If the image $Y' = f(X) \subset X$ misses some $y \in Y$, there exists a ball D, such that $y|D$ does not equal to $y'|Y$. Then the dense holonomy carries D densely over Δ, where 8.5.A applies to the resulting balls D_j and our $Y' \subset Y$ which is smaller than Y on each D_j. Q.E.D.

8.F. Garden of Eden theorem for stable spaces. Now, let both X and Y be stable subproduct spaces of bounded propagation and $f : X \to Y$ be a map (coming from a projective system of maps $f_\Omega : X_\Omega \to Y_\Omega$) of bounded propagation and admitting dense rigid holonomy.

8.F′. *If $\operatorname{ent} X = \operatorname{ent} Y$ (e.g. if $X = Y$) then f is surjective if and only if it is preinjective.*

Proof. If f is surjective, i.e. $f(X) = Y$, then $\operatorname{ent} X = \operatorname{ent} f(X)$ which implies preinjectivity according to 8.E. Conversely, if f is preinjective, then $\operatorname{ent} X = \operatorname{ent} f(X)$ by 8.C′ and then f is surjective by 8.E′.

Notice that 8.F′ generalizes 8.A in three respects. First of all 8.6.A applies to certain subproduct systems which are not product (or full shift) spaces. Second of all we need only partial symmetry of $\{f_\Omega : X_\Omega \to Y_\Omega\}$. And finally the equality $X = Y$ is relaxed to the numerical relation $\operatorname{ent} Y = \operatorname{ent} X$.

8.G. Examples of non-injective preinjective maps. The simplest such map is the difference operator on $X = K^{\mathbb{Z}}$, for any field K, by $f : x(z) \mapsto x(z+1) - x(z)$. Clearly this f is surjective and preinjective but not injective as constants go to zero. More generally, if $\gamma_0 \in \Gamma$ is a non-torsion element in an arbitrary γ_0, then the operator $f : x(\gamma) \mapsto x(\gamma_0 \gamma) - x(\gamma)$ is also preinjective (and surjective but not injective). Furthermore, if $\Gamma = \Gamma_1 \times \mathbb{Z}$, then every "Cauchy operator" has this property, $f : x \mapsto \partial x + F(\gamma_i x)$, where $\partial x = x - z_0 x$, for z_0 being a non-trivial element in $\mathbb{Z}$, acting by translation, i.e. $z_0 x(\gamma) = x(z_0 \gamma) \cdot \gamma_i$, $i = 1, \dots, k$ are some elements in Γ_1, and F is an arbitrary function $K^k \to K$.

Finally, the Laplace operator $\Delta : \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is preinjective but not injective for every (possibly pure torsion) finitely generated group Γ. In fact, if a function $x = x(\gamma)$ with finite support satisfies $\Delta x = 0$, then

$$\langle x, \Delta(x) \rangle_\Gamma \underset{\text{deg}}{=} \sum_{\gamma \in \Gamma} x(\gamma)\, \Delta(x)(\gamma) = 0$$

and then by the standard computation (or by the definition of Δ),

$$\langle x, \Delta(x) \rangle_\Gamma = \langle \operatorname{grad} x, \operatorname{grad} x \rangle_\Gamma$$

where

$$\operatorname{grad} x = (x - \gamma_1 x,\ x - \gamma_2 x, \dots x - \gamma_k x)$$

for a system of generators γ_i of Γ.

Probably, such a Laplace operator, (or the associated diffusion operator) $x \mapsto \sum_{i=1}^{k} \gamma_i x$ is typically preinjective also on $X = K^\Gamma$ for finite fields K. For example, if for every finite subset $B \subset \Gamma$ (serving as the support of x) there is a translate γB which meet the generating ball $\{\gamma_1, \dots \gamma_k\} \subset \Gamma$ at a single point, then clearly, the diffusion operator is preinjective (as every $x : \Gamma \to K$ with support B must vanish).

8.H. Dynamical meaning of preinjectivity. Given a Γ action on a metric space X, the orbits of points x and x' are called *asymptotic*, if $\operatorname{dist}(\gamma(x), \gamma(x')) \to 0$ for $\gamma \to \infty$. Then one may speak of preinjective maps $f : X \to Y$ as those where

$$x \neq x' \Rightarrow f(x) \neq f(x')$$

for the points with asymptotic orbits. Then the Garden of Eden theorem can be generalized to a suitable class of hyperbolic actions but for groups Γ which are much "greater" that $\mathbb{Z}$ most of such actions are, probably, subshifts anyway.

8.I. Remarks on non-amenable groups. (a) Theorem 8.A fails to be true for non-amenable groups (where being residually amenable or residually finite does not help at all unlike our earlier surjunctivity discussion). There are examples in [Ma-Mi] attributed to D.E. Muller of the failure of both implications preinjective $\Rightarrow$ surjective and surjective $\Rightarrow$ preinjective for free products of cyclic groups. Actually Machi conjectured (privately) that such a failure must be characteristic for non-amenability.

(b) A closely related example, pointed out to me by Benjy Weiss, is of a surjective shift map $f : X \to X \times X$ where X is the full shift over the free group. Again the existence of such maps seems probable for all non-amenable groups. More specifically let K be (a finite or infinite) field, Γ a group and $f : (K^p) \overset{\Gamma}{\to} (K^q)^\Gamma$ a Γ-equivariant K-linear map of bounded propagation. If Γ is non-amenable and f is "sufficiently generic", then f is expected to be onto for all $p \geq 2$. (This is easy to show for free groups and, probably for all hyperbolic groups.) Here we are going to prove the following weaker statement.

8.I′. *If Γ a finitely generated non-amenable group, then there exists a subshift of finite type $Z \subset \Phi_+^\Gamma$, for some finite set Φ_+ such that for every set $\underline{X}$ there exists a surjective Γ-equivariant map $f : Z \times \underline{X}^\Gamma \to Z \times (\underline{X} \times \underline{X})^\Gamma$ of bounded propagation.*

Proof. The characteristic (and easy to prove) feature of non-amenability of Γ is the existence of a "compressing vector field" φ, i.e. a map $\varphi : \Gamma \to \Gamma$ where $\varphi(\gamma)\gamma^{-1}$ is contained in a finite subset $\Phi \subset \Gamma$, such that $\operatorname{card} \varphi^{-1}(\gamma) \geq 2$ for all $\gamma \in \Gamma$. In other words φ "compresses" Γ by at least factor of two while the displacement $\operatorname{dist}(\gamma, \varphi(\gamma))$ remains bounded by $\sup\limits_{\gamma \in \Phi} \operatorname{dist}(\mathrm{id}, \gamma)$. Now, given φ, one orders the pull-backs $\varphi^{-1}(\gamma)$ for all $\gamma \in \Gamma$ and assign to each $x(\gamma) \in \underline{X}^\Gamma$ the values of x at the two first (for our ordering) pull-backs $\varphi^{-1}(\gamma) \subset \Gamma$. This gives us a surjective map $f_{\varphi_+} : \underline{X}^\Gamma \to (\underline{X} \times \underline{X})^\Gamma$ of bounded propagation, where φ_+ denotes φ augmented by the ordering. Of course, this f is not Γ-equivariant but the totality of them for *all* φ_+ is equivariant. Namely, we consider the pairs $\varphi_+ = (\varphi, \text{ordering})$, where we allow all φ's with a fixed finite subset $\Phi \subset \Gamma$ and all orderings of the pull-backs of $\varphi^{-1}(\gamma)$, $\gamma \in \Gamma$. Now our map f sends $(\varphi_+, x) \mapsto (\varphi_+, f_{\varphi_+}(x))$ in an equivariant way and the proof is concluded.

Remark. It seems that the above space Z admits no Γ-invariant measure which makes its presence especially annoying. On the other hand, the existence of surjective morphism $X \to X \times X$ must be typical for many subshifts of finite type over non-amenable groups, measure or no measure.

8.J. Question. Does the Garden of Eden theorem generalize to the proalgebraic category?

First, one asks if preinjective $\Rightarrow$ surjective, while the reverse implication needs further modification of definitions.

Here it is worth noticing that the equivalence

$$\text{preinjective} \Leftrightarrow \text{surjective}$$

remains valid for linear maps $f : (K^n)^\Gamma \to (K^n)^\Gamma$ for an arbitrary field K and an amenable group Γ, where instead of the entropy one may use the mean dimension (see [Gro]$_{\mathrm{TIDS}}$) and where instead of Γ one may work over amenable graphs Δ as in 8.F.

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