VOLUME AND BOUNDED COHOMOLOGY

by

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0. Introduction

0.1. Minimal Volume

Take a C^∞-manifold V and consider all complete Riemannian metrics g on V whose sectional curvatures everywhere are bounded in absolute value by one, $|K(g)| \le 1$. We define the minimal volume of V as the lower bound of the total volumes of all such metrics :

$$\text{Min Vol}(V) = \inf_{|K(g)| \le 1} \text{Vol}(V,g) \quad .$$

The minimal volumes of closed connected surfaces V are proportional to the Euler characteristics,

$$\text{Min Vol}(V) = 2\pi |\chi(V)| \quad .$$

Indeed by Gauss-Bonnet one has for $|K| = |K(g)| \le 1$,

$$\text{Vol}(V) \ge \int_V |K \, dv| \ge \left| \int_V K \, dv \right| = 2\pi |\chi| \quad ,$$

with equality for K constant, 1 or -1.

In particular, metrics g of constant curvature ± 1 are extremal : $\text{Vol}(V,g) = \text{Min Vol}(V)$, while the torus and the Klein bottle, who have $\chi = 0$, carry no extremal metrics since their minimal volumes are zero.

The Gauss-Bonnet formula also applies to complete non-compact surfaces V with $|K(V)| \le 1$ and with $\text{Vol}(V) < \infty$. Again, metrics of constant curvature -1 are extremal and so for $\chi(V) < 0$ we get $\text{Min Vol}(V) = -2\pi \, \chi(V)$. In particular $\chi(V) = -\infty$ implies $\text{Min Vol}(V) = \infty$. Furthermore, the cylinder and the Möbius band have zero minimal volume, while for $V = \mathbb{R}^2$ we only obtain in Appendix 1 the following estimate

$$4\pi + 0.01 < \text{Min Vol}(\mathbb{R}^2) \le (2 + 2\sqrt{2})\pi \quad .$$

If $n = \dim V > 2$, then the Gauss-Bonnet formula yields the following inequality for closed manifolds V,

$$\text{Min Vol}(V) \geq \text{const}_n |\chi(V)| \quad \text{for some const}_n > 0 .$$

There are similar inequalities for the Pontryagin numbers p of V,

$$\text{Min Vol}(V) \geq \text{const}_n^1 |p(V)| \quad .$$

In fact by the Chern-Weil theorem there are certain polynomials $P(\Omega)$ in the curvature tensor Ω of V, such that

$$|p| = \left| \int_V P(\Omega) dv \right| \leq \sup \| P(\Omega) \| \, \text{Vol}(V) \quad ,$$

and $|K| \leq 1$ implies $\sup \| P(\Omega) \| \leq \text{const}_n^1$.

Now, if V is an open complete manifold of dimension $n \geq 3$, then the theorem of Gauss-Bonnet does not, in general, apply. In fact, there are manifolds V with zero minimal volume and with non-zero Euler characteristic. For example, for every $\epsilon > 0$ there is a complete metric g_ϵ on $\mathbb{R}^3$ with $|K(g_\epsilon)| \leq 1$ and with $\text{Vol}(\mathbb{R}^3, g_\epsilon) \leq \epsilon$. (See Appendix 2).

Pontryagin classes serve better than the Euler characteristic for open manifolds.

Example : Let V_0 be a closed $4m$-dimensional manifold and let $V = V_0 \times \mathbb{R}$. If V_0 has a non-zero Pontryagin number, then $\text{Min Vol}(V) = \infty$.

Proof : We have a non-trivial integral characterisitc class $p \in H^{4m}(V; \mathbb{R}) = H^{4m}(V_0; \mathbb{R}) = \mathbb{R}$, and for any given metric g on V this p is represented by a $4m$-form $P(\Omega)$ on V which is a Chern-Weil polynomial in Ω of degree $2m$. Then for complete metrics g on V we consider concentric balls $B(R) \subset V$ of radii R around a fixed point $v_0 \in V$ and

we observe that for sufficiently large R_o the boundary spheres $S(R) = \partial B(R)$ for $R \geq R_o$ support a non-trivial class in $H_{4m}(V)$. Therefore

$$\int_{S(R)} \|P(\Omega)\| ds \geq 1 \quad , \quad \text{for} \quad R \geq R_o \quad ,$$

and so

$$\int_{B(R)} \|P(\Omega)\| dv = \int_0^R dR \int_{S(R)} \|P(\Omega)\| ds \geq R - R_o \ .$$

Since $\|P(\Omega)\| \leq \text{const}'_m |K|^{2m}$ we also get

$$\int_{B(R)} |K|^{2m} dv \geq \text{const } R \quad \text{for} \quad R \geq 2R_o \ ,$$

and in particular, for $|K| \leq 1$, we conclude

$$\liminf_{R \to \infty} R^{-1} \text{Vol } B(R) \geq \text{const} > 0 \ .$$

This is even stronger than the required relation $\text{Vol}(V) = \lim\limits_{R \to \infty} \text{Vol}(B(R)) = \infty$.

Remark : If a manifold V' is homeomorphic to the infinite connected sum of copies of the manifold V above, then we have for $R \to \infty$,

$$\int_{S'(R)} \|P(\Omega')\| \to \infty \quad , \quad \text{where} \quad S' = \partial B' \subset V' \ ,$$

and so for $|K(V')| \leq 1$ we obtain

$$\lim_{R \to \infty} R^{-1} \text{Vol}(B'(R)) = \infty \quad .$$

We shall contruct in Appendix 2, for an arbitrary function $\varepsilon(R)$ for which $\lim\limits_{R \to \infty} \varepsilon(R) = 0$, complete metrics g with $|K(g)| \leq 1$ on <u>all</u> smooth manifolds V, such that

$$\lim_{R \to \infty} \varepsilon(R) R^{-1} \text{Vol } B(R) = 0 \quad .$$

Now let us give five examples of manifolds V with $\text{Min Vol}(V) = 0$.

(1) V is compact and it admits a flat Riemannian metric.

(2) V admits a locally free S^1-action. In particular, $\text{Min Vol}(V = V_o \times S^1)=0$, and also $\text{Min Vol}(S^3) = 0$. The latter is the famous example of Berger (see [10]) .

(3) V is a component of the boundary of a compact manifold W whose interior, $\text{Int } W$, either admits a complete locally symmetric metric of non-positive curvature and finite volume, or $\text{Int } W$ admits the structure of a complex quasiprojective (for instance, affine) variety.

(4) V is the product of an arbitrary V_o by a manifold in one of the above examples (1)-(3) .

(5) V is odd dimensional and it is diffeomorphic to a finite or infinite connected sum of manifolds of example (4) . For instance, connected sums of odd dimensional tori have zero minimal volume. Notice that such connected sums admit no non-trivial circle action.

The first example is obvious. For the rest see Appendix 2 .

The main purpose of this paper is to provide new estimates from below for the minimal volume in terms of the simplicial volume defined in section 0.2. In particular, we exhibit in section 0.4. closed odd dimensional manifolds with non-vanishing minimal volume. One's interest in such estimates is motivated, in part, by the following theorem of J. Cheeger [9] which relates the "topological complexity" of a manifold to its geometric size :

Cheeger's finiteness theorem : For any given numbers $D > 0$ and $\varepsilon > 0$ there are at most finitely many diffeomorphism classes of closed Riemannian manifolds V of a fixed dimension n such that

$$|K(V)| \leq 1 \quad ,$$

$$\text{Diameter}(V) \leq D \; ,$$

$$\text{Vol}(V) \geq \varepsilon \qquad .$$

0.2. Simplicial volume

Let X be any topological space. Denote by $C_* = C_*(X)$ the real chain complex of X : a chain $c \in C_*$ is a finite combination $\sum_i r_i \sigma_i$ of singular simplices σ_i in X with real coefficients r_i . We define the <u>simplicial</u> ℓ^1-<u>norm</u> in C_* by setting $\|c\| = \sum_i |r_i|$. This norm gives rise to a pseudo-norm on the homology $H_* = H_*(X; \mathbb{R})$ as follows :

$$\|\alpha\| = \inf_z \|z\| \quad ,$$

where z runs over all singular cycles representing $\alpha \in H_*$.

For a closed manifold V we define its simplicial volume $\|V\|$ as the simplicial norm of its fundamental class. When V is not orientable we pass to the double covering $\widetilde{V}$ and set $\|V\| = \frac{1}{2}\|\widetilde{V}\|$.

If V is open, then its fundamental class is represented by <u>locally finite</u> cycles $c = \sum_{i=1}^{\infty} r_i \sigma_i$, such that each compact subset of V only intersects <u>finitely many</u> (images of) simplices σ_i . Now, the ℓ^1-norm $\|c\| = \sum_{i=1}^{\infty} |r_i|$ may be infinite and the corresponding simplicial volume $\|V\|$ also may be infinite.

Example : For the real line one has $\|\mathbb{R}^1\| = \infty$.

The following functorial property of the simplicial volume is immediate from the definition :

<u>Let</u> $f : V \longrightarrow V'$ <u>be a proper map of degree</u> d . <u>Then</u> $\|V\| \geq d\|V'\|$.

Furthermore, <u>if</u> f <u>is a</u> d-sheeted covering, <u>then</u> $\|V\| = d\|V'\|$.

<u>Corollary</u> : <u>If a closed manifold</u> V <u>admits a self-mapping</u> f <u>of degree</u> $d \geq 2$, <u>then</u> $\|V\| = 0$.

For example, all spheres and tori have zero simplicial volume. In particular closed connected surfaces V with $\chi(V) \geq 0$ have $\|V\| = 0$.

If V is an <u>open</u> manifold which admits a proper self-mapping of degree ≥ 2 , then one can only claim that either $\|V\| = 0$ or $\|V\| = \infty$. In fact, both cases occur, but if V is homeomorphic to the interior of a compact manifold with boundary, $V = \text{Int } \overline{V}$, then the case $\|V\| = \infty$ is excluded. Moreover,

<u>If the boundary</u> $\partial\overline{V}$ <u>of</u> $\overline{V}$ <u>admits a selfmapping</u> f <u>with</u> $|d| = |\deg f| \geq 2$, <u>then</u> $\|\text{Int } \overline{V}\| < \infty$.

<u>Proof</u> : First we represent the fundamental class of $\overline{V}$ by a chain c with boundary b in $\partial\overline{V}$. Then we attach to $\overline{V}$ the cylinder $\partial\overline{V} \times [0,\infty)$, and observe that the resulting manifold V is exactly Int V . Now we extend the chain c to an ℓ^1-cycle of V as follows. Denote by b_1 the image $f_*(b)$ in $\partial\overline{V} \times 1 = \partial\overline{V}$ and let c_1 be a chain in $\partial\overline{V} \times [0,1]$ with the boundary $\partial c_1 = d^{-1}b_1 - b$ for $d = \deg f$. Next we consider the iterates $f^{(k)}$ of f , for $k = 0,1,\ldots$, and take the maps

$$f^k : \partial V \times [0,1] \rightarrow \partial V \times [k,k+1] \subset V \times [0,\infty) \quad ,$$

defined by $f^k : (v,t) \longmapsto (f^{(k)}(v), t+k)$. Then the locally finite chain $\tilde{c} = c + \sum\limits_{k=0}^{\infty} d^{-k} f_*^k(c_1)$ is a fundamental cycle of V of finite ℓ^1-norm,

$$\|\tilde{c}\| = \|c\| + \sum_{k=0}^{\infty} d^{-k} \|c_1\| = \|c\| + \frac{|d|}{|d|-1} \|c_1\| .$$

Corollary : <u>The simplicial volume of Euclidean spaces</u> $\mathbf{R}^n$ <u>is</u> : $\|\mathbf{R}^n\| = 0$,
<u>for</u> $n \geq 2$.

Indeed, the space $\mathbf{R}^n$ has self-mappings of any degree.

Now comes our first interesting example of a closed manifold V
with $\|V\| > 0$, which is a special case of a theorem of Thurston (see
section 0.3).

Example : Let $\|V\|$ be a closed oriented surface with Euler characteristic
$\chi < 0$ of constant negative curvature -1 . The fundamental class is
represented by a cycle $\sum\limits_{i=1}^{q} r_i \sigma_i$, where $\sigma_1,\ldots,$ are singular 2-
simplices. The total (algebraic) volume is

$$\sum_{i=1}^{q} r_i \, \text{Vol}(\sigma_i) = -2\pi\chi \quad .$$

Straighten all singular simplices involved; keeping the vertices fixed, by
lifting to the universal covering, that is the hyperbolic plane $\mathbf{H} = \widetilde{V}$.
Observe moreover that the absolute value of the volume of any straight
triangle in $\mathbf{H}$ with positive angles is majorized (use the excess formula
in hyperbolic geometry) by π . So we find

$$2\pi|\chi| = \sum_{i=1}^{q} r_i \, (\text{Vol straigt } \sigma_i)$$

$$\leq \sum_{i=1}^{q} |r_i| \cdot \pi \quad .$$

Hence $\qquad \|V\| = [\inf \sum_{i=1}^{q} |r_i|] \geq 2|\chi(V)|$

$$\|V\| \geq 2|\chi(V)| \qquad \text{for} \quad \chi < 0 .$$

This is the precise bound. The simplicial volume is

$$\|V\| = 2|\chi(V)|$$

for closed surfaces V of constant negative curvature. In order to see elementarily that $\|V\| \leq 2|\chi|$ we consider the standard model of a fundamental domain in H , a regular k-gon F_o , $k = 2|\chi|+4$. It can be covered by a cycle of $k-2 = 2|\chi|+2$ straight triangles so that in any case $\|V\| \leq 2|\chi|+2$. Then we apply this construction to d-sheeted coverings of V and thus we cover d-times the fundamental class of V by $2d|\chi|+2$ triangles. Hence we find $\|V\| \leq 2|\chi|+2d^{-1}$ and by letting $d \to \infty$ we obtain

$$\|V\| \leq 2|\chi| \quad .$$

Finally, we indicate two general usefull properties of the simplicial volume .

(1) If V_1 is a closed manifold and V_2 is arbitrary, then

$$C\|V_1\| \, \|V_2\| \geq \|V_1 \times V_2\| \geq C^{-1}\|V_1\| \, \|V_2\| \quad , \tag{*}$$

where $C > 0$ is a constant which depends only on $n = \dim(V_1 \times V_2)$.

Observe that the first inequality above is obvious and it also holds if both manifolds, V_1 and V_2 are open. The second inequality is more interesting since it gives an estimate from below for $\|V_1 \times V_2\|$. We prove this inequality in section 1.1. with the bounded cohomology technique. The requirement of V_1 to be closed is essential. Indeed for example $\|\mathbb{R}^1\| = \infty$, while $\|\mathbb{R}^2\| = 0$.

Notice that both inequalities claim nothing whatsoever for the case of $\|V_1\| = 0$ and $\|V_2\| = \infty$. However for all known examples of such manifolds V_1 and V_2 one has $\|V_1 \times V_2\| = 0$.

(2) Connected sums of n-dimensional manifolds satisfy for $n \geq 3$,

$$\|V_1 \# V_2\| = \|V_1\|+\|V_2\| \quad . \tag{**}$$

This is proven in section 3.5 where we also establish the following generalization.

Let V_o be a closed simply connected submanifold of V of codimension one. Then, for $\dim V > 1$, the simplicial volume of V does not change if the submanifold V_o is deleted, $\|V \diagdown V_o\| = \|V\|$.

0.3   Inequalities of Milnor-Sullivan and Thurston

To take the simplicial volume seriously one needs additional examples of manifolds V for which $\|V\| \neq 0$. The following remarkable theorems provide a variety of such examples.

Milnor-Sullivan Theorem ([39],[44])   If a closed manifold V supports an affine flat bundle of dimension $n = \dim V$, then $\|V\| \geq |\chi|$ for the Euler number χ of this bundle.

Recently, Smillie refined Sullivan's argument and proved : $\|V\| \geq 2^n |\chi|$. This is presented in section 1.3, where we also study Pontryagin numbers of non-affine flat bundles.

The theorem of Milnor-Sullivan is only useful for n even since odd dimensional bundles have zero Euler numbers. Our next inequality works for all n .

Thurston's Theorem : Let V be a complete Riemannian manifold of finite volume, $\mathrm{Vol}(V) < \infty$. If the sectional curvature of V satisfies $-\infty < -k \leq K(V) \leq -1$, then $\mathrm{Vol}(V) \leq \mathrm{const}_n \|V\|$.

Thurston's proof is presented in section 1.2. (See also [47] and [55]).

I wish to thank Denis Sullivan who introduced me to these results and to a circle of ideas important for this paper.

0.4. Complements to Thurston's inequality

Characteristic numbers of locally homogeneous manifolds V are proportional to $\text{Vol}(V)$ by Hirzebruch's proportionality principle. In fact characteristic numbers can be obtained following Chern-Weil by integrating over V some universal polynomials in the curvature $\Omega = \Omega(V)$.

Unlike the characteristic numbers the simplicial volume of Riemannian manifolds V cannot be obtained by integration of local invariants of V , but the proportionality phenomenon remains valid (see [47] and also section 2.3).

Proportionality theorem : <u>If the universal coverings of two closed Riemannian manifolds</u> V <u>and</u> V' <u>are isometric, then</u>

$$\|V\|/\text{Vol}(V) = \|V'\|/\text{Vol}(V') \quad .$$

This theorem is clearly true for V and V' with a common <u>finite</u> covering. In fact, the theorem is most useful if the universal covering $\tilde{V}$ of V is a symmetric space and then $\|V\|/\text{Vol}(V) = \text{const} = \text{const}(\tilde{V})$. This constant is, probably, non-zero if $\tilde{V}$ has negative Ricci curvature. Indeed, $\text{const} \neq 0$ if $\tilde{V}$ has negative <u>sectional</u> curvature (rank 1 case) by Thurston's theorem. Also, as we shall prove in section 1.3, this constant is non-zero if some characteristic number p of V does not vanish. Furthermore, R.P. Savage recently proved (see [42]) that $\text{const}(\tilde{V}) \neq 0$ for symmetric spaces $\tilde{V}$ whose isometry groups are special linear groups $\text{Is}(\tilde{V}) = SL_q(\mathbb{R})$ for $n = \dim V = \frac{1}{2}q(q+1)-1$. These manifolds have rank $= q-1$ and all their characteristic numbers vanish for $q > 2$.

Example : Let $\tilde{V}$ be a product of hyperbolic spaces $(K = -1)$ of various dimensions. Then $\|V\| = \text{const Vol}(V)$, and by Thurston's theorem combined with the

inequality (*) of 0.2 we have const $\geq$ const$_n$ > 0 , for n = dim V . Furthermore, if all hyperbolic factors have even dimensions, then

$\|V\|$ = const'$|\chi(V)|$, where the last constant only depends on the dimensions of the factors.

Explicit constants are only known for negative curvature -1, $\tilde{V} = H^n$. They are determined as follows. Take all n-dimensional simplices S in the hyperbolic space H^n of curvature -1 and denote by R_n the upper bound of their volumes,

$$R_n = \sup_{S \subset H^n} Vol(S) \quad .$$

(A simplex is, by definition, the convex hull of n+1 points in general position in H^n).

Then, we have the exact formula :

$$\|V\| = R_n^{-1} Vol(V) \quad .$$

This formula also holds for complete non-compact manifolds of finite volume (see [47] and section 1.2) and the extremal value R_n is always assumed by the regular ideal simplex in H^n with all vertices at infinity. (Milnor [47] for n = 3 , Haagerup and Munkholm [27] for n $\geq$ 4) . Furthermore, $R_2 = \pi$, $R_3 = \frac{3}{2} \sum_{i=1}^{\infty} i^{-2} \sin \frac{2\pi i}{3} \approx 1.0149$, and asymptotically for n $\to \infty$ one has $R_n \approx \frac{\sqrt{n}}{n!}e$. (See [27]) .

Another useful relation for $\|V\|$ comes from the following.

The (Σ b$_i$)-estimate : Let V be a complete connected real analytic manifold with bounded non-positive curvature, $-k^2 \leq K(V) \leq 0$, and let the Ricci tensor of V be negative definite at some point v $\in$ V . Then

$$\sum_{i=0}^{n} b_i(V) \leq \text{const}_n \, k^n \, \text{Vol}(V) \quad,$$

where $n = \dim V$ and b_i are the Betti numbers with any given coefficients (See Appendix 3).

With Thurston's theorem and (**) of 0.2 we come to the following

Corollary : Let V be a connected sum of manifolds of the following two types

(a) Compact locally symmetric spaces with non-zero simplicial volume. For example those which have non-zero Euler characteristics.

(b) Complete manifolds of finite volume with sectional curvature pinched between two negative constants,

$$-k_1 \leq K(V) \leq -k_2 < 0 \quad.$$

Then

$$\sum_{i=0}^{n} b_i(V) \leq \text{const} \, \|V\| \quad,$$

where the constant depends only on $n = \dim V$ and on the ratio k_1/k_2. It is unclear if the constant can be chosen independently of this ratio.

0.5 Estimates from below for the minimal volume

For Riemannian manifolds V we denote by $\text{Ricci}(V)$ the Ricci tensor, and we write $\text{Ricci} \geq -1$ if $\text{Ricci}(\tau,\tau) \geq - <\tau,\tau>$ for all tangent vectors $\tau \in T(V)$. Observe that a bound from below for the sectional curvature, $K(V) \geq -k^2$, implies $\text{Ricci}(V) \geq -(n-1)k^2$, that is $\text{Ricci}(\tau,\tau) \geq -(n-1) \, k^2 <\tau,\tau>$. We prove in section 2.5 the following

Main Inequality : Let V be a complete n-dimensional Riemannian manifold with Ricci $\geq 1/1-n$. Then $\|V\| \leq \text{const}_n \text{Vol}(V)$ for some constant in the interval, $0 < \text{const}_n < n!$.

Corollaries : (A) The estimate for the Minimal Volume : All differentiable manifolds V satisfy

$$\|V\| \leq (n-1)^n \quad n! \text{ Min Vol}(V) \quad .$$

(B) The (Σb_i)-estimate with Ricci curvature : Let V be a complete Riemannian manifold with Ricci $V \geq -k^2$. If V is homeomorphic to a compact hyperbolic manifold, or, more generally, to a connected sum of manifolds (a) and (b) of the last corollary of 0.3, then

$$\sum_{i=0}^{n} b_i(V) \leq \text{const}_n' k^n \text{Vol}(V) \quad .$$

(C) The Volume comparison theorem : Let V and V' be complete Riemannian manifolds of dimension n and let $f : V \longrightarrow V'$ be a continuous proper map. Assume moreover

$$\text{Ricci}(V) \geq 1/1-n \quad ,$$

$$-\infty < -k \leq K(V') \leq -1 \quad ,$$

and Vol $V' < \infty$. Then

$$|\deg(f)| \leq C_n \text{Vol}(V)/\text{Vol}(V') \quad . \tag{$*$}$$

Unfortunately we do not know the explicit value of this constant C_n . The most optimistic conjecture would be $C_n = (n-1)^{-n}$. In fact, the proportionality theorem of 0.3 yields this conjecture when both V and V' have constant negative curvature (notice that Ricci $V = 1/1-n$ here corresponds

to $K(V) = -(1-n)^{-2}$. One may conjecture further that the equality in (*)
with this optimal hypothetic constant C_n only may hold if both manifolds
V and V' do have constant curvature, and then by Thurston's rigidity
theorem (See [47], [25]) the map f is homotopic to a locally isometric
d-sheeted covering.

Now let V be homeomorphic to the interior of a (possibly non-compact)
manifold W whose boundary ∂W is a disjoint union of <u>compact</u> manifolds,
called $\partial_1 W$, $\partial_2 W$,... . We prove in section 2.5 the following estimate from
below for the volumes of balls $B(R) \subset V$ around a fixed point v_o in V .
(Compare   the Example of 0.1) .

<u>The asymptotic inequality</u> : <u>If V is a complete manifold with Ricci V $\geq$ 1/1-n</u>
<u>then</u>

$$\|\partial W\| = \|\partial_1 W\| + \|\partial_2 W\| + .. \leq \text{const}_n \liminf_{R \to \infty} R^{-1} \text{Vol } B(R) ,$$

<u>for</u> $0 < \text{const}_n \leq (n-1)!$.

The estimate for the minimal volume generalizes in section 2.5 to
products $V = V_1 \times V_2$ where V_1 is a <u>closed</u> manifold with a nonzero
Pontryagin number $p = p(V_1)$.

<u>The product Inequality</u> : <u>The product</u> $V = V_1 \times V_2$ <u>satisfies</u>
$p(V_1) \|V_2\| \leq \text{const}'_n \text{Min Vol}(V_1 \times V_2)$, <u>for</u> n = dim V .

<u>Example</u> : Take for V_1 the complex projective plane and let V_2 be a
(compact or not) 3-dimensional manifold with $K(V_2) = -1$ and $\text{Vol}(V_2) < \infty$.
Then $\text{Min Vol}(V_1 \times V_2) \geq \text{const Vol}(V_2)$, for some const > 0 , while
$\|V_1 \times V_2\| = 0$ since V_1 (and so $V_1 \times V_2$) has self mappings of degree
≥ 2 . Notice that all characteristic numbers of this $V_1 \times V_2$ are zero
since n = 7 . We have no examples of closed five dimensional manifolds V

with $\|V\| = 0$ and Min Vol(V) > 0 . On the other hand we shall show in Appendix 2 that for many 3-dimensional manifolds Min Vol(V) $\leq$ const$'\|V\|$.

We shall see in section 3.1 that the simplicial volumes of <u>closed</u> manifolds V are entirely determined by the fundamental groups $\Pi = \pi_1(V)$ and by the classifying maps $V \longrightarrow K(\Pi,1)$. For example, $\|V\| = 0$ for all closed simply connected manifolds V . However we do not know whether the <u>minimal</u> volume is also zero for all closed <u>odd</u> dimensional simply connected manifolds V .

Another interesting problem concerns the sets of values of Min Vol(V) and $\|V\|$, where V runs over all manifolds of a given dimension n . The work of Thurston (see [47][25]) may suggest that both sets are closed countable well ordered non-discrete subsets of the real line, but we do not even know whether the value zero is isolated, i.e. if Min Vol(V) $\leq \varepsilon$ for some $\varepsilon = \varepsilon_n > 0$ implies Min Vol(V) = 0 . The same question is open for the simplicial volume. However, we shall see in section 3.4, that Min Vol(V) $\leq \varepsilon_n$, for some $\varepsilon_n > 0$, does imply $\|V\| = 0$. Moreover, we have the following

<u>Isolation theorem</u> : Let V <u>be a complete</u> n-dimensional manifold with Ricci(V) ≥ -1 , <u>and let the unit ball in</u> V <u>around each point</u> $v \in V$ <u>satis-fy</u> Vol $B_v(1) \leq \varepsilon$ <u>for some sufficiently small positive</u> $\varepsilon = \varepsilon(n)$. <u>Then</u> $\|V\| = 0$. <u>In particular, if</u> Vol(V) $\leq \varepsilon(n)$ <u>then</u> $\|V\| = 0$.

<u>Corollary</u> : <u>If a manifold</u> V <u>with</u> Ricci V ≥ -1 <u>admits a proper map of posi-tive degree onto a manifold</u> V' <u>of negative sectional curvature, for which</u> Vol(V') $< \infty$ <u>and</u> $-\infty < -k_1 \leq K(V') \leq -k_2 < 0$, <u>then for some point</u> $v \in V$ <u>the unit ball has</u> Vol($B_v(1)$) $\geq \varepsilon = \varepsilon(n) > 0$.

Remark : If $V = V'$ and $V \longrightarrow V'$ is the identity map then this corollary reduces to a theorem of Margulis (see [7]) and in this special case one can give an effective estimate for ε , for example, one can take $\varepsilon = \varepsilon_n = \exp(-\exp(\exp n^n))$. Notice, that our proof in section 3.4 depends on the polynomial growth theorem (see [23]) that gives no effective estimate. However, we prove in section 4.3 for manifolds with $|K| \leq 1$ the following more general result with the above effective value $\varepsilon = \varepsilon_n$,

The Injectivity Radius estimate : If $|K(V)| \leq 1$ then the simplicial volume $\|V\|$ is bounded from above by $\text{const}_n \text{Vol}(U_\varepsilon)$, where $U_\varepsilon \subset V$ denotes the set of those points $v \in V$, for which the injectivity radius of V satisfies Inj. $\text{Rad}_v(V) \geq \varepsilon = \varepsilon_n$. In particular, if for all $v \in V$ one has Inj $\text{Rad}_V(V) \leq \varepsilon_n$, then $\|V\| = 0$.

In the general case of Ricci $\geq -k > -\infty$ we prove in section 4.2 the following weaker result.

The Geometric Finiteness Theorem : Let V be a complete manifold with Ricci $V \geq -k > -\infty$ and let the unit balls $B_v(1) \subset V$ satisfy $\text{Vol}(B_v(1)) \to 0$ for $v \to \infty$. Then the simplicial volume is finite, $\|V\| < \infty$.

Corollary : The above manifold V admits no proper map of positive degree onto a connected sum of infinitely many hyperbolic manifolds of finite volumes.

Plan of the paper :

We start section 1 with the translation of our problems to the language of bounded cohomology. Then we prove in this language the theorem of Milnor-Sullivan and Thurston. Next, we construct in section 2 geometric smoothing operators on bounded Borel cochains and prove the Main inequality for compact

manifolds. In fact, we prove this inequality for a modified notion of sim-
plicial volume. The equivalence of the two simplicial volumes is established
in section 3, by means of algebraic averaging operators on bounded cochains
of simplicial multicomplexes. We also prove with these operators the isola-
tion theorem for compact manifolds and the identity $\|V_1 \# V_2\| = \|V_1\| + \|V_2\|$
as well. Finally, in section 4 we return to ℓ^1-chains on open manifolds.
We start with the algebraic diffusion of chains on these manifolds and then
we combine the algebraic diffusion with the geometric smoothing operators.
Thus we prove in section 4.3 the Main Inequality for open manifolds.

In fact, we establish in (D) of 4.3 a sharper result, called Main
Technical Theorem which relates the simplicial norm on homology to the
geometric mass of cycles.

Next, in section 4.4 we refine the simplicial volume for complete non-
compact manifolds by also taking into account the geometric "size" of singular
simplices of fundamental cycles. We prove with this refinement the existence
of extremal manifolds V with $|K(V)| \leq 1$ whose volumes $Vol(V)$ equal
their minimal volumes. In fact we only prove a slightly weaker result for a
modified notion of minimal volume, rather than for the original one. In the final
section 4.5 we study "volumes" of maps between manifolds who themselves may
have infinite volumes. In particular, we generalize the Volume Comparison
Theorem to manifolds of infinite volumes. We also discuss there the Euler characte-
ristic and the signature of complete non-compact manifolds. In the Appendices (1),(2) and
(3) we briefly discuss surfaces, manifolds V with Min Vol (V)=0 and manifolds with $K(V) < 0$
respectively.

1. Bounded cohomology

A general question, arising when a chain complex C_* is equipped with a norm, is the problem of finding a complete set of its chain homotopy invariants in the category of <u>topological</u> vector spaces and <u>bounded</u> operators. (If the topology in C_* is ignored, then the homology groups of C_* give us all these invariants). The problem becomes simpler when we complete C_*. For the simplicial norm of section 0.2 the completion leads to the complex $\overline{C}_*$ of ℓ^1-chains $c = \sum\limits_{i=1}^{\infty} r_i \sigma_i$ with $\|c\| = \sum\limits_{i=1}^{\infty} |r_i| < \infty$, like those we constructed in section 0.2 .

It is more convenient, in some respects, to work with the dual complex $\hat{C}^* = \text{Hom}(\overline{C}_*, \mathbb{R})$ whose elements admit the following independent description.

1.1 Bounded cochains

Denote by Σ the set of all singular simplices $\sigma : \Delta \to X$ and recall that real <u>cochains</u> $c \in C^* = C^*(X)$ are, by definition, certain functions $c : \Sigma \longrightarrow \mathbb{R}$. We define the ℓ^∞-norm of c by setting

$$\|c\|_\infty = \sup_{\sigma \in \Sigma} |c(\sigma)|$$

and we call a cochain c <u>bounded</u> if this norm $\|c\|_\infty$ is finite.

<u>Counterexample</u> : Take a closed n-dimensional manifold V with the oriented volume form ω on V . Then the "standard" singular cocycle c which assigns to each $\sigma : \Delta^n \longrightarrow V$ the integral $c(\sigma) = \int_{\Delta^n} \sigma^*(\omega)$ is not bounded.

Indeed, if the map σ "wraps" Δ^n around V many times, then $c(\sigma)$ becomes arbitrarily large.

For cohomology classes $\beta \in H^*(C_*; \mathbb{R})$ we set $\|\beta\| = \|\beta\|_\infty = \inf_y \|y\|_\infty$ where y runs over all cocycles representing β . Of course, this "norm"

can take infinite values, as will become clear below. We say that a coho-
mology class β is <u>bounded</u> if its "norm" $\|\beta\|_\infty$ is finite.

<u>Cohomological definition of the simplicial volume</u> :

The C^∞-norm in $C^* = \text{Hom}(C_*, \mathbb{R})$ is the dual of the ℓ^1-norm in
C_* . We apply the Hahn-Banach theorem (compare [45]) and conclude that
the norms $\| \ \|_1$ in $H_*(X; \mathbb{R})$ and $\| \ \|_\infty$ in $H^*(X; \mathbb{R})$ are also dual.

<u>Corollary</u> : <u>Let</u> V <u>be a closed oriented</u> n-<u>dimensional manifold with dual</u>
<u>fundamental classes</u> $\alpha \in H_n(V, \mathbb{R})$ <u>and</u> $\beta \in H^n(V; \mathbb{R})$. <u>Then</u>

$$\beta(\alpha) = 1 = \|\beta\|_\infty \|\alpha_1\| = \|\beta\|_\infty \|V\| \quad .$$

<u>Therefore</u> $\|V\|^{-1} = \|\beta\|_\infty ;$ <u>in particular</u> β <u>is bounded iff the simplicial</u>
<u>volume</u> $\|V\|$ <u>does not vanish.</u>

Now, if V is an open manifold then the fundamental cohomology class
β of V is represented by cocycles $c : \Sigma \to \mathbb{R}$ with <u>compact support</u>.
That is for some compact subset $V_o \subset V$ which depends on c , the cochain
$c(\sigma)$ vanishes at those singular simplices $\sigma : \Delta \longrightarrow V$, whose images do not
intersect V_o .

With these cochains we have again our ℓ^∞-norm, $\|\beta\|_\infty$, and we
clearly have the inequality $\|V\| \ \|\beta\|_\infty \leq 1$. However, the equality
$\|V\|^{-1} = \|\beta\|_\infty$ does not, in general, hold.

<u>Example</u> : Let V be the interior of a compact manifold with boundary,
V = Int $\overline{V}$. Then, with the ℓ^1-norm in the relative chain complex

$$C_*(\overline{V}, \partial\overline{V}) = C_*(\overline{V})/C_*(\partial\overline{V}) \quad ,$$

one has the norm of the fundamental class $\overline{\alpha} \in H_*(\overline{V}, \partial\overline{V})$ and then one

puts $\|\overline{V}, \partial\overline{V}\| = \|\overline{\alpha}\|_1$. Now <u>this</u> norm $\| \quad \|_1$ and the corresponding simplicial volume are dual to the ℓ^∞-norm on the cohomology with compact supports in $V = \text{Int } \overline{V}$, and so

$$\|\overline{V}, \partial\overline{V}\|^{-1} = \|\beta\|_\infty \quad .$$

In particular, this simplicial volume $\|\overline{V}, \partial\overline{V}\| = \|\beta\|_\infty^{-1}$ is always <u>finite</u>. However if the simplicial volume $\|\text{Int } \overline{V}\|$ is finite, then clearly the simplicial volume $\|\partial\overline{V}\|$ vanishes, and so one only can claim the inequality

$$\|\text{Int } \overline{V}\| \geq \|\overline{V}, \partial\overline{V}\| \qquad .$$

In order to give a cohomological definition of $\|V\|$ we consider <u>locally finite</u> subsets of singular simplices, $\Phi \subset \Sigma$, such that every compact subset of V intersects only finitely many (images of) simplices in Φ. Then for cochains with compact supports in V we define seminorms $\|c\|_\Phi$ by putting

$$\|c\|_\Phi = \sup_{\sigma \in \Phi} |c(\sigma)| \qquad ,$$

and next we have the corresponding seminorms on the cohomology with compact supports. Finally we take the upper bound of these seminorms over all locally finite sets of singular simplices,

$$\|\beta\|^\infty = \sup_{\Phi \subset \Sigma} \|\beta\|_\Phi \qquad .$$

Now, the Hahn-Banach theorem does apply and for the fundamental class β of V we have $\|\beta\|^\infty = \|V\|^{-1}$.

As the first application we get the second inequality (*) of 0.2,

$$\|V_1 \times V_2\| \geq C^{-1} \|V_1\| \|V_2\|$$

by referring to the dual inequality for the cup-product,

$$\| \beta_1 \cup \beta_2 \|^\infty \leq C \| \beta_1 \|_\infty \| \beta_2 \|^\infty \quad ,$$

which follows from the cup-product formula for singular cochains. Notice that if none of the manifolds V_1 and V_2 is closed, then the product of cochains with compact supports, c_1 in V_1 and c_2 in V_2 , may have non-compact support in $V_1 \times V_2$, and so the failure of the product ine-quality is not surprising.

1.2. Thurston's theorem

We prove three slightly different forms of Thurston's theorem of section 0.3 and we start with the most transparant original version. We consider maps f of simplicial m-dimensional polyhedra P into complete Riemannian manifolds V . For a <u>Lipschitz</u> map f we denote by $mass_m f$ the volume of f counted with geometric multiplicity. This mass equals the total volume of the singular Riemannian metric in P induced by the map $f : P \longrightarrow V$. We denote by $[P]_m$ the number of m-dimensional simplices in P .

(A) <u>Homotopy theorem</u> : <u>If the sectional curvature $K = K(V)$ satisfies $K \leq -1$, and if $m = \dim P \geq 2$, then every continuous map $f_o : P \longrightarrow V$ is homo-topic to a Lipschitz map f such that $mass_m f \leq const_m [P_m]$, for some constant $const_m \leq \pi/(m-1)!$. Furthermore, for manifolds V of constant curvature -1 one has $const_m = R_m \asymp e\sqrt{m}/m!$, where R_m is the volume of the regular ideal simplex in</u> m-dimensional hyperbolic space (see 0.3).

<u>Corollary</u> : <u>If P and V are moreover closed manifolds of the same dimen-sion $m \geq 2$ then there is an upper bound for the degree of maps $f_o : P \longrightarrow V$. In particular, there is no self-mapping $f_o : V \longrightarrow V$ of degree</u> ≥ 2 .

Indeed, $\deg f_o = \deg f \leq const\ mass_m f$, where $const = const(P,V)$ depends on the manifolds P and V .

<u>Proof of the theorem</u> : We construct the required map f by first "straight-ening" the map f_o on all simplices Δ of P and then by estimating volumes of straight simplices in V .

First, take an ordered set of m+1 points $v_o, \ldots, v_m$ in the universal covering $\widetilde{V}$ of V and span these by a <u>straight singular</u> <u>m-simplex</u> $\overline{\sigma}_m : \Delta^m \longrightarrow \widetilde{V}$ by induction : the ℓ-dimensional simplex $\overline{\sigma}_\ell : \Delta^\ell \longrightarrow \widetilde{V}$, $\ell = 1, \ldots, m$, with ordered vertices $v_o, \ldots, v_\ell$, is defined as the geodesic cone from v_ℓ over the $(\ell-1)$-dimensional straight face spanned by the first ℓ vertices $v_o, \ldots, v_{\ell-1}$. Next for an arbitrary simplex $\sigma : \Delta \rightarrow V$ we take a lift $\widetilde{\sigma}$ to $\widetilde{V}$, then we replace this $\widetilde{\sigma}$ by the straight simplex $\overline{\sigma} : \Delta \longrightarrow \widetilde{V}$ spanned by the vertices of $\widetilde{\sigma}$ and finally we project $\overline{\sigma}$ back to V . The resulting simplex $\sigma' : \Delta \longrightarrow V$ is called the <u>straightening</u> <u>of</u> σ and denoted straight σ .

Now, for a map $f_o : P \longrightarrow V$, we order the vertices of P , and then for each simplex $\Delta \subset P$ we straighten the singular simplex $f_o|\Delta : \Delta \rightarrow V$. Since these straightened simplices agree on their common faces, we obtain a map $f : P \longrightarrow V$, which now is straight on all $\Delta \subset P$, and which is homotopic to f_o . Observe, that the only property of V we used here is the uniqueness of geodesic segments joining pairs of points in $\widetilde{V}$.

What is left to prove is the following

<u>Estimate of the volume of a straight simplex</u> : <u>If</u> $K(V) \leq -1$ <u>then straight</u> <u>simplices</u> $\sigma : \Delta^m \longrightarrow \widetilde{V}$ <u>satisfy for</u> $m \geq 2$, $mass_m \sigma \leq \pi/(m-1)!$.

<u>Proof</u> : For K = -1 the sharp result, $mass_m \sigma \leq R_m$, is established in [27] .

This sharp inequality also holds in the general case for geodesic <u>triangles</u> $\Delta^2 \subset \widetilde{V}$. Indeed, the relative curvature of these triangles, that

is their second quadratic form, is non-positive and so the Gauss curvature
of the induced metric in Δ^2 satisfies $G(s) \leq -1$ for all $s \in \Delta^2$.
Then, by Gauss-Bonnet, $\int_{\Delta^2} G(s)ds \geq -\pi$ and so

$$\text{mass } \Delta^2 = \text{area } \Delta^2 \leq \int_{\Delta^2} |G(s)|ds \leq \pi \quad .$$

Now, volumes of geodesic cones over ℓ-dimensional submanifolds in
$\tilde{V}$ satisfy for $K(\tilde{V}) \leq -1$,

$$\text{Vol(Cone)} \leq \ell^{-1} \text{Vol(Base)} \quad ,$$

(see [5][10]) and thus the proof is finished. Compare [35]).

Question : Can one take $\text{const}_m = R_m$?

Example : Let P be a closed connected surface. Then, with an example of
0.2, we get for straight maps $f : P \longrightarrow V$

$$\text{mass}_2 f \leq \pi (2 \mid \chi(P) \mid +2) \quad .$$

In fact, one even can "triangulate" P into exaclty $2|\chi(P)|$ ideal
triangles (see [47]) and thus obtain maps $f : P \longrightarrow V$, in case V is
closed, with $\text{mass}_2 f \leq 2\pi |\chi(P)|$.

One can also proceed in a more traditional way by deforming f_o to a
minimal or to a harmonic map $f : P \longrightarrow V$, and then observe that the
induced Gauss curvature in P is everywhere ≤ -1 . This again yields, via
Gauss-Bonnet, the sharp inequality $\text{mass}_2 f \leq 2\pi |\chi(P)|$.

It is unclear how to obtain sharp inequalities for minimal maps of
general polyhedra (or manifolds) P , but the next version of Thurston's
theorem provides such an estimate for homology.

First for a singular m-dimensional chain $c = \sum\limits_{i} r_i \sigma_i$, for
$\sigma_i : \Delta^m \longrightarrow V$, we put

$$\text{mass}(c) = \sum\limits_{i} |r_i| \text{ mass}_m \sigma_i \quad ,$$

and then for a homology class $\alpha \in H_m(V;\mathbb{R})$ we define mass(α) as the lower
bound of masses of cycles which represent α .

(B) Homology theorem (see [47],[35]): If $K(V) \le -1$ and if $m \ge 2$, then
all $\alpha \in H_m(V;\mathbb{R})$ satisfy

$$\text{mass}(\alpha) \le \text{const}_m \|\alpha\|_1 \quad ,$$

where $\| \ \|_1 = \| \ \|$ is the simplicial norm of section 0.2 and "const$_m$"
is the same as in the homotopy theorem (A).

Proof : First we represent α by a cycle $c = \sum\limits_{i} r_i \sigma_i$ whose norm
$\|c\| = \sum\limits_{i} |r_i|$ is close to $\|\alpha\|_1$ and then we straighten all its singular
simplices σ_i . Straightening commutes with the boundary operator on chains,
and so the straightened chain

$$c' = \sum\limits_{i} r_i \sigma_i' \quad , \quad \text{for } \sigma_i' = \text{straight } \sigma_i ,$$

is, in fact, a cycle homologous to c . Therefore

$$\text{mass}(\alpha) \le \text{mass}(c') = \sum\limits_{i} |r_i| \text{mass}_m \sigma_i'$$

$$\le \text{const}_m \sum\limits_{i} |r_i| = \text{const}_m \|c\| \quad . \qquad \text{Q.E.D.}$$

Remark : The homology theorem, when applied to the fundamental homology class
of a closed n-dimensional manifold V , yields Thurston's theorem of 0.3,

$$\text{Vol}(V) \le \text{const}_n \|V\| \quad .$$

Finally we come to the dual cohomological version of (B) which relates the norm $\| \ \|_\infty$ of section 1.1 to the comass norm on cohomology. Recall that the comass of a differential m-form ω on V is defined as the upper bound of the values of ω at the orthonormal m-frames in V . Then for a class $\beta \in H^m(V; \mathbb{R})$ we define

$$\text{comass } \beta = \inf_\omega \text{comass } \omega \quad ,$$

where ω runs over all closed m-forms representing β .

(C) Cohomology theorem : If $K(V) \leq -1$ and $m \geq 2$ then one has for all $\beta \in H^m(V; \mathbb{R})$,

$$\|\beta\|_\infty \leq \text{const}_m \text{comass } \beta \quad .$$

Proof : First, represent β by a form ω with comass(ω) close to comass(β). Then construct a singular cocycle (!) c representing β by putting, for every singular simplex $\sigma : \Delta^m \longrightarrow V$,

$$c(\sigma) = \int_{\sigma'} \omega \quad , \text{ where } \sigma' = \text{straight } \sigma \ .$$

Since $|c(\sigma)| \leq \text{mass}(\sigma') \text{comass}(\omega)$, we get $|c(\sigma)| \leq \text{const}_m \text{comass}(\omega)$ for all σ . Q.E.D

Remark : One can show with the Hahn-Banach theorem, that the theorems (B) and (C) are, in fact, equivalent.

The relative case : Consider a locally convex subset $V_o \subset V$, for example a totally geodesic submanifold $V_o \subset V$. Then for each singular simplex $\Delta \longrightarrow V_o$ its straightening is also contained in V_o and so the theorems (A),(B) and (C) hold in their respective relative forms.

Now we consider proper geodesics $\gamma : \mathbb{R} \longrightarrow V$ that is both ends ($t \to \pm\infty$ for $t \in \mathbb{R}$) go to infinity in V . We say that V is concave relative to infinity (one might say that the infinity of V is convex) if

no proper geodesic γ extends to a <u>proper</u> map $f : \mathbb{R} \times [0, \infty) \longrightarrow V$ for $f \mid \mathbb{R} \times 0 = \gamma : \mathbb{R} \times 0 = \mathbb{R} \longrightarrow V$.

<u>Examples</u> : If a manifold V_o is closed, then the cylinder $V_o \times \mathbb{R}$ is concave relative to infinity. The space $\mathbb{R}^n$ for $n \geq 2$ is not concave relative to infinity.

If V is concave relative to infinity then every cocycle b in V with <u>compact support</u>"straightens"to a cocycle b' which also has <u>compact support</u>. Furthermore, $b-b' = \delta c$, for some cochain c with <u>compact support</u>. Thus we obtain for the fundamental cohomology class β of V our old estimate

$$\|V\| = (\|\beta\|^{\cap})^{-1} \geq (\|\beta\|_{\infty})^{-1} \geq \mathrm{const}_n^{-1} \mathrm{Vol}(V) \quad .$$

<u>Example</u> : If $-k \leq K(V) \leq -1$ and if $\mathrm{Vol}(V) < \infty$, then V is concave relative to infinity (see Appendix 3) and so

$$\mathrm{Vol}(V) \leq \mathrm{const}_n \|V\| \quad .$$

Thus the proof of Thurston's theorem of 0.3 is concluded.

1.3 <u>The theorem of Milnor-Sullivan</u>

We start with the original geometric version of the theorem (see [39], [44]) .

<u>Let P be a simplicial polyhedron and let $Z \longrightarrow P$ be a flat n-dimensional vector bundle over P. Then the Euler class $\chi(Z) \in H^n(P;\mathbb{Z})$ can be represented by a simplicial cocycle whose value at each simplex of P is $0, 1$ or -1.</u>

<u>Proof</u> : Take a section $f : P \longrightarrow Z$ which does not vanish on the $(n-1)$-skeleton of P. The cocycle, which assigns to each oriented n-dimensional

simplex $\Delta \subset P$ the algebraic number of zeros of f in Δ , is cohomologous to $\chi(Z)$. (This is the definition of χ). Since the bundle Z is flat we can choose the section $P \longrightarrow Z$ piecewise linear, such that its restriction to each simplex Δ in P is the graph of an affine map $\Delta \longrightarrow \mathbb{R}^n$. Now, a generic piecewise linear section has at most one simple zero inside each n-dimensional simplex. Q.E.D.

The following argument due to Smillie, provides a rational cocycle whose value at each n-dimensional simplex equals $\pm 2^{-n}$ for n even and 0 for n odd.

A piecewise linear section $f : P \longrightarrow Z$ is determined over each n-dimensional simplex $\Delta \subset P$ by its values $f_0,\ldots,f_i,\ldots,f_n$ at the vertices of Δ . If we multiply some of f_i by -1, we get a new linear section over Δ . There are 2^{n+1} such sections over Δ , but exactly two (opposite ones) of them have a zero. To see this, study $n+1$ points

$$\varepsilon_0 f_0,\ldots,\varepsilon_n f_n \quad \text{in} \quad \mathbb{R}^n \quad \text{for} \quad \varepsilon_i = \pm 1 ,$$

and examine whether their convex hull contains 0 in the interior. If we assign to each Δ the algebraic average number of zeros while averaging over P , we get the required cocycle representing $\chi(P)$.

Remarks : (a) If P is a closed oriented manifold with fundamental class $[P]$ then we can speak of the Euler numbers, $\chi = <\chi(Z),[P]>$. Any lower bound for the number of n-simplices of a triangulation of P gives us an upper bound for $|\chi|$. One often obtains better estimates by using triangulations of finite coverings of P . For surfaces P , one gets with an example of 0.2

$$|\chi| \leq \frac{1}{2} |\chi(P)| \quad . \tag{$*$}$$

This estimate is sharp. Indeed, by taking a metric of constant curvature in

P , we get $P = H^2/\Pi$ where H^2 is the hyperbolic plane and $\Pi \simeq \pi_1(P)$ is a subgroup in the orientation preserving isometry group $Is(H^2) = PSL_2(\mathbb{R})$ The quotient $PSL_2(\mathbb{R})/\Pi$ is canonically isomorphic to the total space of the unit tangent bundle $S \longrightarrow P$, and since the Euler number of this bundle is even $(= \chi(P))$, there is a double covering $\tilde{S} \longrightarrow S$ such that the pullbacks $\tilde{S}_p \subset \tilde{S}$ of the fibers $S_p = S^1 \subset S$, $p \in P$ are connected. Thus we get another circle bundle

$$\tilde{S} \longrightarrow P \quad \text{(a "square root" of } S)$$

and $\tilde{S} = SL_2(\mathbb{R})/\Pi$ for some lift

$$g : \Pi \longrightarrow SL_2(\mathbb{R}) \longrightarrow PSL_2(\mathbb{R}) = SL_2(\mathbb{R})/(\pm 1) \quad .$$

The vector bundle $\tilde{T}$, associated to $\tilde{S}$, admits a flat structure, since it is also associated to the principle fibration $H^2 \longrightarrow P = H^2/\Pi$ via the linear representation g . Now, $\tilde{T}$ is "the square root" of the tangent bundle $T(V)$ and so

$$\chi(\tilde{T}) = \frac{1}{2}\chi(T(V)) \quad .$$

(b) There is another way, (pointed out by Lusztig) to estimate Euler numbers of n-dimensional flat bundles over a <u>fixed</u> closed n-dimensional manifold P . In fact, flat bundles over P correspond to linear representations $\pi_1(P) \longrightarrow GL_n(\mathbb{R})$. The set of these representations is a real algebraic variety and so it has <u>finitely many</u> connected components. The bundles corresponding to points of any component are topologically equivalent, and in particular, their Euler numbers are equal. Observe furthermore, that the number of different values of $\chi(Z)$ for all flat bundles $Z \longrightarrow P$ is estimated from above by the fundamental group alone, while the <u>maximal value</u> of $|\chi|$ may also depend on the manifold P itself.

Example : Let dim P $\geq$ 4 and let us delete from P an open regular neighbor-hood of the 1-skeleton of some triangulation of P . The boundary B of the resulting manifold P' admits an orientation reversing diffeomorphism D such that the induced homomorphism D_* : $\pi_1(B) \circlearrowleft$ is the identity. Next we take two copies of P' and glue their boundaries with D . We obtain in this way a closed manifold P" which admits a map f : P" $\longrightarrow$ P of degree 2 and such that the induced homomorphism f_* : $\pi_1(P") \longrightarrow \pi_1(P)$ is an isomorphism. Now, for any flat bundle Z $\longrightarrow$ P with Euler number χ , we observe that the induced bundle $f^*(Z)$ over P" has Euler number 2χ , and by repeating this process we obtain flat bundles with arbitrary large χ with-out changing the fundamental group of the underlying manifolds.

The theorem of Milnor-Sullivan-Smillie can be generalized in the following abstract form.

Let Z be an n-dimensional flat bundle over an arbitrary topological space X . Then the Euler class $\chi = \chi(Z) \in H^n(X;\mathbb{R})$ satisfies

$$\|\chi\|_\infty \leq 2^{-n} \quad .$$

(As before $\chi = 0$ for n odd). In particular, one gets the inequality $\|V\| \geq 2^n |\chi|$ of section 0.3.

We shall show in section 3.2 how the geometric version of the theorem, when applied to the geometric realization of a semi-simplicial set of sin-gular simplices in X , yields the abstract theorem.

Our version of the theorem of Milnor-Sullivan-Smillie implies in par-ticular boundedness of the Euler class for affine flat bundles.

A generalization : Let G be an algebraic subgroup in the linear group $GL_m = GL_m(\mathbb{R})$. Take the classifying space BG and consider also the classify-ing space BG^δ for G with the discrete topology.

Consider the natural homomorphism in cohomology, $H^*(BG; \mathbb{R}) \longrightarrow H^*(BG^\delta; \mathbb{R})$, and call an $\alpha \in H^*(BG^\delta, \mathbb{R})$ a <u>characteristic class</u> if it is contained in the image of this homomorphism.

<u>Theorem</u> : <u>Each characteristic class in</u> $H^*(BG^\delta; \mathbb{R})$ <u>is bounded</u>.

This theorem follows, as in the case of the Euler class (see 3.2.) , from the following

<u>Geometric version</u> : <u>Let</u> Z <u>be a flat</u> m-<u>dimensional</u> G-<u>bundle</u> (<u>i.e. a vector bundle with a flat</u> G-<u>structure</u>) <u>over a simplicial polyhedron</u> P. <u>Fix a class</u> $\beta \in H^n(BG; \mathbb{R})$ <u>and consider the characteristic class</u> $\beta^* \in H^n(P; \mathbb{R})$ <u>induced by the classifying map</u> $P \longrightarrow BG$. <u>Then</u> β^* <u>can be represented by a simplicial cocycle which is bounded in absolute value at each</u> n-<u>dimensional simplex in</u> P <u>by a constant depending only on</u> β.

<u>Proof</u> : We assume P of finite dimension $\leq n$ and construct the following finite dimensional "approximation" B_N of the classifying space BG for large N. It will suffice to take $N \geq n+m+2$. Take first the total space K of the canonical principle GL_m-bundle over the Grassmanian $Gr_m(\mathbb{R}^N)$ of m-dimensional subspaces in $\mathbb{R}^N$. The group $G \subset GL_m$ acts on K and the space $B_N = K/G$ will be our approximation to BG. The space B_N now carries the natural structure of a real quasi-projective manifold.

<u>Algebra-Geometric lemma</u> : <u>Let</u> B <u>be a real quasi-projective manifold</u>. <u>Then for each</u> s <u>there is a Zariski closed set</u> $C \subset B$ <u>of dimension</u> s <u>such that the inclusion homomorphism</u> $\tilde{H}_s(C) \longrightarrow \tilde{H}_s(B)$ <u>is surjective</u>, <u>where</u> $\tilde{H}_s$ <u>denotes homology with non-compact supports</u>.

<u>Proof of the lemma</u> : According to Hironaka [30] one can realize B by a Zariski open dense subset in a non-singular projective manifold A. Furthermore, A has a triangulation with the following properties (see [31]) :

(a) each simplex of this triangulation is a semi-algebraic set in A ;

(b) the complement $A \smallsetminus B$ is a closed subcomplex of this triangulation.

Denote by $\overline{C}$ the Zariski closure of the s-skeleton of our triangulation and take for the required C the intersection $\overline{C} \cap B$.

Proof of the theorem : A classifying map $P \longrightarrow B_N$ for any flat G-bundle Z over P now comes from the following construction. Take the trivial bundle $T = P \times \mathbb{R}^N \longrightarrow P$ and an injective homomorphism $Z \longrightarrow T$. Such a homomorphism assigns to each fiber Z_p , $p \in P$, an m-dimensional subspace $Y_p \subset \mathbb{R}^N$ and an isomorphism $Z_p \longrightarrow Y_p$. A subspace Y_p with the G-structure induced from Z_p is interpreted as a point in B_N .

Since Z carries a flat structure we have a notion of piecewise linear homomorphisms $Z \longrightarrow T$ whose corresponding classifying maps $P \dashrightarrow B_N$ are piecewise algebraic. Moreover, these maps are algebraic of degree d on each simplex in P where d depends only on n and of course on the group $G \subset GL_m$.

Now, the class $\widetilde{\beta} \in \widetilde{H}_s(B_N)$, $s = \dim(B_N)-n$, Poincaré dual to $\beta \in H^n(B_N)$, can be realized, according to the lemma, by a cycle $\widetilde{b} = \sum\limits_{i=1}^{k} r_i \widetilde{\Delta}_i$ which is built of k s-dimensional semialgebraic simplices $\widetilde{\Delta}_i$ in C .

For a generic piecewise algebraic map $P \longrightarrow B_N$ the image of each simplex $\Delta \subset P$ intersects each $\widetilde{\Delta}_i \subset B_N$ transversally and only at interior points, whose number is at most $d \cdot \deg(C)$. Therefore, the real intersection number ν_Δ of Δ with $\widetilde{b}$ is at most

$$d \ \deg(C) \sum_{i=1}^{k} |r_i| \leq \text{const}_\beta \qquad .$$

Finally we observe that the cocycle $\Delta \rightarrow \nu_\Delta$ is cohomologous to $\beta^* \in H^n(P;\mathbb{R})$ thus concluding the proof of the theorem.

<u>Remark</u> : In many interesting cases one can realize $\tilde{\beta}$ by a combination of Schubert cycles and then one gets more precise estimates for $\|\beta^*\|_\infty$ like those we obtained for the Euler class.

2. Estimates from above for the simplicial volume

2.1. A preliminary discussion

Consider a closed oriented Riemannian manifold of dimension n with sectional curvature bounded in absolute value by one, $|K(V)| \leq 1$. In order to prove $\|V\| \leq \text{const}_n \text{Vol}(V)$ (see 0.5. we must "cover" the fundamental class of V by at most $\text{const}_n \text{Vol}(V)$ simplices. First, suppose that at all points $v \in V$ the injectivity radius of V satisfies $\text{Inj. Rad}_v(V) \geq \epsilon_0$ for some fixed number $\epsilon_0 > 0$. Then the exponential map $\exp : T_v(V) \longrightarrow V$ at every point $v \in V$ embeds the ϵ_0-ball $B(\epsilon_0)$ in $T_v(V)$ around the origin into V. Furthermore, if ϵ_0 is small then the map $\exp_v : B(\epsilon_0) \longrightarrow V$ is almost isometric, that is its differential D has everywhere a norm close to one. In fact, even without the condition $\text{Inj. Rad}(V) \geq \epsilon_0$ one has $|\|D\|-1| < \epsilon_0^2$ provided $\epsilon_0 \leq 1$ and $|K(V)| \leq 1$ (see [10]). Now if all ϵ_0-balls in a manifold V are roughly Euclidean, there is a triangulation with simplices of size about ϵ_0 and such that the total number of the simplices equals $\text{const}_n \epsilon_0^{-n} \text{Vol}(V)$. In particular we obtain the following

Trivial Inequality : If $|K(V)| \leq 1$ and if $\text{Inj. Rad}_v(V) \geq \epsilon_0$ for all $v \in V$, then

$$\|V\| \leq \text{const}_n \epsilon_0^{-n} \text{Vol}(V) \qquad . \qquad\qquad (*)$$

This inequality gives no interesting estimates from below for $\text{Vol}(V)$ and for $\text{Min.Vol}(V)$ since the condition $\text{Inj. Rad}(V) \geq \epsilon_0$ already implies $\text{Vol}(V) \geq C_n \epsilon_0^n$. However, for manifolds of non-positive curvature one derives the following

Non-trivial Corollary : If $0 \geq K(V) \geq -1$ and if the fundamental group $\pi_1(V)$ is residually finite, then

$$\|V\| \leq \text{const}_n \text{Vol}(V) \qquad . \qquad (**)$$

Proof : Recall, that a group Π is called residually finite if the intersection of all subgroups in Π of finite index is the unit element e in Π . One can also express this condition in our context by saying that for every $\ell > 0$ there is a finite covering $\widetilde{V} \longrightarrow V$ such that every loop in $\widetilde{V}$ of length $\leq \ell$ is contractible. The last condition implies, now for $K(V) \leq 0$, that $\text{Inj. Rad}_v(\widetilde{V}) \geq \frac{1}{2}\ell$ for all $v \in \widetilde{V}$. Then for $\ell \geq 2$, the Trivial Inequality yields $\|\widetilde{V}\| \leq \text{const}_n \text{Vol}(\widetilde{V})$, and as

$$\|\widetilde{V}\| \Big/ \|V\| = \text{Vol}(\widetilde{V})/\text{Vol}(V) = d \qquad ,$$

for d equal the number of sheets of the covering $\widetilde{V} \longrightarrow V$, we also get $\|V\| \leq \text{const}_n \text{Vol}(V)$.

Example : First take a manifold V with a metric g_0 of constant curvature -1 . Then for an arbitrary metric g on V with $0 \geq K(g) \geq -1$ one has

$$\text{Vol}(V, g_0) \leq \text{const}'_n \text{Vol}(V, g) \qquad . \qquad (***)$$

Indeed, manifolds of constant curvature have residually finite fundamental groups. In fact the same holds for all finitely generated subgroups of connected Lie groups, see [41]). Then (***) follows from Thurston's theorem of 0.3 and from (**) .

The arguments above can be extended to cover the general case of manifolds V with $|K(V)| \leq 1$. The idea is to represent the fundamental class of such a V by a cycle $c = \sum_i r_i \sigma_i$ with the following two properties :

(a) all simplices $\sigma_i : \Delta^n \longrightarrow V$ are ε-small for $\varepsilon \simeq (20)^{-n^{-10}}$. That is each σ_i admits a lift $\tilde{\sigma}_i$ to an exponential ε-ball,

$$\tilde{\sigma}_i : \Delta^n \longrightarrow B(\varepsilon) \in T_{v_i}(V) \qquad ,$$

for some point $v_i \in V$, and $\exp_{v_i} \circ \tilde{\sigma}_i = \sigma_i$.

(b) $\|c\| = \sum\limits_{i} |r_i| \leq const_n Vol(V)$.

One can construct such a cycle c by first taking all "ε-cycles" c which satisfy (a) only, and then by minimizing the norm $\|c\|$ among "ε-cycles". In fact, there is the following minimizing procedure which actually diminishes the norm of c, as long as this norm is too large compared to $Vol(V)$. Namely, if $\|c\|$ is large then also a lift $\tilde{c}$ of c to the unit ball $B(1) \subset T_v(V)$ at some point $v \in V$ has a large norm $\|\tilde{c}\|$ compared to $Vol(B(1))$. Therefore, one can replace $\tilde{c}$ by a smaller standard "ε-chain" $\tilde{c}_o$ in $B(1)$ which represents the relative fundamental class of the pair $(B(1), \partial B(1))$. Next, one can construct another chain $\tilde{c}_1$ which "interpolates" between $\tilde{c}$ and $\tilde{c}_o$, in the sense that $\tilde{c}_1$ equals $\tilde{c}$ in a small neighbour-hood of $\partial B(1)$ and such that $\tilde{c}_1 = \tilde{c}_o$ far from the boundary $\partial B(1)$. This $\tilde{c}_1$ has smaller norm than $\tilde{c}$, and by projecting the difference $\tilde{c}_1 - \tilde{c}$ back to V and by adding this projection to the cycle c, we do diminish the norm of c.

A technical difficulty of this argument is the necessity to keep all singular simplices ε-small while constructing the interpolating chain $\tilde{c}_1$. This is the reason for the ridiculously small $\varepsilon \simeq (20)^{-n^{-10}}$. In fact, one gets even worse estimates for "$const_n$" in the inequality $\|V\| \leq const_n Vol(V)$, namely something like $const_n \simeq (100)^{n^{10}}$.

We shall not pursue anymore this line of reasonning and turn to more efficient estimates for $\|V\|$.

2.2 Exact estimates for hyperbolic manifolds

Let V be a closed oriented manifold of constant curvature -1. Then there is the following very efficient way to "cover" the fundamental class of V by immersed straight singular simplices σ_D with all edges of length D. Such a "covering" would yield, for $D \to \infty$, the inequality $\|V\| \leq R_n^{-1} \text{Vol}(V)$, and therefore (see (B) of 1.2.) the exact formula $\|V\| = R_n^{-1} \text{Vol}(V)$ of section 0.3.

We first generalize finite singular chains in V by admitting as a generalized chain any family F of singular simplices σ, parametrized by a manifold F with a finite measure, given by a form $\mu(\sigma)d\sigma$ and total measure the norm (see [47]) $\|F\| = \int_F |\mu(\sigma)| \, d\sigma$ (instead of $\Sigma \, r_i \sigma_i$ with norm $\Sigma |r_i|$). Specifically we take the family F_D of all regular simplices σ_D with the orientations induced from V, that is each σ_D is covered by an embedded regular simplex $\tilde{\sigma}_D$ in the universal covering hyperbolic space H^n. The isometry group $\text{Is}(H^n)$ acts transitively on the simplices $\tilde{\sigma}_D$, and the Haar measure of $\text{Is}(H^n)$ induces a measure $\mu(\sigma)d\sigma$ on $F = F_D$. The generalized chain so defined is a cycle : under the boundary operator the contributions of two sides of any $(n-1)$-face $\tilde{\sigma}'$ cancel each other because these sides are symmetric under the orientation reversing(!) reflection of H^n in the hyperplane spanned by $\tilde{\sigma}'$. This chain F_D represents the fundamental class in case

$$\int \mu(\sigma) \, \text{Vol}(\sigma) \, d\sigma = \text{Vol}(V) \quad ,$$

and then

$$\|F_D\| = \int \mu(\sigma)d\sigma = (\text{Vol } \sigma_D)^{-1} \text{Vol } V \quad .$$

By definition

$$\|V\| \leq \|F_D\| \quad ,$$

and since $\mathrm{Vol}(\sigma_D) \longrightarrow R_n$ for $D \to \infty$, we get $\|V\| \leq R_n^{-1} \mathrm{Vol}\, V$.

Now, we only must show that these generalized cycles F_D give the same value for $\|V\|$ as usual chains. First we take N points in V , each assigned with the weight $\mathrm{Vol}(V)/N$. The resulting atomic measure in V is denoted by μ_N , and eventually for $N \to \infty$, we require the sequence μ_N to converge, in the weak topology, to the Riemannian measure in V . Then we consider all straight simplices, whose edges now may have any length between D and $D+\delta$ for some $\delta > 0$, and whose vertices must be chosen among our N points. Next we consider the (formal) sum Σ of all these simplices and take the normalized chain

$$c = c(N,\delta,D) = \Sigma / \|\Sigma\| \quad .$$

Notice, that this $\|\Sigma\|$ equals the number of the simplices. Now, for fixed D and δ , one has

$$\|\partial c\| \to 0 \quad \text{for } N \to \infty \quad ,$$

and then one can construct chains $c' = c'(N,\delta,D)$ such that $\partial c' = \partial c$ and $\|c'\| \to 0$ for $N \to \infty$. Finally we take <u>cycles</u>

$$c'' = c''(N,\delta,D) = c-c'$$

and observe that

$$\lim_{\delta \to 0} \lim_{N \to \infty} c'' = F_D / \|F_D\| \quad ,$$

and so the cycles $\|F_D\|^{-1} c''$ give in the limit for $N \to \infty$, $\delta \to 0$ and $D \to \infty$, the fundamental class of V , while the norms of these cycles converge to $R_n^{-1} \mathrm{Vol}(V)$. (See [47] for more information).

Nico Kuiper suggested the following direct elegant construction of <u>finite</u> singular cycles F_D' in V , for which $\|F_D'\| \to R_n^{-1} \mathrm{Vol}(V)$ as $D \to \infty$.

Fix a fundamental domain U in the hyperbolic space $H^n = \tilde{V}$ for the Galois action of the group $\Pi = \pi_1(V)$ in the universal covering $\tilde{V}$ of V and let us also fix a point $u \in U$. Then we consider straight simplices $\tilde{\sigma}$ in $\tilde{V}$ with vertices in the Π-orbit of u and we assign to each $\tilde{\sigma}$ with vertices $\gamma_0(u),\ldots,\gamma_n(u)$ a coefficient, $\mu(\tilde{\sigma})$ equal to the (Haar) measure of those regular simplices $\tilde{\sigma}_D$ in $\tilde{V} = H^n$ whose $n+1$ vertices lie in the translates $\gamma_0(U),\ldots,\gamma_n(U)$; precisely one vertex in each set $\gamma_i(U)$, $i = 0,\ldots,n$. As the family $\tilde{F}_D$ of all regular simplices $\tilde{\sigma}_D$ in $\tilde{V}$ is a (generalized) cycle, the Π-invariant chain $\tilde{F}'_D = \Sigma_{\tilde{\sigma}} \mu(\tilde{\sigma})\tilde{\sigma}$ also is a cycle. Furthermore, this cycle $\tilde{F}'_D$ projects to a finite straight singular cycle in V, called F'_D, which is homologous to F_D (i.e. F'_D is a fundamental cycle of V for a proper normalization of the Haar measure in the group $Is(H^n))$, and also $\|F'_D\| = \|F_D\|$. Hence, $\|F'_D\| \to R_n^{-1} \mathrm{Vol}(V)$ as $D \to \infty$.

Remark : Generalized chains F_D can be defined, for $D \leq 1$, in all manifolds V with $K(V) \leq 1$ by taking equilateral geodesic simplices in balls $B(2) \subset T_v(V)$ with the metrics induced from V by the maps $\exp_v : B(2) \longrightarrow V$. However, no canonical measure parametrize these F_D into cycles, though

$$\|\partial F_D\|/\|F_D\| \to 0 \quad \text{for } D \to 0 .$$

It would be interesting to find a small perturbation of F_D to a cycle and thus obtain yet another proof of the inequality $\|V\| \leq \mathrm{const}_n \mathrm{Vol}\, V$.

2.3 Straight invariant cochains

Consider a covering Y of manifold V with Galois group Π and first observe that singular cochains in V lift to those cochains in Y which are invariant under the action of Π. Next, we define a subcomplex, called $C^*(Y:\Pi) \subset C^*(Y)$ of real valued singular cochains c by imposing the following

four conditions.

(1) Cochains $c \in C^*(y:\Pi)$ are <u>straight</u> : the values $c(\sigma)$ for all

$\sigma : \Delta^m \rightarrow Y$ only depend on the vertices $y_o,\ldots,y_i,\ldots,y_m \in Y$ of σ.

The "straight" cochains are, in fact, functions in $m+1$ variables

$y_o,\ldots,y_m \in Y$.

(2) The functions $c = c(y_o,\ldots,y_m)$ are antisymmetric, that is

$$c(y_o,\ldots,y_i,\ldots,y_j,\ldots,y_m) = -c(y_o,\ldots,y_j,\ldots,y_i,\ldots,y_m)$$

for all pairs of variables y_i and y_j .

(3) The functions c are <u>Borel</u> in the variables y_i, that is pullbacks

of Borel subsets in $\mathbb{R}$ are Borel in the Cartesian products $\overset{m}{\underset{o}{\oplus}} Y$.

(4) The functions c are Π-invariant.

<u>Remarks</u> : The condition (3) is purely technical. We could equally well require

continuity (or even smoothness) of c , without changing the homology of

$C^*(Y:\Pi)$.

The condition (2) will be quite important in the further calculations

but again it does not affect the homology of our complex. Indeed, there is

the natural (anti)symmetrization operator which projects all functions to

the antisymmetric ones, namely

$$c(y) \longmapsto \frac{1}{(m+1)!} \underset{\delta}{\Sigma} [\delta] c(\delta(y)) \quad ,$$

where δ runs over all permutations of the variables $y = (y_o,\ldots,y_m)$

and where $[\delta] = 1$ for even permutations δ and $[\delta] = -1$ for odd δ .

This operator induces an isomorphism on the (real !) homology. In fact, it

is a chain homotopy equivalence.

Without the condition (4) our complex would be acyclic (in fact, chain-

contractible). Only the action of Π makes the story interesting. In fact, as we shall see in 3.3,

$$H_*(C^*(Y:\Pi)) \simeq H^*(\Pi;\mathbb{R}) \overset{\text{def}}{=} H^*(K(\Pi,1);\mathbb{R}) \quad,$$

where $K(\Pi,1)$ denotes the Eilenberg MacLane space.

Example : We constructed in the proof of the homotopy theorem of section 1.2 a cocycle $c = \int_{\sigma'} \omega$, which lifts to the universal covering $Y = \tilde{V} \longrightarrow V$ with the properties (1),(3) and (4). In fact, this c is a continuous function in $y_o,\ldots,y_m$ and it even extends to the ideal boundary of $\tilde{V}$. One also can make, with the (anti) symmetrization operator, this cocycle c antisymmetric, without changing its cohomology class and keeping it bounded with the norm $\leq \text{const}_m \text{comass}(\omega)$.

New simplicial volume : The ℓ^∞-norm on functions $c = c(y_o,\ldots,y_m)$ induces a norm in cohomology $H^*(Y:\Pi) \overset{\text{def}}{=} H_*(C^*(Y:\Pi))$. Furthermore, the inclusion $C^*(Y:\Pi) \longrightarrow C^*(V)$ induces a homomorphism $H^*(Y:\Pi) \longrightarrow H^*(V)$. Now let us take for Y the universal covering $\tilde{V} \longrightarrow V$ with $\Pi = \pi_1(V)$ and let us denote by F the homomorphism $H^*(\tilde{V}:\Pi) \longrightarrow H^*(V)$. For all $\beta \in H^*(V)$ we put $\|\beta\|_\infty^{\text{new}} = \inf_\alpha \|\alpha\|_\infty$, where α runs over the pullback $F^{-1}(\beta) \subset H^*(\tilde{V}:\Pi)$. In particular, $\|\beta\|_\infty = \infty$ if β does not come from $H^*(\tilde{V}:\Pi)$. Then, by duality, we define the following new norm for all $\gamma \in H_m(V)$

$$\|\gamma\|^{\text{new}} = \sup_\beta (\|\beta\|_\infty^{\text{new}})^{-1} |\beta(\gamma)|$$

over all $\beta \in H^m(V)$.

It is clear that $\|\gamma\|^{\text{new}} \leq \|\gamma\|^{\text{old}}$ and in particular, for closed manifolds V we have $\|V\|^{\text{new}} \leq \|V\|^{\text{old}}$. In fact, with the remark above, the new norm $\|\gamma\|^{\text{new}}$ equals the old norm of the image of γ under the natural homomorphism $H_*(V) \longrightarrow H_*(K(\Pi,1))$, for $\Pi = \pi_1(V)$. In particular, the new norm (and thus the new simplicial volume) vanishes for simply connected manifolds.

We shall prove in section 3.3 an equivalence theorem claiming the equality $\| \ \|^{new} = \| \ \|^{old}$, but even without this theorem (which requires a bit of abstract machinery) one can use the new norm as efficiently as the old one. Indeed, the principal estimate from below for $\| \ \|^{old}$, namely Thurston's theorem (C) of 1.2, delivers, in fact, the same estimate for the new norm, because the (anti-symmetrized) cocycle $c = \int_{\sigma'} \omega$ is contained in the complex $C^*(Y:\Pi)$. Moreover, the functorial property for maps $f : V \longrightarrow V'$,

$$\| f(\gamma) \| \leq \| \gamma \| \quad .$$

also holds for the new norm and so our main geometric applications in section 0.4 will not suffer if the old norm there, is understood in the new sense. In any case, since the new and the old norms are equal (see 3.3) we do not bother to distinguish them anymore.

Finally, a word of caution : the new simplicial volume has not been yet defined for open manifolds V and so $\|V\|$ must be still understood for such V in the old sense. We return to open manifolds in section 4.

Now, let us observe that the new norm on cohomology $H^*(Y:\Pi)$ also makes sense for an arbitrary locally compact group of homeomorphisms (instead of Π) of any space Y.

Examples : Let Π be a discrete group of isometries of a Riemannian manifold Y. By taking a subgroup of index two we always can make Π orientation preserving. Then the space Y/Π is a pseudo-manifold (in fact it is a rational homology manifold) and if Y/Π is compact of dimension n we have the n-dimensional fundamental class $\beta \in H^n(Y/\Pi; \mathbb{R})$. We represent this β by a singular cocycle $b \in C^*(Y/\Pi)$ and take its pull-back $\tilde{b}$ in the complex $\tilde{C}^*_\Pi(Y)$ of Π-invariant cochains in Y. Finally we consider all cocycles $a \in C^*(Y:\Pi) \subset \tilde{C}^*_\Pi(Y)$ which are cohomologous to $\tilde{b}$ (in $\tilde{C}^*_\Pi(Y)$) and put $\|\beta\|_\infty = \inf_a \|a\|_\infty$. In particular, we may define in this way the simplicial

<u>volume of an orbifold</u> V (see [47]) by taking the "universal covering" $Y = \tilde{V} \longrightarrow V = \tilde{V}/\Gamma$ and by putting $\|V\| = (\|\beta\|_\infty)^{-1}$. For instance, if $\tilde{V}$ has constant curvature -1 then $\|V\| = R_n^{-1} \, \text{Vol}(V)$, as in section 0.3.

Now, let G be the full group of isometries of the manifold Y and let $\mathcal{D}_G^*(Y)$ denote the de Rham complex of G-invariant forms on Y . By integrating forms over singular simplices in Y we get a homomorphism of $\mathcal{D}_G^* = \mathcal{D}_G^*(Y)$ into the complex $\tilde{C}_G^*(Y)$. Let $\tilde{\omega} \in \tilde{C}_G^*(Y)$ denote the image of a form $\omega \in \mathcal{D}_G^*$, and let us define $\|\omega\|_\infty$ as the infimum of ℓ^∞-norms of cocycles $\alpha \in C^*(Y;G)$ which are cohomologous (in $\tilde{C}_G^*(Y)$) to $\tilde{\omega}$. Next we take a discrete subgroup $\Pi \subset G$ and for simplicity we assume that Π acts freely on Y . Then ω also defines a cohomology class in $H^*(Y/\Pi)$, called $[\omega]_\Pi$, and one has

$$\| [\omega]_\Pi \|_\infty \leq \|\omega\|_\infty \qquad . \tag{*}$$

Assume furthermore that the subgroup Π is cocompact in G . Then every Π-invariant cocycle $\alpha \in C^*(Y;\Pi)$ can be averaged over G/Π to a G-invariant cocycle $\bar{\alpha} \in C^*(Y;G)$ with $\|\bar{\alpha}\|_\infty \leq \|\alpha\|_\infty$, and thus the inequality (*) becomes an equality :

$$\| [\omega]_\Pi \|_\infty = \|\omega\|_\infty \qquad . \tag{**}$$

In particular, $\| [\omega]_\Pi \|_\infty$ does not depend on Π as long as Π is cocompact in G .

If Y/Π is a compact manifold, then the equality (**) , when applied to the volume form ω on Y , yields Thurston's proportionality theorem of section 0.3. Indeed, with this ω we, by definition, have

$$\|V = Y/\Pi\| = \text{Vol}(V) (\| [\omega]_\Pi \|_\infty)^{-1} \quad .$$

(See [47] for a "dual" proof in the language of "smeared homology").

2.4. Smoothing of Borel cochains

Our main estimates from above for the simplicial volume are based on the following averaging (smoothing) construction, which is first explained in the geometric language of straight chains. Let, for example, V be an n-dimensional manifold of non-positive curvature which is triangulated into small straight simplices Δ . Then we continuously move the vertices of this triangulation, called $v_i \in V$, $i = 0,\dots,k$, into new positions, $v_i' \in V$, and we observe that any such move $v_i \longrightarrow v_i'$ uniquely extends to a map f of V into itself such that f sends every simplex Δ with some vertices v_i onto the straight simplex Δ' with vertices v_i' . The result of any move $v_i \longrightarrow v_i' = f(v_i)$ can be obtained by going along a geodesic between v_i and v_i' and so the space F of maps $f : V \longrightarrow V$ is parametrized by the Cartesian product of k+1 copies of the universal covering $\tilde{V}$ of V .

Next we take the fundamental class $c = \Sigma \Delta \in C_n(V)$ of our triangulation and then we have a family of cycles $f_*(c) \in C_n(V)$ for all $f \in V$. Now, with some positive normalized (probability) measure μ on F we average this family to the generalized cycle $c_\mu = \int_F f_*(c)d\mu$. (Compare 2.2). As all maps $f \in F$ are homotopic to the identity, the cycle c_μ is homologous to c , while its simplicial norm clearly satisfies :
$\|c_\mu\| \leq \|c\| \int_F d\mu = \|c\|$. In fact, the norm $\|c_\mu\|$ may even become strictly less than $\|c\|$. Indeed, a simplex Δ may be sent by two different maps f in F onto the same geometric simplex Δ' but with opposite orientations and these simplices algebraically cancel.

Example : Let Δ be the oriented (!) unit interval in $\mathbb{R}^1$, $\Delta = [0,1]$. Then the moving pairs of ends v_0' and v_1' are points in $\mathbb{R}^2$ and with the normalized Lebesgue measure μ in the square $\{v_0' \in [-\rho,\rho]$ and $v_1' \in [1-\rho,1+\rho]\}$

$$c_\mu = \rho^{-2} \int_{-\rho}^{\rho} dv_0' \int_{1-\rho}^{1+\rho} dv_1' \left[v_0', v_1' \right] \quad .$$

This chain c_μ is understood as a measure on ordered (!) pairs of points $[v_0', v_1']$, and this measure vanishes at the pairs of points v_0' and v_1' in the interval $[1-\rho, \rho]$, since there are exactly two opposite moves of the interval $\Delta = [0,1]$ onto $\Delta' = [v_0', v_1']$ with equal μ-probabilities. It follows, that indeed $\|c_\mu\| < \|c = \Delta\| = 1$ for $\rho > \frac{1}{2}$, and furthermore, $\|c_\mu\| \to 0$ as $\rho \to \infty$.

We shall return to the averaging (or diffusion) of chains in section 4.3. but now we shall develop a more flexible language of smoothing of of cochains in manifolds V . To do that we consider a covering $Y \longrightarrow V$ with Galois group Π and we construct the following averaging (smoothing) in the subcomplex of bounded cochains in $C^*(Y:\Pi)$, called $\hat{C}^*(Y:\Pi)$. We denote by $M = M(Y)$ the Banach space of finite measures μ on Y with the norm $\|\mu\| = \int_Y |\mu|$, and we denote by $M^+ \subset M$ the cone of positive measures. We observe that every bounded Borel m-cochain $c = c(y_0, \ldots, y_m)$ uniquely extends to a $(m+1)$-linear function on M ,

$$c(\mu_0, \ldots, \mu_m) = \underbrace{\iint}_{m+1} c(y_0, \ldots, y_m) \mu_0(y_0) \cdots \mu_m(y_m) \quad .$$

This extended cochain is again denoted by c . We observe that

$$\sup_{\|\mu_i\| \leq 1} |c(\mu_0, \ldots, \mu_i, \ldots, \mu_m)| = \|c\|_\infty = \sup_{y_i \in Y} |c(y_0, \ldots, y_i, \ldots, y_m)| .$$

Let us call a smooth Π-invariant map $S: Y \longrightarrow M^+$ a smoothing operator. We consider the induced cochains $S^*(c)$ on Y for $c = c(\mu_0, \ldots, \mu_m)$ and we normalize them to

$$S_* c = (S*c)(y_0, \ldots, y_m) = S^*(c)(y_0, \ldots, y_m) / \prod_{i=0}^{m} \|S(y_i)\| .$$

Remark : If we assign to each $y \in Y$ the Dirac δ-measure at y, then for this map $\delta : Y \longrightarrow M$, $\delta : y \longrightarrow \delta_y$, we take $\delta^*(c) = \delta * c = c$ for all $c \in C^*(Y:\Pi)$. However the map δ is not smooth and not even continuous relative to our norm in M.

As the map $S : Y \longrightarrow \overset{*}{M}$ is Π-invariant the smoothing operator $c \longrightarrow S * c$ sends the complex $C^*(Y:\Pi)$ into itself. Furthermore this operator $C^*(Y:\Pi) \rightleftarrows$ is a chain homomorphism commuting with boundary operators, and due to our normalization this operator induces the identity homomorphism on the cohomology $H^*(Y:\Pi)$.

Now let V be a Riemannian manifold. Then for each $y \in Y$ we define the norm of the differential of S,

$$\| D_y S \| = \sup_\tau \| (D_y S)(\tau) \| \quad ,$$

where τ runs over the unit tangent sphere $S_y \subset T_y(Y)$. Then we put

$$[S]_y = \| S(y) \|^{-1} \| D_y S \|$$

and

$$[S] = \sup_{y \in Y} [S_y] \quad .$$

Proposition : Let S be a smoothing with $[S] < \infty$. Then any bounded cocycle $c \in C^m(Y:\Pi)$ is De Rham cohomologous to a closed Π-invariant form ω on Y for which

$$\text{comass}(\omega) \leq m! \, \| c \|_\infty ([S])^m \quad .$$

Proof : First we take $\bar{c} = S * c$ and then we define informally $\omega = \omega(t_1, \ldots, t_m)$ for tangent vectors $t_i \in T_y(Y)$, $y \in Y$, as the limit for $t \to 0$ of

$$m! \, \bar{c}(y, y+t_1, \ldots, y+t_m) / \prod_{i=1}^{m} \| t_i \| \quad .$$

We put the normalizing factor $m!$ to make the integral of ω over the infinitesimal simplex with vertices $y, y+t_1, \ldots, y+t_m$ equal $\bar{c}(y, y+t_1, \ldots, y+t_m)$.

Now, to be precise, we first identify, by parallel translations in M , the tangent spaces $T_\mu(M)$ for all $\mu \in M$ with $T_o(M) = M$. Then we introduce the <u>differential</u> m-form $\tilde{c}$ on M, whose value at the frame of tangent vectors $\mu_1, \ldots, \mu_m$ in $T_\mu(M) = M$ by definition equals $m!c(\mu, \mu_1, \ldots, \mu_m)$. This form $\tilde{c}$ is uniquely characterized by the property that its integral over every linear simplex in M with vertices $\mu, \mu_1, \ldots, \mu_m$ in M equals $c(\mu, \mu_1, \ldots, \mu_m)$.

Observe, that comass $\tilde{c}$, that is the upper bound of the values of $\tilde{c}$ on the frames of unit vectors, does not exceed $m!\|c\|$.

Now, with the map $S: Y \longrightarrow M$ we take the induced form $\tilde{c}^* = S^*(\tilde{c})$ and we define the required form ω by

$$\omega(t_1, \ldots, t_m) = \tilde{c}^*(t_1, \ldots, t_m)/\|S(y)\|^m \quad .$$

for $t_i \in T_y(Y)$, $y \in Y$.

<u>Remark</u> : Since the form $\tilde{c}$ vanishes on the radial tangent field in M one could take first

$$\tilde{S}(y) = \|S(y)\|^{-1} S(y)$$

and then the form ω is induced by $\tilde{S}$,

$$\omega = \tilde{S}^*(\tilde{c}) .$$

Let us slightly sharpen the Proposition for the important case of $m = n = \dim V$, by introducing a new quantity, $[S]^* \le [S]$. Namely, we take the <u>average</u> of $\|D_y S(\tau)\|$ over $\tau \in S_y \subset T_y(Y)$, called $\|D_y S\|^*$,

and then we put

$$[S]_y^* = \|S(y)\|^{-1} \|D_y S\|^*$$

and

$$[S]^* = \sup_{y \in Y} \; [S]_y^* \quad .$$

Example : Let the smoothing $S(y)$ be given by a function $S(y,y')$, such that

$$S * c(y') = \int_Y S(y,y') \, c(y') dy' \quad ,$$

for all functions $c(y')$, $y' \in Y$. Then

$$\|S(y)\| = \int_Y S(y,y') dy' \quad , \quad \text{and}$$

$$\|D_y S\| \leq \int_Y \|\text{grad}_y \, S(y,y')\| dy' \quad .$$

Furthermore, for symmetric functions, $S(y,y') = S(y',y)$,

$$\|D_y S\| \leq \int_Y \|\text{grad}_{y'} \, S(y,y')\| dy' \quad ,$$

and

$$\|D_y\|^* = \int_Y dy' \int_{S_{y'}} |< \tau, \text{grad}_{y'} \, S(y,y')>| d\tau \quad ,$$

where the interior integral of the scalar product is taken with the normalized measure in the sphere $S_{y'}$, and so

$$\|D_y S\|^* = C_n \int_Y \|\text{grad}_{y'} \, S(y,y')\| dy' \quad ,$$

for $C_n = \Gamma(n/2)/\sqrt{\pi} \; \Gamma(\frac{n+1}{2}) < 1$, $n = \dim Y$.

The Integral Inequality : If $m = n = \dim Y$, then the form ω above satisfies

for all fundamental domains $U = U(\Pi) \subset Y$

$$\left| \int_U \omega \right| \le n! \|c\|_\infty \int_U ([S]_y^*)^n dy \le n! \|c\|_\infty ([S]^*)^n \mathrm{Vol}(Y/\Pi) \ .$$

Proof : Indeed, by the geometric arithmetic mean inequality,

$$\mathrm{comass}_y S^*(\widetilde{c}) \le \mathrm{comass}_\mu \widetilde{c} (\|D_y S\|^*)^n \quad ,$$

for all n-forms $\widetilde{c}$ on M and for all $\mu = S(y)$.

Corollary : Every smoothing S in the universal covering Y of a closed Riemannian n-dimensional manifold V satisfies

$$\int_U ([S]_y^*)^n dy \ge \|V\|/n! \quad . \tag{*}$$

Remark : If we introduce the quantity

$$[V]^* = \inf_S \int_U ([S]_y^*)^n dy \quad ,$$

for S running over all smoothing operators S in the universal covering $Y \longrightarrow V$, then (*) reads : $\|V\| \le n! [V]^*$. Notice that the "norm" $[V]^*$ is a conformal invariant of the Riemannian manifold V .

Let us sketch a geometric explanation of (*) . First observe that the cone $M^+ \subset M$ projects to a convex body P^+ , call it a "simplex", in the projective space $P(M) = [M \smallsetminus \{0\}]/\mathbb{R}^*$. Our comass in M induces a comass in P^+ and then we also have the induced comass in the quotient space P^+/Π as well as the dual mass on chains c in P^+/Π :

$$\mathrm{mass}(c) = \sup_\omega \int_c \omega \ ,$$

where ω runs over all forms of comass ≤ 1 . Next, by assigning Dirac δ_y-measures to the points $y \in Y$ we get a canonical map $\delta : V = Y/\Pi \to P^+/\Pi$.

Now, one can show that $\text{mass}(\delta_*[V]) = \|V\|/n!$ where $\delta_*[V] \in H_n(P^+/\Pi)$ denotes the image of the fundamental class of V whose mass is defined as the lower bound of masses of the cycles in $C_n(P^+/\Pi)$ which are homologous to $\delta_*[V]$. On the other hand, the "norm" $[V]^*$ is the lower bound

$$[V]^* = \inf_S \int_S \|(DS)(\tau)\|^n \, d\tau \quad .$$

Here $S = S(V)$ denotes the unite tangent bundle of V and S runs over the smooth maps $V \longrightarrow P^+/\Pi$ homotopic to δ .

Observe that one could equally well use the space of measures on the group Π , rather than the space M of measures on Y . Another interesting space of measures lives on the Fürstenberg boundary of Π (see [14]) and it would be nice to find cycles of least mass in the corresponding space P^+/Π .

2.5. Growth functions and Ricci curvature

Let V be a complete Riemannian manifold and let $Y \longrightarrow V$ be the universal covering of V . For each point $v \in V$ we take a point $y \in Y$ over v and then we consider balls $B_y(R) \subset Y$. Put

$$\ell_v(R) = \log \text{Vol } B_y(R) , \quad \text{for } R \in [0,\infty) ,$$

and

$$\ell_v'(R) = \frac{d\ell_v(R)}{dR} = \text{Vol } \partial B_y(R)/\text{Vol } B_y(R) \quad .$$

Warning : In some exceptionally "irregular" cases the topological boundary $\partial B_y(R)$ does not coincide with the sphere $S_y(R)$ and then the derivative ℓ' is not well defined. To avoid any ambiguity we can use the all-purpose definition

$$\ell_v'(R) \overset{\text{def}}{=} \lim_{\substack{w \to v \\ \epsilon \to 0}} \sup \, \epsilon^{-1}(\ell_w(R) - \ell_w(R-\epsilon))$$

Finally, let

$$\ell' = \ell'(V) = \inf_{R>0} \sup_{v \in V} \ell_v'(R) \qquad .$$

<u>Theorem</u> : <u>Every cohomology class</u> $\beta \in H^m(V)$ <u>satisfies</u>

$$\text{comass}(\beta) \le m!(\ell')^m \|\beta\|_\infty \qquad . \tag{*}$$

<u>Therefore</u>

$$\|\gamma\| \le m!(\ell')^m \text{mass}(\gamma) \quad , \quad \text{for any } \gamma \in H_m(V) . \tag{**}$$

<u>In particular</u>, <u>for closed</u> n-dimensional manifolds V :

$$\|V\| \le n!(\ell')^n \text{Vol}(V) \qquad .$$

<u>Moreover, for</u> $\quad C_n = \Gamma(n/2)/\sqrt{\pi} \; \Gamma(\frac{n+1}{2}) < 1$,

$$\|V\| \le C_n \, n!(\ell')^n \text{Vol}(V) \qquad . \tag{***}$$

<u>Proof</u> : First we consider the smoothing $S = S_R$ which assigns to $y \in Y$ the Riemannian measure dy' in the ball $B_y(R) \subset Y$:

$$S(y,y') = \begin{cases} 1 & \text{for dist}(y,y') \le R \\ \\ 0 & \text{for dist}(y,y') > R \end{cases} .$$

Then, in the "regular" case we have

$$\|S_R(y)\| = \text{Vol } B_y(R) \quad ,$$

$$\|D_y S_R\|^* = C_n \text{ Vol } \partial B_y(R) \qquad ,$$

and

$$\|D_y S_R\| \leq \text{Vol } \partial B_y (R) \qquad .$$

Therefore,

$$\underset{R>0}{\text{Inf}} \ [S_R]^* = C_n \ell' \qquad ,$$

$$\underset{R>0}{\text{Inf}} \ [S_R]^* \leq \ell' \qquad ,$$

and the results of the previous section apply.

To handle possible "irregularities" we slightly refine the definition of the smoothing S_R . Namely we take a small positive function $\varphi : V \longrightarrow (0,R) \subset \mathbb{R}$ and eventually we send this φ to zero in the fine C^1-topology. (Recall, that fundamental neighbourhoods of zero in this topology $U = U_\varepsilon$, are sets of functions φ for which $|\varphi(v)| \leq \varepsilon(v)$ and $\|\text{grad } \varphi(v)\| \leq \varepsilon(v)$ for all positive functions $\varepsilon = \varepsilon(v)$ on V) . Then we define $S_{R,\varphi}$ by the following averaging

$$S_{R,\varphi} (y) = (\varphi(v))^{-1} \int_{R-\varphi(v)}^{R} S_{R'}(y) dR' \qquad ,$$

for the points $y \in Y \longrightarrow V$ over all points $v \in V$. As $\varphi \to 0$ we have

$$\underset{\varphi \to 0}{\text{lim sup}} \ [S_{R,\varphi}]^* \leq C_n \ell' \qquad ,$$

and $\underset{\varphi \to 0}{\text{lim sup}} \ [S_{R,\varphi}] \leq \ell'$. Q.E.D.

Corollary : <u>An estimate for</u> $Ricci \geq -\mathbf{k}^2$. <u>If</u> Ricci $V \geq 1/1-n$ <u>then</u>

$$comass(\beta) \leq m! \, \|\beta\|_\infty \quad , \tag{+}$$

$$\|\gamma\| \leq m! \, mass(\gamma) \quad , \tag{++}$$

$$\|V\| \leq C_n n! \, Vol(V) \quad . \tag{+++}$$

<u>Proof</u> : Bishop's inequality [5] for Ricci $\geq 1/1-n$ implies $\limsup\limits_{R\to\infty} \ell'(R) \leq 1$

This corollary yields all estimates of section 0.4 for closed manifolds V with the exception of the isolation theorem. We have moreover the following generalization of the product inequality of 0.4 .

<u>If</u> $|K(V)| \leq 1$, <u>then an arbitrary product</u> p <u>of Pontryagin classes</u> <u>of V satisfies</u>

$$(p \cup \beta)[V] \leq const_n \|\beta\|_\infty Vol(V) \quad ,$$

<u>where</u> β <u>is an arbitrary cohomology class.</u>

Indeed, this follows from $(p \cup \beta)[V] \leq const_n' \, comass(p) \, comass(\beta) Vol \, V$ with the estimate by Chern-Weil for $|K| \leq 1$,

$$comass(p) \leq C_n' \quad ;$$

and for Ricci $\geq 1-n$,

$$comass(\beta) < (n-1)^n n! \|\beta\|_\infty \, , \text{ by (+) above.}$$

We now can prove one result for <u>open</u> manifolds, namely the asymptotic

estimate of 0.5 . To see this observe that the boundary of the ball B(R)⊆ V
will, for increasing R , eventually support the cycles homologous to
$\partial_1 W$, $\partial_2 W$, and so on. Therefore (++) above yields

$$\lim_{R\to\infty} \inf \text{Vol } \partial B(R) \geq (\|\partial_1 W\|+\|\partial_2 W\|+...)/(n-1)! \quad ,$$

while asymptotically for R → ∞ ,

$$\text{Vol } B(R) = \int_0^R \text{Vol } \partial B(R) dR \sim R \text{ Vol } \partial B(R) \quad .$$

Estimates with the entropy : Let the manifold V be compact. Then there
exists the limit $\lim_{R\to\infty} R^{-1} \ell_v(R)$, called Ent V , which does not depend on
v ∈ V . One can modify the theorem by substituting this entropy for ℓ'
in the inequalities (*)-(***). To do this, fix a number λ > Ent V and
use the following smoothing $S = S_{R,\lambda}$:

$$S(y,y') = \begin{cases} \exp(-\lambda \text{ dist}(y,y'))- \exp(-\lambda R) & \text{, for dist} \leq R . \\ 0 & \text{, for dist}(y,y') > R \end{cases}$$

If R → ∞ , then $[S_{R,\lambda}]^* \to \lambda$ and also the norm [S] is asymptotically
bounded by λ. Hence, the inequalities (*)-(***) hold with any λ > Ent in
place of ℓ' and so with λ = Ent as well.

Recall that Ent V bounds from below the topological entropy of the
geodesic flow of V (see [12],[24],[37]) and so the modified inequality
(***) implies

$$(\text{Top Ent})^n \geq \|V\|/C_n \text{ n! Vol}(V) \quad .$$

This inequality for n = 2 with the sharp constant $1/2\pi < C_2 = 2/\pi$ is due
to Katok [36] .

Finally, if V has negative curvature, then $\ell'_v(R)$ ·Ent = Top Ent as
R → ∞ (see [18][37]) and so the modified version of the theorem follows from
the original one.

3. Bounded cohomology of simplicial multicomplexes

3.0   We denote by $\hat{C}^* = \hat{C}^*(X) \subset C^*(X)$ the complex of the bounded real-valued singular cochains of the space X. A continuous map $f : X \to Y$ induces a homomorphism $\hat{f} : \hat{C}^*(Y) \longrightarrow \hat{C}^*(X)$ which is bounded relative to the ℓ^∞-norms. Indeed $\hat{f}$ is even bounded by one as $\|\hat{f}(c)\|_\infty \leq \|c\|_\infty$ for all $c \in C^*(Y)$.

We denote by $\hat{H}^*(X)$ the homology of the complex $\hat{C}^*(X)$. Observe that the space $\hat{H}^*(X)$ carries a natural pseudo-norm which is also denoted by $\| \ \|_\infty$. ("Pseudo" means that for a non-zero $a \in \hat{H}^*(X)$ one can have $\|a\|_\infty = 0$. This might happen when the image of the coboundary operator $\delta : \hat{C}^*(X) \longrightarrow \hat{C}^*(X)$ is not closed. I do not know if this actually occurs).

A homotopy between two maps $f,g : X \longrightarrow Y$ provides a chain-homotopy h between $\hat{f}$ and $\hat{g}$. The standard construction (see [40] for instance) gives an h which is bounded in each dimension relative to the ℓ^∞-norm, and hence the homomorphisms $\hat{f}^*$ and $\hat{g}^* : \hat{H}^*(Y) \longrightarrow \hat{H}^*(X)$ are equal. In particular, $\hat{H}^*(X)$ depends only on the homotopy type of X. We shall see below that $\hat{H}^*(X)$ depends only on the fundamental group $\pi_1(X)$.

Observe, that $\hat{H}^1(X)$ is always zero. This is clear, because each real-valued 1-cocycle z determines a homomorphism $\pi_1(X) \longrightarrow \mathbb{R}$ (we assume X to be path-connected). When z is bounded this homomorphism is also bounded, and hence trivial. It follows that z is the coboundary of a 0-cocycle.

The results of Milnor-Sullivan and Thurston provide many examples of nontrivial groups $\hat{H}^*$. In fact, Thurston's theorem says that the homomorphisms $\hat{H}^m(X) \longrightarrow H^m(X, \mathbb{R})$ are surjective for $m \geq 2$ if X is a closed manifold of negative curvature. These homomorphisms are in general not

injective. We shall see below for example that the groups $\hat{H}^2$ and $\hat{H}^3$ of the infinite wedge of circles do not vanish.

Bounded cohomology first appeared in the group theoretic context. I learned this notion from Phillip Trauber who explained to me his (unpublished) version of the Theorem of Hirsch-Thurston [33] .

If a group Π is amenable then the bounded cohomology vanishes :

$$\hat{H}^i(\Pi) \overset{def}{=} \hat{H}^i(K(\Pi,1)) = 0 \quad \text{for} \quad i > 0 \; .$$

Recall (see [19]) that a group Π is called amenable if its action on the space of bounded functions $\Pi \longrightarrow \mathbb{R}$ has a left invariant mean (average), that is a Π-invariant projection A of norm one from the space $L^\infty(\Pi)$ onto the (one dimensional) subspace of constant functions.

Examples (see [19]) Abelian groups are amenable. Finite groups are amenable. If a normal subgroup $\Gamma \subset \Pi$ as well as the quotient group Π/Γ is amenable, then Π is itself amenable.

Unions of increasing families of amenable groups are amenable. In particular, if Π is locally solvable (i.e. all finitely generated subgroups in Π are solvable) then it is amenable.

It is unknown whether all (discrete) amenable groups are built out of Abelian and finite group by taking extensions and "increasing unions". Also notice, that subgroups and factor groups of amenable groups are amenable.

The simplest examples of non-amenable groups are free non-abelian groups. All known finitely presented non-amenable groups contain free non-abelian subgroups.

If Π is the fundamental group of a closed Riemannian manifold
V , then non-amenability can be expressed geometrically in terms of the
universal covering $\tilde{V} \longrightarrow V$, by requiring all bounded domains $\Omega \subset \tilde{V}$
to satisfy the inequality $Vol(\Omega) \leq const\ Vol(\partial\Omega)$ for some positive
"const" = const(V) . According to Avez (see [2][21]) this inequality holds,
for example, if V is a non-flat manifold of non-positive curvature,
$K(V) \leq 0$, and then the fundamental group $\Pi = \pi_1(V)$ is non amenable. It is
unknown whether every such group Π contains a free non-abelian subgroup.

One can also express the amenability of $\pi_1(V)$ in terms of the
smoothing operators S of section 2.4 : the group $\Pi = \pi_1(V)$ is amenable
iff for every $\varepsilon > 0$ there exists a Π-invariant smoothing
$S : \tilde{V} \longrightarrow M(\tilde{V})$ for which $[S] \leq \varepsilon$. The existence of such operators S
implies Trauber's vanishing theorem, but the original argument of Trauber
is shorter and also it yields the following more general fact.

Let $f : Y \longrightarrow X$ be a regular covering with an amenable Galois
group Π . Then the induced map $\hat{f}^* : \hat{H}^*(X) \longrightarrow \hat{H}^*(Y)$ is injective
and isometric relative to the norm $\| \ \|_\infty$.

Proof : The Galois group Π acts on Y and thus on $\hat{C}^*(Y)$. Then the
complex $\hat{C}^*(X)$ can be identified with the complex of Π-invariant cochains
in $\hat{C}^*(Y)$. Fix an averaging A on Π and consider the corresponding
averaging in $\hat{C}^*(Y)$, a Π-invariant projection $\hat{C}^*(Y) \longrightarrow \hat{C}^*(X)$. This
projection, call it $\hat{A}$, commutes with differentials and satisfies
$\hat{A} \circ \hat{f} = Id$, where $\hat{f} : C^*(X) \longrightarrow C^*(Y)$ is the cochain homomorphism induced
by f . Now, by the very definition of the averaging, we have $\|\hat{A}\| = 1$
and so $\|\hat{A}^*\| \leq 1$ for $\hat{A}^* : \hat{H}^*(Y) \longrightarrow H^*(X)$. Then with the identity $\hat{A}^* \cdot \hat{f}^* =$
$(\hat{A} \circ \hat{f})^* = Id$ we get $\|\hat{f}^*(\alpha)\|_\infty \geq \|\alpha\|_\infty$ for all $\alpha \in H^*(X)$ and since also
$\|\hat{f}^*\| \leq \|\hat{f}\| = 1$, the map $\hat{f}^*$ is isometric as well as injective.

Examples of non-vanishing $\hat{H}^1$. Let X be a closed surface of genus ≥ 2
and let $f : Y \longrightarrow X$ be an infinite Abelian covering. Since X supports
a metric of negative curvature, the fundamental class α of   X is bounded,
$\alpha \in \hat{H}^2(X)$, and since Abelian groups are amenable the pullback $f^*(\alpha) \in \hat{H}^2(Y)$
is non-zero. Notice that Y is homotopy equivalent to the infinite wedge
of circles.

Also the wedge of two circles has a non-trivial group $\hat{H}^2$. In fact,
this group $\hat{H}^2(S^1 \vee S^1)$ is infinitely generated (see [6]) . Here is a geo-
metric construction of a non-trivial bounded 2-cocycle. Take a complete
surface V of constant negative curvature such that $\mathrm{Vol}(V) = 2\pi < \infty$ and
$\pi_1(V) = \mathbb{Z} * \mathbb{Z}$. Let ω be the volume form and let the cocycle $z \in C^2(V)$
assign to each simplex $\sigma : \Delta^2 \longrightarrow V$ the integral of ω over straight(σ)
as we did it in (C) of 1.2 . Then we have $V = \mathrm{Int}\, \bar{V}$ where the boundary
$\partial \bar{V}$ has self mappings of all degrees and for the ℓ^1-cycle $\tilde{c}$ constructed
in section 0.2 we observe that $z(\tilde{c}) \neq 0$. Therefore z is not a coboundary of
a bounded cochain, as well as $\tilde{c}$ is not the boundary of an ℓ^1-chain.

Jörgensen [34] constructed a closed 3-dimensional manifold V of
constant negative curvature which admits an infinite cyclic covering
$\tilde{V} \longrightarrow V$ such that $\tilde{V}$ is homotopy equivalent to a surface. The pull-back
of the fundamental class of V gives a non-trivial element in $\hat{H}^3(\tilde{V})$ and
thus a non-trivial element in the infinite wedge of circles.

3.1. Vanishing theorems for bounded cohomology

Trauber's theorem implies the vanishing of the bounded cohomology of
$K(\Pi,1)$ spaces with amenable groups Π . In particular, the simplicial
volume of solv-manifolds is zero. By using another averaging procedure in
simplicial models of arbitrary spaces X we prove in section 3.3 the follow-
ing generalization.

The Mapping theorem : Let $f : X_1 \longrightarrow X_2$ be a continuous map such that the induced homomorphism $f_* : \pi_1(X_1) \longrightarrow \pi_1(X_2)$ is a surjection with an amenable kernel. Then the homomorphism $\hat{f}^* : \hat{H}^*(X_2) \longrightarrow \hat{H}^*(X_1)$ is an isometric isomorphism.

Corollaries : (A) If f_* is an isomorphism then $\hat{f}^*$ is also an isomorphism. In particular, if $f : X \longrightarrow K(\Pi,1)$ is the classifying map for $\Pi = \pi_1(X)$, then $\hat{f}^*$ is an isometric isomorphism.

(B) Let X be a closed oriented manifold with fundamental class $[X]$ and let $f : X \longrightarrow K(\Pi,1)$, $\Pi = \pi_1(X)$, denote the classifying map. Then $\|X\| = \|f_*[X]\|$.

(C) If the fundamental group of the manifold X above is amenable, for example if $\pi_1(X) = 0$, then $\|X\| = 0$.

Observe that $(A) \Rightarrow (B) \Rightarrow (C)$.

The proposition (C) is generalized in section 3.3 as follows. First, a subset $Y \overset{i}{\subset} X$ is called "amenable" if for every path connected component Y' of Y the image of the inclusion homomorphism $i_* : \pi_1(Y') \longrightarrow \pi_1(X)$ is an amenable subgroup of $\pi_1(X)$.

The Vanishing theorem : If a manifold X can be covered by some open amenable subsets such that every point $x \in X$ is contained in no more than m subsets, then the homomorphism $\hat{H}^i(X) \longrightarrow H^i(X)$ vanishes for $i \geq m$.

Corollaries : (1) If a closed manifold X can be covered by $m < \dim X$ "amenable" open sets then $\|X\| = 0$.

(2) If the manifold X above can be mapped into a manifold Y with $\dim Y < \dim X$ such that the pullback of every points in Y has an "amenable" neighbourhood in X , then $\|X\| = 0$.

Indeed, one can cover Y by 1+dim Y open subsets, such that each
of them is a union of small disjoint subset. Then the previous corollary
applies to the pullback of this covering.

(3) (Koichi Yano [49]) . If X admits a nontrivial circle action then
$\|X\| = 0$.

Indeed, (2) above applies to the quotient map $X \longrightarrow X/S^1$.

Remark : If the action is free, then the mapping theorem also yields
$\|X\| = 0$. Furthermore, if the action is locally free, then not only $\|X\| = 0$
but also Min Vol $\|X\| = 0$, (see Appendix 2). Finally, if the action is
not locally free, then the classifying map $X \longrightarrow K(\Pi,1)$ sends the fun-
damental class [X] to zero (see Appendix 2) and again by the mapping
theorem, $\|X\| = 0$.

3.2. Simplicial multicomplexes and the isometry $\hat{H}^* \longrightarrow \hat{H}^*_s$

A simplicial multicomplex (for short multicomplex) is defined as a
set K divided into the union of closed affine simplices $\Delta_i \subset K$, $i \in I$,
such that the intersection of any two simplices $\Delta_i \cap \Delta_j$, is a (simplicial)
subcomplex in Δ_i as well as in Δ_j . The set K with the weakest topo-
logy which agrees with the decomposition $K = \bigcup_i \Delta_i$ is denoted by $|K|$.
The union of all m-dimensional simplices in K is called the m-skeleton
of K and denoted by $K^m \subset K$. A map between two multicomplexes,
$K = \bigcup_i \Delta_i \longrightarrow L = \bigcup_j \Delta_j$, is called simplicial if it maps each Δ_i linearly
onto some Δ_j .

Examples : (a) Every simplicial complex is also a multicomplex. The simplest
multicomplex which is not a simplicial complex consists of two one dimen-
sional simplices, $K = \Delta_1^1 \cup \Delta_2^1$ which intersect over their common boundary,

$\partial \Delta_1^1 = \partial \Delta_2^1 = \Delta_1^1 \cap \Delta_2^1$. (See fig. 1.   )
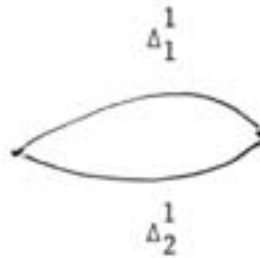

$$\Delta_1^1$$

$$\Delta_2^1$$

Fig. 1.

Also observe that the first barycentric subdivision of any multicomplex is an ordinary simplicial complex.

(b) Every simplicial multicomplex $K = \bigcup_{m=0}^{\infty} K^m$ can be built inductively starting from the discrete set K^0, and then each K^m is obtained from K^{m-1} by attaching m-simplices Δ_ν^m by some simplicial embeddings of the boundaries $\partial \Delta_\nu^m \longrightarrow K^{m-1}$.

(c) Our important example is the following. Take a topological space X and consider the set Σ of all those singular simplices $\sigma : \Delta^m \longrightarrow X$, $m = 0, 1, \ldots$, which are injective on the vertices of Δ^m . Then we take one copy of Δ^m for each σ, call it Δ_σ^m, and put $K(X) = \bigcup_{\sigma \in \Sigma} \Delta_\sigma^m$. This union has in a natural way the structure of a multicomplex such that the canonical map $S : |K(X)| \longrightarrow X$ defined by the condition

$$S|\Delta_\sigma^m = \sigma : \Delta_\sigma^m = \Delta^m \longrightarrow X \ , \quad \text{for all} \ \sigma \in \Sigma \ ,$$

is continuous. Moreover, by the standard argument (see [40]) , this map S is a weak homotopy equivalence.

A multicomplex K is called <u>complete</u> if every continuous map $f : \Delta^m \longrightarrow K$ whose restriction to the boundary $f|\partial \Delta^m : \partial \Delta^m \longrightarrow K$, is a simplicial imbedding, is homotopic relative to $\partial \Delta^m$, to a simplicial

embedding $f' : \Delta^m \longrightarrow K$.

Examples : (a) The complex $K(X)$ is complete.

(b) If K is complete and connected then for any finite set of vertices $\{k_o, \ldots, k_\ell\} \subset K^o$ there is an ℓ-dimensional simplex $\Delta^\ell \subset K$ with vertices $k_o, \ldots, k_\ell$.

(c) If a connected 1-dimensional complex is complete then it consists of a single 1-simplex.

(d) If a connected simplicial complex K is complete as a multicomplex then it equals the simplex spanned by the vertices of K .

The role of the completeness is explained by the following simple fact

(e) Let K be a complete multicomplex with at least $n+1$ vertices in every connected component. Let f be a continuous map of an n-dimensional multi-complex L into K . Then there is a simplicial map f' of the first barycentric subdivision of L into K , homotopic to f and injective on every simplex of the subdivision of L .

Call K large if every component has infinitely many vertices. Then we state the following relative version of (e) .

Homotopy Lemma : Let K be large and complete. Then for any two homotopic simplicial maps $f_o, f_1 : L \longrightarrow K$, both injective on each simplex in L , there exists a simplicial map f of a canonically subdivided cylinder $L \times [0,1]$ into K such that $f_{t=0} = f_o$ and $f_{t=1} = f_1$ and again f is injective on each simplex of the subdivision.

The completeness property is reminiscent of Kan's property in the theory of semi-simplicial sets. Since multicomplexes have no degenerate

simplices they are, from the algebraic point of view less convenient than
semi-simplicial sets, but they better serve our geometric applications. In
any case the standard techniques of semi-simplicial sets (see [40]) apply
to multicomplexes with minor changes. Our exposition is essentially inde-
pendent of [40] as we only need simple facts which can be easily proved
directly. For example we consider simplicial m-cochains on K. They are,
by definition, antisymmetric functions of oriented m-simplices of K.
We take only <u>bounded</u> functions and get the <u>complex of bounded antisymmetric
cochains</u>, called $\hat{C}_a^*(K)$. We denote by $\hat{H}_a^*(K)$ the homology of $\hat{C}_a^*(K)$ with
the ℓ^∞-norm. If the complex K and the maps $f_o, f_1 : L \longrightarrow K$ are as in
the homotopy lemma, then the cylinder $f : L \times [0,1] \longrightarrow K$ provides a
chain homotopy equivalence between the induced maps $f_o^*, f_1^* : \hat{C}_a^*(K) \longrightarrow \hat{C}_a^*(L)$.
This chain homotopy equivalence $\{\hat{C}_a^i(L) \longrightarrow \hat{C}_a^{i-1}(K)\}_{i=0,1,\ldots}$ is bounded
in every dimension i. Therefore the corresponding homomorphisms on homo-
logy, $\hat{f}_0^*$ and $\hat{f}_1^*$ of $\hat{H}_a^*(L)$ to $\hat{H}_a^*(K)$ are equal.

Next we identify $\hat{H}_a^*(K)$ with the bounded <u>singular</u> cohomology
$\hat{H}^*(|K|)$. To do that we start with a natural homomorphism
$h : \hat{C}^*(|K|) \longrightarrow \hat{C}_a^*(K)$, defined as follows. For each $c \in \hat{C}^i(|K|)$ and
for every oriented i-simplex $\Delta \subset K$ we consider all affine isomorphism
δ of the standard i-simplex Δ^i onto Δ. Then we define $c' = h(c)$
by the formula

$$c'(\Delta) = \frac{1}{(i+1)!} \sum_\delta [\delta] \, c(\delta) \, ,$$

where we sum over the affine maps $\delta : \Delta^i \longrightarrow \Delta \subset K$ with $[\delta] = 1$
for the orientation preserving maps and with $[\delta] = -1$ for the reversed
orientation. The desired identification is given in the

<u>Isometry Lemma</u> : <u>For a large complete complex</u> $\check{K}$ <u>the induced homomorphism</u>

<u>on the bounded cohomology,</u>

$$h^* : \hat{H}^*(|K|) \longrightarrow \hat{H}^*_a(K) \quad ,$$

<u>is an isometric isomorphism.</u>

<u>Proof</u> : Fix an integer $N \geq 0$, take the standard simplex Δ^N , and let $\sigma^i_\mu : \Delta^i \longrightarrow \Delta^N$ denote the isomorphisms of the standard simplex Δ^i onto the i-faces of Δ^N , for $i = 0,\dots,N$ and $\mu = 1,\dots,\frac{(N+1)!}{(N-i)!}$. We consider all those singular simplices $\Delta^i \longrightarrow |K| \times \Delta^N$ which project to isomorphisms σ^i_μ . These singular simplices $\Delta^i \longrightarrow |K| \times \Delta^N$ form in a natural way an N-dimensional multicomplex, called $\widetilde{K}_N$. Notice that each i-dimensional simplex Δ in $\widetilde{K}_N$ is <u>canonically</u> isomorphic to Δ^i and thus all $\Delta \subset \widetilde{K}_N$ come with canonical orientations.

Now let us construct a chain homomorphism $A : \hat{C}_a^{i \leq N}(\widetilde{K}_N) \longrightarrow \hat{C}^{i \leq N}(|K|)$ by first considering for every singular simplex $\sigma : \Delta^i \longrightarrow |K|$ the (oriented !) simplices $\Delta_\mu \subset \widetilde{K}_N$, $\mu = 1,\dots,M = \frac{(N+1)!}{(N-i)!}$, which lie in $|K| \times \Delta^N$ over σ .

Then for $c \in \hat{C}^i_a(\widetilde{K}_N)$ we define $c' = A(c)$ according to the formula

$$c'(\sigma) = M^{-1} \sum_{\mu=1}^{M} c(\Delta_\mu) \quad .$$

Next we take the natural map $f : |\widetilde{K}_N| \longrightarrow |K|$. With our assumptions on K we can replace it by a simplicial map $\widetilde{f} : \widetilde{K}_N \longrightarrow K$ which is homotopic to f and injective on all simplices Δ in $\widetilde{K}_N$.

We have now two maps :

$$h : \hat{C}^*(|K|) \longrightarrow \hat{C}^*_a(K)$$

and

$$g = A \circ \widetilde{f}^* : \hat{C}_a^{i \leq N}(K) \longrightarrow \hat{C}^{i \leq N}(|K|) \quad ,$$

both chain homomorphisms of ℓ^{∞}-norm ≤ 1 . The homotopy lemma implies that the composed maps $h \circ g$ and $g \circ h$ are chain homotopic to the identity (in dimensions below N), and so the homomorphism h^{*} is an isomorphic isometry.

An application : the abstract version of the theorem of Milnor-Sullivan-Smillie.

In order to estimate the norm of a characteristic class $\beta \in H^{*}(X)$, we may pass to the multicomplex $K = K(X)$. Then we only need the geometric version of the theorem, but now for the multicomplex K rather than for the simplicial complex P as in 1.3. The arguments of 1.3. immediately apply to all countable multicomplexes (we need "countable" to have a good notion of generic piecewise linear sections). The uncountable case however only requires the following simple lemma.

If for every finite subcomplex K' of K a class $\beta \in H_{a}^{*}(K)$ can be represented by a cocycle $c \in H_{a}(K)$ which is bounded on K' by a given constant b , then $\|\beta\|_{\infty} \leq b$.

3.3. Minimal multicomplexes and their automorphisms.

A simplicial multicomplex K is called minimal (compare [40]) if each continuous map of a simplex Δ into K whose restriction to the boundary is a simplicial embedding is homotopic relative to the boundary $\partial\Delta$ to at most one simplicial embedding $\Delta \longrightarrow K$.

(A) Lemma. Let a multicomplex K be large and complete. Then there is a subcomplex $\tilde{K}$ in K which is complete, minimal and such that the inclusion $\tilde{K} \subset K$ is a homotopy equivalence. Furthermore, one can take $\tilde{K}$ with countably many vertices in every connected component. With this last property the complex

K is uniquely determined, up to simplicial isomorphism, by the homotopy type of K .

Proof . Two i-dimensional simplices Δ_1 and $\Delta_2 \neq \Delta_1$ in K are called homotopic if they have a common geometric boundary, $\partial \Delta_1 = \partial \Delta_2 = \Delta_1 \cap \Delta_2$, and if the sphere $S^i = \Delta_1 \cup \Delta_2 \subset K$ is homotopic to zero.

Now, take countably many vertices in each component of K and denote this set of vertices by $\widetilde{K}^\circ \subset K^\circ \subset K$. Then define $\widetilde{K}$ inductively, starting from the zero skeleton $\widetilde{K}^\circ$, and then by taking for $\widetilde{K}^i$ a maximal system of pairwise non-homotopic simplices in K with boundaries in $\widetilde{K}^{i-1}$.

To prove uniqueness, let $\widetilde{K}_1$ and $\widetilde{K}_2$ be two minimal complete complexes and let $f : \widetilde{K}_1 \longrightarrow \widetilde{K}_2$ be a homotopy equivalence which is bijective on the zero skeletons, $\widetilde{K}_1^\circ \xrightarrow{\sim} \widetilde{K}_2^\circ$. Since $\widetilde{K}_2$ is complete there is a simplicial map $\widetilde{f} : \widetilde{K}_1 \to \widetilde{K}_2$ which is homotopic to f and equal to f on $\widetilde{K}_1^\circ$. Now, the uniqueness of K follows from the following

Sublemma : If a simplicial map $f : \widetilde{K}_1 \to \widetilde{K}_2$ is a homotopy equivalence and if it is bijective on the zero skeletons, then $\widetilde{f}$ is bijective.

Proof : Let us assume, by induction, that $\widetilde{f}$ is bijective on the $(i-1)$-skeletons and let us first show that $\widetilde{f}$ is injective on the i-skeleton of $\widetilde{K}_1$. Indeed, if two i-simplices, say Δ_1 and Δ_2 in $\widetilde{K}_1^i \subset \widetilde{K}_1$, are sent onto the same simplex in $\widetilde{K}_2^i \subset \widetilde{K}_2$, then they have a common boundary, $\partial \Delta_1 = \partial \Delta_2 = \Delta_1 \cap \Delta_2$. The map $\widetilde{f}$ of the sphere $S^i = \Delta_1 \cup \Delta_2 \subset \widetilde{K}_1$ into $\widetilde{K}_2$ is contractible. Since $\widetilde{f}$ is a homotopy equivalence, the sphere $S^i \subset \widetilde{K}_1$ is also contractible and so the simplices Δ_1 and Δ_2 are homotopic. Then Δ_1 and Δ_2 coincide by the minimality of $\widetilde{K}_1$.

To prove the surjectivity of $\widetilde{f}$, we take an i-dimensional simplex $\Delta \subset \widetilde{K}_2^i$ with boundary $S^{i-1} = \partial\Delta \subset \widetilde{K}_2^{i-1}$ and we consider $\widetilde{S}^{i-1} = \widetilde{f}^{-1}(S^{i-1}) \subset \widetilde{K}_1^{i-1}$. Since the sphere S^{i-1} is contractible in $\widetilde{K}_2$, the sphere $\widetilde{S}^{i-1} \subset \widetilde{K}_1$ is also contractible, and by the completeness of $\widetilde{K}_1$ we have a simplex $\Delta \subset \widetilde{K}_1^i$ with $\partial\widetilde{\Delta} = \widetilde{S}^{i-1}$. Then we consider the sphere $S^i = \widetilde{f}(\widetilde{\Delta}) \cup \Delta \subset \widetilde{K}_2$. Since f is a homotopy equivalence and as $\widetilde{K}_1$ is complete, there is a simplex $\Delta' \subset \widetilde{K}_1^i$ with boundary $\widetilde{S}^{i-1}$, such that the sphere $\widetilde{f}(\widetilde{\Delta} \cup \Delta') \subset \widetilde{K}_2$ is homotopic to S^i relative to $\widetilde{\Delta}$. Finally, by the minimality of $\widetilde{K}_2$ we conclude that $\widetilde{f}(\Delta') = \Delta$. Q.E.D.

Corollary : Any space X is weakly homotopy equivalent to a large complete minimal multicomplex.

Indeed, take $\widetilde{K}$ in the multicomplex K(X) defined in the previous section.

Now, denote by $\Gamma = \Gamma(K)$ the group of those simplicial automorphisms of a multicomplex K which are homotopic to the identity. Let $\Gamma_i \subset \Gamma$ be the subgroup which keeps the i-skeleton of K pointwise fixed. Observe that each Γ_i is a normal subgroup in Γ .

Take an (i+1)-dimensional simplex $\Delta_o \subset K^{i+1}$ and denote by $\pi(\Delta_o)$ the set of all (i+1)-simplices $\Delta \subset K$ for which $\partial\Delta = \partial\Delta_o$.

(B) Lemma : If K is a complete minimal multicomplex then the group Γ_i is transitive on every set $\pi(\Delta_o)$.

Proof : Take any (i+1)-simplex $\Delta \subset \pi(\Delta_o)$ and a linear isomorphism $f_o : \Delta_o \rightarrow \Delta$ which keeps fixed the vertices of Δ_o . Then, while keeping fixed the i-skeleton K^i , we extend f_o to a map $\widetilde{f} : K^i \cup \Delta_o \rightarrow K$

which isomorphically sends $K^i \cup \Delta_o$ onto $K^i \cup \Delta$. This map $\tilde{f}$ is homotopic
to the inclusion $K^i \cup \Delta_o \subset K$. In fact, for any multicomplex K there is
a homotopy of $K^i \cup \Delta_o$ to $K^i \cup \Delta$ which keeps fixed K^i minus one i-face
of Δ_o, and moves that i-face over $\Delta \cup \Delta_o$ back to its original position.
Now, we extend $\tilde{f}$ to a continuous map $f : K \rightarrow K$ which is homotopic to the
identity by an extension of the above homotopy. Since K is complete, this
f can be chosen simplicial. Finally, as K is minimal, we apply the sub-
lemma and conclude that f is an automorphism.

(C) Lemma : If K is minimal, then the quotient groups Γ_1/Γ_i are amenable
for all $i = 2,3 \ldots.$

Proof : It suffices to show that the groups Γ_{i-1}/Γ_i are abelian groups
for $i \geq 2$. Take one i-simplex , say $\Delta = \Delta_\alpha, \alpha \in J$, in every orbit of
Γ_{i-1} . The Γ_{i-1}-orbit of every Δ_α is contained in the set $\pi(\Delta_\alpha)$ and
so for every $\gamma \in \Gamma_{i-1}$, we get a sphere $S^i_\gamma = \Delta \cup \gamma(\Delta) \subset K$ that represents
an element of the homotopy group, $[S^i_\gamma] \in \pi_i(K,p_\alpha)$, for some base point
$p_\alpha \in \Delta = \Delta_\alpha$. As all $\gamma \in \Gamma_{i-1}$ are homotopic to the identity, the maps
$H_\alpha : \gamma \longrightarrow [S^i_\gamma]$ are, in fact, homomorphisms $\Gamma_{i-1} \longrightarrow \pi_i(K,p_\alpha)$. Denote by
$H^{(i)} : \Gamma_{i-1} \longrightarrow \Pi^{(i)} = \underset{\alpha \in J}{\oplus} \pi_i(K,p)$ the direct product of the homomorphisms
H_α and observe, as K is minimal, that the kernel of the homomorphism $H^{(i)}$
equals Γ_i . Thus the group Γ_{i-1}/Γ_i is _embedded_ into the _abelian_ group
$\Pi^{(i)}$ for $i \geq 2$. Therefore, the groups Γ_1/Γ_i are solvable and so amenable.
(One can even show that they are nilpotent).

(D) Corollary : If K is large then the homomorphism $I^* : H^*(K/\Gamma_1) \rightarrow H^*(K)$
induced by the quotient map $K \longrightarrow K/\Gamma_1$ is an isometric isomorphism.

Proof : The chain complex $\hat{C}^*_a(K/\Gamma_1)$ is canonically isomorphic to the

subcomplex of Γ_1-invariant cochains in $\hat{C}_a^*(K)$ and hence by averaging over the group Γ_1/Γ_m we get a chain homomorphism $A : \hat{C}^{i \le m}(K) \to \hat{C}_a^{i \le m}(K/\Gamma_1)$ such that $A \circ I = $ id for $I : C_a^*(K/\Gamma_1) \longrightarrow \hat{C}_a^*(K)$. Now, since K is large as well as complete and minimal, the transformation of $\hat{C}_a^*(K)$ induced by any $\gamma \in \Gamma$ is chain homotopic to the identity by the homotopy lemma. Therefore, we also have $I \circ A \sim $ id, and so the homomorphism A is a chain homotopy equivalence for all m. Q.E.D.

(E) Remark : The quotient complex K/Γ_1 is a $K(\Pi,1)$-complex for $\Pi = \pi_1(K)$ and the map $K \to K/\Gamma_1$ induces an isomorphism of fundamental groups.

Proof of the mapping theorem of 3.1. : Since every space X is weakly homotopy equivalent to a large complete and minimal multicomplex, we obtain, with (D) and (E) above, the corollary (A) of the mapping theorem, first for maps of the kind $f : X \to K(\Pi,1)$, and then for all maps $f : X_1 \to X_2$ for which $f_* : \pi_1(X_1) \to \pi_1(X_2)$ is an isomorphism. The mapping theorem itself comes from the following considerations.

Take a large complete and minimal multicomplex K_1 of the $K(\Pi,1)$ type, $\Pi = \pi_1(K_1,1)$, choose a vertex $x \in K_1^\circ \subset K_1$ and let the fundamental group $\pi_1(K_1,x)$ act on K_1 as follows. For every edge e which joins x with another vertex x', and for each $\gamma \in \pi_1(K_1,x)$, we compose the path e with a loop which represents γ and then we take the (unique) edge in K_1, called $\gamma(e)$, which is homotopic to this composition. Thus the direct sum of the groups $\pi_1(K_1,x)$, over all $x \in K_1^\circ$, acts on the one-skeleton of K_1. As K_1 is a $K(\Pi,1)$-complex this action uniquely extends to all of K_1.

Recall, that the direct product of groups G_j, over $j \in J$, consists

by definition of all sequences $\{g_j\}$, $g_j \in G_j$, while in the direct sum

the sequences only have finitely many non-identity entries. The important

point is that direct sums of amenable groups are amenable, while direct

products may be not amenable (see [19]).

Now, with a normal amenable subgroup $\Pi' \subset \Pi$, we have isomorphic

normal subgroups $\Pi'_x \in \pi_1(K_1,x)$ for all $x \in K_1^\circ$ and the direct sum of these

subgroups, called $\widetilde{\Pi}'$, acts on K_1 . The quotient complex $K_1/\widetilde{\Pi}'$ has

$K(\Pi/\Pi',1)$ type and the quotient map $K_1 \to K_1/\widetilde{\Pi}'$ induces the quotient

homomorphism of the fundamental groups $\Pi \to \Pi/\Pi'$. As the group $\widetilde{\Pi}'$ is

amenable, we conclude as before that this quotient map induces isometric

isomorphism of the bounded cohomology groups,

$$\hat{H}_a^*(K_1/\Pi') \xrightarrow{\sim} \hat{H}_a^*(K_1) \ .$$

<div align="right">Q.E.D.</div>

<u>Proof of the vanishing theorem of 3.1.</u> : Take a subset $X_o \overset{I_o}{\subset} X$ and consider

all continuous in t paths of (possibly discontinuous) maps $X_o \to X$,

called $I_t : X_o \to X$, $t \in [0,I]$, with the following three properties :

(1) $I_{t=0} = I_o : X_o \subset X$.

(2) For almost all points $x \in X$ (i.e. only with finitely many

exceptions) $I_t(x) = I_o(x)$ for all $t \in [0,1]$.

(3) The map $I_{t=1}$ sends X_o bijectively onto itself.

Denote by $\pi(X,X_o)$ the group of homotopy classes of the paths I_t .

Observe for a single point $x_o \in X$ that $\Pi(X,x_o) = \pi_1(X,x_o)$; in general,

there is a natural homomorphism of the group $\Pi(X,X_o)$ to the group of

permutations of X_o with finite supports, and the kernel of this homomorphism

is the direct sum of the fundamental groups, $\underset{x \in X_o}{\oplus} \pi_1(X,x)$.

Next, we take a complete minimal multicomplex K and a homotopy equivalence $h : K \to X$ which bijectively maps the zero-skeleton $K_o \subset K$ onto X. Then the group $\Pi(X,X)$ acts, by compositions of paths I_t with edges of K sent by h to X, on the one-skeleton K^1 of K, and the actions of all $\gamma \in \Pi(X,X)$ extend (not uniquely) to automorphisms of K. Therefore, the group $\Pi(X,X)$ acts on the complex $\hat{C}_a^*(K/\Gamma_1) \subset \hat{C}_a^*(K)$ of bounded Γ_1-invariant cochains.

Then, for all subsets $U \subset X$ and $V \subset U$, the group $\Pi(U,V)$ also acts on K^1 via the natural homomorphism $\Pi(U,V) \to \Pi(X,V) \subset \Pi(X,X)$ and if the subset $U \subset X$ is "amenable" (see 3.1) then the image of this homomorphism is an amenable subgroup in $\Pi(X,V)$ and so the resulting action of $\Pi(U,V)$ on the complex $\hat{C}^*(K/\Gamma_1)$ is amenable.

Now, let $U_j \subset X$, $j \in J$, be some amenable open subsets and let $V_j \subset U_j$ be arbitrary mutually disjoint subsets. Then the direct sum of the groups $\Pi(U_j,V_j)$ over $j \in J$ also acts on the complex $\hat{C}_a^*(K/\Gamma_1)$ and with the averaging process we conclude to the following

Proposition : <u>Every bounded cohomology class</u> $\beta \in \hat{H}^*(X)$ <u>can be represented by a cocycle</u> $b \in \hat{C}_a(K/\Gamma_1) \subset C_a^*(K)$ <u>which is also invariant under the action of the direct sum of the groups</u> $\Pi(U_j,V_j)$.

Now, we prove the vanishing theorem for the covering $\{U_j\}$ of X as follows. We take a sufficiently fine triangulation L of the manifold X and we divide the vertices of L into disjoint subsets called V_j, such that the stars of the vertices $v \in V_j$ are contained in the sets U_j. Then, we choose the multicomplex K and the map h such that L becomes the homeomorphic image of a subcomplex L' in K, $h(L') = L$. As no $m+1$ sets U_j intersect, every simplex in L of dimension $\geq m$ has an edge, say

$e \subset \Delta$, which is contained in some set U_j and whose two vertices are in V_j. Now, there is a transformation by the group $\Pi(U_j, V_j)$ which permutes the corresponding two vertices of the simplex $\Delta' \subset L'$, $h(\Delta') = \Delta$, while keeping fixed the other vertices of Δ'. Moreover, there is such a transformation which also sends the one-skeleton of Δ' onto itself. Therefore, every (anti-symmetric !) cocycle b in K, which is invariant under the groups Γ_1 and $\Pi(U_j, V_j)$, vanishes at the simplex Δ'. According to the Proposition, the bounded cocycles can be made invariant under all groups $\Pi(U_j, V_j)$, and then they vanish at all simplices of the complex $L' \approx L$. Therefore, these cocycles are cohomologous to zero. Q.E.D.

The proof of the identity $\| \ \|^{new} = \| \ \|$ of section 2.3. : Let first X be a $K(\Pi, 1)$ space and let $Y \to X$ be the universal covering. Then the complex $\hat{C}^*(X)$ is canonically isomorphic to the complex of bounded Π-invariant singular cochains in Y. Next, we take the complex $\hat{C}^*(Y:\Pi)$ whose i-cochains are bounded functions $c = c(y_o, \ldots, y_i)$ and observe the natural embedding $I : \hat{C}^*(Y:\Pi) \to \hat{C}^*(X) \subset \hat{C}^*(Y)$. Then, as the space Y is contractible, we can Π-equivariantly assign to each $(i+1)$-tuple of points, $(y_o, \ldots, y_i)$, a singular simplex $\sigma : \Delta^i \to Y$ with vertices $y_o, \ldots, y_i$, such that the subsets of $\{y_o, \ldots, y_i\}$ go under this assignment to the corresponding faces of σ. Thus we get a chain homomorphism $S : \hat{C}^*(X) \to \hat{C}^*(Y:\Pi)$. Both homomorphisms, I and S have norms ≤ 1. Furthermore, $S \circ I = id$ and also $I \circ S \sim id$. Indeed, the standard construction usually applied to unbounded cochains (see [40] for instance) gives, a chain homotopy equivalence $I \circ S \sim id$ which is bounded in every dimension. Therefore, the homomorphism I induces isometric isomorphisms on bounded cohomology

$$I^* : \hat{H}^*(Y:\Pi) \overset{\to}{\sim} \hat{H}^*(X).$$

Next, we extend the definitions of $\hat{C}^*(Y:\Pi)$ and of $\hat{H}^*(Y:\Pi)$ to any action of Π on an arbitrary set Y. In particular, for $Y = \Pi$ we get for $\hat{C}^*(\Pi:\Pi)$ the subcomplex of the bounded cochains in the usual complex of (homogeneous) real cochains on Π. More generally, for any set X we may consider the natural action of Π on the product $\Pi \times X$, and we observe for coverings $Y \to X$ the canonical isomorphism $\hat{C}^*(Y:\Pi) = \hat{C}^*((\Pi \times X):\Pi)$. Maps between sets, $X_1 \to X_2$, induce chain homomorphisms,

$$\hat{C}^*((\Pi \times X_2):\Pi) \longrightarrow \hat{C}^*((\Pi \times X_1):\Pi) ,$$

of norm ≤ 1, and again, by the standard argument, we conclude that these maps induce isometric isomorphisms on cohomology. As a conclusion, we obtain an isometric isomorphism

$$\hat{H}^*(X) \approx \hat{H}^*(\Pi) ,$$

where X is a $K(\Pi,1)$-space and $\hat{H}^*(\Pi)$ denotes the cohomology of the complex $\hat{C}^*(\Pi;\mathbb{R}) = \hat{C}^*(\Pi:\Pi)$ of bounded real cochains $c(\gamma_0,\ldots,\gamma_i)$, $\gamma_i \in \Pi$.

Finally, we denote by $\hat{C}_a^*(\Pi) \subset \hat{C}^*(\Pi;\mathbb{R})$ the subcomplex of antisymmetric cochains, and then with the (anti) symmetrization over the permutations of $\{\gamma_0,\ldots,\gamma_i\}$ we again obtain an isometry of cohomology groups,

$$\hat{H}_a^*(\Pi) = \hat{H}^*(\Pi) .$$

Now, let X be any path connected space with $\pi_1(X) = \Pi$ and let $Y \to X$ be the universal covering. We represent Π by an orbit of a point in Y and then we take an arbitrary Π-equivariant map $r : Y \to \Pi \subset Y$. With the discussion above the mapping theorem for maps $X \to K(\Pi,1)$ now claims an isometric isomorphism

$$r^* : \hat{H}{}^*_a(\Pi) \xrightarrow{\sim} \hat{H}{}^*(X) .$$

With this isomorphism the proof of the inequality $\| \ \|_\infty^{new} = \| \ \|_\infty$ is immediate. Indeed, we take first the complex $\widetilde{C}^*(Y:\Pi)$ of bounded Borel cochains, that is the intersection of the complex $C^*(Y:\Pi)$ of section 2.3. with $\hat{C}{}^*(Y:\Pi)$. Then we have a natural homomorphism $\overset{\approx}{F} : \widetilde{H}^*(Y:\Pi) \to \hat{H}{}^*(X)$ (compare with F of 2.3), and with a <u>Borel</u> map r we also have

$$\widetilde{r} : H^*_a(\Pi) \longrightarrow \widetilde{H}^*(Y:\Pi) ,$$

such that $r^* = \overset{\approx}{F} \circ \widetilde{r}$. As $\|\widetilde{r}\|_\infty \leq 1$ and r is an isometry, the homomorphism $\overset{\approx}{F}$ is surjective and such that for any $\overset{\wedge}{\beta} \in \hat{H}{}^*(X)$ we have $\|\overset{\wedge}{\beta}\|_\infty = \inf\|\overset{\wedge}{\alpha}\|_\infty$, where $\overset{\wedge}{\alpha}$ runs over the pullback $\overset{\approx}{F}{}^{-1}(\overset{\wedge}{\beta}) \subset \widetilde{H}^*(Y:\Pi)$. Now, by the definition of $\|\beta\|_\infty^{new}$ of 2.3., we get $\| \ \|_\infty^{new} \leq \| \ \|_\infty$ and since the opposite inequality is obvious, we get the required identity $\| \ \|^{new} = \| \ \|^{old}$.

3.4. The proof of the isolation theorem of section 0.4.

For a metric ball B of radius R in a Riemannian manifold V , we denote by λB , $\lambda > 0$, the concentric ball of radius λR . We denote by $\widetilde{B}$ a ball of radius R in the universal covering $\widetilde{V} \to V$ which projects onto B .

(A). Lemma. <u>For any given positive numbers</u> C <u>and</u> d <u>there is a constant</u> $\mu = \mu(C,d)$ <u>such that the inequalities</u>

$$\text{Vol}(\lambda\widetilde{B})/\text{Vol}(B) \leq C\lambda^d , \text{ for all } \lambda \in [1,\mu]$$

imply "amenability" (see 3.1.) of the ball $B \subset V$.

Proof: Let $\tilde{B} \subset \tilde{V}$ be a ball over $B \subset V$ and consider all those deck transformations $\gamma \in \Pi = \pi_1(V)$ for which the intersection $\tilde{B} \cap \gamma(\tilde{B})$ is not empty. Denote these transformations by $\gamma_1, \ldots, \gamma_i, \ldots, \gamma_m \in \Pi$ and observe that the subgroup $\Pi' \subset \Pi$ generated by γ_i , $i = 1, \ldots, m$, equals the image of $\pi_1(B)$ in $\pi_1(V) = \Pi$. Then we associate for every $\lambda' \geq 1$ the set $\Pi'(\lambda') \subset \Pi'$ of all those $\gamma \in \Pi$ which can be represented by words in $\gamma_1, \ldots, \gamma_m$ of length $\leq \lambda'$, namely

$$\gamma = \gamma_{i_1}^{p_i} , \ldots, \gamma_{i_k}^{p_k} \quad \text{for} \quad |p_1| + \ldots + |p_k| \leq \lambda' .$$

For every $\gamma \in \Gamma'(\lambda')$, the ball $\gamma(\tilde{B})$ is contained in the ball λB for $\lambda = 2\lambda' + 1$ and therefore, the number of elements in $\Pi'(\lambda')$ satisfies

$$\# \Pi'(\lambda') \leq \text{Vol}(\lambda\tilde{B})/\text{Vol}(B) .$$

Finally, if $\# \Pi'(\lambda') \leq C\lambda^d = C(2\lambda'+1)^d$ for all $\lambda' \in [1, \frac{1}{2}\mu - 1]$, then, for a sufficiently large $\mu = \mu(C,d)$, the group Π' , being of initial polynomial growth, contains a nilpotent subgroup of finite index (see the end of [23]) and so Π' is amenable. Q.E.D.

Let us call an open ball in V extremal if it is "amenable" and if all larger concentric balls are not "amenable".

(B) Lemma: For any given number $\rho > 0$ there exists a system of open balls $B_1, \ldots, B_j, \ldots,$ in an arbitrary complete manifold V such that the following four properties hold.

(1) Each ball B_j has radius at most ρ and each concentric ball $4B_j$ is

"amenable". Furthermore, if some ball B_j has a radius strictly less than ρ , then the concentic ball $4B_j$ is extremal.

(2) The balls $\frac{1}{4}B_j \subset V$ are mutually disjoint.

(3) The balls $\frac{3}{4}B_j \subset V$ cover the manifold V .

(4) If two balls B_{j_1} and B_{j_2} intersect then their radii ρ_1 and ρ_2 satisfy,

$$2\rho_1 \geq \rho_2 \geq \frac{1}{2}\rho_1 \quad .$$

Proof : Observe that around each point in V there is exactly one ball which satisfies (1). Furthermore, if two such balls, B of radius ρ and B' of radius $\rho' \leq \rho$, intersect, then the ball $B'' = \frac{2\rho}{\rho'} B'$ of radius 2ρ is contained in the ball $4B$. As the ball $4B$ is "amenable", so is the ball B'' , and since the ball $4B'$ is extremal it contains B'' . Therefore $\rho'' \geq \frac{1}{2}\rho$ and so the property (4) holds for our balls.

Now, let us take a maximal system of balls which satisfies the properties (1) and (2) and prove the property (3) for this system of balls B_j . Indeed, take an arbitrary point $v \in V$ and the ball B around v which satisfies (1). As the system $\{B_j\}$ j=1,..., is maximal, the ball $\frac{1}{4}B$ intersects some ball $\frac{1}{4}B_j$ and then by (4) the concentric ball $\frac{3}{4}B_j$ contains v . Q.E.D.

(C) Lemma : Let V be a complete n-dimensional manifold with Ricci(V) ≥ -1 and let B and B' be some balls in V of radii ρ and ρ' respectively. If the balls B and B' intersect and if their radii ρ and ρ' are less than one, then

$$Vol(B)/Vol(B') \leq const_n \rho^n (\rho')^{-n} \quad .$$

In particular, $\text{Vol}(B) \leq \text{const}_n' \rho^n$.

Proof : This is a special case of Bishop's inequality (see [5] and [22]).

Corollary : Balls B of radius $\rho \leq 1$ contain no more than $\text{const}_n \rho^n (\rho')^{-n}$ of disjoint balls of radii ρ' and therefore, B can be covered by a number not more than $\text{const}_n' \rho^n \varepsilon^{-n}$ of balls of radius ε , for any given positive $\varepsilon < \rho$.

Now, let $B_j \subset V$ of radii ρ_j , $j=1,\ldots,$ be balls which satisfy the properties (2), (3) and (4) of lemma (B) and let us take the following functions $\varphi_j : V \rightarrow \mathbb{R}$ with supports in B_j , $j=1,2,\ldots$. Every function φ_j is zero outside B_j , it equals one on $\frac{3}{4} B_j$ and for $v \in B_j \smallsetminus \frac{3}{4} B_j$ we define it by $\varphi_j(v) = 4\rho^{-1}(1-\text{dist}(v, \frac{3}{4} B_j)$. Observe that the functions φ_j are Lipschitz with $\| \text{grad } \varphi_j \| \leq 4\rho^{-1}$. Then, we assume the covering $\{B_j\}$ to be locally finite and we take the functions $f_j = (\Sigma_j \varphi_j)^{-1} \varphi_j$ which send V into the unit simplex $\overline{\Delta} = \{x_1,\ldots) : \Sigma_j x_j = 1 , x_j \geq 0 \ \forall j\}$ in the Euclidean space with coordinates $x_1,\ldots,x_j,\ldots$. The map $f = (f_1,\ldots,f_j,\ldots)$ sends V into the nerve of the covering $\{B_j\}$, which is realized as a subcomplex P in $\overline{\Delta}$. The dimension of P at any point $v \in V$ equals the multiplicity of the covering B_j . If a point $v \in V$ is covered by at most m balls B_j then the norm of the differential of f satisfies $\| D_v f \| \leq \text{const}_m \rho^{-1}$ where ρ denotes the minimum of the radii of the m balls B_j .

Now, let $\text{Ricci}(V) \geq -1$ and let all balls B_j have radii ≤ 1 . Then by lemma (C), there are at most $N = N(n)$ balls B_j which intersect any given fixed ball B_{j_0} , and in particular $\dim(P) \leq N$. Furthermore, if a point $v \in V$ is contained in some ball B_j then

$$\| D_v f \| \leq 2 \, \text{const}_N \rho_j^{-1} \leq \text{const}_n' \rho_j^{-1}$$

where ρ_j denotes the radius of B_j .

(D) <u>Lemma</u> : There exists a map g of V into the n-skeleton $P^n \subset P$ with the following two properties :

(a) <u>The norm of the differential Dg is bounded on every ball</u> B_j <u>by const</u> ρ_j^{-1} <u>for some</u> const $= \text{const}''_n$.

(b) <u>The pullback under g of the star of every vertex in P is contained in the union of some balls</u> B_j <u>which intersect a certain fixed ball</u> B_{j_0}

<u>Proof</u> : The map $f : V \to P$ satisfies (a) and (b). Let us first construct $f_1 : V \to P^{N-1}$, then $f_2 : V \to P^{N-2}$, and so on until we eventually get $g = f_{N-n}$. We obtain the map f_1 by choosing a point x inside each N-dimensional simplex $\Delta = \Delta^N \subset P$, $x = x(\Delta) \in \text{Int } \Delta$ but not in the image $f(V) \subset P$, and then by projecting the intersections $f(V) \cap \Delta$ radially from x to the boundary of Δ . In order to guarantee the property (a), we choose the points $x \in \text{Int } \Delta$ with dist $(x, f(V)) \geq \epsilon = \epsilon_n > 0$. This is possible since the numbers

$$\epsilon = \epsilon(\Delta) = \sup_{x \in \Delta} \text{dist}(x, f(V)$$

are bounded from below in terms of N . Indeed, as $\dim \Delta = N$ there are at least $k \geq \text{const}^{(1)} \epsilon^{-N}$ disjoint ϵ-balls in Δ with centers in $f(V) \cap \Delta$ for some $\text{const}^{(1)} = \text{const}^{(1)}(N)$. By the property (b) of the map f the pullbacks of these balls to V , called $\tilde{B}_1, \ldots, \tilde{B}_i, \ldots, \tilde{B}_k \subset V$, are contained in the union of some balls $B_{j_1}, \ldots, B_{j_N}$ with non-empty intersection, and by the property (a) all sets $\tilde{B}_i$ contain balls of radii $\geq \text{const}^{(2)} \epsilon \rho_1^{-1}$, where ρ_1 is the radius of B_{j_1} and again $\text{const}^{(2)} = \text{const}^{(2)}(N)$. Now, by

lemma (C) we have $k \geq \text{const}^{(3)} \varepsilon^{-n}$ and thus

$\varepsilon \geq (\text{const}^{(1)}/\text{const}^{(3)})^{1/N-n} \geq \varepsilon(N)$. Q.E.D.

Corollary : If all balls B_j satisfy

$$\text{Vol}(B_j) \leq \delta(n)\rho_j^n$$

for some sufficiently small $\delta(n) > 0$, then, there is a map $g' : V \to P^{n-1} \subset P^n$, which satisfies property (b) of lemma (D).

Proof : Indeed, with the small $\text{Vol}(B_j)$ we conclude according to (a), that no n-simplex in P^n is covered by the image $g(V)$ and so this image can be pushed further to the (n-1)-skeleton P^{n-1} .

Now, we can prove the Isolation theorem. First, choose a sufficiently large constant $C = C(n)$ in lemma (A) and take $d = n$. Next, take ρ of lemma (B) equal to $\frac{1}{4}\mu^{-1}$. Then, we get $\text{Vol}(B_j) \leq \delta(n)\rho_j^n$, such that $\delta(n)$ is small if C is large. For $\rho_j = \rho = \rho(n)$ this follows from the assumption $\text{Vol}(B(1)) \leq \varepsilon(n)$. Then, if the ball $4B_j$ is extremal and so larger concentric balls are not "amenable", we get according to lemma (A)

$$\text{Vol}(B_j) \leq \text{Vol}(4B_j) \leq C^{-1}\lambda^{-n}\text{Vol}(4\lambda\widetilde{B}_j) \quad \text{for some} \quad \lambda \leq \mu .$$

Now, by lemma (C) we have

$$\text{Vol}(4\lambda B_j) \leq \text{const}_n'(4\lambda\rho)^n$$

and so

$$\text{Vol}(B_j) \leq 4^n C^{-1} \text{const}_n' .$$

Finally, we obtain with the property (b) of lemma (D) that the pullback

under g' of every point $x \in K^{n-1}$ is contained in some ball $3B_j$, that
is an "amenable" subset of V , and so the corollary (2) of the vanishing
theorem applies, provided V is a <u>closed</u> manifold. The open case will be
treated later in section 4.2.

<u>Final Remarks</u> : Some complications we met above would disappear if we could
prove that balls of radius $\leq \varepsilon = \varepsilon(n)$ in V with $Ricci(V) \geq -1$ are
"amenable". As the matter stands now, we only have lemma (A) and so we must
be careful in choosing the covering $\{B_j\}$. However, if $|K(V)| \leq 1$, then
by Margulis' lemma small balls are amenable and the proof of this section
can be simplified for manifolds V with $|K(V)| \leq 1$. In fact, we prove
in (C) of section 4.3. the sharper Injectivity Radius estimate of 0.4.

3.5. <u>The proof of the identity</u> $\|V_1 \# V_2\| = \|V_1\| + \|V_2\|$ <u>and some generaliza-
tions</u>.

As the manifolds V_1 and V_2 are assumed of dimension ≥ 3 (see 0.2.)
the (pinching) map $V_1 \# V_2 \to V_1 \vee V_2$ yields an isomorphism of fundamental
groups. If the manifolds are closed, (the open case is studied in section
4.2.) then, by the mapping theorem (see 3.1.) we can replace the connected
sum $V_1 \# V_2$ by the wedge $V_1 \vee V_2$. The homology of this wedge is

$$H_*(V_1 \vee V_2) = H_*(V_1) \oplus H_*(V_2) ,$$

and we must show that $\|[V_1]+[V_2]\| = \|V_1\| + \|V_2\|$ for the fundamental
classes $[V_1]$ and $[V_2]$. Observe that the inequality

$$\|[V_1]+[V_2]\| \leq \|V_1\| + \|V_2\|$$

is immediate from the definition of the norm. To prove the opposite inequality

we first pass to cohomology and then we construct a homomorphism

$Q : \hat{C}^*(V_1) \oplus \hat{C}^*(V_2) \rightarrow \hat{C}^*(V_1 \vee V_2)$, such that $P \circ Q = id$ for the obvious homomorphism

$$P : \hat{C}^*(V_1 \vee V_2) \longrightarrow \hat{C}^*(V_1) \oplus \hat{C}^*(V_2) \ ,$$

and such that $\|Q\|_\infty \le 1$, with the convention $\| c_1 \bigcirc c_2 \|_\infty = \max(\|c_1\|_\infty, \|c_2\|_\infty)$. Given such a homomorphism Q , we have, for every cohomology class $h = (h_1, h_2) \in H^*(V_1 \vee V_2)$, the inequality $\|h\|_\infty \le \max(\|h_1\|_\infty, \|h_2\|_\infty)$, and by duality we get the required relation in homology.

We start the construction of Q with the following combinatorial considerations. Call a simplicial complex K tree-like, if it is simply connected and if it can be decomposed into the union of closed simplices, called Δ_j , $j \in J$, such that any two of them have at most one common vertex. A piecewise linear path in K built of some edges in K is called straight, if it has at most one edge in every simplex Δ_j . Such a path is topologically a closed interval divided into edges, and any two vertices in the (tree-like !) complex K can be joined by a unique straight path.

Now, take $n+1$ vertices in K , called $x_0, \ldots, x_n$, and join every two of them by the straight path called $[x_k, x_\ell]$, for $k, \ell = 0, \ldots, n$ and $k \ne \ell$. Let us intersect all $[x_k, x_\ell]$ with a fixed simplex Δ_j . Observe, that if two paths, call them $[x_{k_0}, x_{\ell_0}]$ and $[x_{k_1}, x_{\ell_1}]$, intersect Δ_j along the edges $[y_{k_0}, y_{\ell_0}]$ and $[y_{k_1}, y_{\ell_1}]$ for some vertices $y_{k_0}, y_{\ell_0}, y_{k_1}$ y_{ℓ_1} of Δ_j , then the path $[x_{k_0}, x_{k_1}]$ intersects Δ_j in the edge $[y_{k_0}, y_{k_1}]$ and therefore the total intersection equals the one-skeleton of some m-dimensional face of Δ_j for $m \le n$. Furthermore, if $m = n$, then the union of all paths $[x_k, x_\ell]$ consists exactly of this one-skeleton spanned by some vertices $y_0, \ldots, y_n$ in Δ_j plus the paths $[x_k, y_k]$.

$k = 0,\ldots,n$. Moreover, there is at most one such "n-dimensional" one-skeleton in K for $n \geq 2$. If there is no such one-skeleton at all, we say that the "simplex" $(x_o,\ldots,x_n)$ is degenerate, and our construction assigns to each non-degenerate "simplex", an actual n-dimensional simplex in K , that is

$$(x_o,\ldots,x_n) \longrightarrow (y_o,\ldots,y_n) .$$

Now, we return to V_1 and V_2 with the fundamental groups Π_1 and Π_2 . We may replace the complexes $\hat{C}^*(V_1)$ and $\hat{C}^*(V_2)$ by $\hat{C}_a^*(\Pi_1)$ and $\hat{C}_a^*(\Pi_2)$, as shown in the proof of the identity $\| \|^{new} = \| \|^{old}$ in section 3.3.

We consider the (infinite dimensional) simplices $\Delta(\Pi_1)$ and $\Delta(\Pi_2)$ spanned by the groups Π_1 and Π_2 as the sets of vertices, and then we interpret our complexes as the complexes of Π_1- and Π_2-invariant simplicial cochains in $\Delta(\Pi_1)$ and $\Delta(\Pi_2)$. The group $\Pi = \Pi_1 * \Pi_2 = \Pi_1(V_1 \vee V_2)$ acts on a tree-like complex K built of infinitely many copies of $\Delta(\Pi_1)$ and $\Delta(\Pi_2)$, as seen by looking at the universal covering of the wedge $V_1 \vee V_2$, a tree-like union of copies of the universal coverings of V_1 and V_2 . The zero-skeleton of K is identified with the zero-skeleton of the simplex $\Delta(\Pi)$ and so every n-face of $\Delta(\Pi)$ becomes an n-dimensional "simplex" $\Delta = (x_o,\ldots,x_n)$. Now, every pair of cochains, $(c_1,c_2) \in \hat{C}_a(\Pi_1) \oplus \hat{C}_a(\Pi_2)$, uniquely defines a simplicial Π-invariant cochain c on K . Then with our correspondence $\Delta \rightarrow \Delta' = (y_o,\ldots,y_n)$ we define $c' = Q(c)$ by putting $c'(\Delta) = c(\Delta')$ in case Δ is a non-degenerate "simplex", and otherwise $c'(\Delta) = 0$. Q.E.D.

A generalization : amalgamated products.

Let a space V be divided into a union of two closed oriented manifolds, $V = V_1 \cup V_2$, where it is assumed for simplicity that V_1, V_2 and the intersection $V_o = V_1 \cap V_2$ are connected and, V_o is supposed to be a subpolyhedron in V_1 as well as in V_2 . We denote $\Pi_o = \pi_1(V_o)$ and we assume the inclusion homomorphisms $\Pi_o \to \Pi_1 = \pi_1(V_1)$ and $\Pi_o \to \Pi_2 = \pi_1(V_2)$ injective. Then the group $\Pi = \pi_1(V)$ is called the amalgamated product of Π_1 and Π_2 . The fundamental homology classes of V_1 and V_2 define a class in V called

$$[V_1] + [V_2] \in H_*(V) \ .$$

Theorem : If the group Π_o is amenable then $\| [V_1] + [V_2] \| = \| V_1 \| + \| V_2 \|$.

Corollary : Let $\dim V_1 = \dim V_2$ and take away from V_1 and V_2 open regular neighbourhoods of V_o , assumed of codimension ≥ 3 . Glue the resulting manifolds by some diffeomorphism (if such exists) of the boundaries. Then the resulting manifold V' satisfies

$$\| V' \| = \| V_1 \| + \| V_2 \| \ .$$

Indeed, as in the case of the connected sum when V_o is only one point, we can apply the mapping theorem since

$$\pi_1(V') = \Pi = \pi_1(V) \ .$$

The proof of the theorem : Look at the universal covering $\widetilde{V}_1 \to V_1$ and observe that $V_o \subset V_1$ lifts to infinitely many disjoint copies of the universal covering of V_o . These copies correspond to the cosets in Π_1/Π_o .

In terms of the simplex $\Delta(\Pi_1)$ we have some faces $\Delta_i(\Pi_o) \subset \Delta(\Pi_1)$ for some set of indices, $i \in I = \Pi_1/\Pi_o$. The same applies to $\Delta(\Pi_2)$, where we have simplices $\Delta_j(\Pi_o)$ for $j \in J = \Pi_2/\Pi_o$ and then we form in a natural way a "tree-like" union of infinitely many copies of $\Delta(\Pi_1)$ and $\Delta(\Pi_2)$, such that any two copies may intersect along at most one copy of the simplex $\Delta(\Pi_o)$. The group Π acts on the resulting "tree-like" complex K' whose zero-skeleton is identified with the zero-skeleton of the simplex $\Delta(\Pi)$.

Now, as the group Π_o is amenable, we can average the bounded cocycles over the group $\Pi(V_o, V_o)$ (see the proof of the vanishing theorem in section 3.3) and then we interpret the averaged cochains in $\hat{C}^*(\Pi_1)$ as Π_1-invariant cochains on the quotient Π_1/Π_o .

In other words, we may work with the cochain complex of bounded Π_1-invariant cochains on the simplex $\Delta(\Pi_1/\Pi_o)$, which may also be obtained by pinching to points the copies $\Delta_i(\Pi_o) \subset \Delta(\Pi_1)$. Finally, by pinching to points all copies of $\Delta(\Pi_o)$ in K' we get a tree-like complex K , which is built of some copies of $\Delta(\Pi_1/\Pi_o)$ and $\Delta(\Pi_2/\Pi_o)$ by glueing these simplices at some vertices. Furthermore, the zero-skeleton of this K is identical to the zero-skeleton of $\Delta(\Pi/\Pi_o)$ and the argument we used above for free products applies.

4. Simplicial volumes of complete manifolds.

If V is an open manifold, then, as was shown in section 1, there
are several non-equivalent notions of simplicial volume. For complete
Riemannian manifolds one can also take into account the geometry of V
at infinity as well as the topology of V , and then more simplicial volumes
emerge.

4.1. Relative bounded cohomology.

For a pair of topological spaces X and X' $\subset$ X , one has a natural
ℓ^1-norm in the relative chain complex $C_*(X,X') = C_*(X)/C_*(X')$. One also
has the dual ℓ^∞-norm on relative cochains in $C^*(X,X')$ induced from the
ℓ^∞-norm in $C^*(X)$ by the inclusion $C^*(X,X') \subset C^*(X)$. The bounded coho-
mology groups $\hat{H}^*(X,X')$ with the pseudo-norm $\| \ \|_\infty$, dual to the
ℓ^1-norm $\| \ \|$ on the relative homology $H_*(X,X')$, then follow as in
section 2. Furthermore, for every $\theta \geq 0$, one defines norms $\| \ \|(\theta)$
on $C_*(X)$ by putting

$$\|c\| \, (\theta) = \|c\| + \theta \, \|\partial c\|$$

Then with the quotient homomorphism $q : C_*(X) \to C_*(X,X')$ one gets the
norm $\|c'\| \, (\theta)$ for all $c' \in C_*(X,X')$ by taking all $c \in q^{-1}(c') \subset C_*(X)$
and setting

$$\|c'\| \, (\theta) = \inf_c \|c\| \, (\theta) \ .$$

The dual norms on relative cochains are denoted by $\| \ \|_\infty(\theta)$. Observe
that all norms $\| \ \|(\theta)$ are equivalent, but not equal to the usual
ℓ^1-norm $\| \ \| = \| \ \|(0)$ and the dual norms $\| \ \|_\infty(\theta)$ are all equivalent
to $\| \ \|_\infty$. Now with these norms on relative chains and cochains we have

the corresponding norms $\| \ \|$ (θ) on $H_*(X,X')$ and the dual norms $\| \ \|_\infty(\theta)$ on the bounded cohomology $\hat{H}^*(X,X')$. Finally we define these norms for all θ in the closed interval $[0,\infty]$ by passing to the limits. Observe that the limit "norm" $\| \ \|$ (∞) may be not equivalent to $\| \ \|$ (θ) for $\theta < \infty$. For example, $\|h\|$ $(\infty) = \infty$ for those $h \in H_*(X,X')$ for which the boundary $\partial h \in H_*(X')$ has non-zero ℓ^1-norm, $\|\partial h\| > 0$.

Remark. Our norm $\| \ \|(\infty)$ equals Thurston's norm $\| \ \|_0$ defined in [47]
Warning : Thurston's approximations $\| \ \|_\varepsilon$ in [47] are different from our $\| \ \|(\theta^{-1})$.
  Thurston proves in [47] the equality $\| \ \| = \| \ \|(\infty)$ for the fundamental classes of compact 3-manifolds whose boundaries are tori. This result generalizes to the following

Equivalence theorem. If the fundamental groups of all path-connected components of the space X' are amenable, then the norms $\| \ \|_\infty(\theta)$ on the groups $\hat{H}^i(X,X')$, for $i \geq 2$, are equal for all $\theta \in [0,\infty]$, and so the dual norms $\| \ \| (\theta)$ on the homology groups $H_i(X,X')$ for $i \geq 2$ are also equal.

Proof. With the averaging technique of section 3 one gets first the following

Relative mapping theorem. Let f be a map of pairs of spaces, $f : (X,X') \to (Y,Y')$ which is bijective on the sets of path-connected components, $\pi_0(X) \xrightarrow{\sim} \pi_0(Y)$ and $\pi_0(X') \xrightarrow{\sim} \pi_0(Y')$. Furthermore, let g be surjective on the fundamental groups of all these components and let the kernels of these surjective homomorphisms be amenable groups for all components of X and of X' . Then the induced homomorphism

$$f^* : \hat{H}^*(Y,Y') \to \hat{H}^*(X,X')$$

is an isometric isomorphism for the norms $\| \; \|_\infty(\theta)$ for all $\theta \in [0,\infty]$. In particular, if the fundamental groups of all components of X and of X' are amenable, then the groups $\hat{H}^i(X,X') = 0$ for $i \geq 2$ and the norms $\| \; \|(\theta)$ on $H_i(X,X')$, for $i \geq 2$, vanish for all $\theta \in [0,\infty]$.

Remark. The group $\hat{H}^1(X,X')$ vanishes if and only if the inclusion map $\pi_o(X') \to \pi_o(X)$ is injective.

Now, by averaging over the amenable group $\Pi(X',X')$, as in section 3.3. , one makes all bounded cochains vanish at all those singular simplices $\Delta \to X$ which have an edge contained in X' . In particular, the values of such cocycles on relative cycles c of dimension ≥ 2 does not depend any more on their boundaries, $\partial c \subset X'$, and thus the equivalence theorem is established.

4.2. Diffusion of chains in open manifolds.

For a sequence of subsets of an open manifold, $U_j \subset V$, we write $U_j \to \infty$ if only finitely many of these sets U_j intersect any given compact subset of V . A subset $U \subset V$ is called large if the complement $V \smallsetminus U$ is relatively compact. Then we have a canonical homomorphism of the homology defined with locally finite singular chains (see 0.2), $\overline{H}_*(V)$, to the relative homology $H_*(V,U)$. For a sequence of large sets $U_j \subset V$ we denote by $h_j \in H_*(V,U_j)$ the values of these homomorphisms on elements $h \in \overline{H}_*(V)$. Then we have for $U_j \to \infty$ the limits $\lim_{j \to \infty} \|h_j\|(\theta)$ for all $\theta \in [0,\infty]$, which depend only on $h \in \overline{H}_*(V)$ and are denoted by $\|h\|(\theta)$. Furthermore, we also have the ℓ^1-norm called $\| \; \|$ on $\overline{H}$, which satisfies

$$\|h\| \geq \|h\|(\theta) \quad \text{for all} \quad \theta \in [0,\infty] \quad .$$

Notice that

$$\| \; \|(\theta_1) \geq \| \; \|(\theta_o) \quad \text{for} \quad \theta_1 \geq \theta_o \; .$$

Observe that for interiors of closed manifolds, $V = \text{Int} \; \overline{V}$, the canonical isomorphism $\overline{H}_*(V) \xrightarrow{\sim} H_*(\overline{V}, \partial \overline{V})$ is <u>isometric</u> in all norms $\| \; \|(\theta)$ and in particular, the simplicial volume $\|\overline{V}, \partial \overline{V}\|$ of section 1.1. equals $\| [V] \| \; (0)$ for the fundamental homology class $[V]$ of V .

The proofs of the following three theorems are given at the end of this section.

First, let us call the manifold V "amenable" at infinity if every large set $U \subset V$ contains another large set, $U' \subset U$, such that U' is an "amenable" (see 3.1.) subset of U .

(1) <u>Equivalence theorem.</u> <u>If V is "amenable" at infinity then the norms</u> <u>on $\overline{H}_i(V)$ for $i \geq 2$ satisfy,</u> $\| \; \|(\theta) = \| \; \|$ <u>for all</u> $\theta \in [0, \infty]$.

Next we consider closed subsets V' in V and for all $h \in \overline{H}_*(V)$ we denote by $h' \in \overline{H}_*(V \smallsetminus V')$ the values of the canonical homomorphism $\overline{H}_*(V) \to \overline{H}_*(V \smallsetminus V')$. Then we say that a sequence of subsets $V'_j \subset V$ is "amenable" at infinity if there is a sequence of large open sets $U_j \subset V$ which goes to infinity, $U_j \to \infty$, such that $V'_j \subset U_j$ and such that for all sufficiently large j each V'_j is an "amenable" subset of U_j .

(2) <u>Cutting of theorem.</u> <u>Let</u> $V' \subset V$ <u>be a union of disjoint compact sub-</u> <u>manifolds (with boundaries),</u> $V' = \underset{j}{\cup} V'_j$, $j = 1, \ldots,$. <u>If the set</u> V' <u>is "amenable" in</u> V <u>and if the sequence</u> V'_j <u>is "amenable" at infinity</u> <u>then for all</u> $h \in \overline{H}_i(V)$, $i \geq 2$, <u>one has the inequalities</u>

$$\|h'\| \geq \|h\| \quad \text{and} \quad \|h'\| \; (\theta) \geq \|h\| \; (\theta) \; ,$$

for all $\theta \in [0,\infty]$. Furthermore, if also all V'_j are closed connected
orientable manifolds of codimension one in V and if the inclusion homomor-
phisms $\pi_1(V'_j) \to \pi_1(V)$ are injective for all $j = 1,\ldots,$ then

$$\|h'\| = \|h\| \quad \text{and} \quad \|h'\| (\theta) = \|h\| (\theta) ,$$

for all $\theta \in [0,\infty]$. For example, for $\dim V \geq 2$, the simplicial volume
does not change if V' is deleted from V .

Corollary (Thurston [47], Soma [43]). Simplicial volumes of 3-manifolds do
not change if the manifolds are cut along spheres and incompressible tori.

Finally, we consider a cover of V by relatively compact open
subsets, $U_j \subset V$, such that $U_j \to \infty$ (i.e. the cover is locally finite).

(3) Vanishing-Finiteness theorem. If the sequence U_j is "amenable" at
infinity and if there is a large set W whose every point, $v \in W$, is
contained in at most m subsets U_j , for some $m = 1,\ldots,$ then the norm
$\| \|$ is finite on the group $\overline{H}_i(V)$ for $i \geq m$. Furthermore, if also
the subsets U_j in V are "amenable" for all $j = 1,\ldots,$ and if every
point $v \in V$ is contained in at most m subsets U_j , then $\|h\| = 0$
for all $h \in \overline{H}_i(V)$ and for $i \geq m$.

Corollaries (A) If $m = \dim V$, then under the first set of assumptions
one has $\|V\| < \infty$ and in the second case $\|V\| = 0$.

(B) The proof of the Isolation and Geometric finiteness theorems of section 0.4
With the covering technique of section 3.4. one satisfies the requirements
of (A) above.

(C) Let P be a piecewise smooth subpolyhedron in V and let us consider
small round spheres $S_v^{n-1} \subset V$, $n = \dim V$, around all points $v \in P$.

We say that P is locally "coamenable" if the fundamental groups of
the differences,

$$S_v^{n-1} \smallsetminus P = S_v^{n-1} \smallsetminus (P \cap S_v^{n-1}) \subset S_v^{n-1} \quad ,$$

are amenable for all $v \in P$.

Examples. If codim $P \geq 3$ then P is "coamenable".

A one dimensional polyhedron P in $\mathbb{R}^3$ is locally "coamenable"
if and only if P is a manifold.

If P is the image of a properly immersed manifold $V_o \to V$ with
normal crossings then P is locally "coamenable".

Let P be a locally "coamenable" polyhedron in V of
dim(P) $\leq$ m-2 for $m \leq n = \dim V$. Then the homology of a small regular
neighbourhood U of P has zero ℓ^1-norm in the dimensions $\geq$ m , that
is $\|h\| = 0$ for all $h \in \bar{H}_{i>m}(P)$. Furthermore, if the manifold V is
compact, then the ℓ^1-norms are finite on the homology groups $\bar{H}_{i>m}(V \smallsetminus P)$.

Indeed, for small round balls $B_v \subset V$ for all $v \in P$ the comple-
ments $B_v \smallsetminus P$ have amenable fundamental groups. Thus we have an "amenable"
covering of $U \smallsetminus P$ by small subsets with multiplicity m-1 and then there
are finer coverings of U and of $V \smallsetminus P$ which satisfy the assumptions of
the vanishing and of the finiteness theorems respectively.

Examples. (a) Let V be the Cartesian product of three open manifolds.
Then V can be realized by a regular neighbourhood of some $P \subset V$ with
codim P $\leq$ 3 and so $\|V\| = 0$. In particular the proportionality theorem

(see 0.3) fails for products of open hyperbolic manifolds of finite volume. (We shall rescue this theorem in section 4.5. by introducing yet another simplicial volume).

(b) Algebra-geometric finiteness theorem. Let V be a non-singular complex quasi-projective algebraic variety. Then the simplicial volume of V is finite. In particular, the interiors of compact manifolds, $V = \text{Int } \overline{V}$, admit no quasi-projective structures if $\|\partial \overline{V}\| \neq 0$.

Proof. By Hironaka's theorem [30], V can be realized as the complement to a subvariety with normal crossing in a non-singular variety.

Remarks. For singular varieties V one also has the fundamental class, $[V] \in \overline{H}_*(V)$, and non-singular resolutions $\widetilde{V} \to V$ are maps of degree one. Therefore $\|[V]\| \leq \|\widetilde{V}\| < \infty$.

Birational maps between non-singular projective varieties have degree one and also they are isomorphic on fundamental groups. It follows that the simplicial volume of such varieties is a birational invariant.

The proof of the theorems (1), (2) and (3). The norms $\| \ \|(\theta)$ on the group $\overline{H}_*(V)$ are limits of the corresponding norms on relative homology groups, and with the dual norms on the bounded cohomology one can use the averaging techniques of section 3. However, for the most interesting ℓ^1-norm $\| \ \|$, there is no apparent bounded cohomology theory and so we must apply the averaging operators to ℓ^1-chains rather than to bounded cochains. Unfortunately, there are no such operators for infinite amenable groups. Indeed, the only invariant ℓ^1-function on an infinite group is zero.

This problem can be resolved by adding to ℓ^1-functions on an infinite group Γ _all_ linear functionals on the space of bounded functions $\Gamma \to \mathbb{R}$. These functionals are, in fact, measures on the Cech-Stone compactification of Γ . In particular, positive Γ-invariant measures of total mass one are exactly our old averaging operators on Γ .

There is an elementary alternative to the Cech-Stone compactification. Namely one can replace averaging operators by _locally finite diffusion operators_ which we define below.

First, we consider a probability measure μ on a group Γ , that is a non-negative function, $\mu = \mu(\gamma)$ for $\gamma \in \Gamma$ for which $\|\mu\|_{\ell^1} = \Sigma_\Gamma |\mu(\gamma)| = 1$. For all $\varphi \in \Gamma$ we define the following "derivatives" $\mu' = D_\varphi \mu$ by setting $\mu'(\gamma) = \mu(\gamma\varphi) - \mu(\gamma)$, and for subsets Φ in Γ we put $\|D_\Phi \mu\| = \sup_{\varphi \in \Phi} \|D_\varphi \mu\|_{\ell^1}$.

Next we let Γ act on a set X and then we have the following (diffusion) operators, $f \to \mu * f$, on ℓ^1-functions $f(x)$,

$$(\mu * f)(x) = \Sigma_\Gamma \mu(\gamma) f(\gamma^{-1}x) .$$

Lemma. Let Γ _be transitive on_ X _and let a subset_ $\Phi \subset \Gamma$ _be transitive on the support of a function_ $f(x)$, _that is for some point_ $x_o \in X$ _the_ Φ-_orbit_ $\Phi x_o = \cup_\Phi \varphi x_o$ _contains the support of_ f . _Then_

$$\|\mu * f\|_{\ell^1} \leq |\Sigma_{x \in X} f(x)| + \|D_\Phi \mu\| \, \|f\|_{\ell^1} .$$

Proof. Let $q : \Gamma \to X$ be the quotient map, $q(\gamma) = \gamma x_o$, and let q_* denote the push-forward operator on function $g = g(\gamma)$,

$$(q_*g)(x) = \sum_{\gamma \in q^{-1}(x)} g(\gamma) \ , \ \text{for} \ \ x \in X \ .$$

Observe that q_* commutes with the diffusions $\mu*$,

$$q_*(\mu*g) = \mu*(q_*g) \ .$$

where $(\mu*g)(\gamma) = \sum_{\lambda \in \Gamma} \mu(\lambda) g(\gamma^{-1}\lambda)$. Now, for the function $f = f(x)$,
there is a function $g = g(\gamma)$ with support in Φ , such that $q_*g = f$,
and so $\| \mu*f \|_{\ell^1} = \| q_*\mu*g \|_{\ell^1} \leq \| \mu*g \|_{\ell^1}$. Next,

$$\| \mu*g \|_{\ell^1} = \sum_{\gamma \in \Gamma} | \sum_{\lambda \in \Gamma} \mu(\lambda) g(\gamma^{-1}\lambda) | =$$

$$= \sum_{\gamma \in \Gamma} | \sum_{\lambda \in \Gamma} \mu(\gamma\lambda) g(\lambda) | \leq \sum_{\gamma \in \Gamma} (| \sum_{\lambda \in \Phi} (\mu(\gamma\lambda) - \mu(\gamma)) g(\lambda) | + | \sum_{\lambda \in \Gamma} \mu(\gamma) g(\lambda) | \leq$$

$$\leq \sum_{\lambda \in \Phi} | g(\lambda) | \ \| D_\lambda \mu \|_{\ell^1} + | \sum_{\lambda \in \Gamma} g(\lambda) | \leq \| D_\Phi \mu \| \ \| g \|_{\ell^1} + | \sum_{\lambda \in \Gamma} g(\lambda) | \ . \ \text{Q.E.D.}$$

(A) <u>Corollary</u> : <u>Let the group</u> Γ <u>be amenable. Then for an arbitrary finite</u>
<u>set</u> $S \subset X$ <u>and for any given</u> $\epsilon > 0$ <u>there exists a probability measure</u>
<u>on</u> Γ <u>with a finite support,</u> μ <u>on</u> Γ , <u>such that all functions</u> $f(x)$
<u>with support in</u> S <u>satisfy</u>

$$\| \mu*f \|_{\ell^1} \leq | \sum_{x \in S} f(x) | + \epsilon$$

<u>Proof</u> : As Γ is amenable, one can find, for every finite set $\Phi \subset \Gamma$ and
for any given $\epsilon > 0$, a finite measure μ on Γ for which $\| D_\Phi \mu \| \leq \epsilon$
(see [19]) and so the lemma applies.

(B) <u>Example</u> : Let K be a simplicial multicomplex and let Γ be a group

of simplicial automorphisms of K. Then Γ also acts on finite simplicial chains $c = \sum_i r_i \sigma_i$ in K,

$$\gamma c = \sum_i r_i \gamma \sigma_i$$

and we write

$$\mu * c = \sum_{\gamma \in \Gamma} \mu(\gamma) \gamma c .$$

Suppose that for each simplex σ_i in some chain $c = \sum_i r_i \sigma_i$ there is a transformation $\gamma_i \in \Gamma$ which sends σ_i onto itself with the reversal orientation, $\gamma_i \sigma_i = \bar{\sigma}_i$. Then, if the group Γ is amenable one can "diffuse" c by some measure μ on Γ, that is one makes $\|\mu * c\| \leq \epsilon$ for some $\mu = \mu(c, \epsilon)$. Indeed, we can write c as a cochain c' on K which is supported on the set of the simplices $\{\sigma_i, \bar{\sigma}_i\}$ and $c'(\sigma_i) = -c'(\bar{\sigma}_i) = \frac{1}{2} r_i$. Then the sums of c' over the Γ-orbits of the oriented (!) simplices σ_i are zero. According to (A), we have a finite measure μ on Γ such that $\|c' = \mu * c'\|_{\ell^1} \leq \epsilon$, and since $\mu * c = \sum_\sigma \tilde{c}'(\sigma) \sigma$, where σ runs over all oriented simplices $\sigma \subset K$, we also have $\|\mu * c\|_{\ell^1} \leq \epsilon$. In particular, if c is an n-dimensional cycle whose homology class $[c]$ is invariant under Γ then $\|[c]\| = 0$. Thus we get an alternative proof of the vanishing theorem of section 3.1. (compare with 3.3) .

Let us generalize (B) to locally finite chains. We start with an arbitrary action of Γ on X with countably many orbits, called $X_j \subset X$, $j = 1, \ldots,$. We say that the action of Γ on X is "locally finite" if there exist subgroups $\Gamma_j \subset \Gamma$, $j = 1, \ldots,$ with the following two properties :

(1) Every group Γ_j is transitive on the set X_j for $j = 1,\ldots,$

(2) The actions of Γ_j are "asymptotically disjoint" : for every
$j_o = 1,\ldots,$ there is an index $k = k(j_o)$ such that the <u>supports</u> of Γ_{j_o}
and Γ_j do not intersect for $j \geq k$, that is every point $x \in X$ is either
kept fixed by all $\gamma \in \Gamma_{j_o}$ or x is kept fixed by all $\gamma \in \Gamma_j$.

Now, we define <u>locally finite</u> diffusion operators on X that are given
by sequences of finite probability measures, μ_j on Γ_j for all
$j = 1,\ldots$. Namely, for a function $f(x)$ we inductively define
$f_1 = \mu_1 * f,\ldots,f_j = \mu_j * f_{j-1}$. According to (2) above, the value $f(x)$ for
every fixed point x does not depend on j for large j and so we put

$$\mu * f = \lim_{j \to \infty} f_j$$

for $\mu = \{\mu_j\}\, j = 1,\ldots$. Observe that this diffusion separately acts on each
orbit X_j . Namely

$$\sum_{x \in X_j} (\mu * f)(x) = \sum_{x \in X_j} f(x) \text{ , for all } j = 1,\ldots,$$

and

$$\| \mu * f | X_j \|_{\ell^1} \leq \| f | X_j \|_{\ell^1} \text{ ,}$$

where

$$\| f | X_j \|_{\ell^1} \overset{\text{def}}{=} \sum_{x \in X_j} |f(x)| \text{ .}$$

Suppose that f is a "locally finite" function, that is its supports
on all orbits X_j are finite sets. Then, clearly, the diffused function
$\mu * f$ is also "locally finite". Furthermore, if $\sum_{x \in X_j} f(x) = 0$ for $j \geq j_o$

and the groups Γ_j are amenable for $j \geq j_o$, then for an arbitrary positive

sequence $\epsilon_{j_o}, \epsilon_{j_o+1}, \ldots,$ there is a locally finite diffusion μ such that

$$\| \mu * f | X_j \|_{\ell^1} \leq \epsilon_j \quad \text{for} \quad j \geq j_o \ .$$

Indeed, this follows from (A) above.

Now, we are ready to extend the techniques of section 3 to open

manifolds V . The modifications needed for the proofs of the theorems

(1), (2) and (3) are quite similar and so we discuss below only the last

case.

The proof of the Vanishing-finiteness theorem : To each set $U_j \subset V$, we

assign a large set $W_j \subset V$ which contains U_j such that for $j > j_o$ each

U_j is an amenable subset in W_j and $W_j \to \infty$. We constructed in section 3.2.

a canonical multicomplex $K(V)$ which consists of all those singular simplices

$\sigma : \Delta \to V$ which are injective on the sets of the vertices of Δ . This set

of vertices is sent by σ into the union of some sets U_j , call them

$U_{j_o}, \ldots, U_{j_m}$, and we say that σ is an admissible simplex if the image

$\sigma(\Delta) \subset V$ is contained in the union $W_{j_o} \cup \ldots \cup W_{j_m}$. Then, we consider a

subcomplex $K \subset K(V)$ with the following four properties :

(1) All simplices in K are admissible.

(2) K is complete : for every admissible simplex $\sigma_o : \Delta \to V$ with boundary

in K there is a homotopy by admissible simplices with fixed boundary,

$\sigma_t : \Delta \to V$ for $t \in [0,1]$, such that σ_1 is contained in K .

(3) The vertices of K lie in the union of some fixed disjoint sets

$V_j \subset U_j$, $j = 1, \ldots$.

(4) K is minimal : it contains no proper subcomplexes, which satisfy
(1) and (2) and which have the same vertices as K .

As every such K is built of some singular simplices $\sigma : \Delta \to V$,
it comes with a canonical map $S : K \to V$. Furthermore, the direct product
of groups $\Pi(V_j, W_j)$ (see 3.3) acts on the 1-skeleton of K and every such
action extends (not uniquely !) to an automorphism $\gamma : K \to K$, such that
the maps $S : K \to V$ and $S \circ \gamma : K \to V$ can be joined by an admissible
homotopy, that is a homotopy of maps which are admissible on all simplices
in K . We denote by $\Gamma(K)$ the group generated by all these automorphisms
γ .

Now, let L be a triangulation of V such that each star of L is
contained in one of the sets U_j , and let K be a multicomplex which
contains L and which satisfies the properties (1)-(4). Denote by $K' \subset K$
the $\Gamma(K)$-orbit of L ,

$$K' = \bigcup_{\gamma \in \Gamma} \gamma(L) .$$

The action of $\Gamma = \Gamma(K)$ is locally finite on the set of all oriented simplices
in K' . Indeed, with every set U_j one first considers the union of all
those $U_{j'}$ which intersect U_j , call this union $\tilde{U}_j$, and then one takes
for $\Gamma_j \subset \Gamma$, for all $j = 1, \ldots,$ the groups of the transformation which are
fixed on those simplices $\Delta \subset K'$ whose images in V do not intersect $\tilde{U}_j$.

Now, under the assumptions of the vanishing theorem, every m-simplex
in L has an edge in one of the sets U_j and so some automorphisms
reverse the orientation of σ , that is $\gamma\sigma = \bar{\sigma}$. Therefore, for every
m-chain

$$c = \Sigma r_i \sigma_i \in C_*(L) \subset C_*(K')$$

the corresponding cochain c' , given by $c'(\sigma_i) = \frac{1}{2} r_i$ and $c'(\sigma_i) = -\frac{1}{2} r_i$,

has zero sums over the orbits of Γ . Next, for the proof of the vanishing

theorem, one may assume all U_j to be "amenable" subsets of W_j and then

the groups Γ_j are amenable for all $j = 1, \ldots,$. Finally, locally finite

diffusions make the ℓ^1-norm of c , or rather of $\mu \ast c$, as small as one

wishes, and now one must only show that the diffused <u>cycles</u> $\mu \ast c$ with

$\|\mu \ast c\|_{\ell^1} \leq \epsilon$ are, in fact, homologous to c , in the sense that $c - \mu \ast c = \partial \tilde{c}$

for some <u>locally finite</u> $(m+1)$-chain $\tilde{c}$. Recall that the diffusion $\mu \ast c$ for

$\mu = \{\mu_j\}$ is the limit of chains $c_j = \mu_j \ast c_{i-1}$ where each operator $\mu_j \ast$ is

obtained by combining some cycles γc_{j-1} for $\gamma \in \Gamma_j$. The transformations

Γ_j are homotopies which are supported in some sets $\tilde{W}_j$ such that

$\tilde{W}_j \to 0$ for $j \to \infty$. Therefore, one has $c_{j-1} - c_j = \partial \tilde{c}_j$ for some locally

finite chains $\tilde{c}_j$ with supports in the sets $\tilde{W}_j$ and then one takes

$$\tilde{c} = \sum_{j=1}^{\infty} \tilde{c}_j .$$

Thus the vanishing theorem is proven.

To prove the finiteness part of the theorem, one also applies diffusion

operators but now only with the groups Γ_j for $j \geq j_o$. This makes chains

c as small outside a fixed compact subset in V , as one wishes.

4.3. <u>Diffusion of chains in Riemannian manifolds.</u>

We prove here the geometric inequalities announced in section 0.5. by

combining the diffusion of chains with the smoothing operators of section 2.

For a singular simplex in a Riemannian manifold, $\sigma : \Delta \to V$, and an

open set $U \subset V$, we denote by mass $(\sigma \cap U)$ the mass of the "intersection"

map $\sigma | \sigma^{-1}(U) : \sigma^{-1}(U) \to U \subset V$. For a singular chain $c = \sum_i r_i \sigma_i$, we put

$$\text{mass}(c \cap U) = \sum_i |r_i| \text{mass}(\sigma_i \cap U) \ .$$

We denote by $B_U(R)$ the R-neighbourhood of U for any $R \geq 0$ and we abbreviate,

$$\text{mass}(c \cap U+R) \overset{\text{def}}{=} \text{mass}(c \cap B_U(R)) \ .$$

Next, we restrict chains $c = \sum_i r_i \sigma_i$ to subsets $U \subset V$ by taking the sum $\sum_i r_i \sigma_i$ only over those indices i for which the images $\sigma_i(\Delta) \subset V$ intersect the set U, and we denote these restricted chains by $c|U$.

Finally, for the growth function $\ell_v(R)$ of section 2 . we put

$$[\ell'(R)|U] = \sup_{v \in B_U(R)} \ell'_v(R) \ .$$

(A). <u>Theorem</u> : <u>Let</u> V <u>be a complete</u> <u>Riemannian manifold of non-positive</u> <u>sectional curvature and let</u> $c = \sum_i r_i \sigma_i$ <u>be a locally finite</u> m-<u>cycle</u> <u>in</u> V . <u>Then for arbitrary positive numbers,</u> $R > 0$ <u>and</u> $\epsilon > 0$ <u>there exists another</u> <u>locally finite</u> m-<u>cycle</u> $c' = c'(R,\epsilon) = \sum_j r'_j \sigma'_j$ <u>in</u> V <u>with the following</u> <u>four properties</u> :

(1) c' <u>is homologous to</u> c .

(2) c' <u>is a straight cycle</u> : <u>the lifts</u> $\tilde{\sigma}'_j$ <u>of simplices</u> σ'_j <u>to the univer-</u> <u>sal covering</u> $\tilde{V}$ <u>of</u> V <u>are straight geodisic simplices</u> (see 1.2.) <u>and</u> <u>each</u> $\tilde{\sigma}'_j$ <u>is contained in a ball of radius</u> $R + \epsilon$.

(3) <u>The cycle</u> c' <u>satisfies for all subsets</u> U <u>of</u> V ,

$$\|c'|U\|_{\ell^1} \leq (1+\epsilon)m! [\ell'(R)|U]^m \text{mass}(c \cap U+R+\epsilon) \ .$$

(4) Furthermore, <u>if</u> c <u>represents the fundamental class of</u> V <u>then the</u> <u>last inequality can be sharpened as follows</u>

$$\|c' \,|\, U\|_{\ell^1} \le C_n (1+\epsilon) \int_B (\ell_v'(R))^n dv \ , \ \text{for} \ B = B_U(R+\epsilon) \ ,$$

<u>and for</u> $C_n = \Gamma(n/2)/\sqrt{\pi}\,\Gamma(\frac{n+1}{2}) < 1$, n = dim V .

<u>Proof</u>. Let first $S : Y \to M^+$ be a Π-invariant smoothing operator (see 2.4.) on the universal covering $Y = \tilde{V} \to V$ for $\Pi = \pi_1(V)$. Suppose furthermore, that the smoothing S is locally finite : there is a discrete subset in Y which supports the measure $S(y)$ for all $y \in Y$, and let also each measure $S(y)$ be supported in the ball $B_y(R) \subset Y$. With such an S , we have the following diffusion operator of straight Π-invariant chains $\tilde{c}$ in Y that is the dual to the smoothing operator on cochains (see 2.4.). Recall (see 1.2.) that every straight simplex $\tilde{\Delta}$ in Y is uniquely determined by the ordered set of its vertices, $y_0,\ldots,y_m \in Y$, and so finite chains of ordered simplices are interpreted as finite measures in the Cartesian product

$$\underbrace{Y \times Y \times \ldots \times Y}_{m+1} \quad .$$

Now, we assign to the simplices $\tilde{\Delta} = \tilde{\Delta}(y_0,\ldots,y_m)$ the normalized Cartesian products of the measures $S(y_0),\ldots,S(y_m)$, that are the chains

$$S_*\tilde{\Delta} = \prod_{i=0}^{m} \| S(y_i) \|_{\ell^1}^{-1} S(y_i) \ .$$

Then, this diffusion extends by linearity to the chains of straight ordered simplices,

$$\tilde{c} = \sum_j r_j \tilde{\Delta}_j \ \longrightarrow \ S_*\tilde{c} = \sum_j r_j S_*\tilde{\Delta}_j \ .$$

Finally, for <u>oriented</u> simplices $\tilde{\Delta}$, we take the $(m+1)!/2$ ordered simplices which have the same vertices as $\tilde{\Delta}$ with orders compatible with the orientations of $\tilde{\Delta}$, called $\tilde{\Delta}_\nu$, $\nu = 1,\ldots,\frac{(m+1)!}{2}$, and we put

$$S*\tilde{\Delta} = 2/(m+1)! \sum_\nu S*\Delta_\nu$$

From now on, we only deal with locally finite chains c of oreinted simplices and the diffusion is denoted by $\tilde{c} \to S*\tilde{c}$.

A small straight m-dimensional sikplex $\tilde{\Delta}$ is Y is called <u>fat</u> if $\text{mass}\Delta \geq \text{const}(\text{diam}\tilde{\Delta})^m$, where for "const" one may use an arbitrarily chosen small positive number which only must be kept fixed as the diameter, diam $\tilde{\Delta}$, goes to zero.

Now, the proposition of section 2.4 . implies the following dual

<u>Proposition</u> : For any given $\delta > 0$ <u>all sufficiently small fat m-dimensional simplices</u> $\tilde{\Delta}$ <u>satisfy</u>

$$\| S*\tilde{\Delta} \|_{\ell^1} \leq (1+\delta)[S]^m m! \ \text{mass} \ \tilde{\Delta} \ .$$

<u>Furthermore, with the integral inequality of 2.4 . one has for</u> $m = n = \dim V$,

$$\| S*\Delta \|_{\ell^1} \leq (1+\delta)([S]^*)^n n! \ \text{mass} \ \tilde{\Delta} \ .$$

<u>Remarks (a)</u> : For non-fat small simplices of diameter $\leq \epsilon$, we only can claim the inequality

$$\| S*\tilde{\Delta} \|_{\ell^1} \leq (1+\delta) \ [S]^m \ \epsilon^m \ .$$

(b) The inequalities of the Proposition and of the remark (b) are

purely local : if the simplices $\tilde{\Delta}$ are supported in a subset $Y' \subset Y$, then one may replace the "norms" $[S]$ and $[S]^*$ above by the supremums

$$\sup_{y \in Y'} [S(y)] \quad \text{and} \quad \sup_{y \in Y'} [S(y)]^* .$$

Now, the theorem is reduced to a locally finite approximation of the smoothings $S_{R,\varphi}$ in the theorem of 2.5 . We construct such an approximation with a discrete sufficiently dense subset Z in V , for example with a set $Z = Z_\varphi$ which intersects the balls $B_v(\varphi(v)) \subset V$ for all $v \in V$. We take the pullback of $Z \subset V$ under the covering map $Y \to V$, called $\tilde{Z} \subset Y$ and we assign to every point $\tilde{z} \in \tilde{Z}$ the measure of the set of those points $y \in Y$ for which $\text{dist}(y,\tilde{z}) \leq \text{dist}(y,\tilde{z}')$, for all points $\tilde{z}' \neq \tilde{z}$ in $\tilde{Z}$. We call this measure $\mu(\tilde{z})$ and finally, we send the points $y \in Y$ to the measures called $\tilde{S}_{R,\varphi}(y)$ which are supported in $\tilde{Z} \subset Y$ for all $y \in Y$ and whose weight at the points $\tilde{z} \in \tilde{Z}$ equal $\mu(\tilde{z})S_{R,\varphi}(y,\tilde{z})$ for all $\tilde{z} \in \tilde{Z}$. This smoothing $y \to S_{R,\varphi}(y)$ satisfies in the limit for $\varphi \to 0$ the same relation as our old $S_{R,\varphi}$ and so the corresponding diffusion of chains of small fat simplices satisfies the requirements (2)-(4).

Next, we modify the original cycle $c = \sum_i r_i \sigma_i$ as follows. First, we make all simplices $\sigma_i : \Delta \to V$ smooth by a small perturbation of c . Then, we subdivide the smoothed σ_i into very small simplices σ_{i_j} such that after straightening up these σ_{i_j} almost all of them become fat : "non-fatness" is allowed only to those σ_{i_j} which are adjacent to the boundary $\partial\sigma_i$ of σ_i . The mass of the straightend chain $\sum_{i_j} r_i \sigma_{i_j}$ is close to mass(c) and the lifts of non-fat simplices to $Y \to V$, called $\tilde{\sigma}_{i_k}$, have

$$\sum_k (\text{diam}\,\sigma_{i_j})^m \leq \epsilon_i \ \text{mass}\sigma_i \ ,$$

where the numbers $\varepsilon_i > 0$ can be made arbitrarily small with sufficiently fine subdivisions of the chain c. Therefore, in the diffusion of the straightend chain $\sum_{ij} r_i \sigma_{ij}$ only the __fat__ simplices substantially contribute to the ℓ^1-norm and thus our theorem is proven.

__(B) A Generalization to__ $K(V) < K^2$. Let the sectional curvature $K(V)$ be bounded from above, $K(V) < K^2$. Then, for every point $v \in V$ the exponential map $\exp_v : T_v \to V$ is an immersion on the ball $B_0(K^{-1}\pi) \subset T_v(V)$ (see [10]) and we equip this ball with the metric induced from V. The resulting (non-complete!) Riemannian manifold is denoted by $\widetilde{B}_v$. The exponential map locally isometrically sends this $\widetilde{B}_v$ to V and the center $\tilde{v}$ of $\widetilde{B}_v$ is sent to v.

Now, every pair of points in the ball $B_{\tilde{v}}(R) \subset \widetilde{B}_v$ for $R \leq \frac{1}{2}K^{-1}\pi$, can be joined by a unique geodesic in $B_{\tilde{v}}(R)$ (see [10]) and so one has straight geodesic simplices in the balls $B_{\tilde{v}}(R)$ for all $v \in V$ and $R \leq \frac{1}{2}K^{-1}\pi$. The projections of these simplices to V are called __straight simplices of radius $\leq R$__.

Next, we modify the definition of $\ell'_v(R)$ by taking the balls $B_{\tilde{v}}(R)$ and by putting $\widetilde{\ell}_v(R) = \log \text{Vol } B_{\tilde{v}}(R)$. Then, we have the corresponding functions $\widetilde{\ell}'_v(R)$ and $[\widetilde{\ell}'(R)|U]$.

Finally, we claim that __the theorem (A) holds for all manifolds__ V __as long as__ $R < \frac{1}{2}K^{-1}\pi$. Namely, there is a cycle c' which is built of straight simplices of radius $\leq R$ and this c' satisfies the properties (1),(3) and (4) with $\widetilde{\ell}'$ in places of ℓ'.

__Proof__ : The smoothing we used in (A) is, in fact, a map which assigns to the points $v \in V$ measures in the balls $B_{\tilde{v}}(R)$ in the universal covering

$Y = \tilde{V} \rightarrow V$, for some lifts v of $\tilde{v}$. With this new interpretation, the smoothing makes perfect sense at the points $v \in V$, with the maps $\exp_v : \tilde{B}_v \rightarrow V$ instead of the universal covering. Finally, as the estimates we used in (A) were local, they extend to the balls $\tilde{B}_v$. Q.E.D.

(C) <u>Manifolds with</u> $|K| \leq 1$: <u>If</u> $K(V) \geq 1$ <u>then Ricci</u> $V \geq 1-n$ <u>and</u> <u>so</u> $\tilde{\ell}'_v(R) \leq \text{const}_n R^{-1}$ <u>for all</u> $R \leq \pi$ <u>and for all</u> $v \in V$. <u>Therefore, (B)</u> <u>above implies the estimate</u> $\text{Min Vol}(V) \leq \text{const}_n \|V\|$ <u>claimed in section</u> 0.5.

Next, we turn to

<u>The proof of the injectivity radius estimate of section 0.5.</u>

<u>Lemma</u> : <u>For every</u> $n = 1, \ldots,$ <u>there exists a positive constant</u> $\epsilon_n \in (0,1)$, <u>called Margulis' constant, such that the balls</u> $B_v(\epsilon)$ <u>in n-dimensional</u> <u>manifolds</u> V <u>with</u> $|K(V)| \leq 1$ <u>are "amenable" subsets in the concentric</u> <u>unite balls</u> $B_v(1) \supset B_v(\epsilon)$ <u>for all</u> $\epsilon \leq \epsilon_n$ <u>and for all</u> $v \in V$.

<u>Proof</u> : Indeed, the image of $\pi_1(B_v(\epsilon))$ in $\pi_1(B_v(1))$ contains a nilpotent subgroup of finite index (see [7]) .

Next, for $|K(V)| \leq 1$, the exponential map $\exp_v B_o(1) \rightarrow V$ covers each point in the ball $B_v(1/2) \subseteq V$ approximately ρ_v^{-1} times for $\rho_v = \text{Inj Rad}_v(V)$ and so $C \rho_v \leq \text{Vol } B_v(1) \leq C^{-1} \rho_v$, for all $v \in V$ and for some universal constant $C = C_n > 0$ (see [11]).

Finally, for manifolds V with $|K(V)| \leq 1$ and for all $\epsilon \geq 0$, the set $U_\epsilon \subseteq V$ of those points $v \in V$ for which $\rho_v \geq \epsilon$ contains the ϵ-neighbourhood of the set U_δ for $\delta \leq C'_n \epsilon$ and for another universal constant $C'_n > 0$. (see [11]) . We apply with these remarks, the covering technique of section 3.4 and we find for some universal positive

$\varepsilon = \varepsilon(n) \in (0, \varepsilon_n)$ a cover of V by small open subsets $U_j, j = 1, \ldots,$ with the following two properties

(1) All U_j are "amenable" subsets in their respective $\frac{1}{2}$-neighbourhood $B_{U_j}(1/2) \supset U_j$.

(2) The inequality $\rho_v \leq \varepsilon$ implies, for all $v \in V$, that the ball $B_v(3\varepsilon) \subset V$ is contained in one of the sets U_j and that at most $n = \dim V$ sets U_j intersect $B_v(3\varepsilon)$.

We represent the fundamental class of V by a cycle c' of the theorem (B) with $R = \varepsilon$ and we observe that with (1) and (2) above this c' splits into a sum, $c' = c_1 + c_2$, where the chain c_1 has its support in the set U_ε and $\|c_1\| \leq \text{const Vol}(U_\varepsilon)$ for some universal $\text{const} = \text{const}(\varepsilon, n)$, while each simplex σ of the chain c_2 intersects no more than n subsets U_j and each σ is contained in one of the sets U_j. Therefore, the locally finite diffusion of section 4.2 makes the ℓ^1-norm of c_2 as small as we wish and so the norm of the diffused cycle c' is bounded from above by $\text{const Vol}(U_\varepsilon)$. Q.E.D.

(D) <u>Manifolds with</u> $\text{Ricci}(V) \geq 1/1-n$. We start with another generalization of the growth function $\ell'(R)$. Take a ball in V of radius $\tilde{R} > R$, called $B_v(\tilde{R}) \subset V$, and let $\tilde{B}_v(\tilde{R}) \to B_v(\tilde{R})$ be the universal covering. Take a point $\tilde{v} \in \tilde{B}_v(\tilde{R})$ over v and consider the radius R ball around $\tilde{v}$, called $B_{\tilde{v}}(R) \subset \tilde{B}_v(\tilde{R})$. Put

$$\ell_v(R, \tilde{R}) = \log \text{Vol } B_{\tilde{v}}(R).$$

Observe that this function is decreasing in $\tilde{R}$ but it is not continuous : the jumps may happen as the topology of the ball $B_v(\tilde{R}) \subset V$ changes.

Next, we define the following "regularized" (compare with 2.5.)
derivative $\frac{d\ell}{dR}$,

$$\ell_v'(R,\widetilde{R}) = \lim_{\substack{\varepsilon\to 0,\widetilde{\varepsilon}\to 0 \\ w\to v}} \sup [\ell_w(R,\widetilde{R}\pm\widetilde{\varepsilon}) - \ell_w(R-\varepsilon,\widetilde{R}\pm\widetilde{\varepsilon})]$$

If $\mathrm{Ricci}\, V \geq 1/1-n$ then Bishop's inequality gives an universal upper
bound for $\ell_v'(R,\widetilde{R})$. Namely, $\ell_v'(R,\widetilde{R}) \leq \ell_o'(R)$ where ℓ_o' is the corresponding
function in the hyperbolic space. In particular, $\ell_v'(R,\widetilde{R}) \leq \mathrm{const}_n R^{-1}$ for
$R \leq 1$ and $\lim_{R\to\infty} \sup \ell_v'(R,\widetilde{R}) \leq 1$.

Then (compare with (A)), we put

$$[\ell'(R,\widetilde{R})|U] = \sup_{v,\overline{R}} \ell_v'(R,\overline{R}) ,$$

where v runs over the R-neighbourhood $B_U(R) \subset V$ of the set U and $\overline{R}$
runs over the interval $[R,\widetilde{R}]$.

Finally, for a singular simplex $\sigma : \Delta \to V$, we write $\mathrm{length}(\sigma) \leq \lambda$
if all edges of σ have length $\leq \lambda$. Furthermore, we write $\mathrm{Rad}\,\sigma \leq \rho$
if the image $\sigma(\Delta) \subset V$ is contained in a ball $B_v(\rho) \subset V$ for some $v \in V$.

Now, we come to our most general relation between the growth function
$\ell' = \ell'(R,\widetilde{R})$ and the ℓ^1-norm on the homology of locally finite chains
$c = \sum_i r_i \sigma_i$ in V.

Main Technical Theorem. Let V be a complete Riemannian manifold and let
$c = \sum_i r_i \sigma_i$ be a locally finite m-cycle in V. Then for arbitrary positive
numbers, $\varepsilon > 0$, $R > 0$ and $\widetilde{R} > R$, there exists another locally finite
m-cycle $c' = \sum_j r_j' \sigma_j'$ in V with the following four properties :

(1) c' is homologous to c .

(2) length $\sigma'_j \leq 2R+\epsilon$ and Rad $\sigma'_j \leq \widetilde{R}+2R+\epsilon$ for all $j = 1,\ldots$

(3) For all open sets $U \subset V$ we have with the notations $c'|U$ and $c \cap U+R+\epsilon$ of (A) and with $\ell' = [\ell'(R,\widetilde{R})|U]$

$$\|c'|U\|_{\ell^1} \leq (1+\epsilon)m!(\ell'+4(\widetilde{R}-R)^{-1})\text{mass}(c \cap U+\widetilde{R}+2R+\epsilon) .$$

(4) If c represents the fundamental class of V , then we have for

$$\overline{\ell}_v = \overline{\ell}_v(R,\widetilde{R}) = \underset{\overline{R}\in[R,\widetilde{R}]}{\text{Sup}} \ell'_v(R,\overline{R})$$

and for the $(\widetilde{R}+2R+\epsilon)$-neighbourhoods $B = B_U(\widetilde{R}+2R+\epsilon)$ of the sets U ,

$$\|c'|U\|_{\ell^1} \leq (1+\epsilon)n! \int_B (C_n\overline{\ell}_v+4(\widetilde{R}-R)^{-1})^n \, dv$$

for

$$C_n = \Gamma(n/2)/\sqrt{\pi} \, \Gamma(\frac{n+1}{2}) < 1 , \quad n = \dim V .$$

Proof : First, we formalize the local conception of smoothing (see (B)) as follows

An étale measure μ over V is defined by the following data :

(i) A collection of étale domains $(\widetilde{U},\widetilde{p})$ over V . These are n-dimensional manifolds U for $n = \dim V$ and immersions $\widetilde{p} : \widetilde{U} \to V$.

(ii) There are maps between some pairs of domains $(\widetilde{U},\widetilde{p})$ and $(\widetilde{U}'\widetilde{p}')$, that are immersions $\widetilde{q} : \widetilde{U} \to \widetilde{U}'$ such that $\widetilde{p} = \widetilde{p}' \circ q$. If such a map $\widetilde{q}$ exists, we say that $\widetilde{U}$ lies over $\widetilde{U}'$. We assume that for any two domains U_1 and

U_2 there is a third domain U which lies over both U_1 and U_2 .

(iii) Finally, each domain $\tilde{U}$ carries a positive measure, called $\tilde{\mu}$ on $\tilde{U}$. We require the measure $\tilde{\mu}'$ in all domains $\tilde{U}'$ over $\tilde{U}$ to be lifts of $\tilde{\mu}$, that is $\tilde{\mu} = \tilde{q}'_* \tilde{\mu}'$, where $\tilde{q}'_*$ denotes the push-forward map on measures for the immersions $\tilde{q}' : \tilde{U}' \to \tilde{U}$. Observe that $\int_{\tilde{U}'} \tilde{\mu}' = \int_{\tilde{U}} q'_* \tilde{\mu}$ and so for positive measures $\tilde{\mu}$ the L^1-norms $\int_{\tilde{U}} |\tilde{\mu}|$ are independent of the particular domain $\tilde{U}$. Then, one may speak of the L^1-norm $\|\mu\|$ of étale measure μ over V .

An étale smoothing $S = S(v)$ over V is a map which sends points $v \in V$ to étale measures $S(v)$ over V . An instance of that are measures defined in (B) on exponential balls $\tilde{B}_v(R) \to V$.

We assume furthermore, that for every points $v_o \in V$ there is an étale domain $\tilde{U} \to V$ such that the measures $\tilde{S}(v) = S(v)|\tilde{U}$ are simultaneously defined on $\tilde{U}$ for all v in some neighbourhood $U_o \subset V$ of v_o .

The map $\tilde{S}$ sends points of U_o to ordinary measures on $\tilde{U}$, and if this $\tilde{S}$ is smooth, then we define as in section 2.4. the quantities $[\tilde{S}(v)] = [S(v)|\tilde{U}]$ and $[\tilde{S}(v)]^* = [S(v)|\tilde{U}]^*$ for all $v \in U_o$. We call these $\tilde{U}$ domains of definition of S near v_o and we finally define $[S(v_o)]$ and $[S(v_o)]^*$ as $\inf_{\tilde{U}}[S(v_o)|\tilde{U}]$ and $\inf_{\tilde{U}}[S(v_o)|\tilde{U}]^*$ where $\tilde{U}$ runs over all domains of definition.

Next, we assign to each point $v \in V$ a neighbourhood of v , say $W_v \subset V$, and then we call a singular simplex $\sigma : \Delta \to V$ admissible if it has distinct vertices, $v_o,\ldots,v_m \in V$ and its image $\sigma(\Delta) \subset V$ is contained in the union $W_{v_o} \cup \ldots \cup W_{v_m}$. We take a subcomplex K in the canonical simplicial multicomplex $K(V)$ (compare with the proof of the vanishing theorem

in 3.3.)   with the following three properties :

(1) All simplices in K are admissible.

(2) K is complete : every admissible simplex with boundary in K is "admissibly" homotopic to a simplex in K .

(3) K is minimal : it contains no proper subcomplexes with the properties (1) and (2) .

Every such complex K is uniquely determined, up to an admissible isomorphism, by its zero skeleton $K^\circ \subset V$: an isomorphism $I : K \to K'$ is admissible if the canonical map $S : K \to V$ (see 3.2.) is "admissibly" homotopic to $S' : K' \to V$ with the zero-skeleton kept fixed. Observe, that the group Γ_1 of those admissible automorphisms of K which keep the one-skeleton fixed, is amenable on all skeletons of K (compare 3.3.) .

Finally, we associate to étale smoothings S over V diffusions of small simplices to chains in K . To do this, we need all domains of definition of the étale measures $S(v)$ to be connected and simply connected, for all $v \in V$ and also, we require the images $\widetilde{p}(\widetilde{U}) \subset V$ of all these domains to lie in the intersections $\bigcap_v W_v$ for v running over $\widetilde{p}(\widetilde{U})$.

Let Δ be a small straight simplex in V and let $\widetilde{\Delta} \subset \widetilde{U}$ be a lift of Δ to an étale domain, such that the measures $\widetilde{S}(v) = S(v)|\widetilde{U}$ are simultaneously defined for all $v \in \Delta$. Then the vertices $\widetilde{v}_o, \ldots \widetilde{v}_m$ of $\widetilde{\Delta}$ are diffused to the points $\widetilde{v}'_o, \ldots, \widetilde{v}'_m$ in $\widetilde{U}$ weighted with some measures according to $\widetilde{S}$. Each vertex $\widetilde{v}_k$, $k = 1, \ldots, m$ is joined with $\widetilde{v}'_k$ by a path called $\widetilde{e}_k$ and every edge $[\widetilde{v}_k, \widetilde{v}_\ell]$ of $\widetilde{\Delta}$ goes to the composition of paths

$$[\widetilde{v}'_k, \widetilde{v}'_\ell] = \widetilde{e}_k * [\widetilde{v}_k \widetilde{v}_\ell] * \widetilde{e}_\ell .$$

The projection of this path back to V , is an admissible singular one-simplex and so there is a unique one-simplex in K , called $[v'_k, v'_\ell]$ homotopic to the projected path $[\tilde{v}'_k, \tilde{v}'_\ell]$. Thus, we diffused Δ into a measure on the set of m-simplices in the quotient complex K/Γ_1 , that is a "smeared chain" in K/Γ_1 . It would be an actual finite chain if the measures $S(v_k)$, k = 0,...,m were finite. We agree to normalize this smeared chain by dividing it by the product $\prod_{k=0}^{m} \| S(v_k) \|$ and the result is denoted by $S*\Delta/\Gamma_1$.

Remark : Among the diffused simplices, there may be some with coinciding vertices and those do not belong to K . One could add such degenerate simplices to K , but we just ignore them since their total mass is zero for measures S(v) with continuous densities and this mass remains negligibly small for our further approximations of continuous measures by discrete ones.

Now, let $c = \sum_i r_i \Delta_i$ be a chain of small simplices Δ_i in V . Then with an étale smoothing S over V , we have the diffusion

$$c \longrightarrow S*c/\Gamma_1 = \sum_i r_i S*\Delta_i/\Gamma_1 .$$

We suppose that the measures $\tilde{S}(v)$ have continuous densities on $\tilde{U}$ and we approximate the diffusion $S*$ by discrete diffusions S_o^* . Namely, we consider discrete sets $Z \subset V$ with the weights $\mu(z)$ attached to all $z \in Z$ as in the proof of (A) and we replace the measures $\tilde{S}(v)$ on the domains $\tilde{p} : \tilde{U} \to V$ by the measures $S_o(v)$ supported on $\tilde{Z} = \tilde{p}^{-1}(Z) \subset \tilde{U}$ with weights $\tilde{S}_o(v, \tilde{z}) = \mu(z)\tilde{S}(v, \tilde{z})$ for all z in $\tilde{Z}$ over z . The diffused chains $c_o = S_o*c/\Gamma$ are countable chains in K/Γ_1 and they lift (not uniquely) to chains in K, as the homotopies of projected edges $[\tilde{v}'_k, \tilde{v}'_\ell]$

extend to admissible homotopies of Δ to simplices in K . Furthermore, one can choose these extended homotopies such that <u>cycles</u> in K/Γ_1 lift to <u>cycles</u> in K .

Observe that the lifted chains $\bar{c}_o$ lie in the subcomplex K_Z in K spanned by the set Z as the zero skeleton, $K_Z^o = Z$. If we assume $W_z \to \infty$ for $z \to \infty$, then the chains $\bar{c}_o$ are locally finite, that is their images in V under the canonical map $K_Z \to V$ are locally finite, provided the original chain c is locally finite.

Finally, for small fat m-simplices Δ near a point $v \in V$, the total mass of the chain $S_o * \Delta$ in K/Γ_1 is estimated from above by $(1+\delta)[S_o(v)]^m m!$ mass Δ as in the Proposition of (A) , provided the set $Z \subset V$ is sufficiently dense, and for all ϵ-simplices we have $\| S_o * \Delta \|_{\ell^1} \leq (1+\delta)[S(v)]^m \epsilon^m$ regardless of their fatness. We claim, that these estimates also hold for appropriate lifts of the chains c_o to K . Indeed, as the group Γ_1 is amenable in the dimension m (that is Γ_1/Γ_{m+1} is an amenable group see 3.3.) we can construct locally finite diffusion operators in K_Z (see 4.2.) , which send chains $\bar{c} = \sum_{j_v} \bar{r}_{j_v} \bar{\sigma}_{j_v}$ in K_Z over chains $\sum_j r_j \sigma_j$ in K_Z/Γ_1 with $\sum_v \bar{r}_j = r_j$, $j = 1,\ldots,$ to diffused chains $\hat{c} = \sum_{j_\mu} \hat{r}_{j_\mu} \hat{\sigma}_{j_\mu}$ over $\sum_j r_j \sigma_j$ such that the norms $\| \hat{c} | \sigma_j \|_{\ell^j} = \sum_\mu |\hat{r}_{j_\mu}|$ become as close to $|r_j| = |\sum_\mu \hat{r}_{j_\mu}|$ for all $j = 1 \ldots$. as we want.

The proof of the theorem is now reduced to the construction of étale smoothing operators S whose "norms" $[S]$ and $[S]^*$ are controlled by the growth function $\ell'(R,\tilde{R})$.

We take for étale domains the universal coverings of the open balls $B_v(R') \subset V$ for all $v \in V$ and $R' \in (R+\tilde{R}, R+\tilde{R}+\epsilon/2)$. We fix, in every such

covering $\widetilde{B}_v(R') \to B_v(R')$ a point $\widetilde{v} \in \widetilde{B}_v(R')$ over v . Then, for pairs of balls $B_v(R')$ and $B_{v''}(R'') \subset B_v(R')$ for $R + \widetilde{R} < R'' < R' - \text{dist}(v,v'')$ we allow for étale maps $\widetilde{B}_{v''}(R'') \to \widetilde{B}_v(R')$ only those which send the point $\widetilde{v}'' \in \widetilde{B}_{v''}(R'')$ to $\widetilde{v}' \in \widetilde{B}_v(R)$ with $\text{dist}(\widetilde{v}',\widetilde{v}) = \text{dist}(v'',v)$. We always assume these distances to be so small that this étale map is unique.

Next, we construct the following family of measures, called $\overline{S}_{R,\overline{R}}(v) = \overline{S}_{\overline{R}}(v)$ in the étale domains $\widetilde{B}_v(R')$ for all $\overline{R} \in [R,\widetilde{R}]$, by first taking the universal coverings of the balls $B_{\widetilde{v}}(\overline{R}) \subset \widetilde{B}_v(R')$, then by taking balls $B_{\overline{v}}(R)$ in these coverings $\widetilde{B}_{\widetilde{v}}(\overline{R})$ with the centers $\overline{v}$ over $\widetilde{v}$, and finally by considering the measure $\overline{S}$ on $\widetilde{B}_{\widetilde{v}}(\overline{R})$ which is given by the characteristic function ψ of the ball $B_{\overline{v}}(R) \subset \widetilde{B}_v(\overline{R})$, that is ψ equals one on the ball $B_{\overline{v}}(R)$ and $\psi = 0$ on $\widetilde{B}_{\widetilde{v}}(\overline{R}) \smallsetminus B_{\overline{v}}(R)$. The push-forward of $\overline{S}$ to $\widetilde{B}_v(R')$ is our measure $\overline{S}_{\overline{R}}(v)$. In order to have this $\overline{S}$ as an étale measure, we also take all its push-forwards under the étale maps of the coverings of the balls. Thus, we get the étale "smoothing" $\overline{S}_{\overline{R}}(v)$ for all $v \in V$; yet it is not smooth, not even continuous : the measures $\overline{S}_{\overline{R}}(v)$ may jump as the topology of the balls $B_{\widetilde{v}}(\overline{R})$ changes.

We regularize the "smoothings" $\overline{S}_{\overline{R}}$ by the following averaging :

$$S(v) = S_{R,\widetilde{R}}(v) = (\widetilde{R}-R)^{-1} \int_R^{\widetilde{R}} \overline{S}_{\overline{R}}(v)(a+b\overline{R})d\overline{R} \ ,$$

for a and b chosen such that $a + bR = 1$ and $a + b\widetilde{R} = 0$. Observe that the measures $\overline{S}_{\overline{R}}(v)$ are decreasing in $\overline{R}$ and so the measures $S_{R,\widetilde{R}}(V)$ are decreasing in $\widetilde{R}$ (but increasing in R) . Therefore,

$$\left\| \frac{d}{d\widetilde{R}} S_{R,\widetilde{R}}(v) \right\|_{L^1} = -\frac{d}{d\widetilde{R}} \left\| S_{R,\widetilde{R}}(v) \right\|_{L^1}$$

and a straightforward calculation yields

$$- \frac{d}{d\tilde{R}} \| S_{R,\tilde{R}}(v) \|_{L^1} \leq 4(\tilde{R}-R)^{-1} \| S_{R,\tilde{R}}(v) \|_{L^1} .$$

Then, we observe that $\| \bar{S}_{\bar{R}}(v) \|_{L^1} = \text{Vol } B_{\bar{v}}(R)$, and that

$$\| DS_{R,\tilde{R}}(v) \|_{L^1} \leq \| \frac{d}{dR} S_{R,\tilde{R}}(v) \|_{L^1} + \| \frac{d}{dR} S_{R,\tilde{R}}(v) \|_{L^1} .$$

Finally, we conclude to the desired inequality

$$[S_{R,\tilde{R}}(v)] \leq 4(\tilde{R}-R)^{-1} + \bar{\ell}_v(R,\tilde{R})$$

(see (4) of the theorem for the notations), and in the same way we get

$$[S_{R,\tilde{R}}(v)]^\bullet \leq 4(\tilde{R}-R)^{-1} + C_n \bar{\ell}_v(R,\tilde{R}) .$$

This smoothing only needs a minor change : a regularization in R with a fast decaying function $\varphi(v)$ (see the proof of the theorem in 2.5.) and then we are ready to conclude the proof. We take the balls $B_v(R+\tilde{R}+\epsilon/2) \subset V$ for the sets W_v and we take a complex K such that every one-simplex σ in K which is admissibly homotopic to a simplex of length $2R+\epsilon$, itself has length $< 2R+\epsilon$. Then, diffusions of small $\epsilon/2$-simplices Δ will have length $< 2R+\epsilon$ and the "admissibility" of simplices in K make their radii $\leq \tilde{R}+2R+\epsilon$.

Next, before diffusing the cycle c ,we first subdivide and then straighten its simplices, thus making the majority of them small and fat, and finally we diffuse them to the subcomplex $K_Z \subset K$ for a sufficiently

dense discrete set $Z \subset V$.

Corollary: <u>Proof of the main inequality of section 0.5.</u> If Ricci $\geq 1/1-n$
then $\lim\limits_{R \to \infty} \sup \ell'(R,\tilde{R}) \leq 1$ and if also $\tilde{R}-R \longrightarrow \infty$, then for $U = V$ the
inequality (4) of the theorem implies for $\epsilon \longrightarrow 0$,

$$\| V \| \leq C_n \ n! \ \text{Vol}(V) < n! \text{Vol}(V) \ .$$

4.4. Geometric simplicial volumes.

Suppose, we have fixed a notion of "size" for singular simplices
$\sigma : \Delta \longrightarrow V$ for Riemannian manifolds V . We have already met such "sizes"
as $\text{mass}(\sigma)$, $\text{Rad}(\sigma)$ and $\text{length}(\sigma)$. Furthermore, one may consider lifts of
Δ to some covering of V (or of open subsets of V which contain $\sigma(\Delta)$)
and then we have "Rad" for these lifts. Another class of sizes appears if
we fix a metric in the standard simplex Δ . Then we have the Lipschitz
constant of $\sigma : \Delta \longrightarrow V$, called $\text{Dil}(\sigma)$ that is the "sup"-norm of the
differential of σ . One also has such norms of the exterior powers
$\Lambda^k D\sigma$, $k = 1,\ldots, \dim \Delta$. Furthermore, there are the Dirichlet functionals
that are the integrals $\int\limits_{\Delta} \| \Lambda^k D\sigma(x) \|^{\ell} dx$.

For a chain $c = \sum\limits_i r_i \sigma_i$, we put "size" $(c) = \sup\limits_i$ "size" (σ_i) and
for all real $s > 0$ we define

$$\| V | \text{ "size" } \leq s \| = \inf\limits_c \| c \|_{\ell^1}$$

where c runs over all locally finite cycles of "size" $\leq s$ which represent
the fundamental class of V . We set

$$\| V | \text{"size"} < \infty \| = \inf\limits_{s>0} \| V | \text{"size"} \leq s \|$$

This simplicial volume is not a topological invariant any more; however it is invariant under uniformly Lipschitz homeomorphisms. Moreover, for proper Lipschitz maps $f : V \to V'$ for which $\text{dist}(f(x),f(y)) \leq \text{const dist}(x,y)$ for all x and y in V the basic functorial inequality remains true

$$\| V |"size" < \infty \| \geq |\deg f| \; \| V' |"size" < \infty \| \; .$$

It is clear that $\| V \| \leq \| V |"size" < \infty \|$, and that for any reasonable "size", we have the equality for compact V . However, these geometric simplicial volumes may lead to new topological invariant of compact manifolds. Namely, fix any sequence of positive numbers, $\Lambda = \{\lambda_i\}$ $i = 1,\ldots,$ and then for a compact manifold V with a metric g denote by V^Λ the disjoint union of the Riemannian manifolds $(V,\lambda_i g)$. Since the Lipschitz class of V^Λ only depends on Λ but not on g , we have the whole spectrum of invariants

$$\| V^\Lambda |"size" < \infty \| \; ,$$

one invariant for every Λ and every "size". In fact, these invariants tend to take only two values : zero and infinity, and then the actual invariant is the partition of all sequences Λ into the two classes.

The theorems (A) and (B) of the previous section give upper bounds not only for the topological simplicial volume, but also for $\| V |"size" \leq s \|$ for $"size"(\sigma) = \text{Di}\ell(\sigma)$ while the more general theorem (D) applies to a "weaker size", namely to length + Rad.

Now, let us consider only those fundamental cycles $c = \sum_i r_i \sigma_i$ for which $"size"\sigma_i \to 0$ for $i \to \infty$. By taking the infimum of the norms

$\|c\|_{\ell^1} = \Sigma_i |r_i|$ over all these cycles c we now get another Lipschitz

invariant simplicial volume, called

$$\| V \| \text{ "size"} \to 0 \| \geq \| V | \text{ "size"} < \infty \| \quad .$$

The arguments of the previous section can be extended to this simplicial

volume as well. Namely, instead of our smoothing S_R with a fixed R , one

could take a function $R(v)$ with a small gradient and such that $R(v) \to 0$

for $v \to \infty$. We do not need here this generalization, but we are going to

use the simplicial volume $\|V|\text{Rad} \to 0\|$ for proving the existence of com-

plete manifolds V of all dimensions n that are __extremal__ in the following

sense : extremal manifolds $V = (V,g)$ by definition have metrics g of

class C^1 with measurable curvature $K(g)$ and $|K(V)| \overset{\text{def}}{=} \| K(g) \|_{L^\infty} \leq 1$.

Furthermore, $\text{Vol}(V) < \infty$ and all complete manifolds V' which admit proper

Lipschitz maps $V' \to V$ of non-zero degrees abide : $\text{Vol}(V') \geq \text{Vol}(V)$.

In particular, there is no bounded deformations of the metric g of V

which keep $|K| \leq 1$ and which decrease the volume.

__Theorem__ : There exist extremal manifolds of all dimensions n .

__Remarks__. (a) It is unclear if there are compact, not just complete, extremal

manifolds of all dimensions. Nor do we know how many different extremal

manifolds there are for a given dimension.

(b) It appears more natural to define extremal manifolds with arbitrary

homeomorphisms $f : V' \to V$ rather than with Lipschitz maps. This notion

has been discussed for surfaces in section 0.1. In this discussion we omitted

surfaces homeomorphic to $\mathbb{R}^2$. In fact, such surfaces do admit extremal

metrics, but they are never smooth. Indeed, at every point where the curvature

is continuous, it must be itself extremal, that is $K = \pm 1$, and so there must be a jump from $+1$ to -1 .

It is unclear if there are such topological extremal metrics on manifolds of dimensions > 2 . Nor is it clear how to relax the condition

$|K| \leq 1$ to Ricci ≥ -1 .

Proof : The theorem is an immediate corollary of the following

Sub-theorem : For an arbitrary number $h > 0$ there exists a complete Riemannian manifold $\overline{V}$ of a given dimension n with the following three properties :

(a) $|K(\overline{V})| \leq 1$ and $\mathrm{Vol}(\overline{V}) < \infty$,

(b) $\|\overline{V} | \mathrm{Rad} \to 0\| \geq h$,

(c) All complete n-dimensional manifolds V with $|K(V)| \leq 1$ and with $\|V | \mathrm{rad} \to 0\| \geq h$ abide :

$$\mathrm{Vol}(V) \geq \mathrm{Vol}(\overline{V}) .$$

Proof : Consider all complete n-dimensional manifolds V for which

$$|K(V)| \leq 1 \text{ and } \|V | \mathrm{Rad} \to 0\| \geq h$$

and let us show that the functional $V \to \mathrm{Vol}\, V$ assumes its minimum on this set of manifolds. Let Vol_h denotes the infimum of this functional and let V_i be a minimizing sequence, that is $\mathrm{Vol}(V_i) \to \mathrm{Vol}_h$ for $i \to \infty$. As $|K(V_i)| \leq 1$ and $\mathrm{Vol}(V_i) \leq \mathrm{const}$, one may assume (by taking a subsequence

if necessary) that the sequence V_i has the "limit", that is a complete manifold $\overline{V}$ with the following property (see [25]) :

There is an increasing sequence of relatively compact open subsets $\overline{U}_i \subset \overline{V}$, such that $\underset{i}{\cup}\, \overline{U}_i = \overline{V}$, and a sequence of maps $\overline{g}_i : \overline{U}_i \rightarrow V_i$ whose distortions converge to zero,

$$\sup_{x,y \in U_i} \left| \frac{\text{dist}(\overline{g}_i(x), \overline{g}_i(y))}{\text{dist}(x,y)} - 1 \right| \underset{i \to \infty}{\longrightarrow} 0 .$$

and such that the injectivity radii of the complements to the images of the maps $\overline{g}_i$ go to zero,

$$\sup_{v \in W_i} \text{Inj. Rad}_v(V_i) \underset{i \to \infty}{\longrightarrow} 0 , \quad \text{for} \quad W_i = V_i \smallsetminus \overline{g}_i(\overline{U}_i)$$

This limit manifold $\overline{V}$ clearly has $\text{Vol}(\overline{V}) \leq \underset{i \to \infty}{\lim} \text{Vol}(V_i) = \text{Vol}_h$, and also $|K(V)| \leq 1$, though this $\overline{V}$ may not be C^2-smooth. Observe that the manifold $\overline{V}$ may be empty : this happens if $\text{Inj. Rad}(V_i) \to 0$ for $i \to \infty$.

Our sub-theorem now is reduced to the inequality

$$\| \overline{V} | \text{Rad} \to 0 \| \geq \underset{i \to \infty}{\lim} \| V_i | \text{Rad} \to 0 \| . \tag{$*$}$$

To prove ($*$), take a fundamental cycle $\overline{c} = \underset{k}{\Sigma}\, \overline{r}_k\, \overline{\sigma}_k$ in $\overline{V}$ for which $\text{Rad}(\sigma_k) \underset{k \to \infty}{\longrightarrow} 0$ and let us show that all manifolds V_i for $i \geq j = j(\overline{c})$ support fundamental cycles $c(i) = \underset{\ell}{\Sigma}\, r_\ell(i)\, \sigma_\ell(i)$, also with $\text{Rad}\, \sigma_\ell(i) \underset{\ell \to \infty}{\longrightarrow} 0$, such that $\| c(i) \|_{\ell^1} \leq \| \overline{c} \|_{\ell^1} + i^{-1}$ for all $i \geq j$. Since all manifolds V_i have $\text{Inj. Rad}_v(V_i) \underset{v \to \infty}{\longrightarrow} 0$ they admit coverings by open subsets $U_\mu(i) \subset V_i$ with the following three properties (compare with (C) of 4.3.) :

(1) Every set $U_\mu(i)$ is contained in a ball $B_\mu(i)$ in V_i of radius $R_\mu(i)$ and all $U_\mu(i)$ are amenable subsets in $B_\mu(i)$. Furthermore, for each i we have $R_\mu(i) \to 0$ as $\mu \to \infty$.

(2) For every $i \geq i_o = i_o(\{\overline{U}_i, \overline{g}_i\})$ the multiplicity of the covering $\{U_\mu(i)\}$ outside the image $\overline{g}_i(U_{i_o}) \subset V_i$ does not exceed $n = \dim V = \dim V_i$, that is every point $v \in V_i \smallsetminus g_i(U_{i_o})$, for all $i \geq i_o$, is contained in at most n subsets $U_\mu(i)$.

(3) If a simplex $\overline{\sigma}_k$ of the chain $\overline{c}$ is not contained in the set $\overline{U}_{i_1} \subset \overline{V}$, for some large $i_1 = i_1(\overline{c})$, then the maps $\overline{g}_i$, for all $i \geq i_1$, send $\overline{\sigma}_k$ to one of the subsets $U_\mu(i) \subset V_i$ and the image $\overline{g}_i(\overline{\sigma}_k) \subset V_i$ intersects at most n subsets $U_\mu(i)$.

Now, take a set $\overline{U}_{i_2} \subset \overline{V}$ for some i_2 much greater than i_o and i_1 and send the restricted chain $c|\overline{U}_{i_2}$ to the manifolds V_i, for large $i \geq i_2$, by the maps $\overline{g}_i$. We extend the resulting chains $\overline{g}_i(c|U_{i_2})$ to the fundamental cycles $c'(i)$ in V_i such that all new simplices of $c'(i)$, that are the simplices σ of the difference $c'(i) - g_i(\overline{c}|\overline{U}_{i_2})$, lie far away from $g_i(U_{i_o}) \subset V_i$, and such that every new simplex σ is contained in one of the sets $U_\mu(i)$ for all $i \geq i_2$ and σ intersect no more than n sets $U_\mu(i)$.

We apply the locally finite diffusion (see 4.2.) to the cycle $c'(i)$ in V_i for each $i \geq i_2$. This diffusion "annihilates" all new simplices σ of the chains $c'(i)$ (compare (C) of 4.3.) while the norms of the chains $g_i(c,|U_{i_2})$ can only decrease under the diffusion. In other words, one can diffuse the cycles $c'(i)$ to cycles $c(i)$ such that

$$\|c(i)\|_\ell \leq \|\overline{g}_i(c|U_{i_2})\| + \varepsilon_i$$

for arbitrary $\varepsilon_i > 0$ and for all $i \geq i_2$. Since $\| \bar{g}_i (c | \bar{U}_{i_2}) \| \leq \| \bar{c} \|$, we make $\| c(i) \|_{\ell^1} \leq \| \bar{c} \|_{\ell^1} + \varepsilon_i$ for $\varepsilon_i = i^{-1}$. Finally, according to (1) above, we have $R_\mu (i) \underset{\mu \to \infty}{\longrightarrow} 0$ and so the radii of the simplices of all chains $c(i)$ for $i \geq i_2$ also go to zero. Q.E.D.

4.5. Specific simplicial volumes.

We now want to apply the results of section 4.3. to complete manifolds of _infinite_ volume. First, for an open subset $U \subset V$ and for a positive number $R > 0$, we define the R-interior of U , denoted $U-R \subset U$, as the set of all those points $u \in U$ for which the closed balls $B_u(R) \subset V$ are contained in U . Observe, that this definition uses only the induced Riemannian structure of U and so one may speak of R-interiors of all abstractly defined non-complete manifolds U . In fact, if U is a _complete_ manifold, then $U-R = U$ for all $R \geq 0$.

Next, manifolds U of finite total volumes are called R-stable if $\mathrm{Vol}(U-R) > (1-R^{-1}) \mathrm{Vol}\, U$. Furthermore, a sequence of manifolds U_i, $i = 1, \ldots,$ is called stable if U_i are R_i-stable for $R_i \to \infty$ as $i \to \infty$.

Examples. (a) The sequences of Euclidean balls of radii $\to \infty$ are stable, while such sequences of balls in hyperbolic spaces are not stable.

(b) Let V be a closed manifold and let $\tilde{V} \to V$ be a regular covering with the Galois group Π . Then $\tilde{V}$ admits a stable sequence of open subsets, $\tilde{U}_i \subset \tilde{V}$, if and only if the group Π is amenable (see [21]). Furthermore, let V be an arbitrary manifold and let the group Π be amenable. Then $\tilde{V}$ admits a stable sequence if and only if V does. In fact, if $\tilde{U} \to U$ is an artibrary locally isometric map and if $\tilde{U}$ is R-stable, then U contains an R-stable open subset. Indeed, let $\tilde{U}_k \subset U$ be the sets

of those points in U which have at least k-points in their pullbacks in $\tilde{U}$. Then $(\tilde{U}-R)_k \subset \tilde{U}_k - R$ and so

$$\sum_k \text{Vol}(\tilde{U}_k - R) \geq \sum_k \text{Vol}(\tilde{U}-R)_k =$$

$$= \text{Vol}(\tilde{U}-R) \geq (1-R^{-1})\text{Vol }\tilde{U} = (1-R^{-1}) \sum_k \text{Vol }\tilde{U}_k \quad .$$

Therefore, one of the sets $\tilde{U}_k \subset \tilde{U}$ is R-stable.

Now, for a given map between two n-dimensional manifolds, $f : V_1 \to V_2$, we want to define "the ratio of the volumes", $\text{Vol}(V_1)/\text{Vol}(V_2)$, in the case of infinite volumes, $\text{Vol}(V_1)$ and (or) $\text{Vol}(V_2)$. First, for all relatively compact open subsets $U \subset V_2$, we put

$$(U,R) = \text{Vol}(f^{-1}(U-R))/\text{Vol}(U) ,$$

and then

$$\text{Vol}_+(V_1:V_2;f) = \inf_{R>0} \sup_{U \subset V} (U,R) .$$

Next, we define $\text{Vol}_-(V_1:V_2;f)$ as the lower bound of those numbers $\nu \geq 0$ for which there exists a <u>stable</u> sequence $U_i \subset V$ of relatively compact open subsets with the property

$$\inf_{R \geq 0} \sup_{i=1,2,\ldots} (U_i,R) \leq \nu$$

This definition is only meaningful if stable sequences do exist.

<u>Examples</u> : If V_1 and V_2 are closed manifolds then

$$\text{Vol}_+ = \text{Vol}_- = \text{Vol}(V_1)/\text{Vol}(V_2) .$$

For the identity map, $V \to V$, one has $Vol_+ = Vol_- = 1$ if V admits a stable sequence of open subsets, $U_i \subset V$. Otherwise, $Vol_+ = 0$.

For linear maps $f : \mathbb{R}^n \to \mathbb{R}^n$ we have $Vol_+ = Vol_- = (Det\ f)^{-1}$.

If $f : V_1 \to V_2$ is a proper locally diffeomorphic map (i.e. a finite covering) whose Jacobian is bounded everywhere from below by $J_o > 0$, then $Vol_+ \leq (\deg f)/J_o$.

Also observe that Vol_+ and Vol_- are stable under bounded perturbations of maps : if $dist(f(x), f'(x)) \leq const$ for all $x \in V_1$ then the maps f and f' have equal "volumes".

Our main purpose is to estimate Vol_+ and Vol_- from below in terms of appropriate specific (simplicial) volumes, which are defined as follows : First we define for fundamental cycles $c = \sum_j r_j \sigma_j$ of a Riemannian manifold V the quantity (see 4.3. for the notations)

$$\|c : Vol\|_+ = \inf_{R > 0} \sup_{U \subset V} (\|c|U-R\|_{\ell^1} (Vol\ U)^{-1}).$$

Then we define $\|c : Vol\|_-$ by taking only the "sup" over some stable sequences of sets U, namely we take for this $\|\ \|_-$ the lower bound of those numbers ν for which there exists a stable sequence of relatively compact open subsets $U_i \subset V$, such that

$$\inf_{R > 0} \sup_i (\|c|U_i - R\|_{\ell^1} (Vol\ U_i)^{-1}) \leq \nu.$$

Finally, we fix some notion of "size" (see 4.4.) and then we minimize the above norms over all cycles c with "size"$(c) = \sup_j$ "size"$(\sigma_j) < \infty$,

$$\|V : Vol\|_+ = \inf_c \|c : Vol\|_+.$$

and

$$\|V : Vol\|_- = \inf_c \|c: Vol\|_- \ .$$

Examples : If V is a closed manifold then

$$\|V : Vol\|_+ = \|V : Vol\|_- = \|V\| \ (Vol \ V)^{-1} \ .$$

If V is a complete manifold of finite volume, then

$$\|V : Vol\|_+ \geq \|V|"size" < \infty \|/Vol \ V \geq \|V : Vol\|_- \ .$$

Specific volumes of proper Lipschitz maps $f : V_1 \to V_2$ have the following functorial properties

$$|\deg(f)| \ \|V_2 : Vol\|_+ \leq Vol_+(V_1:V_2;f)\|V_1 : Vol\|_+$$

and

$$|\deg(f)| \ \|V_2 : Vol\|_- \leq Vol_-(V_1:V_2;f)\|V_1 : Vol\|_+ \ .$$

If the manifolds V_1 and V_2 are closed, then both inequalities are equivalent to

$$|\deg(f)|\|V_2\| \leq \|V_1\| \ .$$

To prove the above inequalities, we notice that Lipschitz maps f preserve chains of "size" $< \infty$ and that

$$f^{-1}(U-R) \subset f^{-1}(U) - LR$$

for $L = L_f$ = the lipschitz constant of f .

Observe that the second inequality may not hold with $\| \ \|_-$ on the left hand side.

These functorial inequalities yield, with some estimates for specific volumes, lower bounds for Vol_+ and Vol_-. For example, if a specific volume of a Riemannian manifold V is different from zero as well as from infinity, then every bi-Lipschitz self-homeomorphism $f : V \longrightarrow V$ with a constant Jacobian $J = J_f$ satisfies $|J| = 1$.

We start our estimates of $\| \ \|_+$ and $\| \ \|_-$ with several elementary observations.

Let $g : V \longrightarrow V'$ be a locally isometric covering map and suppose that lifts to V of chains c' in V' with "size"$(c') < \infty$ also have finite "size" in V. For example, this happens to "size" = dil . Then, clearly

$$\| V : Vol \|_+ \leq \| V' : Vol \|_+ .$$

Furthermore, if g is a regular covering with an amenable Galois group, then also

$$\| V : Vol \|_- \leq \| V' : Vol \|_- .$$

Next, let V' be a closed manifold and let $g : V \longrightarrow V'$ be a Lipschitz map whose Jacobian satisfies for some stable sequence of subsets $U_i \subset V$:

$$\lim_{i \to \infty} ((Vol \ U_i)^{-1} \int_{U_i} J_g) = J_o > 0 .$$

Then for "size" = dil we claim :

$$\| V : \text{Vol} \|_+ \geq J_o \| V' \| / \text{Vol}(V') \ .$$

Furthermore, <u>if all stable sequences</u> U_i <u>satisfy</u>

$$\liminf_{i \to \infty} (\text{Vol } U_i)^{-1} \int_{U_i} J_g \geq J_o \ ,$$

then also

$$\| V : \text{Vol} \|_- \geq J_o \| V' \| / \text{Vol } V' \ .$$

<u>In particular, for regular coverings</u> $g : V \to V'$ <u>with amenable Galois</u>
<u>groups, we get for this</u> "size" = dil ,

$$\| V : \text{Vol} \|_+ = \| V : \text{Vol} \|_- = \| V' \| / \text{Vol}(V') \ .$$

<u>Proof</u> : Take a fundamental cycle $c = \sum_j r_j \sigma_j$ of V with $\sup_j \text{dil } \sigma_j < \infty$
and then restrict it to the sets $U_i - R$ for some large R . We send these
restricted chains to V' and call the images $c'(i)$. Since the sequence
U_i is stable and since all simplices of the chains $c'(i)$ have uniformly
bounded "size" = dil , we can extend these $c'(i)$ to <u>cycles</u> $c(i)$, also
with $\sup_i \text{dil}(c(i)) < \infty$, such that

$$\| c(i) - c'(i) \|_{\ell^1} / \text{Vol}(U_i) \underset{i \to \infty}{\to} 0 \ . \tag{$*$}$$

Now, the homology classes $h_i = \left[\left(\int_{U_i} J_g \right)^{-1} c(i) \right]$ converge to $[V'] / \text{Vol}(V')$
and so

$$\liminf_{i \to \infty} \| h_i \|_{\ell^1} \geq \| [V'] \|_{\ell^1} / \text{Vol}(V') = \| V' \| / \text{Vol}(V') \ .$$

Finally, as

$$[c(i)/\text{Vol}(U_i)] = J_o h_i ,$$

and as $\|c|U_i\|_{\ell^1} = \|c'(i)\|_{\ell^1}$, we conclude with (+) above to

$$\liminf_{i \to \infty} \|c| U_i - R \|_{\ell^1}/\text{Vol } U_i \geq \| V' \| /\text{Vol } V' ,$$

and so, $\| V : \text{Vol}\|_+ \geq \|V'\|/\text{Vol}(V')$. As the same argument applies to $\| \|_-$, the proof is finished.

Let us indicate a generalization of the equality concerning the amenable coverings $g : V \to V'$. First, we call a manifold V <u>stable at infinity</u> if it contains a stable sequence of relatively compact open subsets $U_i \subset V$.

<u>Example</u>. Let V be complete and let $B_v(R)$ be concentric balls around a fixed point $v \in V$. If

$$\liminf_{R \to \infty} R^{-1} \log B_v(R) = 0 ,$$

then V is stable at infinity. Indeed, $B_v(R-R') \subset B_v(R)-R'$ for all $R' < R$ and so some sequence of balls is stable.

<u>Proportionality theorem</u> : <u>If the universal coverings of manifolds</u> V <u>and</u> V' <u>are isometric and if</u> V' <u>is a closed manifold, then, for "size" = dil,</u>

$$\| V : \text{Vol}\|_+ \leq \|V'\|/\text{Vol}(V') .$$

<u>Furthermore, if</u> V <u>is stable at infinity then</u>

$$\| V : \text{Vol} \|_+ = \| V : \text{Vol} \|_- = \| V' \| / \text{Vol}(V') \ .$$

Proof : Let G denote the isometry group of the universal covering $\tilde{V} = \tilde{V}'$. Then every finite singular chain in V as well as an V' can be smeared (see [47]) with the Haar measure in G to a generalized G-invariant cycle in V (see 2.2.) . In particular, fundamental cycles c' of V' smear to cycles $\tilde{c}'$ in $\tilde{V}$ and these define smeared fundamental cycles c in V which can be approximated by usual cycles as in (2.2.) with their ℓ^1-norms on all sets $U \subset V$ roughly proportional to $\| c' \| \text{Vol}(U)/\text{Vol}(V')$. This proves the inequality $\| V : \text{Vol} \|_+ \leq \| V' \|/\text{Vol} V'$.

Next, we take a stable sequence $U_i \subset V$ and a fundamental cycle c of V for which the limit

$$\lim_{i \to \infty} (\text{Vol } U_i)^{-1} \| c | U_i - R \|_{\ell^1}$$

is close to $\| V : \text{Vol} \|_-$. Then, the chains $(\text{Vol}(U_i))^{-1}(c|U_i-R)$ in V are first smeared to G-invariant chains in $\tilde{V}$ and then as before we have chains c'(i) in V' whose boundaries $\partial c'(i)$ have $\| \partial c'(i) \|_{\ell^1} \xrightarrow[i \to \infty]{} 0$. As the chains c(i) have uniformly bounded "size" = dil(c'(i)) , one can extend them, as we did it above (also see 2.2.) to cycles c(i) such that [c(i)] = [V']/Vol V' and such that the norms $\| c(i) \|_{\ell^1}$ are close to the norms $\| c'(i) \|_{\ell^1}$ which are in their turn close to the norms $\| c | U_i - R \|_{\ell^1} / \text{Vol } U_i$. Thus, in the limit, we come to the required inequality $\| V' \|/\text{Vol } V' \leq \| V : \text{Vol} \|_-$. Q.E.D.

Remarks and corollaries. (1) If the manifolds V_1 and V_2 are both stable at infinity and if their universal coverings are isometric, then we can

conclude to the equality

$$\|V_1 : Vol\|_+ = \|V_2 : Vol\|_+ = \|V_1 : Vol\|_- = \|V_2 : Vol\|_- .$$

provided, there is a third manifold V' which is <u>closed</u> and whose universal covering is isometric to those of V_1 and V_2 . This last condition (the existence of V') cannot be dropped but probably it can be somehow relaxed.

(2) The isometry between the universal coverings, $\tilde{V} \simeq \tilde{V}'$, can be replaced by the following weaker condition : the existence of a locally isometric map of $\tilde{V}$ to $\tilde{V}'$. With this new condition, the proportionality theorem even holds for non-complete manifolds V . In fact, our proof for complete manifolds extends to the general case only with minor changes.

<u>Digression : Euler's characteristic and signature.</u>

For a subset $U \subset V$ we denote by $H_c(U;F)$ the cohomology with compact supports for some coefficient field F and we denote by $b^m(U \subset V;F)$ the ranks of the inclusion homomorphisms $H_c^m(U;F) \rightarrow H^m(V;F)$ for $m = 0,1,\ldots,n = \dim V$. We denote by $b_m(U \subset V;F)$ the ranks of the homomor- phisms $H_m(U;F) \rightarrow H_m(V;F)$ and we abbreviate : $b^m(U;F) = b(U \subset U;F)$ and $b_m(U;F) = b_m(U \subset U;F)$.

Now, we assume that the universal covering of the manifold V admits a locally isometric map into a closed manifold V' and we take a stable sequence of relatively compact open subsets $U_i \subset V$, $i = 1,\ldots$. We assume furthermore that the sequence U_i has <u>bounded geometry</u> : there is a number $\epsilon \in (0,1)$ such that $\mathrm{Inj.Rad}_v(V) \geq \epsilon$ for all $v \in U_i - \epsilon^{-1}$ and for all $i = 1,\ldots$.

<u>Proportionality Proposition.</u> The "<u>Euler characteristics</u>"
$$\chi(i) = \sum_{m=0}^{n} (-1) b_m((U_i - R) \subset V;F) \quad \text{and} \quad \bar{\chi}(i) = \sum_{m=0}^{n} (-1)^m b^m((U_i - R) \subset V;F) \quad \text{satisfy}$$

for every $R \geq 2\epsilon^{-1}$,

$$\lim_{i \to \infty} \chi(i)/\text{Vol}(U_i) = \lim_{i \to \infty} \overline{\chi}(i)/\text{Vol}(U_i) = \chi(V')/\text{Vol}(V') .$$

Proof. As the geometry of U_i is bounded there are compact submanifolds U_i' in V such that $U_i - R \subset U_i' \subset U_i - R/2$, and such that the boundaries $\partial U_i'$ are smooth hypersurfaces in V whose principal curvatures are bounded in absolute value by a constant $C = C(V', \epsilon)$. It follows that $\text{Vol}(U_i')/\text{Vol}\ U_i \to 0$ for $i \to \infty$ and we obtain with the Gauss-Bonnet theorem for manifolds with boundaries,

$$\chi(U_i')/\text{Vol}(U_i) \longrightarrow \chi(V')/\text{Vol}(V') \quad \text{as} \quad i \to \infty .$$

Next, we observe that the geometry of the closed manifolds $\partial U_i'$ is bounded and so $b_m(\partial U_i'; F) \leq \text{const}\ \text{Vol}(\partial U_i')$, for some constant which only depends on V' and C . As the images of the inclusion homomorphisms $H_m(\partial U_i'; F) \to H_m(U_i; F)$ contain the kernels of the homomorphisms $H_m(U_i'; F) \to H_m(V; F)$, we get

$$(b_m(U_i'; F) - b_m(U_i' \subset V; F))/\text{Vol}(U_i) \to 0 ,$$

and so for every $R \geq 2\epsilon^{-1}$,

$$[b_m(U_i'; F) - b_m((U_i - R) \subset V; F))]/\text{Vol}\ U_i \to 0 .$$

This implies $\lim\limits_{i \to \infty} \chi(i)/\text{Vol}(U_i) = \chi(V')/\text{Vol}(V')$. We obtain in the same way,

$$[b_m(U_i'; F) - b^m((U_i - R) \subset V; F))]/\text{Vol}(U_i) \to 0 .$$

and conclude to the second relation

$$\lim \bar{\chi}(i)/\mathrm{Vol}(U_i) = \chi(V')/\mathrm{Vol}(V') \; .$$

Remark. Our proof is a slight modification of an unpublished argument of Morgan and Phillips.

Corollaries. Let V be an infinite Galois covering of V'. Denote by $\chi^{ev}(V)$ the sum of the Betti numbers $b^m(V;F)$ over all even numbers m and let χ^{odd} be the sum of the odd Betti numbers. Observe that the inequality $b^m(V;F) \neq 0$ for some m implies $b^m(V;F) = \infty$. Moreover, for any stable sequence U_i this inequality implies : $\lim_{i \to \infty} b^m(U_i \subset V;F)/\mathrm{Vol}(U_i) > 0$.

Now, let the covering $V \to V'$ have an amenable Galois group. Then the proposition implies,

(A). If $\chi(V') > 0$, then $\chi^{ev}(V) = \infty$; if $\chi(V') < 0$, then $\chi^{odd}(V) = \infty$

Furthermore, with the observation above we get

(A') If $\chi(V') = 0$ and $\chi^{ev}(V) \neq 0$ then $\chi^{odd}(V) = \infty$.

The corollaries (A) and (A') hold for an arbitrary finite polyhedron P' in place of the manifold V. In fact, the argument is even shorter in the combinatorial case. As an application we obtain a short proof of the following result of Baumslug-Pride [3].

(B) Theorem. Let an abstract group Γ be presented by p generators and q relations $\Gamma = \{\gamma_1, \ldots, \gamma_p | w_1, \ldots, w_q\}$. If $q \leq p-2$ then the group Γ is large : it contains a subgroup of finite index, $\Gamma' \subset \Gamma$, such that Γ' admits a surjective homomorphism onto the free group with two generators.

Proof. Let P' be a two dimensional cell complex with one zero-cell, with
p one-cells and with q 2-cells, such that $\pi_1(P') = \Gamma$. As $q < p$, there
exists an infinite cyclic covering $P \to P'$ and as $\chi(P') < 0$ we have
$b^1(P;\mathbb{R}) = \infty$ by the corollary (A). It follows, that some finite cyclic cove-
ring $\widetilde{P} \to P$ possesses a pair of non-trivial (i.e. not cohomologous to zero)
integral cocycles with disjoint supports. This pair defines a map of $\widetilde{P}$ to
the wedge of two circles such that the induced homomorphism
$\pi_1(\widetilde{P}) \to \pi_1(S^1 \vee S^1)$ is surjective. Q.E.D.

Next, we solve a problem posed by Baumslug and Pride in [4] .

(B') Theorem. If $q \leq p-1$ and if the word w_1 is a proper power, $w_1 = (w_0)^\ell$
for some prime number $\ell \geq 2$, then the group Γ is large.

Proof. First, let us show that $H_2(P;\mathbb{Z}_\ell) \neq 0$. Indeed, the 2-cell in P'
which corresponds to the word w_1 gives us a $\mathbb{Z}_\ell$-cycle in P' and this
cycle lifts to P. Now, as $\dim P' = 2$, the translates of this lift generate
an infinite subgroup in $H_2(P;\mathbb{Z}_\ell)$ and so $b^2(P;\mathbb{Z}_\ell) = \infty$. Therefore,
$b^1(P;\mathbb{Z}_\ell) = \infty$ by corollary (A'), and so some finite cyclic covering $\widetilde{P}$ of
P possesses a pair of non trivial $\mathbb{Z}_\ell$-cocycles with disjoint supports.
Then, we have a surjective homomorphism of $\pi_1(\widetilde{P})$ onto the free product
$\mathbb{Z}_\ell * \mathbb{Z}_\ell$. Q.E.D.

Remark. In the algebraic theorems above, one could avoid any reference to
infinite coverings. Indeed, let $\widetilde{P}_k \to P'$ be a sequence of cyclic coverings
of orders $k = 1,2,\ldots$, which converges to an infinite cyclic covering
$P \to P'$. If P' is a finite polyhedron and if $b_m(\widetilde{P}_k;F) \to \infty$ as $k \to \infty$
(for some fixed m), then all $\widetilde{P}_k$, for large k, possess pairs of non-
trivial m-dimensional F-cocycles with disjoint supports. In particular, if

$b_i(P_k;F) \rightarrow \infty$ for some field F , then the group $\pi_1(P')$ is large. As a corollary, we obtain the following

(B") <u>Theorem</u>. <u>Let</u> V' <u>be a closed oriented 4-dimensional manifold whose Euler characteristic</u> χ <u>and signature</u> σ <u>satisfy</u> : $\chi < |\sigma|$. <u>Then the fundamental group</u> $\pi_1(V')$ <u>is large</u>.

<u>Proof</u>. The inequality $\chi \leq |\sigma|$ implies $\chi \leq b_2(V;\mathbb{R})$ and so $b_1(V;\mathbb{R}) > 0$. Then, we have our cyclic coverings $\tilde{V}_k \rightarrow V'$. As $\sigma(\tilde{V}_k) = k\sigma$ and $\chi(\tilde{V}_k) = k\chi$, the inequality $\chi \leq |\sigma|-1$ implies $\chi(\tilde{V}_k) \leq b_2(\tilde{V}_k;\mathbb{R})-k$, and so $b_1(\tilde{V}_k;\mathbb{R}) \geq k$. Q.E.D.

<u>Remarks</u>. (1) This proof is similar to an argument of <u>Winkelkemper</u> who shows in [48] that the group $\pi_1(V)$ is non-abelian.

(2) Theorem (B) follows from (B"). Indeed, the two dimensional polyhedron P' embeds into $\mathbb{R}^5$ and then (B") applies to the boundary V' of the regular neighbourhood of $P' \subset \mathbb{R}^5$.

The major drawback of our proportionality proposition for the Euler characteristic is the bounded geometry assumption. In the following special case this assumption can be omitted.

Let the manifold V be <u>complete</u> and let $\{U_i\}$ be a stable <u>increasing</u> sequence which <u>exhausts</u> V , that is $U_1 \subset U_2 \subset ... \subset U_i \subset$ and $\underset{i}{\cup} U_i = V$. We assume as before that the universal covering of V is isometric to the universal covering of a compact manifold V' .

(C) <u>Theorem</u>. <u>The proportionality relation</u> $\lim_{i\to\infty} \chi(i)/Vol(U_i) = \chi(V')/Vol(V')$ <u>holds if the following two conditions are satisfied</u>,

(i) <u>the manifold</u> V <u>is homotopy equivalent to a finite polyhedron</u>,

(ii) <u>the fundamental group</u> $\pi_1(V)$ <u>is residually finite</u>.

<u>Remarks and Corollaries</u>. (a) It is unclear whether the conditions (i) and (ii) are essential.

(b) The theorem is equivalent, because of (i), to

$$\chi(V)/\text{Vol}(V) = \lim_{i \to \infty} \chi(V)/\text{Vol}(U_i) = \chi(V')/\text{Vol}(V') . \text{ If } \text{Vol}(V) = \infty , \text{ then the}$$

theorem only claims that $\chi(V') = 0$

<u>Example</u>. <u>Let</u> V <u>be a complete manifold of constant negative curvature which satisfies the following three conditions</u> :

(1) V <u>is homotopy equivalent to a finite polyhedron</u>.

(2) V <u>has sub-exponential growth</u> : <u>balls</u> $B_v(R) \subset V$ <u>around a fixed point</u> $v \in V$ <u>satisfy</u> $\lim_{R \to \infty} \inf R^{-1} \log \text{Vol } B_v(R) = 0$.

(3) <u>The dimension of</u> V <u>is even</u>.

<u>Then</u> $\text{Vol}(V) < \infty$.

Notice that none of the conditions (1) - (3) can be dropped.

(c) If the universal covering V of V is homogeneous, then the condition $\text{Vol}(V) < \infty$ implies (i) and (ii) (see [41]) and so we conclude to the following result of <u>Harder</u> [28] ,

(C') <u>Theorem</u>. <u>If</u> V <u>is a locally homogeneous Riemannian manifold of finite volume, then</u> $\chi(V) = \int_V G(v)dv$, <u>where</u> $G(v)$ <u>is the Gauss-Bonnet integrand</u>.

<u>The proof of (C)</u>. According to (ii) there is a sequence of finite coverings

$\tilde{V}(i) \rightarrow V$ for which the corresponding coverings $\tilde{U}_i(i) \subset \tilde{V}(i)$ of $U_i \subset V$ have bounded geometry. Then the proportionality proposition applies to the sequence $\{U_i(i)\}$ and with (i) we get $\lim_{i \to \infty} \chi(\tilde{V}(i)/\text{Vol } U_i(i) = \chi(V')/\text{Vol } V'$.

Therefore, $\lim_{i \to \infty} \chi(V)/\text{Vol } U_i = \chi(V')/\text{Vol } V'$. Q.E.D.

The proportionality proposition and its corollaries probably hold for the signature which is understood for open 4k-dimensional manifolds U as the signature σ of the (possibly degenerate) intersection form on the homology $H_{2k}(U; \mathbb{R})$. The following fact supports this conjecture.

Proposition. Let V' be a closed 4k-dimensional manifold with non-zero signature and let $V \rightarrow V'$ be a regular covering with an infinite amenable Galois group Π. Then the homology $H_{2k}(V; \mathbb{R})$ is infinite dimensional.

Proof. By the index theorem for infinite coverings (see [1]) there exists a harmonic L_2-form on V of degree 2k. Since the group Π is amenable, there also exists a non-exact closed 2k-form on V with compact support. As the group Π is infinite, it follows that $b^{2k}(V; \mathbb{R}) = \infty$. Q.E.D.

The Ricci curvature and volume provide the following upper bound for the signature and for the Euler characteristic.

(D) Theorem. Let V be a closed Riemannian manifold with Ricci $V \geq -1$. Let V' be a closed manifold with contractible universal covering, $\dim V' = n = \dim V$, and let $f : V \rightarrow V'$ be a map of degree $d \neq 0$. If the fundamental group $\Pi' = \pi_1(V')$ is residually finite then

$$|\chi(V')| \leq \text{const}_n \text{ Vol } V$$

as well as

$$|\sigma(V')| \leq \text{const}_n \text{ Vol } V ,$$

for some universal constant const_n .

Remarks. (a) The "residually finite" condition probably is unnecessary.

(b) The topological assumptions of (D) imply, for all known examples, $|\chi(V')| + |\sigma(V')| \leq \text{const}'_n \|V'\|$, and so all (known) specific applications of (D) also follow from the main inequality of 0.5.

Proof. Since Ricci $V \geq -1$, there exists (see 3.4), for every positive $\delta \leq 1$, a map $h = h_\delta$ of V into some n-dimensional polyhedron $Q = Q_\delta$ (which is in fact, a subpolyhedron of the nerve of some cover of V by ε-balls for $\varepsilon \approx \frac{1}{n} \delta$) with the following two properties :

(i) The pullback of every point, $h^{-1}(x) \subset V$, $x \in Q$, has diameter $< \delta$.

(ii) The total number of n-dimensional simplices in Q is at most $C_n \delta^{-n} \text{ Vol } V$, for some universal constant C_n . Moreover, the image $g_*[V] \in H_n(Q;\mathbb{Z})$ can be represented by an integral cycle with ℓ^1-norm $\leq C_n \delta^{-n} \text{ Vol } V$.

These properties imply the following

Sublemma. If all closed curves in V of length $< 6\delta$ are sent by the map $f = V \to V'$ into contractible curves then the Betti numbers of V' satisfy

$$\sum_{0}^{n} b_i(V';\mathbb{R}) \leq \text{const}_n \delta^{-n} \text{ Vol } V .$$

Proof. Let us first show that the map f extends to the cylinder $Y = Y_h \supset V$

of the map h . Assume without loss of generality that the map $h : V \to Q$ is simplicial in some subdivisions of V and Q and then we triangulate Y into very small simplices. Next, we send each vertex $y \in Y$ to the "nearest" point in V and then, for every edge in Y , the corresponding points in V can be joined, according to (i), by a path of length $< 2\delta$. Thus, with the map $f : V \to V'$, we obtain a map of the one-skeleton of Y to V' . As the boundary of every 2-simplex in Y goes through a curve of length $< 6\delta$ in V , this map extends to the 2-skeleton of Y . Finally, as the manifold V' is aspherical, we get the extension to the all of Y .

Now, with (ii), we represent the d-times fundamental class of V' by an integral cycle of norm $\leq C_n \delta^{-n}$ Vol V and by Poincaré duality

$$\Sigma \, b_i(V',\mathbb{R}) \leq 2^n C_n \delta^{-n} \text{ Vol } V \, .$$

Remark. If a prime number p does not divide d then this inequality also holds for $b_i(V';\mathbb{Z}_p)$.

To prove (D), we take a sequence of subgroups $\Pi'_j \subset \Pi'$ whose intersection is the identity element and by applying the sublemma to the corresponding finite coverings $\tilde{V}'_j \to V'$ and $\tilde{V}_j \to V$ we get
$\lim_{j \to \infty} \sup \, [\Sigma_i \, b_i(\tilde{V}_j;\mathbb{R})] \, / \text{Vol } \tilde{V}_j \leq 2^n C_n \delta^{-n}$ for $\delta = 1$. Finally, as the numbers $\chi(V')$ and $\sigma(V')$ are multiplicative for finite coverings, they are estimated from above by $2^n C_n$ Vol V . Q.E.D.

There is yet another characteristic number, the $\hat{A}$-genus $\hat{A}(V)$, which also can be estimated from above if Ricci $V \geq -1$, but now with the diameter of V rather than with the volume (compare [15],[16],[17]).

(E) Theorem. Closed spin-manifolds V (i.e. the Stiefel class $w_2=0$) with

Ricci V $\geq$ -1 <u>satisfy</u>

$$\hat{A}(V) \leq 2^{n/2} + (Diam)^n (const_n)^{1+Diam} , \qquad (*)$$

<u>for some universal constant const</u>$_n$.

<u>Proof</u>. By Lichnerowicz formula (see [26] for more details) the square of the Dirac operator is

$$D^2 = \Delta' + \frac{1}{4} S ,$$

where Δ' is the Laplacian on the spin bundle and S is the scalar curvature of V . Therefore, the index $Ind\ D = \hat{A}(V)$ is bounded from above by the number of those eigenvalues of the operator Δ' which do not exceed $-\inf\limits_{v \in V} S(v)$. These eigenvalues λ_i' are related to the eigenvalues λ_i of the ordinary Laplace operator by <u>Kato's</u> inequality (see [29]) ,

$$\sum_i exp-\lambda_i' t \leq 2^{n/2} \sum_i exp-\lambda_i\ t ,$$

for all $t > 0$, and the application of the following inequality of [22] concludes the proof

$$\lambda_i \geq (Diam)^{-2} (c_n^{1+Diam})_i {}^{2/n} \qquad .$$

<u>Corollary</u>. <u>For</u> $n = \dim V = 4$ <u>the signature of a spin manifold</u> V <u>with</u> Ricci V $\geq$ -1 <u>satisfies</u>

$$|\sigma(V)| \leq 32 + (Diam)^4 (const)^{1+Diam} .$$

<u>Remark</u>. The argument above when applied to the Hodge-Laplace operator on

forms gives a lower bound for the eigenvalues in terms of Diam V and of the lower bound on the curvature operator (compare [15],[16],[17]).

Finally, let V be a closed 4-dimensional <u>Einstein manifold</u> with Ricci V = -1 (and so with $S(v) \equiv -2$). Then by a formula of Berger-Thorpe-Hitchin (see [32]),

$$\chi(V) \geq \theta |\sigma(V)| + \theta' \text{ Vol } V ,$$

for every θ in the interval $[0,3/2]$ and for $\theta' = 1/48\pi^2 - \theta^2/108\pi^2$.

Therefore, the main inequality of 0.5. implies the following

Theorem. <u>All closed 4-dimensional Einstein manifolds satisfy for above</u> θ <u>and</u> θ' ,

$$\chi(V) \geq \theta |\sigma(V)| + \theta' \|V\|/216$$

Example. Take a Cartesian square $X \times X$ of a closed surface of genus $q \geq 2$. Delete p open ball from $X \times X$ and take the double, say V, of the resulting manifold with boundary. This V has $\chi(V) = 8(q-1)^2 - 2p$ and $\sigma(V) = 0$, while by Milnor-Sullivan-Thurston inequality of section 0.3.,

$$\|V\| = 2\|X \times X\| \geq 32(q-1)^2 .$$

It follows that for $8(q-1)^2 - 2p < (q-1)^2/324\pi^2$ the manifold V admits no Einstein metric.

Now, we return to our main topic and we extend Thurston's inequality of section 0.3. to the specific volume $\| \|_-$.

(A)   Let V be a Riemannian manifold with sectional curvature ≤ -1 .
If V is stable at infinity then it satisfies for "size" = dil ,

$$\|V : \text{Vol}\|_- \geq \text{const}_n^{-1} \geq (n-1)!/\pi .$$

Proof. With Thurston's straightening argument (see 1.2.), we obtain for
all stable sequences $U_i \subset V$ and for all fundamental cycles c with
dil(c) $< \infty$,

$$\liminf_{i \to \infty} \text{Vol}(U_i)/\|c\,|U_i-R\| \geq \text{const}_n$$

for all $R \geq 0$, where const_n is the constant of section 1.2. Observe, that
the manifold V does not even need to be complete.

Next, we observe the following estimate for $\|\ \|_+$ from above :

(B)   All n-dimensional manifolds V with Ricci(V) $\geq 1/1-n$ satisfy
for "size" = length + Rad ,

$$\|V : \text{Vol}\|_+ \leq C_n\, n! , \tag{*}$$

for the constant $C_n < 1$ of (D) in 4.3.

Proof. For complete manifolds V this follows from (D) of 4.3. and Bishop's
inequality [5] . If V is not complete, we just extend the induced metric
in V - R for some R > 0 to a complete metric in $V \supset V - R$ and then (D)
of 4.3. applies

(B') Remark. If V has non-positive sectional curvature, then according
to (A) of 4.3. the inequality (*) holds with a sharper "size" , namely

with "size"(σ) = dil(σ) , that is the Lipschitz constant of singular sim-
plices σ . Furthermore, the inequality (*) for "size" = dil holds for all
manifolds V with $|K(V)| \leq 1$, but now with another constant $C'_n > C_n$ in
place of C_n (see (c) of 4.3.).

Finally, we come to the following

Volume comparison theorem. Let V_1 and V_2 be n-dimensional manifolds,
such that Ricci$(V_1) \geq 1/1-n$ and $K(V_2) \leq -1$. Suppose furthermore, that the
manifold V_2 is stable at infinity. Then all proper Lipschitz maps
$f : V_1 \rightarrow V_2$ satisfy

$$\text{Vol}_-(V_1 : V_2; f) \geq A_n \ \deg(f)$$

for $A_n = (\text{const}_n \ C_n \ n!)^{-1} > (n\pi)^{-1}$.

Corollaries and examples.

(1) If V_2 is a complete manifold of subexponential growth, then
the pullbacks of R-balls $B(R) \subset V_2$ around a fixed point satisfy for
$R \rightarrow \infty$,

$$\lim_{R \to \infty} \sup \text{Vol} \ f^{-1}(B(R))/\text{Vol} \ B(R) \geq \text{Vol}_- >(\deg f)/n\pi \quad .$$

In particular, if V_2 has finite volume, then

$$\text{Vol}(V_1)/\text{Vol}(V_2) >(\deg f)/n\pi$$

(2) If both manifolds V_1 and V_2 are hyperbolic manifolds of
sectional curvature -1 , then with the exact estimate of section 2.2.
one gets the sharp inequality

$$\mathrm{Vol}_-(V_1 : V_2; f) \geq \deg f \;.$$

It would be interesting to analyse the case of equality in the spirit of Thurston's rigidity theorem (see [47],[25]) . Recall, that by a theorem of Sullivan [46], a bi-Lipschitz homeomorphism between hyperbolic manifolds of subexponential growths (in fact, between manifolds for which $\lim \sup R^{-1} \log \mathrm{Vol}\, B(R) < n-1$) is homotopic to an isometry.

Proof. Let first $K(V_1) \leq 0$. Then, by (B') we have the upper bound for $\|V_1 : \mathrm{Vol}\|_+$ with "size" = dil , while (A) gives the lower bound for $\|V_2 : \mathrm{Vol}\|_-$ with the same "size" . Therefore, the functorial properties of these specific volumes yield the theorem.

Now, if V_1 only has $\mathrm{Riccci}\, V_1 \geq -1$, this argument does not work since the theorem (B) applies to the "size" = length + Rad which is "weaker" then dil . In order to close the gap, we first state the following sharpening of the main technical theorem of section 4.3. Namely, for singular simplices $\sigma : \Delta \to V$ we introduce a new "size" called $\widetilde{\mathrm{Rad}}\, \sigma$: a simplex σ has $\widetilde{\mathrm{Rad}}\, \sigma < \rho$ if, first, it has $\mathrm{Rad}\, \sigma \leq \rho$ and so it is contained in ball $B_v(\rho) \subset V$. We require furthermore, that the lift of σ to the universal covering $\widetilde{B}_v(\rho)$, of $B_v(\rho)$ call it $\tilde{\sigma} : \Delta \to \widetilde{B}_v(\rho)$, has its image $\tilde{\sigma}(\Delta)$ in the ball $B_{\tilde{v}}(\rho) \subset \widetilde{B}_v(\rho)$ with the center $\tilde{v}$ in $B_v(\rho)$ over $v \in V$. The minimal ρ for which σ lifts to such a ball $B_{\tilde{v}}(\rho)$ is our new "size" = $\widetilde{\mathrm{Rad}}(\sigma)$.

(1) Claim. The main technical theorem of (D) in 4.3. holds with $\widetilde{\mathrm{Rad}}$ substituted for Rad.

We shall prove this later. Now, we sharpen the theorem (A) for manifolds

V of negative curvature. If such V is complete, we denote by $\overline{Rad}\,\sigma$ for

$\sigma : \Delta \to V$ the radius of a lift of this simplex σ to the universal cove-

rings V . If V is not complete, we only take into account the simplices

contained in $V - R$ for large R and then lift them, whenever possible, to

exponential balls $\exp_v : B(\rho) \to V$ for $v \in V - R$ and for $\rho \leq R$. The mini-

mal ρ for which such a lift exists is called $\overline{Rad}\,\sigma$. Then for a chain

$c = \Sigma r_i \sigma_i$ we write $\overline{Rad}\,c \leq \rho$ if all simplices σ_i which are contained

in the R-interior $V - R$ for a sufficiently large R , have $\overline{Rad}\,\sigma \leq \rho$

(2) <u>Claim</u>. <u>Thurston's theorem</u> (A) <u>holds for this weaker</u> "size"(c) $= \overline{Rad}(c)$.

Indeed, simplices in exponential balls can be straightened as before,

while the simplices outside $V - R$ do not matter anyway.

Now, let V_2 be a complete manifold. Then, for maps $f : V_1 \to V_2$

with the Lipschitz constant L_f we have $\overline{Rad}(f_*(c)) \leq L_f \overset{\frown}{Rad}(c)$ for all

chains c in V_1 . Thus our theorem for complete manifolds V_2 is reduced

to the first claim. We postpone the treatment of non-complete manifolds

until the very end. Now we come to

<u>The proof of the Claim</u> (1) . We must diffuse all simplices σ_i of the chain

$c = \Sigma r_i c_i$, or rather the small fat simplices Δ of a subdivision of c ,

to some simplices of $\overset{\frown}{Rad} \leq \tilde{R} + 2R + \epsilon$ (compare with proof in (D) of 4.3.).

To do that, we modify the complex K only by admitting those simplices σ

with the vertices $v_o, \dots, v_m$ in V , whose lifts $\tilde{\sigma}$ to the universal

covering of the union of the balls $B_k = B_{v_k}(R+\tilde{R}+\epsilon/2)$, $k = 0, \dots, m$, are

contained in the union of the balls $B_{\tilde{v}_k}(R+\tilde{R}+\epsilon/2)$ in this covering $\overset{\frown}{\underset{k}{U\,B}_k}$,

where the centers $\tilde{v}_k$ of these new balls are exactly the vertices

$\tilde{v}_o, \dots, \tilde{v}_m$ of σ . With these new admissible simplices $\tilde{\sigma}$ we build a new

complex, called $\widetilde{K}$ with the properties (i) - (iii) of (D) 4.3. and this $\widetilde{K}$ has the same formal properties as the old K. But now, we have a problem with the operators $\overline{S}_{R,\overline{R}}$: these operators when applied to the edges of simplices Δ, are only defined up to homotopies in the coverings $\widetilde{B}_v(R')$, rather than within some balls in these coverings, and so these homotopies may not be admissible in the new sense. Therefore, we must replace the étale domains $\widetilde{B}_v(R')$ by some new domains which simultaneously enjoy the two properties : they are simply connected and they are "balls" of radius R'. Here is the construction. For an arbitrary open ball $B_v(R)$ in a Riemannian manifold, we first take the universal covering $\widetilde{B}_v(R)$ of $B_v(R)$ and then the ball $B_{\widetilde{v}}(R) \subset \widetilde{B}_v(R)$ around some point $\widetilde{v} \in \widetilde{B}_v(R)$ over v, called $B'_v(R) = B_{\widetilde{v}}(R) \rightarrow B_v(R)$. Then by the iteration, we get the following projective system of balls and of locally isometric maps

$$B_v(R) \xleftarrow{\quad P_1 \quad} B'_v(R) \xleftarrow{\quad P_2 \quad} B''(R) \leftarrow \cdots \xleftarrow{\quad P_i \quad} B_v^{(i)}(R) \leftarrow \cdots$$

This projective system converges over all balls $B_v(R_o) \subset B_v(R)$ for $R_o < R$. Namely for all $i \geq i_o = i_o(R_o)$ the maps P_i become bijective on the concentric R_o-balls, that is for every point $\widetilde{w} \subset B^{(i-1)}(R)$ which projects on $B_v(R_o)$, there is only one point in the pullback $(p_i)^{-1}(\widetilde{w})$. Therefore, the projective system $\{p_i : B^{(i)} \rightarrow B^{(i-1)}\}$ has a limit, called $\hat{p} : \hat{B}'_v(R) \rightarrow B_v(R)$, where $\hat{B}_v(R)$ is a <u>simply connected</u> Riemannian manifold and $\hat{p}$ is a locally isometric map. Furthermore, this B is <u>a ball</u> relative to a unique lift $\hat{v} \in \hat{B}$ of $v \in V$ as the center. Namely the exponential map at $\hat{v}$ is defined on the Euclidean ball $B_o(R) \subset T_{\hat{v}}(B)$ and the map $\exp_{\hat{v}} : B_o(R) \rightarrow \hat{B}_v(R)$ is surjective. Now, we substitute the domains $\widetilde{B}_v(R')$ by the balls $\widehat{B}_v(R')$ and the rest of the proof of the technical theorem goes without any problem. Thus the volume comparison theorem is established

for complete manifolds V_2 .

Now, let V_2 be an arbitrary manifold of negative curvature. Over all points $v \in V_2 - R$, we have the exponential (étale) balls, call them $p_v = \exp_v : \overline{B}_v(\rho) \to V_2$ for $\rho < R$, and with the map $f : V_1 \to V_2$, we lift this (étale) system of the balls $\overline{B}_v(\rho)$ to a system of étale domains $\overline{B}_w^*$ over V_1 for all $w \in V_1$. Namely, each domain $\overline{B}_w^*$ is defined as the set of those pairs $(w',v') \in V_1 \times B_v$ for which $f(w') = p_v(v')$ for all w in the pull-back $f^{-1}(v) \subset V_1$, while the étale map $p_w^* : \overline{B}_w^* \to V_1$ is induced by the projection $V_1 \times \overline{B}_w^* \to V_1$. Furthermore, all étale maps between the domains $\overline{B}_v$ lift to étale maps between $\overline{B}_w^*$.

Next, we take a fundamental cycle $c = \sum_i r_i \Delta_i$ of V_1 which is built of small fat simplices Δ_i and we directly diffuse this c into a fundamental class of V_2 . We do that with an étale smoothing S in V_1 over the domains $\overline{B}_w^*$ as follows. Simplices Δ in V_1 are lifted to simplices $\overline{\Delta}$ in $\overline{B}_w^*$ and then the smoothing S sends the vertices of $\overline{\Delta}$ to some points $\overline{w}_0', \ldots, \overline{w}_n'$ in $\overline{B}_w^*$, assigned with certain weights. Next, with the map f we have a unique map of $\overline{B}_w^*$ to an exponential ball $\overline{B}_v$ over V_2 for $v = f(w)$ and thus we send $\overline{w}_0', \ldots, \overline{w}_n'$ to some points, called $\overline{v}_0', \ldots, \overline{v}_n'$, in $\overline{B}_v$ which carry the same weight as $(\overline{w}_0', \ldots, \overline{w}_n')$. Finally, these points in $\overline{B}_v$ over V_2 span unique straight simplex whose projection on V_2 is exactly what we need. So, we have sent Δ to a straight chain in V_2 , called $S^f * \Delta$, and the chain c is sent to the fundamental cycle $S^f * c = \sum_i r_i S^f * \Delta_i$ in V_2 .

Now, we choose the étale smoothing $S = S(w)$ for all $w \in V_1$ in the same way we did it before . Namely, we use balls in the domains $\overline{B}_w^*$ over V_1 : if $\overline{B}_w^*$ is sent by f to an exponential ball $\overline{B}_v(\rho)$ over

V_2 , for $v = f(w)$, then we take the point $\bar{w} = (w,\bar{v}) \in \bar{B}_w^*$, where $\bar{v} \in \bar{B}_v(\rho)$ is the center of $\bar{B}_v(\rho)$. Next, we take the ball $B_{\bar{w}}(\rho') \subset \bar{B}_w^*$ of the radius $\rho' < (L_f)^{-1}\rho$, where L_f is the Lipschitz constant of f and finally, we take our old measure with density one in $B_{\bar{w}}(\rho')$ and zero outside $B_{\bar{w}}(\rho')$. This measure is regularized with a fast decaying function φ on V_2 and then it is made discrete with a subset $Z \subset V_1$ (see (D) of 4.3.). We estimate the norm $[S(w)]^*$ as before and for $\rho' \to \infty$ we get

$$\| S^f * c | U-R \|_{\ell^1} \leq C_n \, n! \; \mathrm{Vol}(f^{-1}(U))$$

for all $U \subset V_1$ and for $R > \rho > L_f \rho'$. This estimates combined with (A) implies our theorem.

Final Remarks. We want to indicate possible candidates for specific volumes for compact manifolds. Let us start with an example. Recall, that the singular homology theory can be built with singular cubes, $\delta : \square \to X$, rather than with simplices, and for this cubical singular homology we define as before the cubical ℓ^1-norm on the homology $H_*(X)$, called $\| \; \|^\square$. Then, for all $h \in H_m(X)$, we consider the ratio $\| h \|^\square / \| h \|$, called $\| h; \square : \Delta \|$. We cannot go further since we do not know the answer to the following

Question. Does, actually, the "norm" $\| h; \square : \Delta \|$ depend on X and h or $\| h ; \square : \Delta \| = \mathrm{const} = \mathrm{const}(m)$.

In fact, there is a whole spectrum of norms on $H_*(X)$. Indeed, to each space X we associated in section 3.2. a complete minimal multi-complex K . This K is unique if all its components have countably many vertices and this K comes with a homotopy equivalence $S : K \to X$.

All simplices in K can be identified with the faces of the standard

infinite dimensional simplex Δ^∞ and thus with every metric G in Δ^∞ which is invariant under automorphisms of Δ^∞ we obtain a metric in K . Now, we can speak of masses of homology classes in K and we put

$$\| h ; G \| = \mathrm{mass}(S_*^{-1} h) \quad \text{for all} \quad h \in H_*(X)$$

and for all spaces X .

Again, we do not know if the ratio $\| h ; G \| / \| h \|$ depends on h or

$$\| h;G \| / \| h \| = \mathrm{const}(G, \dim h) \ .$$

Example. Let Δ^∞ be realized by the "projective simplex" $P^+ \subset P(M)$ (see 2.4) for M the space of measures on the set of the vertices of Δ^∞ . The ℓ^1-metric in M ,

$$\mathrm{dist}(\mu_1,\mu_2) = \| \mu_1 - \mu_2 \|_{\ell^1}$$

induces a metric in the projective space $P(M)$: lengths of curves in $P(M)$ are defined as the lower bounds of the lengths of their lifts to M . Then, with this metric G on P^+ (or rather with the mass defined in 2.4) one can prove for all $h \in H_m(X)$,

$$\| h;G \| = \| h \| / m! \ \ .$$

Appendix 1. On Min Vol $\mathbb{R}^2$.

Let V be a complete simply connected surface with $K(V) \leq 1$.

Proposition. There exist two points, x and y in V , with the following two properties :

(1) $\text{dist}(x,y) = \pi$,

(2) $\text{Inj. Rad}_x(V) + \text{Inj. Rad}_y(V) \geq \pi$.

Proof. If there is some point $v_0 \in V$, for which $R_0 = \text{Inj. Rad}_{v_0}(V) \geq \pi$, then the proof is finished. If $R_0 < \pi$, there is a simple geodisic loop, L_0 in V , at v_0 such that $\text{length}(L_0) = 2R_0$ and a unique point $v_0' \in L_0$ with $\text{dist}(v_0, v_0') = R_0$. The loop L_0 bounds a disk D_0 , and the cut-locus of V relative to v_0 contains, by a theorem of Meyers (see [18]) , a conjugate point v in D_0 . Since $\text{dist}(v_0, v) \geq \pi$, there is a point $v_1 \in D_0$ for which $\text{dist}(v_0, v_1) = \pi$. Let R_1 denote the injectivity radius at v_1 and let L_1 be the corresponding loop at v_1 . If the loops L_0 and L_1 intersect, then $L_0 + L_1 = 2(R_0 + R_1) \geq 2\pi$, and the proof is finished. If the intersection of these loops is empty, then L_1 bounds a disk D_1 inside D_0 . Then again, we have a point $v_2 \in D_1$ with $\text{dist}(v_1, v_2) = \pi$. By repeating this process, we arrive eventually at a pair of points $x = v_i$ and $y = v_{i+1}$, $i \leq \infty$, for which the loops L_i and L_{i+1} do intersect and the proof is finished.

Now, let γ be a minimal geodesic between x and y and let $z \in \gamma$ be a point for which $\rho' = \text{dist}(z,x) \leq \text{Inj} \cdot \text{Rad}_y(V)$ and $\rho = \text{dist}(z,y) \leq \text{Inj. Rad}_x(V)$. Then the ball $B = B_z(\pi) \subset V$ contains two disjoint open injective balls $B_x(\rho)$ and $B_y(\rho')$ with $\rho + \rho' = \pi$. Therefore, $\text{Vol}(B) \geq 4\pi$. Suppose that V is homeomorphic to $\mathbb{R}^2$ and let moreover $|K(V)| \leq 1$. Take points $w \in V$ far from B .

Then by Bishop's inequality ([5]) for increasing $R = \text{dist}(w, B)$,

$$\limsup_{R \to \infty} \text{Vol } B_w(R)/\text{Vol } B_w(R+2\pi) \geq e^{-2\pi} \ ,$$

and so $\text{Vol}(V) \geq 4\pi(1+e^{-2\pi})$. This implies our assertion in section 0.1. :

$$\text{Min Vol}(\mathbb{R}^2) \geq 4\pi(1+e^{-2\pi}) > 4\pi + 0.01 \ .$$

Finally, take the largest of the two balls in the standard sphere S^2 whose boundary circle has geodesic curvature -1 and then extend the metric of this ball B_o to a complete C^1-metric in $\mathbb{R}^2$ with Gaussian curvature -1 outside B_o. This metric has $|K| = 1 \leq 1$, $\int K \, d\sigma = 2\pi$, $\int_{K>o} d\sigma = (2+\sqrt{2})\pi$, $\int |K \, d\sigma| = \text{Vol} = (2+2\sqrt{2})\pi$. Therefore,

$$\text{Min Vol}(\mathbb{R}^2) \leq (2+2\sqrt{2})\pi \ .$$

Appendix 2. Manifolds with Min. Vol = 0 .

Let V be a compact Riemannian manifold with a non-vanishing Killing field X . The Riemannian metric g of V then splits into orthogonal components, $g = g_0 + g_1$, where the quadratic form g_0 of rank = 1 is zero on the orthogonal complement to X , while the form g_1 of rank = n-1 is zero on X . Then the metrics $g_\epsilon = \epsilon^2 g_0 + g_1$ have uniformly bounded curvatures , $|k(V,g_\epsilon)| \leq$ const for $\epsilon \to 0$, while $Vol(V,g_\epsilon) \to 0$.

For a compact manifold with a locally free S^1-action, we have, with some S^1-invariant metric on V , such a field X and so Min Vol V = 0 as claimed in section 0.1. Moreover, if V is not compact, then the above argument still applies to an appropriate S^1-invariant metric and to some S^1-invariant function ϵ on V . Then again Min Vol V = 0 .

Now, we introduce a notion of a T-manifold (or a T-structure) which generalizes manifolds with locally free S^1-action. Suppose that V is covered by open subsets U_i; i = 1,..., and let every manifold U_i be endowed with a locally free S^1-action, called A_i on U_i for i = 1,..., . The collection $\{U_i, A_i\}$ is called a T-structure on V if all intersections $U_{i_1} \cap ... \cap U_{i_k}$ of the sets U_i are invariant under the actions $A_{i_1},...,A_{i_k}$ and these actions commute whenever they are simultaneously defined.

Examples. (1) A 3-manifold V is called a graph manifold if there are some disjoint embedded 2-tori T_k in V , such that the complement of an open tabular neighbourhood of the union $\underset{k}{\cup} T_k$ admits a free circle action.

Clearly, graph manifolds are exactly all 3-dimensional T-manifolds .

Connected sums of graph manifolds also are graph manifolds (see [40]).
Furthermore, the argument of [40] applies to all odd dimensions :

(2) Connected sums of odd dimensional T-manifolds are T-manifolds.

(3) Observe that $\mathbb{R}^3$ is a graph manifold. As the product $V \times V_o$
for a T-manifold V is also a T-manifold, we conclude that products
$\mathbb{R}^3 \times V_o$ are T-manifolds for all manifolds V_o .

(4) Let $f : W_o \rightarrow W$ be a proper immersion with transversal self-
intersections. If codim W_o = 2 then the boundary of the regular neighbour-
hood of the image $f(W_o) \subset W$ is a T-manifold.

Indeed, local models of transversal intersections are systems of
mutually orthogonal subspaces in $\mathbb{R}^2$ of codimensions 2 . Rotations around
these subspaces give commuting S^1-actions on some neighbourhoods U_i of
our boundary, and small perturbations of these actions form a T-structure.

(5) Corollary. If the interior of a compact manifold X admits a complex
quasi-projective structure, then with the theorem of Hironaka (see [30])
one gets a T-structure on the boundary of X .

Now, we claim that all T-manifolds V have Min.Vol(V) = 0 .

Indeed, for an appropriate metric g on V which is A_i-invariant
on the neighbourhoods U_i for all i , there are some ϵ_i-squeezings of
g in the A_i-directions that keep the curvature bounded and make
Vol(V) $\rightarrow$ 0 for $\epsilon_i \rightarrow$ 0 .

Let us now exhibit T-manifolds without S^1-action. First, we recall
some elementary facts (see [8] for more information) on actions of the group

S^1 on closed connected manifolds V . For each point $v \in V$ the orbit $S^1 v : S^1 \to V$ gives us an element in the fundamental group, called $\sigma \in \pi_1(V, v)$. Clearly, this σ is in the center of $\pi_1(V)$. Furthermore, it acts trivially on all homotopy groups $\pi_i(V)$. We denote by $C \supset V$ the cylinder of the quotient map $q : V \to V/S^1$ and we observe for <u>locally free</u> actions, following an unpublished remark of Sullivan, that the pair (C, V) is a rational homology manifold with boundary $\partial C = V$. In particular, all characteristic numbers of these manifolds V vanish. (Of course, this is also true for all T-manifolds V as $\mathrm{Min.Vol}(V) = 0$) . Next, we denote by $\sigma_v' \in \pi_1(V, v)$ the homotopy classes of the images of the orbits S_v^1 in V and we observe that the element σ is a multiple of the elements σ_v' for all $v \in V$.

Now, take a $K(\Pi, 1)$ space for some group Π and consider maps $f : V \to K(\Pi, 1)$. The following remark is also due to Sullivan :

<u>If all "geometric orbits" σ_v' are sent to the identity in</u> $\pi_1(K(\Pi, 1)) = \Pi$, <u>by a map</u> f , <u>then the map</u> $f : V \to K(\Pi, 1)$ <u>extends to the cylinder</u> $C \supset V$. <u>In particular, the fundamental class</u> $[V] \in H_n(V; \mathbb{R})$ <u>goes to zero under the map</u> f .

Indeed, the pullback of any point under the map $q : V \to V/S^1$ is path connected : it is just a circle or a point. Therefore, the map q is surjective on the fundamental groups and the kernel of this surjective homomorphism equals the normal subgroup N spanned by the fundamental groups of the pullbacks $q^{-1}(x)$ for all $x \in V/S^1$. This is seen by looking at the covering $\tilde{V} \to V$ which has Galois group $\pi_1(V)/N$.

Now, $\pi_1(C) = \pi_1(V/S^1)$, and so the inclusion homomorphism is surjective. The Kernel spanned by $\sigma_v' \in \pi_1(V, v)$ for $v \in V$. It follows that the homo-

morphism $f : \pi_1(V) \rightarrow \Pi$ factors through $\pi_1(C)$ and so the map f extends to C.

Corollary. <u>Let the action of S^1 on V be locally free. Then all Pontryagin numbers of the map f vanish. That is for all $h \in H^*(K(\Pi,1);\mathbb{R})$ and for all Pontryagin classes $p = p(V)$ one has</u>

$$<p(V) \cup f^*(h) , [V]> = 0 .$$

Indeed, Pontryagin numbers are bordism invariants for rational homology manifolds.

Example. <u>Let V_1 be a $K(\Pi,1)$ manifold such that the group Π has no center. If a manifold V_2 has a non-zero Pontryagin number</u> , then the product $V = V_1 \times V_2$ admits no locally free S^1-actions.

Remark. If we only require the (multiple!) orbit $\sigma \in \pi_1(V)$ to go to the identity (rather than all σ_v'), then f may not extend to C but the fundamental class of V goes to zero anyway, $f_*([V]) = 0$.

Indeed, as we work with H_* with real coefficients the difference between σ_v' and σ who is a multiple of all σ_v' is not essential. Geometrically speaking, one does not extend the map f itself, but a collection of lifts of f to finite coverings of S^1-invariant neighbourhoods of all S^1-orbits. These coverings unwrap all multiple orbits, so that we have coherent extensions of these lifts to the corresponding cylinders. These extensions add up algebraically to a chain in our $K(\Pi,1)$, whose boundary is a multiple of $f_*[V]$.

Corollary (Sullivan, unpublished). <u>If the S^1-action has a fixed point in</u>

V , then all maps f of V to $K(\Pi,1)$-spaces have $f_*([V]) = 0$.

Indeed, if there is a fixed point, then σ goes to the identity.

Let us call a manifold V essential, if the classifying map
$V \to K(\Pi,1)$ for $\Pi = \pi_1(V)$ sends the fundamental class $[V]$ to a non-
zero element in $H_*(K(\Pi,1);\mathbb{R})$. If V is not orientable, we apply the defi-
nition to an oriented double covering of V .

For example, all $K(\Pi,1)$-manifolds V are essential. If $V_1 \to V_2$
is a map of non-zero degree and if V_2 is essential then also V_1 is essen-
tial. On the other hand, manifolds with finite fundamental groups are not
essential.

Now, if V is an essential manifold which admits an S^1-action, then
there is a (non trivial!) element σ in the center of the fundamental group
and also the action of this σ on all homotopy groups $\pi_i(V)$, $i = 1,2,\ldots$,
is trivial.

Example. A connected sum $V = V_1 \# V_2$, where V_1 is a $K(\Pi,1)$-manifold
and V is not a homotopy sphere, has no nontrivial S^1-action, as claimed
in section 0.1.

There is a natural generalization of T-manifolds . Namely, instead
of a covering $\{U_i\}$ we use an étale covering, that is a system of locally
diffeomorphic maps $p_i : \tilde{U}_i \to V$ (see (D) of 4.3.) whose images cover V .
The intersections of two étale domains, say of $p_i : \tilde{U}_i \to V$ and $p_j : \tilde{U}_j \to V$,
is the set of pairs (u,u') for $u \in U_i$ and $u' \in U_j$ such that
$p_i(u) = p_j(u')$.

We say that S^1-actions Λ_i on $\tilde{U}_i$ define an F-structure on V if

for the intersections of all finite subsystems, $U_{i_1}, \ldots, U_{i_k}$, the actions $A_{i_1}, \ldots, A_{i_k}$ lift to these intersections and the lifted actions commute.

Example. Let V be a flat manifold and let $U \to V$ be a covering torus. Then any isometric S^1-action on U defines an F-structure on V , although there is no obvious T-structures on flat manifolds.

One sees as before, that F-manifolds V have Min Vol(V) = 0 . Furthermore, interiors of compact manifolds $V = \text{Int } \overline{V}$, whose boundaries $\partial\overline{V}$ admit F-structures, have Min Vol(V) < ∞ . Probably, all manifolds V with Min Vol(V) = 0 admit F-structures. This is not difficult to show for 3-manifolds. In fact, many 3-manifolds V can be cut along some 2-spheres and incompressible tori into simpler manifolds V_i , such that some of these V_i are T-manifolds and the remaining ones admit complete hyperbolic structures of finite volumes (see [47]). Let Vol_h denotes the total volume of these hyperbolic pieces. Then one can show by an argument as above that $\text{Min Vol}(V) \leq \text{Vol}_h$. Observe (see 0.3.) that $V_h = R_3 \| V \|$ where R_3 is the volume of the ideal regular simplex in the hyperbolic 3-space. Probably, $\text{Min Vol}(V) = R_3 \| V \|$ for these manifolds V .

We conclude with a simple construction of complete metrics g on open manifolds V , such that the curvatures $K_v(g)$ at points $v \in V$ rapidly decay for $v \to \infty$, while the volumes of balls $B_{v_o}(R) \subset V$ have slow growth for $R \to \infty$.

We start with an arbitrary complete metric g on V and we indicate two closely related modifications of g . The first modification makes curvatures small at infinity, and the second takes care of $\text{Vol } B_{v_o}(R)$.

(1) Conical telescope. Fix a reference point $v_o \in V$ and let $S_1, \ldots, S_i \ldots$

be smoothed spheres around v_o of radii $1,\ldots,i\ldots$. Then by a small perturbation of

g we make small tubular neighbourhoods, $U_i = [0,\epsilon_i] \times S_i$ of S_i, _conical_,

keeping the name "g" unchanged. Recall, that conical metrics are characte-

rized by the following property : the maps $[0,\epsilon_i/2] \times S_i \rightarrow [0,\epsilon_i] \times S_i$

given by $(t,s) \rightarrow (2t,s)$ are expanding homotheties with factors we choose

to be 2 for all $i = 1,\ldots$. Now, take the region in V outside some

$S_i = 0 \times S_i$ and multiply the metric g by a number $\lambda_i > 0$. This metric

$\lambda_i g$ can be joined with the old g inside S_i over the homothetic cone

$[0,\lambda_i\epsilon_i] \times S_i$. Then we apply this operation to all "joints" S_i with large

numbers λ_i , first making the sectional curvature K outside $\lambda_1\epsilon_1 \times S_1$

as small as we wish, then even smaller outside $\lambda_2\epsilon_2 \times S_2$ and so on.

Observe, that this operation also makes $\text{Inj Rad}_v(V) \rightarrow \infty$ for $v \rightarrow \infty$. The

fastest decay of K_v one gets is

$$|K_v| \le \text{const } (\text{dist}(v,v_o))^{-2}, \text{ for } v \rightarrow \infty ,$$

and for some $\text{const} = \text{const}_n$, $n = \dim V$.

Notice, that necessarily, $\text{Vol } B_{v_o} (R) \approx \text{const}' R^n$ for these metrics.

(2) _Cylindrical telescope._ Now, let the metric g be the product metric

in the neighbourhoods $[0,\epsilon_i] \times S_i$. Then, by just making the cylinders

longer, $[0,\epsilon_i] \longrightarrow [0,\lambda_i]$, for large λ_i we obtain $\text{Vol } B_{v_o} (R) \le C(R)R$

for any given function $C(R) \underset{R \to \infty}{\rightarrow} \infty$, without changing the curvatures of V .

Furthermore, if the cylindrical telescope is preceded by a conical one, then

$$(RC(R))^{-1} \text{Vol } B_{v_o} (R) \longrightarrow 0 , \text{ for } R \rightarrow 0 ,$$

$$K_v(V)) \longrightarrow 0 , \text{ for } v \longrightarrow \infty ,$$

$$\text{Inj Rad}_v(V) \longrightarrow \infty , \text{ for } v \longrightarrow \infty .$$

Finally, observe that the argument of the example of 0.1 also shows
for topological products $V = V_o \times \mathbb{R}$:

If the sectional curvatures at all points $v \in V$ satisfy

$$|K_v(V)| \leq \mathrm{const}(\mathrm{dist}(v, v_o))^{-2-\varepsilon}$$

for a fixed point $v_o \in V$ and for some $\varepsilon > 0$, then all Pontryagin numbers
of V_o vanish.

Appendix 3 : Manifolds of non-positive curvature.

First, let V be a complete Riemannian manifold of strictly negative curvature $K(V)$, and $-k^2 \le K(V) \le -1$. If the manifold V has $\mathrm{Vol}(V) < \infty$, then the geometry of V at infinity is fairly simple. The complement of some compact submanifold V_o with boundary is a union of finitely many cusps. These are described as follows. First one shows that the manifold V only has finitely many ends. Furthermore, the distance between any two rays going to one end is bounded. To get a better picture, we pass to the universal covering $\tilde{V} \to V$ and lift a ray corresponding to an end of V, to a ray $\gamma \subset \tilde{V}$. Then, we consider the horofunction corresponding to γ, called $h(v) = h_\gamma(v)$ defined for all $v \in \tilde{V}$ by $h(v) = \lim_{x \to \infty} (\mathrm{dist}(v,x) - \mathrm{dist}(v_o,x))$, where x are points on γ and v_o denotes the initial point of γ. There is a unique maximal subgroup Π_γ in the fundamental group of V which keeps this function h invariant. This group Π_γ necessarily contains a nilpotent subgroup of finite index whose rank equals $n-1$ for $n = \dim V$. Next, we consider the sets $h^{-1}(-\infty, -t)$, for $t \in \mathbb{R}$, called horoballs $H(t) \subset \tilde{V}$. These horoballs are geodesicly convex and invariant under Π_γ. One shows that for a sufficiently large t the quotient $H(t)/\Pi(\gamma)$ imbeds isometrically into V under the covering map $\tilde{V} \to V$ and this is exactly the cusp which corresponds to γ. It is clear now, that V is indeed concave at infinity as we claimed in section 1.2. For proofs of the properties above see [13].

Now, let $-k^2 \le K(V) \le 0$. Then, the geometry at infinity may be more complicated, even for products of manifolds of strictly negative curvature. However, if the volume $\mathrm{Vol}(V)$ is finite, and if the Riemannian metric in V is real analytic, then the manifold V has "finite topological type" [20] : it is homeomorphic to the interior of a compact manifold with boundary, $V \approx \mathrm{Int}\, V$. Furthermore, if V is a locally symmetric space, then the

boundary $\overline{V}$ has a natural F-structure, as is clear from the analysis performed in [41].

Finally, if V is real analytic and the universal covering $\tilde{V}$ of V has no Euclidean factors in the De Rham decomposition, $\tilde{V} \neq V_o \times \mathbb{R}^1$, then

$$\sum_{i=0}^{n} b_i(V) \leq \text{const}_n k^n \text{Vol}(V) ,$$

for Betti numbers b_i with arbitrary coefficients. The proof will appear elsewhere.

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