GENERALIZATION OF THE SPHERICAL ISOPERIMETRIC INEQUALITY TO THE UNIFORMLY CONVEX BANACH SPACES

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0. In this paper we generalize the classical isoperimetric inequality on S^n to non-invariant measures and prove as a corollary the concentration of measure on spheres $S(X)$ of uniformely convex Banach spaces X. Our argument avoids symmetrization and (or) the calculus of variation by a direct appeal to Cavaliery's principle similar to that used in Hadwiger's proof of the Brunn-Minkowski theorem [H]. In fact, we use the localized Brunn's theorem at the final stage of our proof, though a slight rearrangment of our argument would imply this theorem. (In the appendix, we give, for the completeness sake, a short proof of Brunn's theorem). One of the application is the lower exponential bound on the dimension of ℓ_∞ admitting a symmetric map $S(X) \to S(\ell_\infty)$ with a fixed Lipschitz constant.

In order to keep the presentation transparant we did not attemt to state the most general isoperimetric inequality serving all possible applications. This has unavoidably let to repetitions of some arguments at different places in the paper as some readers may notice.

1. Let μ be some measure on the Euclidean sphere S^n and let A and B be two disjoint subsets in S^n. We seek an upper bound on dist (A,B) in terms of the measures $\mu(A)$ and $\mu(B)$, where "dist" is some metric on S^n. If B is the complement of the ε-neighbourhood of A, then for $\varepsilon \to 0$ our question reduces to the <u>isoperimetric problem</u>.

2. To formulate our main results we have to introduce some notions.

2.1. Consider an open arc $\sigma \subset S^n$ between two opposite points t_+ and t_- in S^n and call a subset Σ σ-admissible if it is an union of open arcs between t_+ and t_- and if every point t in σ lies in the interior of Σ. Next divide σ into three subintervals, say $\sigma = (t_+, a) \cup (a,b) \cup [b, t_-)$, called α_1, α_2 and α_3 respectively and let A_i, $i = 1,2,3$ be open subsets in S^n such that $A_i \cap \sigma = \alpha_i$. Finally take a Borel measure μ on S^n and define the <u>relative canonical measure</u> $\mu_\sigma(\alpha_2/\alpha_i)$ for $i = 1,3$ by

$$\mu_\sigma(\alpha_2/\alpha_i) = \inf \liminf_{\Sigma \to \sigma} \frac{\mu(A_2 \cap \Sigma)}{\mu(A_i \cap \Sigma)} \tag{2.1}$$

where inf is taken over all above tripples (A_1, A_2, A_3) and where we assume $\frac{\infty}{\infty} = \infty$ and $\frac{o}{o} = o$.

2.2. In the case of "good" measure μ the definition (2.1) simplifies as follows.

Consider the family of all non-negative measures on S^n with a continuous density function. We will call such measures <u>regular</u>. So, for every regular measure μ there exists $f_\mu(t) \in C(S^n)$ such that for any Borel set $A \subset S^n$, $\mu(A) = \int_A f_\mu(t)\,dt$. Take two opposite points t and $-t$ on S^n and consider all maximal arcs σ between t and $-t$. This gives a partition $\mathcal{H}_t$ of $S^n \setminus \{t; -t\}$ and hence every regular measure μ induces a measure (defined up to a constant) on every $\sigma \in \mathcal{H}_t$, called μ_σ. We call such partitions <u>canonical partitions</u>. Clearly, in this case (2.1) may be rewritten as

$$\mu_\sigma(\alpha_2/\alpha_i) = \frac{\mu_\sigma(\alpha_2)}{\mu_\sigma(\alpha_i)} .$$

2.3. <u>Examples</u> : **a.** If μ is the standard measure on S^n then, obviously $\mu_\sigma = \text{Const.}(\text{Sin}\ \theta)^{n-1}\, d\sigma$ for θ - the angle from $[o, \pi]$ parametrizing σ.

b. Let $S^n_+ \subset S^n$ be a hemisphere and let $S^n_+ \simeq \mathbb{R}^n$ be a projective isomorphism. Then, in case t and $-t$ lie on ∂S^n_+, arcs σ are straight lines in $\mathbb{R}^n$ and so the $\mathbb{R}^n$-invariant measure μ on $\mathbb{R}^n \simeq S^n_+$ gives the Lebesgue measure dt on σ's .

2.4. Next, let A_1 and A_3 be closed subsets in S^n, let A_2^+ be the union of all arcs in S^n between A_1 and A_3 (e.g. $A_2^+ = \mathrm{Conv}\, A_1 \cup A_3$ in the case of convex sets A_1 and A_3) and let $A_2 = A_1 * A_3$ be defined as

(2.2) $$A_2 = A_1 * A_3 \overset{\mathrm{def}}{=} A_2^+ \setminus (A_1 \cup A_3) .$$

Assume the μ-measures of A_1 and A_3 to be in (o,∞) and let $\lambda = \mu(A_1)/\mu(A_3)$. Call a pair of points $a \in A_1$ and $b \in A_3$ extremal if there is a maximal arc σ in S^n which contains a and b such that the open interval $(a,b) \subset \sigma$ misses A_1 and A_3. Then define

$$\lambda - \mathrm{dist}_\mu(a,b) = \inf_\sigma \max (\lambda\, \mu_\sigma(\alpha_2/\alpha_1)\, , \mu_\sigma(\alpha_2/\alpha_3))$$

and

$$\lambda - \mathrm{dist}_\mu(A_1,A_3) = \inf(\lambda - \mathrm{dist}_\mu(a,b))$$

over all extremal pairs (a,b) and all σ. In what follows we abbreviate : $\lambda - \mathrm{dist} = \mathrm{dist}$.

3. Theorem.

$$\mu(A_2) \geqslant \mathrm{dist}_\mu(A_1,A_3)\mu(A_3) = \lambda^{-1}\, \mathrm{dist}_\mu(A_1,A_3)\mu(A_1) .$$

3.1. Example : If μ is the $O(n)$ -invariant measure, then the explicit formula for the canonical measure (see Example 2.3.a) gives a sharp lower bound for the spherical distance between A_1 and A_3 with the equality for balls around opposite points in S^n. Thus we recapture the classical isoperimetric inequality for S^n.

3.2. The proof of the theorem involves a few constructions which we consider to be of an independent interest. By an obvious approximation argument and the definition (2.1). We may (and shall) assume the measure μ is positive regular which means, in addition to the regularity condition, that $\mu(A) > o$ for every open subset A in S^n.

Not important Remark. One could eliminate λ from the story by multiplying the measure by λ on A_3 and thus reducing the problem to the case $\lambda = 1$. However we prefer to keep λ.

Take an open hemisphere S^n_+ and fix a projective isomorphism $S^n_+ \leftrightarrow \mathbb{R}^n$ sending every straight line of $\mathbb{R}^n$ to a maximal arc on S^n_+ (and conversely). This provides a one-to-one correspondence between positive regular measures on $\mathbb{R}^n$ and S^n_+, thus identifying measures on S^n_+ and $\mathbb{R}^n$ which we denote by the same μ.

4. Convex restrictions of measures. Take an affine (i.e. a translate of a linear) subspace $E \subset \mathbb{R}^n$, fix a projection $p : \mathbb{R}^n \to E$ and consider decreasing sequences of convex subsets K_i, $i \in \mathbb{N}$ in $\mathbb{R}^n$, such that $M = \bigcap_i K_i \subset E$. By restricting a given measure μ to each K_i and then by projecting to E we obtain a sequence of measures μ_i on E. Call a non identicly zero measure ν on E a convex restriction of μ to M if for some sequence of above K_i and some sequence of real numbers λ_i,

$$\nu = \lim \lambda_i \mu_i ,$$

for the weak limit of measures.

If a measure ν' on a convex subset M' in an affine subspace $E' \subset E$ is a convex restriction of ν, then obviously, ν' also is a convex restriction of the original μ.

5. First step. Use of Brunn's Theorem. Take a k-dimensional subspace $E \subset \mathbb{R}^n$ and a convex body $K \subset \mathbb{R}^n$. Observe that the $(n-k)$-dimensional symmetrization $S_E K$ is convex by Brunn's Theorem (see Appendix).

5.1. Lemma.: Let a decreasing family of convex sets $\{K_i\}$ define a convex restriction measure $\mu_M (M = \bigcap K_i \subset E)$ of μ. Then the family $\{S_E K_i\}$ defines the same measure μ_M.

Proof. This follows from the definition of $S_E K_i$ and the uniqueness of the Radon-Nicodim derivative f_μ which is a continuous function in our case and therefore well defined on M.

5.2. Remark. If μ is absolutely continuous (rather than regular) with respect to Lebesque measure then the Lemma only holds true for almost every k-dimensional subspace $E \subset \mathbb{R}^n$.

6. Convex partitions of S^n_+ and $\mathbb{R}^n$.

A set $A \subset S^n_+$ is called convex if it contains the arc (in S^n_+) between a and b for all a and b in A. Clearly A is convex iff the corresponding by the projective isomorphism set in $\mathbb{R}^n$ is the convex set in $\mathbb{R}^n$.

6.1. Definition : We say that $\mathcal{a}$ is a k-dimensional convex partition of $\mathbb{R}^n$ if

i) every $A \in \mathcal{a}$ is convex and k-dimensional, i.e. there exists a k-dimensional affine subspace E such that $A \subset E$ and the interior $\mathring{A}$ of A in E is not emply.

ii) there exist a family of convex open neighbourhood K_i of $\mathring{A}$ such that $\cap K_i = \mathring{A}$ and every $K_i = \cup \mathring{A}_\alpha$ for some $A_\alpha \in \mathcal{a}$.
The image of $\mathcal{a}$ on S^n_+ by a projective isomorphism is called the k-dimensional convex partition of S^n_+.

6.2. Consider a 1-dimensional convex partition $\mathcal{a}$ of $\mathbb{R}^n$.
By the Radon-Nicodim Theorem $\mathcal{a}$ and a regular measure μ induce a measure μ_I on I (well defined up to a constant multiple) for every $I \in \mathcal{a}$. Take a family of convex bodies $\{K_i\}$ such that $\cap K_i = \mathring{I}$ and every $K_i = \cup \mathring{I}_\alpha$ for some $I_\alpha \in \mathcal{a}$ (such family exists by 6.1, ii)).

Clearly μ_I equals (up to constant multiple) to the convex restriction of μ defined by $\{K_i\}$. Therefore, (use 5.1) μ_I is the convex restriction of μ defined by the family $\{S_I K_i\}$ obtained from $\{K_i\}$ by the symmetrization around I.

7. Second step. Construction of 1-dimensional partition of S^n; use of Borsuk-Ulam Theorem.

We consider a regular positive measure μ on S^n. Let A_1, A_3 and $A_2 = A_1 * A_3 \subset S^n$ be subsets from 2.4. Let for $i = 1,3$

$$\lambda_i = \mu(A_2)/\mu(A_i)$$

and

$$\mu(A_1) = \lambda\mu(A_3) \quad . \text{(So} \quad \lambda = \lambda_3/\lambda_1 \text{ .)}$$

Define $H_x^+ = \{y \in S^n : (y,x) \geq 0\}$ and $H_x^- = -H_x^+$. Note that $\overset{\circ}{H}{}_x^+ = S_+^n$.
We consider a map $\varphi : S^n \to \mathbb{R}^2$ such that

$$\varphi(x) = \left(\frac{\mu(A_2 \cap H_x^*)}{\mu(A_1 \cap H_x^+)} \; ; \; \frac{\mu(A_2 \cap H_x^+)}{\mu(A_3 \cap H_x^+)}\right) .$$

By Borsuk-Ulam Theorem there exists x_o such that $\varphi(x_o) = \varphi(-x_o)$ which means that for $i = 1$ and 3

$$\lambda_i = \mu(A_2 \cap H_{x_o}^+) \;/\; \mu(A_i \cap H_{x_o}^+)$$

and, as a consequence $\lambda = \mu(A_1 \cap H_{x_o}^+) \;/\; \mu(A_3 \cap H_{x_o}^+)$.

First, we fix one such x_o and let $S_+^n = \overset{\circ}{H}{}_{x_o}^+$ and $A_i^+ = S_+^n \cap A_i$.

Next we define a convex partition of S_+^n by induction as follows.
If $M \subset S_+^n$ is one of the convex sets from the proceeding inductive step, then, by assumption

$$\mu(A_2^+ \cap M)/\mu(A_i^+ \cap M) = \lambda_i$$

for $i=1$ and 3 ; next we construct a map $\varphi : S^n \to \mathbb{R}^2$ using $A_i^+ \cap M$ as above (instead of A_i) . The map φ determines how to divide M into two convex pieces M^+ and M^- by a hyperplane in such a way that $\mu(A_2^+ \cap M^+)/\mu(A_i^+ \cap M^+) = \lambda_i$ (for $i=1$ and 3), again. The same holds for the intersections $A_i^+ \cap M^-$.
We continue to refine our partitions and end up with a new partition $\mathcal{a}_{n-1}$ whose elements are of a strictly lesser dimension than n (using that μ is positive). By the Radon-Nicodim Theorem our measure μ defines (up to factor) a measure μ_α on every $M_\alpha \in \mathcal{a}_{n-1}$ (we use the fact that μ is regular). The construction implies that $\mu_\alpha(A_2 \cap M_\alpha)/\mu_\alpha(A_i \cap M_\alpha) = \lambda_i$ for $i=1,\ 3$.

Then we may continue the same procedure with every M_α if $\dim M_\alpha \geq 2$.
The last condition is important when Borsuk-Ulam Theorem is used.

Finally, we construct a partition α of S^n_+ such that for every $I \in \alpha$

i) $\dim I = 1$

ii) $\mu_I(A_2 \cap I) / \mu_I(A_i \cap I) = \lambda_i$ for $i = 1$ and 3,

where μ_I is a measure induced by the partition α on I which is defined by the Radon-Nicodim Theorem up to a factor.

iii) μ_I is a convex restriction measure of μ.

The last property follows from 4.

8. The conclusion of the proof. We regard α constructed in section 7 as a partition of $\mathbb{R}^n \sim S^n_+$ into straight intervals $I \subset \mathbb{R}^n$. Then, by the property iii) of μ_I (see 7) and Remark 6.2, the measure μ_I on I is a convex restriction of μ defined by a family $\{K_i\}_{i \in \mathbb{N}}$ where K_i are convex sets having the $(n-1)$-dimensional symmetry in the direction perpendicular to I centered at I. Therefore, μ_I is defined by some family of shrinking convex sets $\{T_i\}_{i \in \mathbb{N}}$ in $\mathbb{R}^2$. It is easily seen (see Fig. 1) that we may replace T_i by a cone obtained by rotation C_i around I (or in the degenerate case by a cylinder) such that the convex restriction measure μ_C defined on I by μ and the family $C = \{C_i\}_{i \in \mathbb{N}}$ satisfies

i) $\mu_C(A_2 \cap I) / \mu_C(A_i \cap I) \leqslant \lambda_i$

for $i = 1$ and 3.

Also it is clear that a family of symmetric cones centered on the straight line containing I is defined by a canonical partition (see 2.2). Therefore

ii) $\mu_C = \mu_\sigma$ up to a constant factor for some σ containing I.

Let now $\rho = \mathrm{dist}_\mu(A_1, A_3)$. Then for extremal points a and b on I and any σ, $I \subset \sigma$ we have either

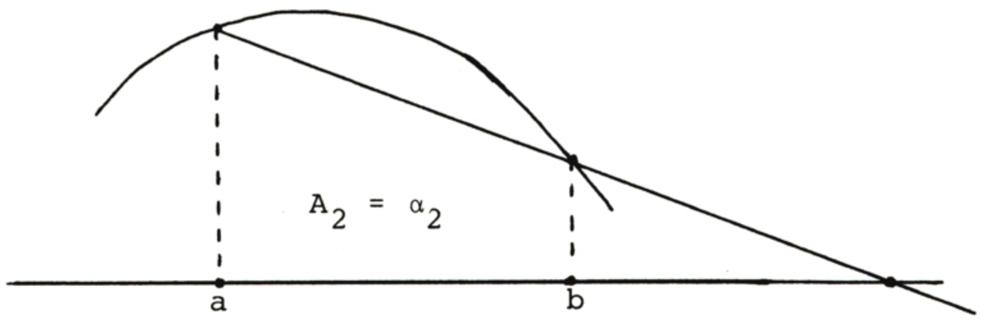

Fig. 1

$$\rho \leqslant \lambda\mu_\sigma(\alpha_2)/\mu_\sigma(\alpha_1)$$

or

$$\rho \leqslant \mu_\sigma(\alpha_2)/\mu_\sigma(\alpha_3) \;,$$

where $\alpha_1=(-t,a]$ and $\alpha_3=[b,t)$ and the points $\pm t$ are joined by the arc σ .

Start with the case when $A_2 \cap I$ is a single interval (e.g. the sets A_1 and A_3 are convex). Then $A_2 \cap I = \alpha_2$ and by i) $\mu_\sigma(\alpha_2)/\mu_\sigma(\alpha_i) \leqq \lambda_i$ for $i = 1$ and 3 . Therefore in both cases $\rho \leqq \lambda.\lambda_1=\lambda_3$ which implies Theorem 3 in this special case.

In the general case we have again to prove that

(8.1)
$$\mu_I(A_2 \cap I) \geqslant \frac{\rho}{\lambda} \min\{\mu_I(A_1 \cap I) \;;\; \lambda\mu_I(A_3 \cap I)\}$$
$$(= \frac{\rho}{\lambda} \mu_I(A_1 \cap I) = \rho\mu_I(A_3 \cap I)) \;.$$

By an approximation argument we way assume that $A_2 \cap I$ contains a finite number of intervals. Let A_2'' be one of such intervals and A_2' be the union of all intervals from $A_2 \cap I$ on the one side (say on the left) of A_2'' . Call also A_i' the part of $A_i \cap I$ on the same left part of A_2'' for $i=1$ and 3 . Let, for example, A_2'' be joined from the right by an interval from $A_1 \cap I$, called A_1'' . We will assume that

(8.2)
$$\mu_I(A_2') \geqslant \frac{\rho}{\lambda} \min\{\mu_I(A_1') \;;\; \lambda\mu_I(A_3')\}$$

(i.e. (8.1) is satisfied for the sets A_i'), and we prove (8.1) for the sets $A_i' \cup A_i''$. (We leave for the reader to check the starting point of such induction which will be finished after a finite number of steps and will prove (8.1)). Let $\alpha_2=A_2'' \subset I$. We choose a maximal arc $\sigma \supset I$ (= a straight line under the projective isomorphism $S_+^n \sim \mathbb{R}^n$) in the same way (see Fig. 1) as we did above for the convex case with $\alpha_2=A_2''$ playing the role of the entire A_2 . Then we choose the interval on the right of α_2 to be α_1 (which corresponds to our assumption that A_1'' joins A_2'' from the right) and the remaining interval from the left to be α_3 . By the definition of the λ-dist. ρ

$$\mu_\sigma(A_2'') \geqslant \frac{\rho}{\lambda} \min\{\mu_\sigma(\alpha_1) \;;\; \lambda\mu_\sigma(\alpha_3)\}$$

By the construction of σ (see again Fig. 1 and the explanation) we have

$$\mu_I(A''_2) \geqslant \mu_\sigma(A''_2) \quad \text{and} \quad \mu_I(A''_1) \leqslant \mu_\sigma(A''_1) \leqslant \mu_\sigma(\alpha_1) \ ,$$

$$\mu_I(A'_3) \leqslant \mu_\sigma(A'_3) \leqslant \mu_\sigma(\alpha_3) \ .$$

Therefore

$$\mu_I(A''_2) \geqslant \frac{\rho}{\lambda}\{\min \ \mu_I(A''_1) \ ; \lambda\, \mu_I(A'_3)\}. \tag{8.3}$$

Adding (8.2) and (8.3) we have

$$\mu_I(A'_2 \cup A''_2) \geqslant \frac{\rho}{\lambda} \min\{\mu_I(A'_1 \cup A''_1) \ ; \ \mu_I(A''_1)+\lambda\mu_I(A'_3) \ ;$$

$$\lambda\mu_I(A'_3)+\mu_I(A'_1) \ ; 2\lambda\mu_I(A'_3)\} \geqslant \frac{\rho}{\lambda}\{\min \ \mu_I(A'_1 \cup A''_1) \ ; \lambda\mu_I(A'_3)\}.$$

So, as we wanted, (8.1) is proved for the sets $A'_i \cup A''_i$ (note that A''_3 is empty in our case before).

□

9. <u>Remark</u>. The 1-dimensional partition constructed in Section 7 is not necessarily a convex partition (we passed through intermediate dimensions). We indicate below how to modify this construction to obtain a 1-dimensional convex partition satisfying property ii) from 7.

Let $M \subset S^n_+$ is a convex n-dimensional body from the intermediate step of construction. Then $\mu(A^+_2 \cap M)/\mu(A^+_i \cap M)=\lambda_i$ for $i=1, 3$. Take a triple of points $x_1, x_2, x_3 \subset M$ which maximize the determinant $|(x_i,x_j)_{i=1,2,3}|$. Define the map $\varphi\colon S^2 \to \mathbb{R}^2$ using $A^+_i \cap M$ as above for $S^2 \subset \mathbb{R}^3 = \text{span}\{x_1,x_2,x_3\}$. Using this map subdivide M into two pieces M^+ and M^- by a hyperplane such that the numbers λ_1 and λ_3 coincide with the above. It is clear that these refinements of our partitions will directly lead to a 1-dimensional convex partition $\mathfrak{a}$ with the property ii) from 7 .

10. The preceeding construction can be adjusted to various results related to the isoperimetric inequalities.

10.1. Theorem. Let A and B be closed subsets of S^n, μ a regular positive measure on S^n and $f(x,y)$ any continuous function on $(S^n \times S^n) - \{(t,-t), t \in S^n\}$. There exists a maximal open arc σ and disjoined sets $\alpha_i \subset \sigma$, i=1, 2, 3 (where $\alpha_2 = \alpha_1 * \alpha_3$) such that

(10.1) $$\inf\{f(a,b) : a \in \alpha_1, b \in \alpha_3\} \geqslant \inf\{f(x,y) : x \in A, y \in \beta, x \neq -y\}$$

and for $C = A * B$

(10.2) $$\frac{\mu_\sigma(\alpha_2)}{\mu_\sigma(\alpha_1)} \leqslant \frac{\mu(C)}{\mu(A)} \; ; \; \frac{\mu_\sigma(\alpha_2)}{\mu_\sigma(\alpha_3)} \leqslant \frac{\mu(C)}{\mu(B)} \quad .$$

Proof of the theorem follows from 8, i) and ii).

Remark 10.2. An important case is when $f(x,y)$ is a distance function on S^n.

Remark 10.3. We say that $f(x,y)$ is monotone if for any maximal arc σ and for every $x,y,z \subset \sigma$, $y \in (x,z) \subset \sigma$,

$$\max\{f(x,y) \; , \; f(y,z)\} \leqslant f(x,z) .$$

Now, if in the theorem a function $f(x,y)$ is monotone, then there exists a maximal open arc σ (joined some points $\pm t \in S^n$) and a partition σ on three intervals : $\alpha_1 = (-t,a]$, $\alpha_2 = (a,b)$, $\alpha_3 = [b,t)$ such that $f(a,b) \geqslant \inf\{f(x,y) : x \in A \; , \; y \in B \; , \; x \neq -y\}$ and (10.2) is satisfied for the above σ and $\alpha_i \subset \sigma$.

□

The theorem below follows from the section 7.

Theorem 10.4. Let A and B be closed subsets of S^n, μ a regular positive measure on S^n and $C = A * B$.

i) There exists a maximal open arc σ and a convex restriction measure ν on σ such that

$$\nu(C \cap \sigma)/\nu(A \cap \sigma) = \mu(C)/\mu(A)$$

and

$$\nu(C \cap \sigma)/\nu(B \cap \sigma) = \mu(C)/\mu(B) .$$

ii) if, in addition, μ is a probability measure then there exists a maximal open arc σ and a convex restriction measure ν on σ such that

$$\nu(A \cap \sigma) = \mu(A) \quad \text{and} \quad \nu(B \cap \sigma) = \mu(B) .$$

11. Concentration phenomena on the unit sphere of a uniformly convex normed space. Let a normed space $X = (\mathbb{R}^{n+1}, \|.\|)$ have for fixed $\varepsilon > 0$ the modulus of convexity at least $\delta(\varepsilon) > 0$. It means that for every two points x, y in X, $\|x\| = \|y\| = \|$ and $\|x-y\| \geqslant \varepsilon$,

$$1 - \frac{\|x+y\|}{2} \geqslant \delta(\varepsilon) .$$

11.1. Linear functionals. Take a vector $f \in X^*$, $\|f\| = 1$. Define $K = \{x \in X , \|x\| \leqslant 1\}$, $S(X) = \partial K = \{x \in X , \|x\| = 1\}$ and $K_\lambda = K \cap \{x : f(x) = \lambda\}$. Clearly, $\mathrm{Vol}_n K_\lambda = \mathrm{Vol}_n K_{-\lambda}$ and $(K_\lambda + K_{-\lambda})/2 = A \subset K_o$. By Brunn-Minkowski inequality.

$$\mathrm{Vol}_n (K_\lambda + K_{-\lambda})^{1/n} \geqslant (\mathrm{Vol}_n K_\lambda)^{1/n} + (\mathrm{Vol}_n K_{-\lambda})^{1/n}$$

and therefore $\mathrm{Vol}_n K_\lambda \leqslant \mathrm{Vol}_n A$. Note also that for any $x \in K_\lambda$ and $y \in K_{-\lambda}$ we have $\|x-y\| \geqslant 2\lambda$ and, consequently, $\|x+y\|/2 \leqslant 1-\delta(2\lambda)$. Therefore (see [GM2])

Lemma. $\mathrm{Vol}_n K_\lambda \leqslant (1-\delta(2\lambda))^n \, \mathrm{Vol}_n K_o$.

So, we see, that the volume of the levels of a linear functional are exponentially concentrated at the zero level. We continued this direction in [GM2] to show that (see [GM2], Theorem 3.2).

<u>Theorem</u>. For every $f \in X^*$, $\| f \|^* = 1$,

$$\mathrm{Vol}_{n+1}\{x \in K \ , \ |f(x)| \geq \varepsilon\} \leq (n+1)(1-\varepsilon)e^{-n\delta(2\varepsilon)/2}\ \mathrm{Vol}_{n+1}K \ .$$

We will extend the results from 11.1 to an arbitrary 1-Lipschitz function on $S(X)$

11.2. <u>A measure on $S(X)$</u>. A standard $(n+1)$-dimensional volume on $\mathbb{R}^{n+1}$ induces the probability measure μ on $S(X)$: for any Borel set $A \subset S(X)$,

$$(11.1) \qquad \mu(A) \overset{\mathrm{def}}{=} \mathrm{Vol}_{n+1}\{\cup\, tA \ , \ o \leq t \leq 1\}/\mathrm{Vol}\ K \ .$$

To apply Theorem 3 to this measure μ on $S(X)$ we have to estimate μ_σ for any maximal arc σ . However our estimate will also work for any convex restriction measure and the application of Theorem 10.4 will be easier.

Note that Theorem 3 and Theorem 10.4 concern measures on S^n in projective sense and applicable for μ on $S(X)$ (the reader who feels uncomfortable at this point, may choose <u>any</u> euclidean sphere S^n and, using the radial projection of $S(X)$ to S^n, transport all constructions and results from $S(X)$ to S^n and vice versa).

Fix $z \in S(X)$. Let $f \in X^*$, $\| f \| = 1$, be the support functional at z , i.e. $f(z) = 1$. Consider $\mathrm{Ker}\, f \cap S(X) = S_o$. Take any $x \in S_o$. We study σ which joins $\pm z$ and pass through x (i.e. a "half" of the two dimensional sphere $S(X) \cap \mathrm{span}\{x;z\}$) . Parametrize points on σ by $t \in (-1,+1)$ such that $x_t = \sigma \cap \{x : f(x) = t\}$.

<u>Proposition 11.3</u>. Let $\delta(\varepsilon)$ be as in 11. If ν is a convex restriction probability measure on σ then there exists t_o such that for any $\theta > 0$

$$(11.2) \qquad \begin{cases} \nu[-z, x_{t_o-\theta}] \leq \dfrac{[1-\delta(\theta)]^{n-1}}{1-[1-\delta(\theta)]^{n-1}}\, \nu[x_{t_o-\theta}, x_{t_o}] & (\text{for } -1 < t_o - \theta) \\ \text{and} & \\ \nu[x_{t_o+\theta}, z] \leq \dfrac{[1-\delta(\theta)]^{n-1}}{1-[1-\delta(\theta)]^{n-1}}\, \nu[x_{t_o}, x_{t_o+\theta}] & (\text{for } t_o + \theta < 1) \end{cases}$$

where one may choose t_o as the maximum point of the density $f_\nu(t)$ of the measure ν. The function $[f_\nu(t)]^{1/(n-1)}$ is concave (i.e. $\frown$).

It is following from (11.2) that

(11.3) $$\nu\{(-z,x_{t_o-\theta})\cup(x_{t_o+\theta},z)\}\leqslant(1-\delta(\theta))^{n-1}\simeq e^{-\delta(\theta)(n-1)}$$

Proof. We use an argument similar to that of 11.1 where a concentration property of linear functionals was proved. Define $\Delta\nu(x_t)$ to be an infinitesimal (n-1)-dimensional volume of infinitesimal convex neighborhood Δ_t of x_t in $S(X)\cap\{x:f(x)=t\}$ which induces the density $f_\nu(t)$ of the probability convex restriction measure ν on σ at the point x_t. Then by Brunn-Minkowski inequality for any θ such that $t+\theta\leqslant 1$ and $t-\theta\geqslant -1$.

$$\mathrm{Vol}\left(\frac{\Delta_{t+\theta}+\Delta_{t-\theta}}{2}\right)^{1/(n-1)}\geqslant\frac{\Delta\nu(x_{t+\theta})^{1/(n-1)}+\Delta\nu(x_{t-\theta})^{1/(n-1)}}{2}\geqslant$$

$$\geqslant\min\{\Delta\nu(x_{t\pm\theta})\}^{1/(n-1)}$$

Also for any $y_{t+\theta}\in\Delta_{t+\theta}$ and $y_{t-\theta}\in\Delta_{t-\theta}$ we have

$$\frac{y_{t+\theta}+y_{t-\theta}}{2}=\lambda y_t$$

for some $o<\lambda<1-\delta(2\theta)$ where $y_t\in S(X)\cap\{x:f(x)=t\}$ and y_t belongs to the arc joining $y_{t-\theta}$ and $y_{t+\theta}$, i.e. y_t belongs to any convex neighborhood of the arc $[x_{t-\theta},x_{t-\theta}]$ which is contained $\Delta_{t\pm\theta}$. Therefore

$$\mathrm{Vol}\left(\frac{\Delta_{t+\theta}+\Delta_{t-\theta}}{2}\right)\leqslant[1-\delta(2\theta)]^{n-1}\ \mathrm{Vol}\ \Delta_t$$

(where as before $\mathrm{Vol}\ \Delta_t\overset{\text{def}}{\equiv}\Delta\nu(x_t)$) and, together with the above inequality ,

$$\frac{[f_\nu(t+\theta)]^{1/(n-1)}+[f_\nu(t-\theta)]^{1/(n-1)}}{2}\leqslant(1-\delta(2\theta))f_\nu(t)^{1/(n-1)}\leqslant f_\nu(t)^{1/(n-1)}$$

and

(11.4) $$\min f_\nu(t\pm\theta)\leqslant(1-\delta(2\theta))^{n-1}f_\nu(t)\ .$$

It follows that $[f_\nu(t)]^{1/(n-1)}$ is <u>concave</u>.
Let f_ν attain the maximum at t_o. Then

$$f_\nu(t_o \pm 2\theta) \leqslant (1-\delta(2\theta))^{n-1} f_\nu(t_o \pm \theta) \leqslant (1-\delta(2\theta))^{n-1} f_\nu(t_o)$$

and for any $t > t_o$ (or $t < t_o$) for $\theta < t - t_o$ and $t+\theta < 1$ (similarly $\theta < t_o - t$ and $t - \theta \geqslant -1$)

$$(11.5) \qquad f_\nu(t+\theta) \leqslant (1-\delta(2\theta))^{n-1} f_\nu(t)$$

(or similarly $f_\nu(t-\theta) \leqslant (1-\delta(2\theta))^{n-1} f_\nu(t)$) .
Now integrate (11.5) from $t_o+\theta$ to $1-\theta$ and obtain

$$\chi(\theta) \overset{def}{=} \nu[t_o+2\theta;1] \leqslant (1-\delta(2\theta))^{n-1} \nu[t_o+\theta\,;\,1-\theta] \leqslant$$

$$\leqslant (1-\delta(2\theta))^{n-1}\{\nu[t_o+\theta;t_o+2\theta]+\chi(\theta)\}$$

(to simplify notation we write $\nu[t,\tau]$ instead of $\nu[x_t,x_\tau]$) . Therefore

$$\chi(\theta) \leqslant \frac{[1-\delta(2\theta)]^{n-1}}{1-[1-\delta(2\theta)]^{n-1}} \nu[t_o+\theta;t_o+2\theta]$$

for $\theta > 0$. Similarly we deal with the comparison of $\nu[-1;t_o-2\theta]$ and $\nu[t_o-2\theta;t_o-\theta]$. Then the statement of the proposition follows.

□

<u>Corollary 11.4</u>. Let $I_\varepsilon(x_{t_o}) = \{x \in \sigma : \| x - x_{t_o} \| \leqslant 2\varepsilon\}$

Then $$\nu(\sigma - I_\varepsilon) \leqslant (1-\delta(\varepsilon))^{n-1} \simeq e^{-\delta(\varepsilon)(n-1)} .$$

<u>Proof</u>. In fact, this corollary follows from the <u>proof</u> of the proposition 11.3. To establish it we have to choose a new parametrization of the arc σ. Take $x_\theta \in \sigma$, such that $\theta = \rho(x_\theta, -z) = \int_{-z}^{x_\theta} \|dx_t\|$, i.e. θ is the length of the arc $(-z,x_\theta)$. It is known [S] that $a=\rho(z,-z)$ changes between $3 \leqq a \leqq 4$ (a is the "π" of a normed space $E=\text{span}\{z;x\}$) and for $\theta \leqq t;\ \theta \leqq a$

$$(11.6) \qquad \| x_t - x_\theta \| \leqslant \rho(x_t,x_\theta) \leqslant 2\| x_t - x_\theta \| ,$$

Take $t \in (o,a)$ and consider $x_t \in \sigma$. Choose $\theta_\varepsilon(t)$ such that $\| x_{t-\theta_\varepsilon(t)} - x_{t+\theta_\varepsilon(t)} \| = 2\varepsilon$ (we assume that ε is small enough to guarantee that $\theta_\varepsilon(t) < t$ and $t+\theta_\varepsilon(t) < a)$. Let $\theta_\varepsilon = \max\{\theta_\varepsilon(t)$ by all admissible $t\}$. Clearly from (11.6)

(11.7) $$\varepsilon \leqslant \theta_\varepsilon \leqslant 2\varepsilon .$$

The proof of Proposition 11.3 shows that

(11.4') $$\min \varphi_\nu(t \pm \theta_\varepsilon) \leqslant (1-\delta(2\varepsilon))^{n-1} \varphi_\nu(t)$$

where $\varphi_\nu(t)$ is the same density function f_ν as in the Proposition but now parametrized in the new way. Using majoration (11.7) we also have (by concavity of $\varphi_\nu^{1/(n-1)}$)

$$\min \varphi_\nu(t \pm 2\varepsilon) \leqslant (1-\delta(2\varepsilon))^{n-1} \varphi_\nu(t) .$$

The last inequality leads to the new form of (11.2) and (11.3) with the new parametrization of x_t which is precisely Corollary 11.4.

□

The next Corollary is a more direct consequence of Proposition 11.3.

Corollary 11.5. Let an arc $[a,b] \subset \sigma$, where σ is a maximal arc joined points $\pm z$, and $\| a-b \| \geqslant \varepsilon > 0$. Then there exists a number $\lambda(\varepsilon) > 0$ depending only on $\delta_X(\varepsilon) > 0$ such that for any convex measure ν induced by μ either

$$\frac{\nu(-z,a]}{\nu(a,b)} \leqslant e^{-\lambda(\varepsilon)n}$$

or

$$\frac{\nu[b,z)}{\nu(a,b)} \leqslant e^{-\lambda(\varepsilon)n} .$$

Proof. Integrate (11.3) starting from suitable t (or, if $b \in (x_{t_o}, -z)$, integrate the similar inequality for $f_\nu(t-\theta))$.

Theorem 11.6. Let $\delta_X(\varepsilon)$ be the modulus of convexity of a normed $(n+1)$-dimensional space X and μ be the probability measure (11.1) on $S(X)$. Let $a(\varepsilon) = \delta(\frac{\varepsilon}{2} - 2\theta_n)$ and $\theta_n = 1-(1/2)^{1/(n-1)} \simeq \ln 2/(n-1)$. Then for every Borel set

$A \subset S(X)$, $\mu(A) \geq 1/2$ and every $\varepsilon > 0$

$$\mu(A_\varepsilon) \geq 1-e^{-a(\varepsilon)n}$$

where $A_\varepsilon = \{x \in S(X) \; ; \; \rho(x,A_\varepsilon) \leq \varepsilon\}$ and $\rho(x,y) = \|x-y\|$.

Proof. Let $B=(A_\varepsilon)^c$. We use Theorem 10.4, ii) to estimate $\mu(B)$ from above. By this theorem there exists a maximal arc σ and a convex restriction measure ν such that

$$\nu(A \cap \sigma) = \mu(A) \geq 1/2 \quad .$$

By Proposition 11.3 some "small" neighborhood of $A \cap \sigma$ contains the point x_{t_o} where the maximum of the density function of ν is attained. Therefore, ε-neighborhood of $A \cap \sigma$ contains a θ-interval around x_{t_o} . Hence, $\mu(B)=\nu(B \cap \sigma)$ is exponentially small.

Now we will have a precise estimate. Take θ_o such that $\nu(I_{\theta_o}(x_{t_o}))=\nu\{x \in \sigma : \|x-x_{t_o}\| \leq 2\theta_o\} = 1/2$. Then, by Corollary 11.4

$$[1-\delta(\theta_o)]^{n-1} \geq 1/2 \; .$$

It means that

$$\delta(\theta_o) \leq 1-(1/2)^{1/n-1} \simeq \frac{\ell n 2}{n-1} \; .$$

Let θ_n be such that $\delta(\theta_n)=1-(1/2)^{1/(n-1)} \simeq \frac{\ell n 2}{n-1}$. Then $\nu[I_{\theta_n}(x_{t_o})] \geq 1/2$.

Therefore, if $\nu(A \cap \sigma) \geq 1/2$ then there exists $x_t \in A \cap \sigma$ and $\|x_t - x_{t_o}\| \leq 2\theta_n$. Now, take ε-neighborhood of $\{x_t\}$ and let $\varepsilon = 2\theta + 4\theta_n$. Then $A_\varepsilon \cap \sigma \supset \{x_t\}_\varepsilon \supset \{x_{t_o}\}_{2\theta}$. Therefore, by Corollary 11.4 and Theorem 10.4, ii), we have

$$\mu(B) = \nu(B \cap \sigma) \leq \nu(\sigma - I_\theta(x_{t_o})) \leq (1-\delta(\theta))^{n-1} \simeq$$

$$\simeq e^{-\delta(\theta)(n-1)} = e^{-\delta(\frac{\varepsilon}{2} - 2\theta_n)(n-1)} \quad .$$

Remark 11.7. Note that Theorem 11.5 shows that any family of finite dimensional spaces $\{X_n, \dim X_n \to \infty\}$, such that $\delta_{X_n}(\varepsilon) \geq \delta(\varepsilon) > 0$ for $\varepsilon > 0$, is a Levy family (see definition and a number of related examples [GM1], [AM]).

Let $f(x)$ be a continuous function on $S(X)$, $\dim X=n+1$ where $S(X)$ is the unit sphere of X. We call L_f the median of $f(x)$ (or Levy mean) if

$$\mu\{x \in S(X) : f(x) \geq L_f\} \geq 1/2 \quad \text{and} \quad \mu\{x \in S(X) : f \leq L_f\} \geq 1/2 .$$

Let $\omega_f(\varepsilon)$ be the modulus of continuity of the function $f(x)$.

It follows from Theorem 11.6 that

(11.8) $$\mu\{x \in S(X) : |f(x)-L_f| \leq \omega_f(\varepsilon)\} \geq 1-2\, e^{-a(\varepsilon)n} .$$

12. Application to a Lipschitz embedding problem. Let $X=(\mathbb{R}^n, \|\cdot\|)$ be a uniformly convex space with the modulus of convexity $\delta(\varepsilon) > 0$ (for $\varepsilon > 0$). Let $S(X)=\{x \in X : \|x\|=1\}$ and, similarly, $S(\ell_\infty^N)$ be the unit sphere of the space ℓ_∞^N of dimension N.

Theorem. Fix $1 > \varepsilon > 0$. If $N < \frac{1}{2} e^{a(\varepsilon)n}$ where $a(\varepsilon) > 0$ was defined in Theorem 11.6, then there exists no 1-Lipschitz antipodal (i.e., $\varphi(-x) = -\varphi(x)$) map

$$\varphi : S(X) \to S(\ell_\infty^N)$$

Proof. Assume φ exists. Let $f_i(x)$, $i=1, \cdots N$, be the i-th coordinate in ℓ_∞^N of $\varphi(x)$, i.e. $\varphi(x) = (f_i(x))_{i=1}^N \in S(\ell_\infty^N)$. Then

i) $\max_i |f_i(x)| = 1$ for $x \in S(X)$,

ii) $f_i(-x) = -f_i(x)$ for any $i=1,\dots,N$ and $x \in S(X)$

iii) $|f_i(x)-f_i(y)| \leq \| x-y \|$, $i=1,\dots,N$; $x,y \in S(X)$

(because $\| \varphi(x)-\varphi(y)\| \leq \| x-y \|$ implies $\max_i |f_i(x)-f_i(y)| \leq \| x-y \|$)

Define $A_i = \{ x \in S(X) : |f_i(x)| \leq \varepsilon \}$. By ii) , 0 is the median of f_i for every $i \in \{1,\dots,N\}$ (see 11.7 for the definition). Also, by iii), and (11.8),

$$\mu(A_i) \geq 1-2e^{-a(\varepsilon)n} .$$

Then $\mu(\bigcap_{i=1}^{N} A_i) \geq 1-2N e^{-a(\varepsilon)n}$ and, in the case of $N < \frac{1}{2} e^{a(\varepsilon)n}$, there exists $x \in \bigcap_i^N A_i$. Hence, $|f_i(x)| \leq \varepsilon$ for every $i=1,\dots,N$ which contradicts i) .

□

Note that in the case of a linear embedding $\varphi : X \to \ell_\infty^N$, dim X=n , the above Theorem was proved by Pisier [P].

APPENDIX

Let $P : \mathbb{R}^n \to \mathbb{R}^{n-k}$ be a linear projection and let K be a convex subset in $\mathbb{R}^n$. We study the k-dimensional volume of the intersections $K \cap P^{-1}(x)$ for $x \in \mathbb{R}^{n-k}$.

Brunn's Theorem. Let $\varphi(x) = \mathrm{Vol}_k\, K \cap P^{-1}(x)$. Then the function $\varphi^{1/k}$ is concave on the image $P(K) \subset \mathbb{R}^{n-k}$

We will prove a more general statement.

Definition. We say that a function $f : K \to \mathbb{R}$ is α-concave $(\alpha > 0)$ if

i) K is a convex set in $\mathbb{R}^n$,

ii) $f(x) \geq 0$ for $x \in K$

iii) $f^{1/\alpha}$ is concave on K , i.e.

$$f^{1/\alpha}\left(\frac{x_1 + x_2}{2}\right) \geq \frac{f^{1/\alpha}(x_1) + f^{1/\alpha}(x_2)}{2}$$

for any $x_1 , x_2 \in K$.

Lemma 1. Let f be α-concave, g be β-concave and let $\mathrm{Dom}\, f = \mathrm{Dom}\, g = K \subset \mathbb{R}^n$. Then the product fg is $(\alpha+\beta)$-concave.

Proof. Let $x_1, x_2 \in K$. Then

$$\frac{[f(x_1)g(x_1)^{1/(\alpha+\beta)} + f(x_2)g(x_2)]^{1/(\alpha+\beta)}}{2} \leq \left[\frac{f(x_1)^{1/\alpha} + f(x_2)^{1/\alpha}}{2}\right]^{\frac{\alpha}{\alpha+\beta}} .$$

$$. \left[\frac{g(x_1)^{1/\beta} + g(x_2)^{1/\beta}}{2}\right]^{\beta/(\alpha+\beta)}$$ (by Hölder inequality for $p = \frac{\alpha+\beta}{\alpha}$

and $q = \frac{\alpha+\beta}{\beta}$) $\leq \left[f\left(\frac{x_1 + x_2}{2}\right)\right]^{\frac{1}{\alpha+\beta}} \left[g\left(\frac{x_1 + x_2}{2}\right)\right]^{\frac{1}{\alpha+\beta}}$ (by α- and β-concavity of the functions f and g) .

□

Consider a linear projection $P : \mathbb{R}^m \xrightarrow{\text{onto}} \mathbb{R}^{m-1}$. Then $\mathbb{R}^m = \mathbb{R}^{m-1} + \operatorname{Ker} P$. Let K be a convex set $K \subset \mathbb{R}^m$. We have for every $x \in K$: $x = y+t$, where $y \in PK \subset \mathbb{R}^{m-1}$ and $t \in I_y = \{x \in K : Px = y\}$ is an interval in $y + \operatorname{Ker} P$. Let f be a function, $\operatorname{Dom} f = K$. We will write $f(x) = f(y;t)$ where $x = y+t$, $y \in PK$ and $t \in I_y$. Define a <u>projection</u> Pf of a function f as the function with $\operatorname{Dom} Pf = P \operatorname{Dom} f (=PK)$ and

$$(Pf)(y) \overset{\text{def}}{\equiv} \int_{t \in I_y} f(y;t)\,dt \; .$$

<u>Lemma 2</u>. If f is α-concave then Pf is $(1+\alpha)$-concave.

<u>Proof</u>. It is sufficient, by the definition of concavity to consider the case $m=2$. Then PK is an interval. Let $x_1, x_2 \in PK$ and $x = (x_1+x_2)/2$ and let $I_{x_i} = [a_i, b_i]$, $i=1,2$. We may assume that $I_x = [\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2}]$. Let $c_i \in I_{x_i}$ $(i=1,2)$ satisfy

$$\int_{a_i}^{b_i} f(t,x_i)\,dt = 2\int_{a_i}^{c_i} f(t,x_i)\,dt = 2\int_{c_i}^{b_i} f(t,x_i)\,dt \; .$$

Let K_{up} be the convex hull of the two intervals (for $i=1,2$) $[(c_i;x_i),(b_i;x_i)] \subset K$ and K_{down} be the convex hull of the intervals (again for $i=1,2$) $[(a_i;x_i),(c_i;x_i)] \subset K$. Let $f_1 = f|_{K_{up}}$ and $f_2 = f|_{K_{down}}$. It is easy to check (we leave it to the reader) that if $\varphi_i = Pf_i$ $(i=1,2)$ are $(1+\alpha)$-concave then the same is true for the original projection Pf. Therefore, our problem is reduced to the $(1+\alpha)$-concavity of the functions φ_1 and φ_2. We continue this procedure and build the partitions of the intervals $[a_i,b_i]$ for $i=1,2$: $t_{o,i} = a_i < t_{1,i} < \ldots < t_{n-1,i} < b_i = t_{n,i}$, such that

$$\int_{a_i}^{b_i} f(t,x_i)\,dt = n\int_{t_{p-1,i}}^{t_{p,i}} f(t,x_i)\,dt$$

for every $p=1,\ldots,n$ and $i=1,2$. Let $K_p \subset K$ be a trapez which is the convex hull of the two intervals $[(t_{p-1,i};x_i),(t_{p,i};x_i)]$ $(i=1,2)$. Then, by the above remark one only needs to check that the functions $P(f|_{K_p})$ are $(1+\alpha)$-concave for every $p=1,\ldots,n$. By the obvious approximation argument, the problem is reduced now to the following observation :

Let $t_i \in I_{x_i}$ and $\Delta_i > 0$ $(i=1,2)$; for $x = \lambda x_1 + (1-\lambda) x_2$. Denote $t(x) = \lambda t_1 + (1-\lambda) t_2$ and $\Delta(x) = \lambda \Delta_1 + (1-\lambda)\Delta_2$. Then, by Lemma 1, the function $f(t(x),x).\Delta(x)$ is $(1+\alpha)$-concave because f is α-concave and the linear function $\Delta(x)$ is 1-concave.

□

Now we prove Brunn's theorem as follows. We start from the characteristic function $\chi_K(x)$ of the set K which is α-concave for every $\alpha > 0$. After k consequent projections we come to the function $\varphi(x)$ on $\widetilde{K}$ which is, by Lemma 2, $(k+\alpha)$-concave for every $\alpha > 0$ and, therefore, k-concave.

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