# Supersymmetric quantum theory and non-commutative geometry

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# Supersymmetric Quantum Theory

# and

# Non-Commutative Geometry

by

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**Abstract**

Classical differential geometry can be encoded in spectral data, such as Connes' spectral triples, involving supersymmetry algebras. In this paper, we formulate non-commutative geometry in terms of supersymmetric spectral data. This leads to generalizations of Connes' non-commutative spin geometry encompassing non-commutative Riemannian, symplectic, complex-Hermitian and (Hyper-) Kähler geometry. A general framework for non-commutative geometry is developed from the point of view of supersymmetry and illustrated in terms of examples. In particular, the non-commutative torus and the non-commutative 3-sphere are studied in some detail.

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## Contents

## 1. Introduction

The study of highly singular geometrical spaces, such as the space of leaves of certain foliations, of discrete spaces, and the study of quantum theory have led A. Connes to develop a general theory of non-commutative geometry, involving non-commutative measure theory, cyclic cohomology, non-commutative differential topology and spectral calculus, [*Co1 − 5*]. A broad exposition of his theory and a rich variety of interesting examples can be found in his book [*Co1*]. Historically, the first examples of non-commutative spaces carrying geometrical structure emerged from non-relativistic quantum mechanics, as discovered by Heisenberg, Born, Jordan, Schrödinger and Dirac. Mathematically speaking, non-relativistic quantum mechanics is the theory of quantum phase spaces, which are non-commutative deformations of certain classical phase spaces (i.e., of certain symplectic manifolds), and it is the theory of dynamics on quantum phase spaces. Geometrical aspects of quantum phase spaces and supersymmetry entered the scene *implicitly* in Pauli's theory of the non-relativistic, spinning electron and in the theory of non-relativistic positronium. Later on, the mathematicians discovered Pauli's and Dirac's theories of the electron as a powerful tool in algebraic topology and differential geometry.

In a companion paper [*FGR1*], hereafter referred to as I, we have described a formulation of classical differential geometry in terms of the spectral data of non-relativistic, supersymmetric quantum theory, in particular in terms of the quantum theory of the non-relativistic electron and of positronium propagating on a general (spin$^c$) Riemannian manifold. The work in I is inspired by Connes' fundamental work [*Co1 − 5*], and by Witten's work on supersymmetric quantum theory and its applications in algebraic topology [*Wi1, 2*]; it attempts to merge these two threads of thought. Additional inspiration has come from the work in [*AG, FW, AGF, HKLR*] on the relation between index theory and supersymmetric quantum theory and on supersymmetric non-linear $\sigma$-models, as well as from the work by Jaffe and co-workers on connections between supersymmetry and cyclic cohomology [*Ja1 − 3*]. To elucidate the roots of some of these ideas in Pauli's non-relativistic quantum theory of the electron and of positronium has proven useful and suggestive of various generalizations.

The work described in the present paper has its origins in an attempt to apply the methods of non-commutative geometry to exploring the geometry of string theory, in particular of superstring vacua; see [*CF, FG*]. In trying to combine quantum theory with the theory of gravitation, one observes that it is impossible to localize events in space-time arbitrarily precisely, and that, in a compact region of space-time, one can only resolve a finite number of distinct events [*DFR*]. One may then argue, heuristically, that space-time itself must have quantum-mechanical features at distance scales of the order of the Planck length, and that space-time and matter should be merged into a fundamental quantum theory of space-time-matter. Superstring theory [*GSW*] is a theoretical framework incorporating some of the features necessary for a unification of quantum theory and the theory of gravitation. Superstring vacua are described by certain superconformal field theories, see e.g. [*GSW*]. The intention of the program formulated in [*CF, FG*] is to reconstruct space-time geometry from algebraic data of superconformal field theory. In the study of concrete examples, one observes that, in general, the target spaces (space-times) of superconformal field theories are *non-commutative* geometrical spaces, and the tools of Connes' non-commutative geometry become essential in describing their geometry. This observation has been confirmed more recently in the theory of D-branes [*Pol*, *Wi4*].

The purpose of this paper is to cast some of the tools of non-commutative (differential) geometry into a form that makes connections to supersymmetric quantum theory manifest and that is particularly useful for applications to superconformal field theory. The methods and results of this paper are mathematically precise. Applications to physics are not treated here; but see e.g. [*FGR2*]. Instead, the general formalism developed in this paper is illustrated by an analysis of the geometry of the non-commutative torus and of the fuzzy 3-sphere; more details can be found in [*Gr*].

Next, we sketch some of the key ideas underlying our approach to non-commutative geometry; for further background see also part I and [*FGR2*].

Connes has shown how to formulate classical geometry in terms of algebraic data, so-called *spectral triples*, involving a commutative algebra $\mathcal{A} = C^\infty(M)$ of (smooth) functions on the smooth manifold $M$ under consideration, a Hilbert space $\mathcal{H}$ of spinors over $M$ on which the algebra $\mathcal{A}$ acts by bounded operators, and a self-adjoint Dirac operator $D$ on $\mathcal{H}$ satisfying certain properties with respect to $\mathcal{A}$. As explained in [*Co1*], it is possible to extract complete geometrical information about $M$ from the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.
The definition of spectral triples involves, in the classical case, a Clifford action on certain vector bundles over $M$, e.g. the spinor bundle or the bundle of differential forms. As was recalled in ref. I, the latter bundle actually carries two anti-commuting Clifford actions – which can be used to define two Dirac-Kähler operators, $\mathcal{D}$ and $\overline{\mathcal{D}}$. It turns out that the algebraic relations between these operators are precisely those of the two supercharges of $N = (1,1)$ supersymmetric quantum mechanics (see part I, especially section 3, for the precise meaning of the terminology): These relations are $\{\mathcal{D}, \overline{\mathcal{D}}\} = 0$ and $\mathcal{D}^2 = \overline{\mathcal{D}}^2$. The commutators $[\mathcal{D}, a]$ and $[\overline{\mathcal{D}}, a]$, for arbitrary $a \in \mathcal{A}$, extend to bounded operators (anti-commuting sections of two Clifford bundles) acting on the Hilbert space $\mathcal{H}$ of square-integrable differential forms. Furthermore, if the underlying manifold $M$ is compact, the operator $\exp(-\varepsilon\mathcal{D}^2)$ is trace-class for any $\varepsilon > 0$. One may then introduce a nilpotent operator

$$d := \mathcal{D} - i\overline{\mathcal{D}} \ ,$$

which turns out to correspond to exterior differentiation of differential forms.
From the $N = (1,1)$ supersymmetric spectral data $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \overline{\mathcal{D}})$ just described, one can reconstruct the de Rham-Hodge theory and the Riemannian geometry of smooth (compact) Riemannian manifolds.

$N = (1,1)$ supersymmetric spectral data are a variant of Connes' approach involving spectral triples. They are very natural from the point of view of supersymmetric quantum theory and encode the differential geometry of Riemannian manifolds (not required to be spin$^c$ manifolds).

In a formulation of differential geometry in terms of spectral data $(\mathcal{A}, \mathcal{H}, \mathcal{D}, \overline{\mathcal{D}}, \ldots)$ with supersymmetry, additional geometrical structures, e.g. a symplectic or complex structure, appear in the form of *global gauge symmetries* commuting with the elements of $\mathcal{A}$ but acting non-trivially on the Dirac-Kähler operators $\mathcal{D}$ and $\overline{\mathcal{D}}$; see part I. For example, a global gauge symmetry group containing $\mathrm{U}(1) \times \mathrm{U}(1)$ generates four Dirac-Kähler operators – the "supercharges" of $N = (2,2)$ supersymmetry – from $\mathcal{D}$ and $\overline{\mathcal{D}}$ and identifies the underlying manifold $M$ as a *Kähler manifold*. A global gauge symmetry group containing $\mathrm{SU}(2) \times \mathrm{SU}(2)$ leads to eight supercharges generating an $N = (4,4)$ supersymmetry algebra and is characteristic of Hyperkähler geometry; see also [*AGF*, *HKLR*]. Complex-Hermitian and symplectic geometry are encoded in $N = (2,2)$ supersymmetric spectral data with partially *broken* supersymmetry. A systematic classification of different types of differential geometry in terms of supersymmetric spectral data extending the $N = (1,1)$ data of Riemannian geometry has been described in I (see section I 3 for an overview, and [*FGR2*]).

In this paper, we generalize these results from classical to non-commutative geometry, starting from the simple prescription to replace the commutative algebra of functions $C^\infty(M)$ over a classical manifold by a general, possibly non-commutative $^*$-algebra $\mathcal{A}$ satisfying certain properties. Section 2 contains general definitions and introduces various kinds of spectral data: We start with an exposition of Connes' non-commutative spin geometry; most of the material can be found in [*Co1*], but we add some details on metric aspects ranging from connections over curvature and torsion to non-commutative Cartan structure equations. In subsection 2.2, we introduce spectral data with $N = (1,1)$ supersymmetry that naturally lead to a non-commutative analogue of the de Rham complex of differential forms. Moreover, this "Riemannian" formulation of non-commutative geometry allows for immediate specializations to spectral data with extended supersymmetry – which, in the classical case, correspond to manifolds carrying complex, Kähler, Hyperkähler or symplectic structures. Spectral data with higher supersymmetry are treated in subsections 2.3 – 2.5. In subsection 2.2.5, we discuss the relationship between spectral triples, as defined by Connes, and spectral data with $N = (1,1)$ supersymmetry: Whereas in the classical case, one can always pass from one description of a smooth manifold to the other, the situation is not quite as clear in the non-commutative framework. We propose a procedure how to construct $N = (1,1)$ data from a spectral triple – heavily relying on Connes' notion of a real structure [*Co4*] –, but the construction is not complete for general spectral triples. Furthermore, subsection 2.2.6 contains proposals for definitions of non-commutative manifolds and non-commutative phase spaces, as suggested by the study of $N = (1,1)$ spectral data and by notions from quantum physics.

In sections 3 and 4 we discuss two examples of non-commutative spaces, namely the "fuzzy 3-sphere" and the non-commutative torus. The choice of the latter example does not require further explanation since it is one of the classic examples of a non-commutative space; see e.g. [*Co1*, *Co5*, *Ri*]. Here we add a description of the non-commutative 2-torus in terms of spectral data with $N = (1,1)$ and $N = (2,2)$ supersymmetry, thus showing that this space can be endowed with a non-commutative Riemannian and a non-commutative Kähler structure. This is not too surprising, since the non-commutative torus can be regarded as a deformation of the classical flat torus. The calculations in section 4 also provide an example where the general ideas of subsection 2.2.5 on how to construct $N = (1,1)$ from $N = 1$ spectral data can be carried out completely.

The other example, the non-commutative 3-sphere discussed in section 3 (see also [*Gr*]), represents a generalization of another prototype non-commutative geometrical space, namely the fuzzy 2-sphere [*Ber*, *Ho*, *Ma*, *GKP*]. We choose to study the 3-sphere for the following reasons: First, in contrast to the fuzzy 2-sphere and the non-commutative torus, it cannot be viewed as a quantization of a classical phase space. Second, it is the simplest example of a series of quantized spaces arising from so-called Wess-Zumino-Witten-models – conformal field theories associated to non-linear $\sigma$-models with compact simple Lie groups as target manifolds, see [*Wi3*]. There is reason to expect that the spectral data arising from other WZW-models – see [*FG*, *FGR2*] for a discussion – can be treated essentially by the same methods as the fuzzy 3-sphere associated to the group SU(2).
In view of the conformal field theory origin, one is led to conjecture that, as a non-commutative space, the non-commutative 3-sphere describes the non-commutative geometry of the quantum group $U_q(sl_2)$, for $q = \exp(2\pi i/k + 2)$ where $k \in \mathbb{Z}_+$ is the level of the WZW-model. The parameter $k$ appears in the spectral data of the non-commutative 3-sphere in a natural way. One may expect that the fuzzy 3-sphere can actually be defined for arbitrary values of this parameter, since the same is true for the quantum group. As in the example of the non-commutative torus with rational deformation parameter, a truncation of the algebra of "functions" occurs for the special values $k \in \mathbb{Z}_+$, leading to the finite-dimensional matrix algebras used in section 3.

In section 5, we conclude with a list of open problems arising naturally from our discussion. In particular, we briefly comment on other, string theory motivated applications of non-commutative geometry; see also [*FG*, *FGR2*].

The present text is meant as a companion paper to I: Now and then, we will permit ourselves to refer to [*FGR1*] for technical details of proofs which proceed analogously to the classical case. More importantly, the study of classical geometry in part I provides the best justification – besides the one of naturality – of the expectation that our classification of (non-commutative) geometries according to the supersymmetry content of the spectral data leads to useful and fruitful definitions of non-commutative geometrical structure.

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## 2. Spectral data of non-commutative geometry

In the following, we generalize the notions of part I from classical differential geometry to the non-commutative setting. The classification of geometrical structure according to the "supersymmetry content" of the relevant spectral data, which was uncovered in [*FGR1*], will be our guiding principle. In the first part, we review Connes' formulation of non-commutative geometry using a single generalized Dirac operator, whereas, in the following subsections, spectral data with realizations of some genuine supersymmetry algebras will be introduced, allowing us to define non-commutative generalizations of Riemannian, complex, Kähler and Hyperkähler, as well as of symplectic geometry.

### 2.1 The $N = 1$ formulation of non-commutative geometry

This section is devoted to the non-commutative generalization of an algebraic description of spin geometry – and, according to the results of section I2, of general Riemannian geometry – following the ideas of Connes [*Co1*]. The first two subsections contain the definition of abstract $N = 1$ spectral data and of differential forms. In subsection 2.1.3, we describe a notion of integration which leads us to a definition of square integrable differential forms. After having introduced vector bundles and Hermitian structures in subsection 2.1.4, we show in subsection 2.1.5 that the module of square integrable forms always carries a generalized Hermitian structure. We then define connections, torsion, and Riemannian, Ricci and scalar curvature in the next two subsections. Finally, in 2.1.8, we derive non-commutative Cartan structure equations. Although much of the material in section 2.1 is contained (partly in much greater detail) in Connes' book [*Co1*], it is reproduced here because it is basic for our analysis in later sections and because we wish to make this paper accessible to non-experts.

#### 2.1.1 The $N = 1$ spectral data

**Definition 2.1** A quadruple $(\mathcal{A}, \mathcal{H}, D, \gamma)$ will be called a set of $N = 1$ *(even) spectral data* if

1) $\mathcal{H}$ is a separable Hilbert space;
2) $\mathcal{A}$ is a unital $^*$-algebra acting faithfully on $\mathcal{H}$ by bounded operators;
3) $D$ is a self-adjoint operator on $\mathcal{H}$ such that
   *i)* for each $a \in \mathcal{A}$, the commutator $[\,D, a\,]$ defines a bounded operator on $\mathcal{H}$,
   *ii)* the operator $\exp(-\varepsilon D^2)$ is trace class for all $\varepsilon > 0$;
4) $\gamma$ is a $\mathbb{Z}_2$-grading on $\mathcal{H}$, i.e., $\gamma = \gamma^* = \gamma^{-1}$, such that

$$\{\,\gamma, D\,\} = 0\ , \quad [\,\gamma, a\,] = 0 \quad \text{for all} \;\; a \in \mathcal{A}.$$

As mentioned before, in non-commutative geometry $\mathcal{A}$ plays the role of the "algebra of functions over a non-commutative space". The existence of a unit in $\mathcal{A}$, together with property $3\,ii)$ above, reflects the fact that we are dealing with "compact" non-commutative spaces. Note that if the Hilbert space $\mathcal{H}$ is infinite-dimensional, condition $3\,ii)$ implies that the operator $D$ is unbounded. By analogy with classical differential geometry, $D$ is interpreted as a (generalized) Dirac operator.

Also note that the fourth condition in Definition 2.1 does not impose any restriction on $N = 1$ spectral data: In fact, given a triple $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D})$ satisfying the properties 1 - 3 from above, we can define a set of $N = 1$ *even* spectral data $(\mathcal{A}, \mathcal{H}, D, \gamma)$ by setting

$$\mathcal{H} = \widetilde{\mathcal{H}} \otimes \mathbb{C}^2 , \qquad \mathcal{A} = \widetilde{\mathcal{A}} \otimes \mathbf{1}_2 ,$$
$$D = \widetilde{D} \otimes \tau_1 , \qquad \gamma = \mathbf{1}_{\widetilde{\mathcal{H}}} \otimes \tau_3 ,$$

where $\tau_i$ are the Pauli matrices acting on $\mathbb{C}^2$.

### 2.1.2 Differential forms

The construction of differential forms follows the same lines as in classical differential geometry: We define the unital, graded, differential $^*$-algebra of universal forms, $\Omega^\bullet(\mathcal{A})$, as in $[Co1, CoK]$:

$$\Omega^\bullet(\mathcal{A}) = \bigoplus_{k=0}^{\infty} \Omega^k(\mathcal{A}) , \quad \Omega^k(\mathcal{A}) := \{ \sum_{i=1}^{N} a_0^i \delta a_1^i \cdots \delta a_k^i \,|\, N \in \mathbb{N},\ a_j^i \in \mathcal{A} \} , \tag{2.1a}$$

where $\delta$ is an abstract linear operator satisfying $\delta^2 = 0$ and the Leibniz rule. Note that, even in the classical case where $\mathcal{A} = C^\infty(M)$ for some smooth manifold $M$, no relations ensuring (graded) commutativity of $\Omega^\bullet(\mathcal{A})$ are imposed. The complex conjugation of functions over $M$ is now to be replaced by the $^*$-operation of $\mathcal{A}$. We define

$$(\delta a)^* = -\delta(a^*) \tag{2.1b}$$

for all $a \in \mathcal{A}$. With the help of the (self-adjoint) generalized Dirac operator $D$, we introduce a $^*$-representation $\pi$ of $\Omega^\bullet(\mathcal{A})$ on $\mathcal{H}$,

$$\pi(a) = a , \qquad \pi(\delta a) = [\, D, a \,] ,$$

cf. $[Co1]$ or eq. (I2.12). A graded $^*$-ideal $J$ of $\Omega^\bullet(\mathcal{A})$ is defined by

$$J := \bigoplus_{k=0}^{\infty} J^k , \qquad J^k := \ker \pi \,|_{\Omega^k(\mathcal{A})} . \tag{2.2}$$

Since $J$ is not a differential ideal, the graded quotient $\Omega^\bullet(\mathcal{A})/J$ does not define a differential algebra and thus does not yield a satisfactory definition of the algebra of differential forms. This problem is solved as in the classical case.

**Proposition 2.2** [*Co1*] The graded sub-complex

$$J + \delta J = \bigoplus_{k=0}^{\infty} \left( J^k + \delta J^{k-1} \right) ,$$

where $J^{-1} := 0$ and $\delta$ is the universal differential in $\Omega^\bullet(\mathcal{A})$, is a two-sided graded differential $^*$-ideal of $\Omega^\bullet(\mathcal{A})$.

We define the unital graded differential $^*$-algebra of differential forms, $\Omega_D^\bullet(\mathcal{A})$, as the graded quotient $\Omega^\bullet(\mathcal{A})/(J + \delta J)$, i.e.,

$$\Omega_D^\bullet(\mathcal{A}) := \bigoplus_{k=0}^{\infty} \Omega_D^k(\mathcal{A}) \, , \qquad \Omega_D^k(\mathcal{A}) := \Omega^k(\mathcal{A})/(J^k + \delta J^{k-1}) \, . \tag{2.3}$$

Since $\Omega_D^\bullet(\mathcal{A})$ is a graded algebra, each $\Omega_D^k(\mathcal{A})$ is, in particular, a bi-module over $\mathcal{A} = \Omega_D^0(\mathcal{A})$.

Note that $\pi$ does *not* determine a representation of the algebra (or, for that matter, of the space) of differential forms $\Omega_D^\bullet(\mathcal{A})$ on the Hilbert space $\mathcal{H}$: A differential $k$-form is an equivalence class $[\omega] \in \Omega_D^k(\mathcal{A})$ with some representative $\omega \in \Omega^k(\mathcal{A})$, and $\pi$ maps this class to a *set* of bounded operators on $\mathcal{H}$, namely

$$\pi\big([\omega]\big) = \pi(\omega) + \pi\big(\delta J^{k-1}\big) \, .$$

In general, the only subspaces where we do not meet this complication are $\pi\big(\Omega_D^0(\mathcal{A})\big) = \mathcal{A}$ and $\pi\big(\Omega_D^1(\mathcal{A})\big) \cong \pi\big(\Omega^1(\mathcal{A})\big)$. However, the image of $\Omega_D^\bullet(\mathcal{A})$ under $\pi$ is $\mathbb{Z}_2$-graded,

$$\pi\big(\Omega_D^\bullet(\mathcal{A})\big) = \pi\Big(\bigoplus_{k=0}^{\infty} \Omega_D^{2k}(\mathcal{A})\Big) \oplus \pi\Big(\bigoplus_{k=0}^{\infty} \Omega_D^{2k+1}(\mathcal{A})\Big) \, ,$$

because of the (anti-)commutation properties of the $\mathbb{Z}_2$-grading $\gamma$ on $\mathcal{H}$, see Definition 2.1.

### 2.1.3 Integration

Property 3*ii*) of the Dirac operator in Definition 2.1 allows us to define the notion of integration over a non-commutative space in the same way as in the classical case, see part I. Note that, for certain sets of $N = 1$ spectral data, we could use the Dixmier trace, as Connes originally proposed; but the definition given below, first introduced in [*CFF*], works in greater generality (cf. the remarks in section I 2.1.3). Moreover, it is closer to notions coming up naturally in quantum field theory.

**Definition 2.3** The *integral* over the non-commutative space described by the $N=1$ spectral data $(\mathcal{A},\mathcal{H},D,\gamma)$ is a state $⨍$ on $\pi\big(\Omega^\bullet(\mathcal{A})\big)$ defined by

$$⨍ : \begin{cases} \pi\big(\Omega^\bullet(\mathcal{A})\big) \longrightarrow \mathbb{C} \\ \qquad \omega \qquad \longmapsto ⨍\omega := \operatorname*{Lim}_{\varepsilon\to 0^+} \dfrac{\mathrm{Tr}_{\mathcal{H}}\big(\omega e^{-\varepsilon D^2}\big)}{\mathrm{Tr}_{\mathcal{H}}\big(e^{-\varepsilon D^2}\big)} \end{cases},$$

where $\mathrm{Lim}_{\varepsilon\to 0^+}$ denotes some limiting procedure making the functional $⨍$ linear and positive semi-definite; the existence of such a procedure can be shown analogously to [*Co1, 3*], where the Dixmier trace is discussed.

For this integral $⨍$ to be a useful tool, we need an additional property that must be checked in each example:

**Assumption 2.4** The state $⨍$ on $\pi\big(\Omega^\bullet(\mathcal{A})\big)$ is *cyclic*, i.e.,

$$⨍\omega\,\eta^* = ⨍\eta^*\,\omega$$

for all $\omega,\eta\in\pi\big(\Omega^\bullet(\mathcal{A})\big)$.

The state $⨍$ determines a positive semi-definite sesqui-linear form on $\Omega^\bullet(\mathcal{A})$ by setting

$$(\omega,\eta) := ⨍\pi(\omega)\,\pi(\eta)^* \tag{2.4}$$

for all $\omega,\eta\in\Omega^\bullet(\mathcal{A})$. In the formulas below, we will often drop the representation symbol $\pi$ under the integral, as there is no danger of confusion.
Note that the commutation relations of the grading $\gamma$ with the Dirac operator imply that forms of odd degree are orthogonal to those of even degree with respect to $(\cdot,\cdot)$.
By $K^k$ we denote the kernel of this sesqui-linear form restricted to $\Omega^k(\mathcal{A})$. More precisely we set

$$K := \bigoplus_{k=0}^{\infty} K^k\ ,\qquad K^k := \{\,\omega\in\Omega^k(\mathcal{A})\,|\,(\omega,\omega\,)=0\,\}\ . \tag{2.5}$$

Obviously, $K^k$ contains the ideal $J^k$ defined in eq. (2.2); in the classical case they coincide. Assumption 2.4 is needed to show that $K$ is a two-sided ideal of the algebra of universal forms, so that we can pass to the quotient algebra.

**Proposition 2.5** The set $K$ is a two-sided graded $^*$-ideal of $\Omega^\bullet(\mathcal{A})$.

PROOF: The Cauchy-Schwarz inequality for states implies that $K$ is a vector space. If $\omega\in K^k$, then Assumption 2.4 gives

$$(\omega^*,\omega^*) = ⨍\pi(\omega)^*\pi(\omega) = ⨍\pi(\omega)\pi(\omega)^* = 0\ ,$$

i.e. that $K$ is closed under the involution $*$. With $\omega$ as above and $\eta \in \Omega^p(\mathcal{A})$, we have that

$$
\begin{aligned}
(\eta\omega, \eta\omega) &= \int\!\!\!\!\!\!- \; \pi(\eta)\pi(\omega)\pi(\omega)^*\pi(\eta)^* = \int\!\!\!\!\!\!- \; \pi(\omega)^*\pi(\eta)^*\pi(\eta)\pi(\omega) \\
&\leq \|\pi(\eta)\|^2_{\mathcal{H}} \int\!\!\!\!\!\!- \; \pi(\omega)^*\pi(\omega) = 0
\end{aligned}
$$

where $\|\cdot\|_{\mathcal{H}}$ is the operator norm on $\mathcal{B}(\mathcal{H})$. On the other hand, we have that

$$
(\omega\eta, \omega\eta) = \int\!\!\!\!\!\!- \; \pi(\omega)\pi(\eta)\pi(\eta)^*\pi(\omega)^* \leq \|\pi(\eta)\|^2_{\mathcal{H}} \int\!\!\!\!\!\!- \; \pi(\omega)\pi(\omega)^* = 0 \;,
$$

and it follows that both $\omega\,\eta$ and $\eta\,\omega$ are elements of $K$, i.e., $K$ is a two-sided ideal. ∎

We now define

$$
\widetilde{\Omega}^\bullet(\mathcal{A}) := \bigoplus_{k=0}^{\infty} \widetilde{\Omega}^k(\mathcal{A}) \;, \quad \widetilde{\Omega}^k(\mathcal{A}) := \Omega^k(\mathcal{A})/K^k \;. \tag{2.6}
$$

The sesqui-linear form $(\cdot,\cdot)$ descends to a positive definite scalar product on $\widetilde{\Omega}^k(\mathcal{A})$, and we denote by $\widetilde{\mathcal{H}}^k$ the Hilbert space completion of this space with respect to the scalar product,

$$
\widetilde{\mathcal{H}}^\bullet := \bigoplus_{k=0}^{\infty} \widetilde{\mathcal{H}}^k \;, \quad \widetilde{\mathcal{H}}^k := \overline{\widetilde{\Omega}^k(\mathcal{A})}^{(\cdot,\cdot)} \;. \tag{2.7}
$$

$\widetilde{\mathcal{H}}^k$ is to be interpreted as the *space of square-integrable k-forms*. Note that $\widetilde{\mathcal{H}}^\bullet$ does not in general coincide with the Hilbert space that would arise from a GNS construction using the state $\int\!\!\!\!\!-$ on $\widetilde{\Omega}^\bullet(\mathcal{A})$: Whereas in $\widetilde{\mathcal{H}}^\bullet$, orthogonality of forms of different degree is installed by definition, there may exist forms of even degree (or odd forms) in the GNS Hilbert space that have different degrees but are not orthogonal.

**Corollary 2.6** The space $\widetilde{\Omega}^\bullet(\mathcal{A})$ is a unital graded $^*$-algebra. For any $\omega \in \widetilde{\Omega}^k(\mathcal{A})$, the left and right actions of $\omega$ on $\widetilde{\Omega}^p(\mathcal{A})$ with values in $\widetilde{\Omega}^{p+k}(\mathcal{A})$,

$$
m_L(\omega)\eta := \omega\eta \;, \quad m_R(\omega)\eta := \eta\omega \;,
$$

are continuous in the norm given by $(\cdot,\cdot)$. In particular, the Hilbert space $\widetilde{\mathcal{H}}^\bullet$ is a bi-module over $\widetilde{\Omega}^\bullet(\mathcal{A})$ with continuous actions.

Proof: The claim follows immediately from the two estimates given in the proof of the previous proposition, applied to $\omega \in \widetilde{\Omega}^k(\mathcal{A})$ and $\eta \in \widetilde{\Omega}^p(\mathcal{A})$. ∎

This remark shows that $\widetilde{\Omega}^\bullet(\mathcal{A})$ and $\widetilde{\mathcal{H}}^\bullet$ are "well-behaved" with respect to the $\widetilde{\Omega}^\bullet(\mathcal{A})$-action. Furthermore, Corollary 2.6 will be useful for our discussion of curvature and torsion in sections 2.1.7 and 2.1.8.

Since the algebra $\widetilde{\Omega}^\bullet(\mathcal{A})$ may fail to be differential, we introduce the unital graded differential $^*$-algebra of square-integrable differential forms $\widetilde{\Omega}^\bullet_D(\mathcal{A})$ as the graded quotient of $\Omega^\bullet(\mathcal{A})$ by $K + \delta K$,

$$\widetilde{\Omega}^\bullet_D(\mathcal{A}) := \bigoplus_{k=0}^{\infty} \widetilde{\Omega}^k_D(\mathcal{A}) \ , \quad \widetilde{\Omega}^k_D(\mathcal{A}) := \Omega^k(\mathcal{A})/(K^k + \delta K^{k-1}) \cong \widetilde{\Omega}^k(\mathcal{A})/\delta K^{k-1} \ . \tag{2.8}$$

In order to show that $\widetilde{\Omega}^\bullet_D(\mathcal{A})$ has the stated properties, one repeats the proof of Proposition 2.2. Note that we can regard the $\mathcal{A}$-bi-module $\widetilde{\Omega}^\bullet_D(\mathcal{A})$ as a "smaller version" of $\Omega^\bullet_D(\mathcal{A})$ in the sense that there exists a projection from the latter onto the former; whenever one deals with a concrete set of $N = 1$ spectral data that satisfy Assumption 2.4, it will be advantageous to work with the "smaller" algebra of square-integrable differential forms. The algebra $\Omega^\bullet_D(\mathcal{A})$, on the other hand, can be defined for arbitrary data.

In the classical case, differential forms are identified with the orthogonal complement of $Cl^{(k-2)}$ within $Cl^{(k)}$, see [*Co1* ] and the remarks in part I, after eq. (I2.15). Now, we use the scalar product $(\cdot,\cdot)$ on $\widetilde{\mathcal{H}}^k$ to introduce, for each $k \geq 1$, the orthogonal projection

$$P_{\delta K^{k-1}} \ : \ \widetilde{\mathcal{H}}^k \longrightarrow \widetilde{\mathcal{H}}^k \tag{2.9}$$

onto the image of $\delta K^{k-1}$ in $\widetilde{\mathcal{H}}^k$, and we set

$$\omega^\perp := (1 - P_{\delta K^{k-1}})\,\omega \in \widetilde{\mathcal{H}}^k \tag{2.10}$$

for each element $[\omega] \in \widetilde{\Omega}^k_D(\mathcal{A})$. This allows us to define a positive definite scalar product on $\widetilde{\Omega}^k_D(\mathcal{A})$ via the representative $\omega^\perp$:

$$(\,[\omega],[\eta]\,) := (\,\omega^\perp,\eta^\perp\,) \tag{2.11}$$

for all $[\omega],[\eta] \in \widetilde{\Omega}^k_D(\mathcal{A})$. In the classical case, this is just the usual inner product on the space of square-integrable $k$-forms.

### 2.1.4 Vector bundles and Hermitian structures

Again, we simply follow the algebraic formulation of classical differential geometry in order to generalize the notion of a vector bundle to the non-commutative case:

**Definition 2.7** [*Co1* ] A *vector bundle* $\mathcal{E}$ over the non-commutative space described by the $N = 1$ spectral data $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is a finitely generated projective left $\mathcal{A}$-module.

Recall that a module $\mathcal{E}$ is *projective* if there exists another module $\mathcal{F}$ such that the direct sum $\mathcal{E} \oplus \mathcal{F}$ is *free*, i.e., $\mathcal{E} \oplus \mathcal{F} \cong \mathcal{A}^n$ as left $\mathcal{A}$-modules,for some $n \in \mathbb{N}$. Since $\mathcal{A}$ is an algebra, every $\mathcal{A}$-module is a vector space; therefore, left $\mathcal{A}$-modules are representations of the algebra $\mathcal{A}$, and $\mathcal{E}$ is projective iff there exists a module $\mathcal{F}$ such that $\mathcal{E} \oplus \mathcal{F}$ is isomorphic to a multiple of the left-regular representation.

By Swan's Lemma [*Sw*], a finitely generated projective left module corresponds, in the commutative case, to the space of sections of a vector bundle. With this in mind, it is straightforward to define the notion of a Hermitian structure over a vector bundle:

**Definition 2.8** [*Co1*] A *Hermitian structure* over a vector bundle $\mathcal{E}$ is a sesqui-linear map (linear in the first argument)

$$\langle \cdot , \cdot \rangle \, : \, \mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{A}$$

such that for all $a, b \in \mathcal{A}$ and all $s, t \in \mathcal{E}$

1) $\langle\, as, bt \,\rangle = a \,\langle\, s, t \,\rangle\, b^*$ ;
2) $\langle\, s, s \,\rangle \geq 0$ ;
3) the $\mathcal{A}$-linear map
$$g : \begin{cases} \mathcal{E} \longrightarrow \mathcal{E}^*_R \\ s \longmapsto \langle\, s, \cdot \,\rangle \end{cases} ,$$
where $\mathcal{E}^*_R := \{\phi \in \mathrm{Hom}(\mathcal{E},\mathcal{A}) \,|\, \phi(as) = \phi(s)a^* \}$, is an isomorphism of left $\mathcal{A}$-modules, i.e., $g$ can be regarded as a metric on $\mathcal{E}$.

### 2.1.5 Generalized Hermitian structure on $\widetilde{\Omega}^k(\mathcal{A})$

In this section we show that the $\mathcal{A}$-bi-modules $\widetilde{\Omega}^k(\mathcal{A})$ carry Hermitian structures in a slightly generalized sense. Let $\overline{\mathcal{A}}$ be the weak closure of the algebra $\mathcal{A}$ acting on $\widetilde{\mathcal{H}}^0$, i.e., $\overline{\mathcal{A}}$ is the von Neumann algebra generated by $\widetilde{\Omega}^0(\mathcal{A})$ acting on the Hilbert space $\widetilde{\mathcal{H}}^0$.

**Theorem 2.9** There is a canonically defined sesqui-linear map

$$\langle \cdot , \cdot \rangle_D \, : \, \widetilde{\Omega}^k(\mathcal{A}) \times \widetilde{\Omega}^k(\mathcal{A}) \longrightarrow \overline{\mathcal{A}}$$

such that for all $a, b \in \mathcal{A}$ and all $\omega, \eta \in \widetilde{\Omega}^k(\mathcal{A})$

1) $\langle\, a\omega, b\eta \,\rangle_D = a \,\langle\, \omega, \eta \,\rangle_D\, b^*$ ;
2) $\langle\, \omega, \omega \,\rangle_D \geq 0$ ;
3) $\langle\, \omega\, a, \eta \,\rangle_D = \langle\, \omega, \eta\, a^* \,\rangle_D$ .

We call $\langle \cdot , \cdot \rangle_D$ a *generalized Hermitian structure on* $\widetilde{\Omega}^k(\mathcal{A})$. It is the non-commutative analogue of the Riemannian metric on the bundle of differential forms. Note that $\langle \cdot , \cdot \rangle_D$ takes values in $\overline{\mathcal{A}}$ and thus property 3) of Definition 2.8 is not directly applicable.

Proof: Let $\omega, \eta \in \widetilde{\Omega}^k(\mathcal{A})$ and define the $\mathbb{C}$-linear map

$$\varphi_{\omega,\eta}(a) = \int\!\!\!\!\!\!- \, a\,\eta\,\omega^* \ ,$$

for all $a \in \widetilde{\Omega}^0(\mathcal{A})$. Note that $a$ on the rhs actually is a representative in $\mathcal{A}$ of the class $a \in \widetilde{\Omega}^0(\mathcal{A})$, and analogously for $\omega$ and $\eta$ (and we have omitted the representation symbol $\pi$). The value of the integral is, however, independent of the choice of these representatives, which is why we used the same letters. The map $\varphi$ satisfies

$$|\varphi_{\omega,\eta}(a)| \le \Big| ⨍ aa^* \Big|^{\frac{1}{2}} \Big| ⨍ \omega\eta^*\eta\omega^* \Big|^{\frac{1}{2}} \le (a,a)^{\frac{1}{2}} \Big| ⨍ \omega\eta^*\eta\omega^* \Big|^{\frac{1}{2}} .$$

Therefore, $\varphi_{\omega,\eta}$ extends to a bounded linear functional on $\widetilde{\mathcal{H}}^0$, and there exists an element $\langle\, \omega, \eta\, \rangle_D \in \widetilde{\mathcal{H}}^0$ such that

$$\varphi_{\omega,\eta}(x) = (x, \langle\, \omega, \eta\, \rangle_D\, )$$

for all $x \in \widetilde{\mathcal{H}}^0$; since $(\cdot,\cdot)$ is non-degenerate, $\langle\, \omega, \eta\, \rangle_D$ is a well-defined element; but it remains to show that it also acts as a bounded operator on this Hilbert space. To this end, choose a net $\{a_\iota\} \subset \widetilde{\Omega}^0(\mathcal{A})$ which converges to $\langle\, \omega, \eta\, \rangle_D$. Then, for all $b, c \in \widetilde{\Omega}^0(\mathcal{A})$,

$$\begin{aligned} (\, \langle\, \omega, \eta\, \rangle_D b, c\, ) &= \lim_{\iota\to\infty} (\, a_\iota b, c\, ) = \lim_{\iota\to\infty} ⨍ a_\iota b c^* = \lim_{\iota\to\infty} ⨍ a_\iota (cb^*)^* \\ &= \lim_{\iota\to\infty} (\, a_\iota, cb^*\, ) = (\, \langle\, \omega, \eta\, \rangle_D, cb^*\, ) \ , \end{aligned}$$

and it follows that

$$\begin{aligned} |(\, \langle\, \omega, \eta\, \rangle_D b, c\, )| &= |(\, \langle\, \omega, \eta\, \rangle_D, cb^*\, )| = |(\, cb^*, \langle\, \omega, \eta\, \rangle_D\, )| \\ &= \Big| ⨍ cb^*\eta\,\omega^* \Big| = \Big| ⨍ \omega^* cb^*\eta \Big| = \Big| ⨍ b^*\eta\,\omega^* c \Big| \\ &\le \Big| ⨍ b^*b \Big|^{\frac{1}{2}} \Big| ⨍ c^*\omega\,\eta^*\eta\omega^* c \Big|^{\frac{1}{2}} \le \|\,\omega\eta^*\|_{\mathcal{H}} \Big| ⨍ b^*b \Big|^{\frac{1}{2}} \Big| ⨍ c^*c \Big|^{\frac{1}{2}} \\ &\le \|\,\omega\eta^*\|_{\mathcal{H}}\, (b,b)^{\frac{1}{2}} (c,c)^{\frac{1}{2}} \ . \end{aligned}$$

In the third line, we first use the Cauchy-Schwarz inequality for the positive state $⨍$ , and then an estimate which is true for all positive operators on a Hilbert space; the upper bound $\|\,\omega\eta^*\|_{\mathcal{H}}$ again involves representatives $\omega, \eta \in \pi\big(\Omega^k(\mathcal{A})\big)$, which was not explicitly indicated above, since any two will do.
As $\widetilde{\Omega}^0(\mathcal{A})$ is dense in $\widetilde{\mathcal{H}}^0$, we see that $\langle\, \omega, \eta\, \rangle_D$ indeed defines a bounded operator in $\widetilde{\mathcal{H}}^0$, which, by definition, is the weak limit of elements in $\widetilde{\Omega}^0(\mathcal{A})$, i.e., it belongs to $\overline{\mathcal{A}}$. Properties 1-3 of $\langle \cdot,\cdot \rangle_D$ are easy to verify. ∎

Note that the definition of the metric $\langle \cdot,\cdot \rangle_D$ given here differs slightly from the one of refs. $[CFF, CFG]$. One can, however, show that in the $N = 1$ case both definitions agree; moreover, the present one is better suited for the $N = (1,1)$ formulation to be introduced later.

### 2.1.6 Connections

**Definition 2.10** A *connection* $\nabla$ on a vector bundle $\mathcal{E}$ over a non-commutative space is a $\mathbb{C}$-linear map

$$\nabla \,:\, \mathcal{E} \longrightarrow \widetilde{\Omega}^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}$$

such that

$$\nabla(as) = \delta a \otimes s + a\nabla s$$

for all $a \in \mathcal{A}$ and all $s \in \mathcal{E}$.

Given a vector bundle $\mathcal{E}$, we define a space of $\mathcal{E}$-valued differential forms by

$$\widetilde{\Omega}^\bullet_D(\mathcal{E}) := \widetilde{\Omega}^\bullet_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E} \;;$$

if $\nabla$ is a connection on $\mathcal{E}$, then it extends uniquely to a $\mathbb{C}$-linear map, again denoted $\nabla$,

$$\nabla \,:\, \widetilde{\Omega}^\bullet_D(\mathcal{E}) \longrightarrow \widetilde{\Omega}^{\bullet+1}_D(\mathcal{E}) \tag{2.12}$$

such that

$$\nabla(\omega s) = \delta\omega\, s + (-1)^k \omega\, \nabla s \tag{2.13}$$

for all $\omega \in \widetilde{\Omega}^k_D(\mathcal{A})$ and all $s \in \widetilde{\Omega}^\bullet_D(\mathcal{E})$.

**Definition 2.11** The *curvature* of a connection $\nabla$ on a vector bundle $\mathcal{E}$ is given by

$$\mathrm{R}\,(\nabla) = -\nabla^2 \,:\, \mathcal{E} \longrightarrow \widetilde{\Omega}^2_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{E}\ .$$

Note that the curvature extends to a map

$$\mathrm{R}\,(\nabla) \,:\, \widetilde{\Omega}^\bullet_D(\mathcal{E}) \longrightarrow \widetilde{\Omega}^{\bullet+2}_D(\mathcal{E})$$

which is left $\mathcal{A}$-linear, as follows easily from eq. (2.12) and Definition 2.10.

**Definition 2.12** A connection $\nabla$ on a Hermitian vector bundle $(\mathcal{E}, \langle\cdot,\cdot\rangle)$ is called *unitary* if

$$\delta\,\langle\, s,t\,\rangle = \langle\, \nabla s, t\,\rangle - \langle\, s, \nabla t\,\rangle$$

for all $s, t \in \mathcal{E}$, where the rhs of this equation is defined by

$$\langle\, \omega \otimes s, t\,\rangle = \omega\,\langle\, s,t\,\rangle\ , \qquad \langle\, s, \eta \otimes t\,\rangle = \langle\, s,t\,\rangle\, \eta^* \tag{2.14}$$

for all $\omega, \eta \in \widetilde{\Omega}^1_D(\mathcal{A})$ and all $s, t \in \mathcal{E}$.

### 2.1.7 Riemannian curvature and torsion

Throughout this section, we make three additional assumptions which limit the generality of our results, but turn out to be fulfilled in interesting examples.

**Assumption 2.13** We assume that the $N = 1$ spectral data under consideration have the following additional properties:

1) $K^0 = 0$. (This implies that $\widetilde{\Omega}^0_D(\mathcal{A}) = \mathcal{A}$ and $\widetilde{\Omega}^1_D(\mathcal{A}) = \widetilde{\Omega}^1(\mathcal{A})$, thus $\widetilde{\Omega}^1_D(\mathcal{A})$ carries a generalized Hermitian structure.)
2) $\widetilde{\Omega}^1_D(\mathcal{A})$ is a vector bundle, called the *cotangent bundle over* $\mathcal{A}$. ($\widetilde{\Omega}^1_D(\mathcal{A})$ is always a left $\mathcal{A}$-module. Here, we assume, in addition, that it is *finitely generated* and *projective.*)
3) The generalized metric $\langle\cdot,\cdot\rangle_D$ on $\widetilde{\Omega}^1_D(\mathcal{A})$ defines an isomorphism of left $\mathcal{A}$-modules between $\widetilde{\Omega}^1_D(\mathcal{A})$ and the space of $\mathcal{A}$-anti-linear maps from $\widetilde{\Omega}^1_D(\mathcal{A})$ to $\mathcal{A}$, i.e., for each $\mathcal{A}$-anti-linear map
$$\phi\,:\,\widetilde{\Omega}^1_D(\mathcal{A}) \longrightarrow \mathcal{A}$$
satisfying $\phi(a\omega) = \phi(\omega)a^*$ for all $\omega \in \widetilde{\Omega}^1_D(\mathcal{A})$ and all $a \in \mathcal{A}$, there is a unique $\eta_\phi \in \widetilde{\Omega}^1_D(\mathcal{A})$ with
$$\phi(\omega) = \langle\,\eta_\phi,\omega\,\rangle_D\ .$$

If $N = 1$ spectral data $(\mathcal{A},\mathcal{H},D,\gamma)$ satisfy these assumptions, we are able to define non-commutative generalizations of classical notions like torsion and curvature. Whereas torsion and Riemann curvature can be introduced whenever $\widetilde{\Omega}^1_D(\mathcal{A})$ is a vector bundle, the last assumption in 2.13 will provide a substitute for the procedure of "contracting indices" leading to Ricci and scalar curvature.

**Definition 2.14** Let $\nabla$ be a connection on the cotangent bundle $\widetilde{\Omega}^1_D(\mathcal{A})$ over a non-commutative space $(\mathcal{A},\mathcal{H},D,\gamma)$ satisfying Assumption 2.13. The *torsion* of $\nabla$ is the $\mathcal{A}$-linear map
$$\mathrm{T}(\nabla) := \delta - m\circ\nabla\,:\,\widetilde{\Omega}^1_D(\mathcal{A}) \longrightarrow \widetilde{\Omega}^2_D(\mathcal{A})$$
where $m\,:\,\widetilde{\Omega}^1_D(\mathcal{A})\otimes_{\mathcal{A}}\widetilde{\Omega}^1_D(\mathcal{A}) \longrightarrow \widetilde{\Omega}^2_D(\mathcal{A})$ denotes the product of 1-forms in $\widetilde{\Omega}^\bullet_D(\mathcal{A})$.

Using the definition of a connection, $\mathcal{A}$-linearity of torsion is easy to verify. In analogy to the classical case, a unitary connection $\nabla$ with $\mathrm{T}(\nabla) = 0$ is called a *Levi-Civita connection.* In the classical case, there is exactly one Levi-Civita connection that, in addition, is a real operator on the complexified bundle of differential forms. In contrast, for a given set of non-commutative spectral data, there may be several (real) Levi-Civita connections – or none at all.

Since we assume that $\widetilde{\Omega}^1_D(\mathcal{A})$ is a vector bundle, we can define the *Riemannian curvature* of a connection $\nabla$ on the cotangent bundle as a specialization of Definition 2.11. To proceed further, we make use of part 2) of Assumption 2.13, which implies that there exists a finite set of generators $\{ E^A \}$ of $\widetilde{\Omega}^1_D(\mathcal{A})$ and an associated "dual basis" $\{ \varepsilon_A \} \subset \widetilde{\Omega}^1_D(\mathcal{A})^*$,

$$\widetilde{\Omega}^1_D(\mathcal{A})^* := \{ \phi : \widetilde{\Omega}^1_D(\mathcal{A}) \longrightarrow \mathcal{A} \,|\, \phi(a\omega) = a\phi(\omega) \quad \text{for all } a \in \mathcal{A}, \omega \in \widetilde{\Omega}^1_D(\mathcal{A}) \} ,$$

such that each $\omega \in \widetilde{\Omega}^1_D(\mathcal{A})$ can be written as $\omega = \varepsilon_A(\omega) E^A$, see e.g. [*Jac*]. Because the curvature is $\mathcal{A}$-linear, there is a family of elements $\{ \mathrm{R}^A{}_B \} \subset \widetilde{\Omega}^2_D(\mathcal{A})$ with

$$\mathrm{R}\,(\nabla) = \varepsilon_A \otimes \mathrm{R}^A{}_B \otimes E^B ; \tag{2.15}$$

here and in the following the summation convention is used. Put differently, we have applied the canonical isomorphism of vector spaces

$$\mathrm{Hom}_{\mathcal{A}}\big(\widetilde{\Omega}^1_D(\mathcal{A}),\ \widetilde{\Omega}^2_D(\mathcal{A}) \otimes_{\mathcal{A}} \widetilde{\Omega}^1_D(\mathcal{A})\big) \ \cong\ \widetilde{\Omega}^1_D(\mathcal{A})^* \otimes_{\mathcal{A}} \widetilde{\Omega}^2_D(\mathcal{A}) \otimes_{\mathcal{A}} \widetilde{\Omega}^1_D(\mathcal{A})$$

– which is valid because $\widetilde{\Omega}^1_D(\mathcal{A})$ is projective – and chosen explicit generators $E^A, \varepsilon_A$. Then we have that $\mathrm{R}\,(\nabla)\,\omega = \varepsilon_A(\omega)\,\mathrm{R}^A{}_B \otimes E^B$ for any 1-form $\omega \in \widetilde{\Omega}^1_D(\mathcal{A})$.
Note that although the components $\mathrm{R}^A{}_B$ need not be unique, the element on the rhs of eq. (2.15) is well-defined. Likewise, the Ricci and scalar curvature, to be introduced below, will be *invariant* combinations of those components, as long as we make sure that all maps we use have the correct "tensorial properties" with respect to the $\mathcal{A}$-action.
The last part of Assumption 2.13 guarantees, furthermore, that to each $\varepsilon_A$ there exists a unique 1-form $e_A \in \widetilde{\Omega}^1_D(\mathcal{A})$ such that

$$\varepsilon_A(\omega) = \langle\, \omega, e_A \,\rangle_D$$

for all $\omega \in \widetilde{\Omega}^1_D(\mathcal{A})$. By Corollary 2.6, every such $e_A$ determines a bounded operator $m_L(e_A) : \widetilde{\mathcal{H}}^1 \longrightarrow \widetilde{\mathcal{H}}^2$ acting on $\widetilde{\mathcal{H}}^1$ by left multiplication with $e_A$. The adjoint of this operator with respect to the scalar product $(\cdot,\cdot)$ on $\widetilde{\mathcal{H}}^\bullet$ is denoted by

$$e_A^{\mathrm{ad}} : \widetilde{\mathcal{H}}^2 \longrightarrow \widetilde{\mathcal{H}}^1 . \tag{2.16}$$

$e_A^{\mathrm{ad}}$ is a map of right $\mathcal{A}$-modules, and it is easy to see that also the correspondence $\varepsilon_A \mapsto e_A^{\mathrm{ad}}$ is right $\mathcal{A}$-linear: For all $b \in \mathcal{A},\ \omega \in \widetilde{\Omega}^1_D(\mathcal{A})$, we have

$$(\varepsilon_A \cdot b)(\omega) = \varepsilon_A(\omega) \cdot b = \langle\, \omega, e_A \,\rangle\, b = \langle\, \omega, b^* e_A \,\rangle ,$$

and, furthermore, for all $\xi_1 \in \widetilde{\mathcal{H}}^1,\ \xi_2 \in \widetilde{\mathcal{H}}^2$,

$$(\, b^* e_A(\xi_1), \xi_2 \,) = (\, e_A(\xi_1), b\xi_2 \,) = (\, \xi_1, e_A^{\mathrm{ad}}(b\xi_2) \,) ,$$

where scalar products have to be taken in the appropriate spaces $\widetilde{\mathcal{H}}^k$. Altogether, the asserted right $\mathcal{A}$-linearity follows. Therefore, the map

$$\varepsilon_A \otimes \mathrm{R}^A{}_B \otimes E^B \longmapsto e_A^{\mathrm{ad}} \otimes \mathrm{R}^A{}_B \otimes E^B$$

is well-defined and has the desired tensorial properties. The definition of Ricci curvature involves another operation which we require to be similarly well-behaved:

**Lemma 2.15** The orthogonal projections $P_{\delta K^{k-1}}$ on $\widetilde{\mathcal{H}}^k$, see eq. (2.9), satisfy

$$P_{\delta K^{k-1}}(axb) = aP_{\delta K^{k-1}}(x)b$$

for all $a, b \in \mathcal{A}$ and all $x \in \widetilde{\mathcal{H}}^k$.

PROOF: Set $P := P_{\delta K^{k-1}}$, and let $y \in P\widetilde{\mathcal{H}}^k$. Then

$$(\, P(axb), y\,) = (\, axb, P(y)\,) = (\, axb, y\,) = (\, x, a^*yb^*\,) = (\, x, P(a^*yb^*)\,) = (\, aP(x)b, y\,)\ ,$$

where we have used that $P$ is self-adjoint with respect to $(\cdot,\cdot)$, that $Py = y$, and that the image of $P$ is an $\mathcal{A}$-bi-module. ∎

This lemma shows that projecting onto the "2-form part" of $\mathtt{R}^A{}_B$ is an $\mathcal{A}$-bi-module map, i.e., we may apply

$$e_A^{\mathrm{ad}} \otimes \mathtt{R}^A{}_B \otimes E^B \longmapsto e_A^{\mathrm{ad}} \otimes \left(\mathtt{R}^A{}_B\right)^{\perp} \otimes E^B$$

with $\left(\mathtt{R}^A{}_B\right)^{\perp} = (1 - P_{\delta K^1})\,\mathtt{R}^A{}_B$ as in eq. (2.10). Altogether, we arrive at the following definition of the *Ricci curvature*,

$$\mathtt{Ric}(\nabla) = e_A^{\mathrm{ad}}\Big(\left(\mathtt{R}^A{}_B\right)^{\perp}\Big) \otimes E^B \in \widetilde{\mathcal{H}}^1 \otimes_{\mathcal{A}} \widetilde{\Omega}^1_D(\mathcal{A})\ ,$$

which is in fact independent of any choices. In the following, we will also use the abbreviation

$$\mathtt{Ric}_B := e_A^{\mathrm{ad}}\Big(\left(\mathtt{R}^A{}_B\right)^{\perp}\Big)$$

for the components (which, again, are not uniquely defined).

From the components $\mathtt{Ric}_B$ we can pass to scalar curvature. Again, we have to make sure that all maps occurring in this process are $\mathcal{A}$-covariant so as to obtain an invariant definition. For any 1-form $\omega \in \widetilde{\Omega}^1_D(\mathcal{A})$, right multiplication on $\widetilde{\mathcal{H}}^0$ with $\omega$ defines a bounded operator $m_R(\omega)\ :\ \widetilde{\mathcal{H}}^0 \longrightarrow \widetilde{\mathcal{H}}^1$, and we denote by

$$\omega_R^{\mathrm{ad}}\ :\ \widetilde{\mathcal{H}}^1 \longrightarrow \widetilde{\mathcal{H}}^0 \tag{2.17}$$

the adjoint of this operator. In a similar fashion as above, one establishes that

$$(\omega a)_R^{\mathrm{ad}}(x) = \omega_R^{\mathrm{ad}}(xa^*)$$

for all $x \in \widetilde{\mathcal{H}}^1$ and $a \in \mathcal{A}$. This makes it possible to define the *scalar curvature* $\mathtt{r}\,(\nabla)$ of a connection $\nabla$ as

$$\mathtt{r}\,(\nabla) = \left(E^{B*}\right)_R^{\mathrm{ad}}(\mathtt{Ric}_B) \in \widetilde{\mathcal{H}}^0\ .$$

As was the case for the Ricci tensor, acting with the adjoint of $m_R\big(E^{B*}\big)$ serves as an analogue for "contraction of indices". We summarize our results in the following

**Definition 2.16** Let $\nabla$ be a connection on the cotangent bundle $\widetilde{\Omega}^1_D(\mathcal{A})$ over a non-commutative space $(\mathcal{A}, \mathcal{H}, D, \gamma)$ satisfying Assumption 2.13. The *Riemannian curvature* $\mathtt{R}(\nabla)$ is the left $\mathcal{A}$-linear map

$$\mathtt{R}(\nabla) = -\nabla^2 \,:\, \widetilde{\Omega}^1_D(\mathcal{A}) \longrightarrow \widetilde{\Omega}^2_D(\mathcal{A}) \otimes_{\mathcal{A}} \widetilde{\Omega}^1_D(\mathcal{A}) \,.$$

Choosing a set of generators $E^A$ of $\widetilde{\Omega}^1_D(\mathcal{A})$ and dual generators $\varepsilon_A$ of $\widetilde{\Omega}^1_D(\mathcal{A})^*$, and writing $\mathtt{R}(\nabla) = \varepsilon_A \otimes \mathtt{R}^A{}_B \otimes E^B$ as above, the *Ricci tensor* $\mathtt{Ric}\,(\nabla)$ is given by

$$\mathtt{Ric}(\nabla) = \mathtt{Ric}_B \otimes E^B \in \widetilde{\mathcal{H}}^1 \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) \,,$$

where $\mathtt{Ric}_B := e^{\mathrm{ad}}_A\Big(\big(\mathtt{R}^A{}_B\big)^\perp\Big)$, see eqs. (2.10) and (2.16). Finally, the *scalar curvature* $\mathtt{r}\,(\nabla)$ of the connection $\nabla$ is defined as

$$\mathtt{r}\,(\nabla) = \big(E^{B*}\big)^{\mathrm{ad}}_R(\mathtt{Ric}_B) \in \widetilde{\mathcal{H}}^0 \,,$$

with the notation of eq. (2.17). (Note that, in the classical case, our definition of the scalar curvature differs from the usual one by a sign.) Both $\mathtt{Ric}(\nabla)$ and $\mathtt{r}\,(\nabla)$ do not depend on the choice of generators.

### 2.1.8 Non-commutative Cartan structure equations

The classical Cartan structure equations are an important tool for explicit calculations in differential geometry. Non-commutative analogues of those equations were obtained in [*CFF*, *CFG*]. Since proofs were only sketched in these references, we will give a rather detailed account of their results in the following. Throughout this section, we assume that the space $\widetilde{\Omega}^1_D(\mathcal{A})$ is a vector bundle over $\mathcal{A}$. In fact, no other properties of this space are used. Therefore all the statements on the non-commutative Cartan structure equations for the curvature will hold for any finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$; the torsion tensor, on the other hand, is defined only on the cotangent bundle over a non-commutative space.

Let $\nabla$ be a connection on the vector bundle $\widetilde{\Omega}^1_D(\mathcal{A})$, then the curvature and the torsion of $\nabla$ are the left $\mathcal{A}$-linear maps given in Definitions 2.16 and 2.14,

$$\begin{aligned} \mathtt{R}(\nabla) \,&:\, \widetilde{\Omega}^1_D(\mathcal{A}) \longrightarrow \widetilde{\Omega}^2_D(\mathcal{A}) \otimes_{\mathcal{A}} \widetilde{\Omega}^1_D(\mathcal{A}) \,, \\ \mathtt{T}(\nabla) \,&:\, \widetilde{\Omega}^1_D(\mathcal{A}) \longrightarrow \widetilde{\Omega}^2_D(\mathcal{A}) \,. \end{aligned}$$

Since the left $\mathcal{A}$-module $\widetilde{\Omega}^1_D(\mathcal{A})$ is finitely generated, we can choose a finite set of generators $\{\,E^A\,\}_{A=1,\ldots,N} \subset \widetilde{\Omega}^1_D(\mathcal{A})$, and define the components $\Omega^A{}_B \in \widetilde{\Omega}^1_D(\mathcal{A})$, $\mathtt{R}^A{}_B \in \widetilde{\Omega}^2_D(\mathcal{A})$ and $\mathtt{T}^A \in \widetilde{\Omega}^2_D(\mathcal{A})$ of connection, curvature and torsion, resp., by setting

$$\nabla E^A = -\,\Omega^A{}_B \otimes E^B \,, \tag{2.18}$$

$$\mathtt{R}(\nabla)\, E^A = \mathtt{R}^A{}_B \otimes E^B \,, \tag{2.19}$$

$$\mathtt{T}(\nabla)\, E^A = \mathtt{T}^A \,. \tag{2.20}$$

Note that the components $\Omega^A{}_B$ and $\mathrm{R}^A{}_B$ are *not* uniquely defined if $\widetilde{\Omega}^1_D(\mathcal{A})$ is not a free module. Using Definitions 2.16 and 2.14, the components of the curvature and torsion tensors can be expressed in terms of the connection components:

$$\mathrm{R}^A{}_B = \delta\,\Omega^A{}_B + \Omega^A{}_C\,\Omega^C{}_B \ , \tag{2.21}$$

$$\mathrm{T}^A \;= \delta E^A + \Omega^A{}_B E^B \ . \tag{2.22}$$

As they stand, eqs. (2.21) and (2.22) cannot be applied for solving typical problems like finding a connection without torsion, because the connection components $\Omega^A{}_B$ cannot be chosen at will unless $\widetilde{\Omega}^1_D(\mathcal{A})$ is free. We obtain more useful Cartan structure equations if we can relate the components $\Omega^A{}_B$ to those of a connection $\widetilde{\nabla}$ on a free module $\mathcal{A}^N$. To this end, we employ some general constructions valid for any finitely generated projective left $\mathcal{A}$-module $\mathcal{E}$.

Let $\{\,\widetilde{E}^A\,\}_{A=1,\dots,N}$ be the canonical basis of the standard module $\mathcal{A}^N$, and define a left $\mathcal{A}$-module homomorphism

$$p : \begin{cases} \mathcal{A}^N \longrightarrow \widetilde{\Omega}^1_D(\mathcal{A}) \\ a_A\widetilde{E}^A \longmapsto a_A E^A \end{cases} \tag{2.23}$$

for all $a_A \in \mathcal{A}$. Since $\widetilde{\Omega}^1_D(\mathcal{A})$ is projective there exists a left $\mathcal{A}$-module $\mathcal{F}$ such that

$$\widetilde{\Omega}^1_D(\mathcal{A}) \oplus \mathcal{F} \cong \mathcal{A}^N \ . \tag{2.24}$$

Denote by $i\,:\,\widetilde{\Omega}^1_D(\mathcal{A}) \longrightarrow \mathcal{A}^N$ the inclusion map determined by the isomorphism (2.24), which satisfies $p \circ i = \mathrm{id}$ on $\widetilde{\Omega}^1_D(\mathcal{A})$. For each $A = 1,\dots,N$, we define a left $\mathcal{A}$-linear map

$$\widetilde{\varepsilon}_A : \begin{cases} \mathcal{A}^N \longrightarrow \mathcal{A} \\ a_B\widetilde{E}^B \longmapsto a_A \end{cases} . \tag{2.25}$$

It is clear that $\widetilde{\varepsilon}_A(\omega)\widetilde{E}^A = \omega$ for all $\omega \in \mathcal{A}^N$. With the help of the inclusion $i$, we can introduce the left $\mathcal{A}$-linear maps

$$\varepsilon_A : \begin{cases} \widetilde{\Omega}^1_D(\mathcal{A}) \longrightarrow \mathcal{A} \\ \omega \longmapsto \widetilde{\varepsilon}_A\big(i(\omega)\big) \end{cases} \tag{2.26}$$

for all $A = 1,\dots,N$. With these, $\omega \in \widetilde{\Omega}^1_D(\mathcal{A})$ can be written as

$$\omega = p\big(i(\omega)\big) = p\big(\widetilde{\varepsilon}_A(i(\omega))\widetilde{E}^A\big) = \varepsilon_A(\omega)E^A \ , \tag{2.27}$$

and we see that $\{\,\varepsilon_A\,\}$ is the dual basis already used in section 2.1.7. The first step towards the non-commutative Cartan structure equations is the following result; see also [*Kar*].

**Proposition 2.17** Every connection $\widetilde{\nabla}$ on $\mathcal{A}^N$

$$\widetilde{\nabla} : \mathcal{A}^N \longrightarrow \widetilde{\Omega}^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^N$$

determines a connection $\nabla$ on $\widetilde{\Omega}^1_D(\mathcal{A})$ by

$$\nabla = (\mathrm{id} \otimes p) \circ \widetilde{\nabla} \circ i, \tag{2.28}$$

and every connection on $\widetilde{\Omega}^1_D(\mathcal{A})$ is of this form.

PROOF: Let $\widetilde{\nabla}$ be a connection on $\mathcal{A}^N$ – which always exists (see the remarks after the proof). Clearly, $\nabla = (\mathrm{id} \otimes p) \circ \widetilde{\nabla} \circ i$ is a well-defined map, and it satisfies

$$\nabla(a\omega) = (\mathrm{id} \otimes p)\big(\widetilde{\nabla}(a\, i(\omega))\big) = (\mathrm{id} \otimes p)\big(\delta a \otimes i(\omega) + a\widetilde{\nabla} i(\omega)\big) = \delta a \otimes \omega + a\nabla\omega$$

for all $a \in \mathcal{A}$ and all $\omega \in \widetilde{\Omega}^1_D(\mathcal{A})$. This proves that $\nabla$ is a connection on $\widetilde{\Omega}^1_D(\mathcal{A})$.
If $\nabla'$ is any other connection on $\widetilde{\Omega}^1_D(\mathcal{A})$, then

$$\nabla' - \nabla \in \mathrm{Hom}_{\mathcal{A}}\big(\widetilde{\Omega}^1_D(\mathcal{A}),\ \widetilde{\Omega}^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \widetilde{\Omega}^1_D(\mathcal{A})\big)\ ,$$

where $\mathrm{Hom}_{\mathcal{A}}$ denotes the space of homomorphisms of left $\mathcal{A}$-modules. Since

$$\mathrm{id} \otimes p\ :\ \widetilde{\Omega}^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^N \longrightarrow\ \widetilde{\Omega}^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \widetilde{\Omega}^1_D(\mathcal{A})$$

is surjective and $\widetilde{\Omega}^1_D(\mathcal{A})$ is a projective module, there exists a module map

$$\varphi\ :\ \widetilde{\Omega}^1_D(\mathcal{A}) \longrightarrow \widetilde{\Omega}^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^N$$

with

$$\nabla' - \nabla = (\mathrm{id} \otimes p) \circ \varphi\ .$$

Then $\widetilde{\varphi} := \varphi \circ p \in \mathrm{Hom}_{\mathcal{A}}\big(\mathcal{A}^N, \widetilde{\Omega}^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{A}^N\big)$, and $\widetilde{\nabla} + \widetilde{\varphi}$ is a connection on $\mathcal{A}^N$ whose associated connection on $\widetilde{\Omega}^1_D(\mathcal{A})$ is given by $\nabla'$:

$$(\mathrm{id} \otimes p) \circ (\widetilde{\nabla} + \widetilde{\varphi}) \circ i = \nabla + (\mathrm{id} \otimes p) \circ \varphi = \nabla'\ .$$

This proves that every connection on $\widetilde{\Omega}^1_D(\mathcal{A})$ comes from a connection on $\mathcal{A}^N$. ∎

The importance of this proposition lies in the fact that an arbitrary collection of 1-forms $\{\widetilde{\Omega}^A{}_B\}_{A,B=1,\ldots,N} \subset \widetilde{\Omega}^1_D(\mathcal{A})$ defines a connection $\widetilde{\nabla}$ on $\mathcal{A}^N$ by the formula

$$\widetilde{\nabla}\left(a_A \widetilde{E}^A\right) = \delta a_A \otimes \widetilde{E}^A - a_A \widetilde{\Omega}^A{}_B \otimes \widetilde{E}^B\ ,$$

and conversely. Thus, not only the existence of connections on $\mathcal{A}^N$ and $\widetilde{\Omega}^1_D(\mathcal{A})$ is guaranteed, but eq. (2.28) allows us to compute the components $\Omega^A{}_B$ of the induced connection $\nabla$ on $\widetilde{\Omega}^1_D(\mathcal{A})$. The action of $\nabla$ on the generators is

$$\begin{aligned}
\nabla E^A &= (\mathrm{id} \otimes p)\big(\widetilde{\nabla}\, i(E^A)\big) = (\mathrm{id} \otimes p)\big(\widetilde{\nabla}\, \widetilde{\varepsilon}_B(i(E^A))\widetilde{E}^B\big) = (\mathrm{id} \otimes p)\big(\widetilde{\nabla}\, \varepsilon_B(E^A)\widetilde{E}^B\big) \\
&= (\mathrm{id} \otimes p)\big(\delta\varepsilon_B(E^A) \otimes \widetilde{E}^B - \varepsilon_B(E^A)\, \widetilde{\Omega}^B{}_C \otimes \widetilde{E}^C\big) \\
&= \delta\varepsilon_B(E^A) \otimes E^B - \varepsilon_C(E^A)\, \widetilde{\Omega}^C{}_B \otimes E^B\ ,
\end{aligned}$$

where we have used some of the general properties listed before. In short, we get the relation

$$\Omega^A{}_B = \varepsilon_C(E^A)\,\widetilde{\Omega}^C{}_B - \delta\varepsilon_B(E^A) \tag{2.29}$$

expressing the components of the connection $\nabla$ on $\widetilde{\Omega}^1_D(\mathcal{A})$ in terms of the components of the connection $\widetilde{\nabla}$ on $\mathcal{A}^N$.

Upon inserting (2.29) into (2.21,22), one arrives at Cartan structure equations which express torsion and curvature in terms of these unrestricted components. We can, however, obtain equations of a simpler form if we exploit the fact that the map $\widetilde{\nabla} \mapsto \nabla$ is many-to-one; this allows us to impose some extra symmetry relations on the components of the connection $\widetilde{\nabla}$.

**Proposition 2.18** Let $\widetilde{\Omega}^A{}_B$ be the coefficients of a connection $\widetilde{\nabla}$ on $\mathcal{A}^N$, and denote by $\approx\!\!\!\nabla$ the connection on $\mathcal{A}^N$ whose components are given by $\widetilde{\widetilde{\Omega}}^A{}_B := \varepsilon_C(E^A)\,\widetilde{\Omega}^C{}_D\,\varepsilon_B(E^D)$. Then, these components enjoy the symmetry relations

$$\varepsilon_C(E^A)\,\widetilde{\widetilde{\Omega}}^C{}_B = \widetilde{\widetilde{\Omega}}^A{}_B \qquad \widetilde{\widetilde{\Omega}}^A{}_C\,\varepsilon_B(E^C) = \widetilde{\widetilde{\Omega}}^A{}_B\ , \tag{2.30}$$

and $\widetilde{\widetilde{\nabla}}$ and $\widetilde{\nabla}$ induce the same connection on $\widetilde{\Omega}^1_D(\mathcal{A})$. In particular, every connection on $\widetilde{\Omega}^1_D(\mathcal{A})$ is induced by a connection on $\mathcal{A}^N$ that satisfies (2.30).

PROOF: We explicitly compute the action of the connection $\nabla$ induced by $\widetilde{\widetilde{\nabla}}$ on a generator, using eqs. (2.27,28) and the fact that all maps and the tensor product are $\mathcal{A}$-linear:

$$\begin{aligned}
\nabla E^A &= -\,\Omega^A{}_B \otimes E^B = \delta\varepsilon_B(E^A)\otimes E^B - \varepsilon_C(E^A)\,\widetilde{\widetilde{\Omega}}^C{}_B \otimes E^B \\
&= \delta\varepsilon_B(E^A)\otimes E^B - \varepsilon_C(E^A)\varepsilon_D(E^C)\,\widetilde{\Omega}^D{}_F\varepsilon_B(E^F)\otimes E^B \\
&= \delta\varepsilon_B(E^A)\otimes E^B - \varepsilon_D\big(\varepsilon_C(E^A)E^C\big)\,\widetilde{\Omega}^D{}_F \otimes \varepsilon_B(E^F)E^B \\
&= \delta\varepsilon_B(E^A)\otimes E^B - \varepsilon_D(E^A)\,\widetilde{\Omega}^D{}_F \otimes E^F
\end{aligned}$$

This shows that $\nabla$ is identical to the connection induced by $\widetilde{\nabla}$. The symmetry relations (2.30) follow directly from $\mathcal{A}$-linearity and (2.27). ∎

We are now in a position to state the *Cartan structure equations* in a simple form.

**Theorem 2.19** Let $\widetilde{\Omega}^A{}_B$ and $\widetilde{\widetilde{\Omega}}^A{}_B$ be as in Proposition 2.18. Then the curvature and torsion components of the induced connection on $\widetilde{\Omega}^1_D(\mathcal{A})$ are given by

$$\begin{aligned}
\mathrm{R}^A{}_B &= \varepsilon_C(E^A)\,\delta\,\widetilde{\widetilde{\Omega}}^C{}_B + \widetilde{\widetilde{\Omega}}^A{}_C\,\widetilde{\widetilde{\Omega}}^C{}_B + \delta\varepsilon_C(E^A)\,\delta\,\varepsilon_B(E^C)\ , \\
\mathrm{T}^A\ &= \varepsilon_B(E^A)\,\delta E^B + \widetilde{\widetilde{\Omega}}^A{}_B E^B\ .
\end{aligned}$$

PROOF: With eqs. (2.21,29,30) and the Leibniz rule, we get

$$\begin{aligned}
\mathrm{R}^A{}_B &= \delta\,\widetilde{\widetilde{\Omega}}^A{}_B + \big(\widetilde{\widetilde{\Omega}}^A{}_C - \delta\varepsilon_C(E^A)\big)\big(\widetilde{\widetilde{\Omega}}^C{}_B - \delta\varepsilon_B(E^C)\big) \\
&= \delta\left(\varepsilon_C(E^A)\,\widetilde{\widetilde{\Omega}}^C{}_B\right) + \big(\widetilde{\widetilde{\Omega}}^A{}_C - \delta\varepsilon_C(E^A)\big)\big(\widetilde{\widetilde{\Omega}}^C{}_B - \delta\varepsilon_B(E^C)\big) \\
&= \varepsilon_C(E^A)\delta\,\widetilde{\widetilde{\Omega}}^C{}_B + \widetilde{\widetilde{\Omega}}^A{}_C\,\widetilde{\widetilde{\Omega}}^C{}_B + \delta\varepsilon_C(E^A)\,\delta\,\varepsilon_B(E^C) - \widetilde{\widetilde{\Omega}}^A{}_C\delta\varepsilon_B(E^C)\ .
\end{aligned}$$

The last term does in fact not contribute to the curvature, as can be seen after tensoring with $E^B$:

$$\widetilde{\widetilde{\Omega}}^A{}_C \delta\varepsilon_B(E^C) \otimes E^B = -\delta\big(\widetilde{\widetilde{\Omega}}^A{}_B\big) \otimes E^B + \big(\delta\widetilde{\widetilde{\Omega}}^A{}_C\big) \otimes \varepsilon_B(E^C)E^B = 0 \ ,$$

where we have used the Leibniz rule, the relations (2.30) and $\mathcal{A}$-linearity of the tensor product.
To compute the components of the torsion, we use eqs. (2.22,29) analogously,

$$\mathbf{T}^A = \delta E^A + \widetilde{\widetilde{\Omega}}^A{}_B E^B - \delta\varepsilon_B(E^A)E^B = \delta E^A + \widetilde{\widetilde{\Omega}}^A{}_B E^B - \delta E^A + \varepsilon_B(E^A)\delta E^B \ ,$$

which gives the result. ∎

The Cartan structure equations of Theorem 2.19 are considerably simpler than those one would get directly from (2.29) and (2.21,22). The price to be paid is that the components $\widetilde{\widetilde{\Omega}}^A{}_B$ are not quite independent from each other, but of course they can easily be expressed in terms of the arbitrary components $\widetilde{\Omega}^A{}_B$ according to Proposition 2.18. Therefore, the equations of Theorem 2.19 are useful e.g. for determining connections on $\widetilde{\Omega}^1_D(\mathcal{A})$ with special properties. We refer the reader to [*CFG*] for an explicit application of the Cartan structure equations.

## 2.2 The $N = (1,1)$ formulation of non-commutative geometry

In this section, we introduce the non-commutative generalization of the description of Riemannian geometry by a set of $N = (1,1)$ spectral data, which was presented, for the classical case, in section 2.2 of part I. The advantage over the $N = 1$ formulation is that now the algebra of differential forms is naturally represented on the Hilbert space $\mathcal{H}$. Therefore, calculations in concrete examples and also the study of cohomology rings will become much easier. There is the drawback that the algebra of differential forms is no longer closed under the $^*$-operation on $\mathcal{H}$, but we will introduce an alternative involution below and add further remarks in section 5.
The $N = (1,1)$ framework explained in the following will also provide the basis for the definition of various types of complex non-commutative geometries in sections 2.3 and 2.4.

### 2.2.1 The $N = (1,1)$ spectral data

**Definition 2.20** A quintuple $(\mathcal{A}, \mathcal{H}, \mathrm{d}, \gamma, *)$ is called a *set of* $N = (1,1)$ *spectral data* if

1) $\mathcal{H}$ is a separable Hilbert space;
2) $\mathcal{A}$ is a unital $^*$-algebra acting faithfully on $\mathcal{H}$ by bounded operators;
3) $\mathrm{d}$ is a densely defined closed operator on $\mathcal{H}$ such that
   *i*) $\mathrm{d}^2 = 0\,$,

*ii*) for each $a \in \mathcal{A}$, the commutator $[\,\mathtt{d}, a\,]$ extends uniquely to a bounded operator on $\mathcal{H}$,

*iii*) the operator $\exp(-\varepsilon\triangle)$ with $\triangle = \mathtt{d}\mathtt{d}^* + \mathtt{d}^*\mathtt{d}$ is trace class for all $\varepsilon > 0$;

4) $\gamma$ is a $\mathbb{Z}_2$-grading on $\mathcal{H}$, i.e., $\gamma = \gamma^* = \gamma^{-1}$, such that

*i*) $[\,\gamma, a\,] = 0$ for all $a \in \mathcal{A}$,

*ii*) $\{\,\gamma, \mathtt{d}\,\} = 0$;

5) $*$ is a unitary operator on $\mathcal{H}$ such that

*i*) $*\,\mathtt{d} = \zeta\,\mathtt{d}^*\,*$ for some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$,

*ii*) $[\,*, a\,] = 0$ for all $a \in \mathcal{A}$.

Several remarks are in order. First of all, note that we can introduce the two operators

$$\mathcal{D} = \mathtt{d} + \mathtt{d}^* \ , \quad \overline{\mathcal{D}} = i\,(\mathtt{d} - \mathtt{d}^*\,)$$

on $\mathcal{H}$ which satisfy the relations

$$\mathcal{D}^2 = \overline{\mathcal{D}}^2 \ , \qquad \{\,\mathcal{D}, \overline{\mathcal{D}}\,\} = 0 \ ,$$

cf. Definition I 2.6. Thus, our notion of $N = (1,1)$ spectral data is an immediate generalization of a classical $N = (1,1)$ Dirac bundle – except for the boundedness conditions to be required on infinite-dimensional Hilbert spaces, and the existence of the additional operator $*$ (see the comments below).

As in the $N = 1$ case, the $\mathbb{Z}_2$-grading $\gamma$ may always be introduced if not given from the start, simply by "doubling" the Hilbert space – see the remarks following Definition 2.1.

Moreover, if $(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{\mathtt{d}}, \tilde{\gamma})$ is a quadruple satisfying conditions 1-4 of Definition 2.20, we obtain a full set of $N = (1,1)$ spectral data by setting

$$\begin{aligned}
\mathcal{H} &= \widetilde{\mathcal{H}} \otimes \mathbb{C}^2 \ , \qquad\qquad \mathcal{A} = \widetilde{\mathcal{A}} \otimes \mathbf{1}_2 \ , \\
\mathtt{d} &= \widetilde{\mathtt{d}} \otimes \frac{1}{2}\,(\mathbf{1}_2 + \tau_3) - \widetilde{\mathtt{d}}^* \otimes \frac{1}{2}\,(\mathbf{1}_2 - \tau_3) \ , \\
\gamma &= \tilde{\gamma} \otimes \mathbf{1}_2 \ , \qquad\qquad * = \mathbf{1}_{\tilde{\mathcal{H}}} \otimes \tau_1
\end{aligned}$$

with the Pauli matrices $\tau_i$ as usual. Note that, in this example, $\zeta = -1$, and the $*$-operator additionally satisfies $*^2 = 1$ as well as $[\,\gamma, *\,] = 0$.

The unitary operator $*$ was not present in our algebraic formulation of classical Riemannian geometry. But for a compact oriented manifold, the usual Hodge $*$-operator acting on differential forms satisfies all the properties listed above, after appropriate rescaling in each degree. (Moreover, one can always achieve $*^2 = 1$ *or* $\zeta = -1$.) For a non-orientable manifold, we can apply the construction of the previous paragraph to obtain a description of the differential forms in terms of $N = (1,1)$ spectral data including a Hodge operator. In our approach to the non-commutative case, we will make essential use of the existence of $*$, which we will also call *Hodge operator*, in analogy to the classical case.

### 2.2.2 Differential forms

We first introduce an involution, $\natural$, called *complex conjugation*, on the algebra of universal forms:

$$\natural : \Omega^\bullet(\mathcal{A}) \longrightarrow \Omega^\bullet(\mathcal{A})$$

is the unique $\mathbb{C}$-anti-linear anti-automorphism such that

$$\natural(a) \equiv a^\natural := a^* \ , \qquad \natural(\delta a) \equiv (\delta a)^\natural := \delta(a^*) \tag{2.31}$$

for all $a \in \mathcal{A}$. Here we choose a sign convention that differs from the $N = 1$ case, eq. (2.1). If we write $\hat{\gamma}$ for the mod 2 reduction of the canonical $\mathbb{Z}$-grading on $\Omega^\bullet(\mathcal{A})$, we have

$$\delta \natural \hat{\gamma} = \natural \delta \ . \tag{2.32}$$

We define a representation of $\Omega^\bullet(\mathcal{A})$ on $\mathcal{H}$, again denoted by $\pi$, by

$$\pi(a) := a \ , \qquad \pi(\delta a) := [\, \mathtt{d}, a\,] \tag{2.33}$$

for all $a \in \mathcal{A}$. The map $\pi$ is a $\mathbb{Z}_2$-graded representation in the sense that

$$\pi(\hat{\gamma}\omega\hat{\gamma}) = \gamma\pi(\omega)\gamma \tag{2.34}$$

for all $\omega \in \Omega^\bullet(\mathcal{A})$.

Although the abstract algebra of universal forms is the same as in the $N = 1$ setting, the interpretation of the universal differential $\delta$ has changed: In the $N = (1,1)$ framework, it is represented on $\mathcal{H}$ by the nilpotent operator $\mathtt{d}$, instead of the self-adjoint Dirac operator $D$, as before. In particular, we now have

$$\pi(\delta\omega) = [\, \mathtt{d}, \pi(\omega)\,]_g \tag{2.35}$$

for all $\omega \in \Omega^\bullet(\mathcal{A})$, where $[\cdot,\cdot]_g$ denotes the graded commutator (defined with the canonical $\mathbb{Z}_2$-grading on $\pi(\Omega^\bullet(\mathcal{A}))$, see (2.34)). The validity of eq. (2.35) is the main difference between the $N = (1,1)$ and the $N = 1$ formalism. It ensures that there do not exist any forms $\omega \in \Omega^p(\mathcal{A})$ with $\pi(\omega) = 0$ but $\pi(\delta\omega) \neq 0$, in other words:

**Proposition 2.21** The graded vector space

$$J = \bigoplus_{k=0}^{\infty} J^k \ , \quad J^k := \ker \pi \,|_{\Omega^k(\mathcal{A})}$$

with $\pi$ defined in (2.33) is a two-sided graded *differential* $\natural$-ideal of $\Omega^\bullet(\mathcal{A})$.

Proof: The first two properties are obvious, the third one is the content of eq. (2.35). Using (2.31) and the relations satisfied by the Hodge $*$-operator according to part 5) of Definition 2.20, we find that

$$\begin{aligned}\pi\big((\delta a)^\natural\big) &= \pi(\delta(a^*)) = [\, \mathtt{d}, a^*\,] = [\, a, \mathtt{d}^*\,]^* = \zeta\,[\, a, * \,\mathtt{d} *^{-1}\,]^* \\ &= \zeta * [\, a, \mathtt{d}\,]^* *^{-1} = -\zeta * \pi(\delta a)^* *^{-1} \ ,\end{aligned}$$

which implies

$$\pi(\omega^\natural) = (-\zeta)^k \, * \, \pi(\omega)^* *^{-1} \tag{2.36}$$

for all $\omega \in \Omega^k(\mathcal{A})$. In particular, $J = \ker \pi$ is a $\natural$-ideal. ∎

As a consequence of this proposition, the algebra of differential forms

$$\Omega_{\mathrm{d}}^\bullet(\mathcal{A}) := \bigoplus_{k=0}^{\infty} \Omega_{\mathrm{d}}^k(\mathcal{A}) \ , \qquad \Omega_{\mathrm{d}}^k(\mathcal{A}) := \Omega^k(\mathcal{A})/J^k \ , \tag{2.37}$$

is represented on the Hilbert space $\mathcal{H}$ via $\pi$. For later purposes, we will also need an involution on $\Omega_{\mathrm{d}}^\bullet(\mathcal{A})$, and according to Proposition 2.21, this is given by the anti-linear map $\natural$ of (2.31). Note that the "natural" involution $\omega \mapsto \omega^*$, see eq. (2.1), which is inherited from $\mathcal{H}$ and was used in the $N = 1$ case, is no longer available here: The space $\pi(\Omega^k(\mathcal{A}))$ is not closed under taking adjoints, because $\mathtt{d}$ is not self-adjoint.
In summary, the space $\Omega_{\mathrm{d}}^\bullet(\mathcal{A})$ is a unital graded differential $\natural$-algebra and the representation $\pi$ of $\Omega^\bullet(\mathcal{A})$ determines a representation of $\Omega_{\mathrm{d}}^\bullet(\mathcal{A})$ on $\mathcal{H}$ as a unital differential algebra.

### 2.2.3 Integration

The integration theory follows the same lines as in the $N = 1$ case: The state $⨍$ is given as in Definition 2.3 with $\mathcal{D}^2$ written as $\triangle = \mathtt{d}\mathtt{d}^* + \mathtt{d}^*\mathtt{d}$. Again, we make Assumption 2.4 about the cyclicity of the integral. This yields a sesqui-linear form on $\Omega_{\mathrm{d}}^\bullet(\mathcal{A})$ as before:

$$(\omega, \eta) = ⨍ \omega\, \eta^* \tag{2.38}$$

for all $\omega, \eta \in \Omega_{\mathrm{d}}^\bullet(\mathcal{A})$, where we have dropped the representation symbols $\pi$ under the integral.
Because of the presence of the Hodge $*$-operator, the form $(\cdot, \cdot)$ has an additional feature in the $N = (1, 1)$ setting:

**Proposition 2.22** If the phase in part 5) of Definition 2.20 is $\zeta = \pm 1$, then the inner product defined in eq. (2.38) behaves like a real functional with respect to the involution $\natural$, i.e., for $\omega, \eta \in \Omega_{\mathrm{d}}^\bullet(\mathcal{A})$ we have

$$(\, \omega^\natural, \eta^\natural \,) = \overline{(\omega, \eta)}$$

where the bar denotes ordinary complex conjugation.

PROOF: First, observe that the Hodge operator commutes with the Laplacian, which is verified e.g. by taking the adjoint of the relation $* \mathtt{d} = \zeta\, \mathtt{d}^* *$. Then the claim follows immediately using eq. (2.36), unitarity of the Hodge operator, and cyclicity of the trace on $\mathcal{H}$: Let $\omega \in \Omega_{\mathrm{d}}^p(\mathcal{A})$, $\eta \in \Omega_{\mathrm{d}}^q(\mathcal{A})$, then

$$\begin{aligned}(\, \omega^\natural, \eta^\natural \,) &= ⨍ \omega^\natural (\eta^\natural)^* = (-\zeta)^p (-\bar\zeta)^q ⨍ \, * \, \omega^* *^{-1} * \eta *^{-1} = (-\zeta)^{p-q} ⨍ \omega^* \eta \\ &= (-\zeta)^{p-q} ⨍ \eta\, \omega^* = (-\zeta)^{p-q}\, \overline{(\omega, \eta)} \ ;\end{aligned}$$

again, we have suppressed the representation symbol $\pi$. The claim follows since the $\mathbb{Z}_2$-grading implies $(\omega, \eta) = 0$ unless $p - q \equiv 0 \pmod 2$. ∎

Note that, in examples, $p$- and $q$-forms for $p \neq q$ are often orthogonal with respect to the inner product $(\cdot,\cdot)$; then Proposition 2.22 holds independently of the value of $\zeta$.

Since $\Omega_{\mathrm{d}}^\bullet(\mathcal{A})$ is a $\natural$- and *not* a $^*$-algebra, Proposition 2.5 is to be replaced by

**Proposition 2.23** The graded kernel $K$, see eq. (2.5), of the sesqui-linear form $(\cdot,\cdot)$ is a two-sided graded $\natural$-ideal of $\Omega_{\mathrm{d}}^\bullet(\mathcal{A})$.

PROOF: The proof that $K$ is a two-sided graded ideal is identical to the one of Proposition 2.5. That $K$ is closed under $\natural$ follows immediately from the proof of Proposition 2.22. ∎

The remainder of section 2.1.3 carries over to the $N = (1,1)$ case, with the only differences that $\widetilde{\Omega}^\bullet(\mathcal{A})$ is a $\natural$-algebra and that the quotients $\Omega^k(\mathcal{A})/\big(K^k + \delta K^{k-1}\big) \cong \widetilde{\Omega}^k(\mathcal{A})/\delta K^{k-1}$ are denoted by $\widetilde{\Omega}_{\mathrm{d}}^k(\mathcal{A})$.

While $\Omega_{\mathrm{d}}^\bullet(\mathcal{A})$ is a differential algebra (by construction), $\widetilde{\Omega}^\bullet(\mathcal{A})$ is *not*, in general, a differential algebra, because the ideal $K$ may not be a differential ideal (i.e. there may exist $\omega \in K^{k-1}$ with $\delta\omega \notin K^k$). However, $K$ is trivial in many interesting examples. If $K$ is trivial then the algebra $\widetilde{\Omega}^\bullet(\mathcal{A})$ of square-integrable forms is a differential algebra which is faithfully represented on $\widetilde{\mathcal{H}}^\bullet$.

### 2.2.4 Unitary connections and scalar curvature

Except for the notions of unitary connections and scalar curvature, all definitions and results of sections 2.1.4-8 literally apply to the $N = (1,1)$ case as well. The two exceptions explicitly involve the $^*$-involution on the algebra of differential forms, which is no longer available now. Therefore, we have to modify the definitions for $N = (1,1)$ non-commutative geometry as follows:

**Definition 2.24** A connection $\nabla$ on a Hermitian vector bundle $\big(\mathcal{E}, \langle\cdot,\cdot\rangle\big)$ over an $N = (1,1)$ non-commutative space is called *unitary* if

$$\mathtt{d}\,\langle\, s, t\,\rangle = \langle\, \nabla s, t\,\rangle + \langle\, s, \nabla t\,\rangle$$

for all $s, t \in \mathcal{E}$; the Hermitian structure on the rhs is extended to $\mathcal{E}$-valued differential forms by

$$\langle\, \omega \otimes s, t\,\rangle = \omega\,\langle\, s, t\,\rangle\ , \qquad \langle\, s, \eta \otimes t\,\rangle = \langle\, s, t\,\rangle\,\eta^\natural$$

for all $\omega, \eta \in \widetilde{\Omega}_{\mathrm{d}}^\bullet(\mathcal{A})$ and $s, t \in \mathcal{E}$.

**Definition 2.25** The *scalar curvature* of a connection $\nabla$ on $\widetilde{\Omega}_{\mathrm{d}}^1(\mathcal{A})$ is defined by

$$\mathtt{r}\,(\nabla) = \big(E^{B\,\natural}\big)^{\mathrm{ad}}_R(\mathtt{Ric}_B) \in \widetilde{\mathcal{H}}_0\ .$$

### 2.2.5 Remarks on the relation of $N = 1$ and $N = (1, 1)$ spectral data

The definitions of $N = 1$ and $N = (1, 1)$ non-commutative spectral data provide two different generalizations of classical Riemannian differential geometry. In the latter context, one can always find an $N = (1, 1)$ description of a manifold originally given by an $N = 1$ set of data (see part I), whereas a non-commutative set of $N = (1, 1)$ spectral data seems to require a different mathematical structure than a spectral triple, because of the additional generalized Dirac operator which must be given on the Hilbert space. Thus, it is a natural and important question under which conditions on an $N = 1$ spectral triple $(\mathcal{A}, \mathcal{H}, D)$ there exists an associated $N = (1, 1)$ set of data $(\mathcal{A}, \widetilde{\mathcal{H}}, \mathrm{d}, *)$ over the same non-commutative space $\mathcal{A}$.

We have not been able yet to answer the question of how to pass from $N = 1$ to $N = (1, 1)$ data in a general way. But in the following we present a procedure that might lead to a solution. Our guideline is the classical case, where the main step in passing from $N = 1$ to $N = (1, 1)$ data is to replace the Hilbert space $\mathcal{H} = L^2(S)$ by $\widetilde{\mathcal{H}} = L^2(S) \otimes_{\mathcal{A}} L^2(S)$ carrying two actions of the Clifford algebra and therefore two anti-commuting Dirac operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ – which yield a description equivalent to the one involving the nilpotent differential $\mathrm{d}$, see the remark after Definition 2.20.

It is plausible that there are other approaches to this question, in particular approaches of a more operator algebraic nature, e.g. using a "Kasparov product of spectral triples", but we will not enter these matters here.

The first problem one meets when trying to copy the classical step from $N = 1$ to $N = (1, 1)$ is that $\mathcal{H}$ should be an $\mathcal{A}$-bi-module. To ensure this, we require that the set of $N = 1$ (even) spectral data $(\mathcal{A}, \mathcal{H}, D, \gamma)$ is endowed with a *real structure* [*Co4*], i.e. that there exists an anti-unitary operator $J$ on $\mathcal{H}$ such that

$$J^2 = \epsilon \mathbf{1} , \qquad J\gamma = \epsilon' \gamma J , \qquad JD = DJ$$

for some (independent) signs $\epsilon, \epsilon' = \pm 1$, and such that, in addition,

$$JaJ^* \quad \text{commutes with } b \text{ and } [\,D, b\,] \text{ for all } a, b \in \mathcal{A} .$$

This definition of a real structure was introduced by Connes in [*Co4*]; $J$ is of course a variant of Tomita's modular conjugation (cf. the next subsection).

In the present context, $J$ provides a canonical right $\mathcal{A}$-module structure on $\mathcal{H}$ by defining

$$\xi \cdot a := J a^* J^* \xi$$

for all $a \in \mathcal{A}$, $\xi \in \mathcal{H}$, see [*Co4*]. We can extend this to a right action of $\Omega^1_D(\mathcal{A})$ on $\mathcal{H}$ if we set

$$\xi \cdot \omega := J \omega^* J^* \xi$$

for all $\omega \in \Omega^1_D(\mathcal{A})$ and $\xi \in \mathcal{H}$; for simplicity, the representation symbol $\pi$ has been omitted. Note that by the assumptions on $J$, the right action commutes with the left action of $\mathcal{A}$. Thus $\mathcal{H}$ is an $\mathcal{A}$-bi-module, and we can form tensor products of bi-modules *over the algebra*

$\mathcal{A}$ just as in the classical case. If $\mathcal{H}$ carries a Hermitian structure, see Definition 2.8, then $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{H}$ can be endowed with a natural inner product.

The real structure $J$ in addition allows us to define the anti-linear "flip" operator

$$\Psi : \begin{cases} \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) \\ \omega \otimes \xi \longmapsto J\xi \otimes \omega^* \end{cases} .$$

It is straightforward to verify that $\Psi$ is well-defined and that it satisfies

$$\Psi(a\, s) = \Psi(s)\, a^*$$

for all $a \in \mathcal{A}$, $s \in \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H}$.

From now on, we assume furthermore that $\mathcal{H}$ is a *projective* left $\mathcal{A}$-module. Then it admits connections

$$\nabla : \mathcal{H} \longrightarrow \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} ,$$

i.e. $\mathbb{C}$-linear maps such that

$$\nabla(a\xi) = \delta a \otimes \xi + a \nabla \xi$$

for all $a \in \mathcal{A}$ and $\xi \in \mathcal{H}$. We assume that $\nabla$ commutes with the grading $\gamma$ on $\mathcal{H}$, i.e. $\nabla \gamma \xi = (\mathbf{1} \otimes \gamma) \nabla \xi$ for all $\xi \in \mathcal{H}$. For each connection $\nabla$ on $\mathcal{H}$, there is an "associated right-connection" $\overline{\nabla}$ defined with the help of the flip $\Psi$:

$$\overline{\nabla} : \begin{cases} \mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) \\ \xi \longmapsto -\Psi(\nabla J^* \xi) \end{cases}$$

$\overline{\nabla}$ is again $\mathbb{C}$-linear and satisfies

$$\overline{\nabla}(\xi a) = \xi \otimes \delta a + (\overline{\nabla}\xi) a .$$

A connection $\nabla$ on $\mathcal{H}$, together with its associated right connection $\overline{\nabla}$, induces a $\mathbb{C}$-linear "tensor product connection" $\widetilde{\nabla}$ on $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{H}$ of the form

$$\widetilde{\nabla} : \begin{cases} \mathcal{H} \otimes_{\mathcal{A}} \mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \\ \xi_1 \otimes \xi_2 \longmapsto \overline{\nabla}\xi_1 \otimes \xi_2 + \xi_1 \otimes \nabla \xi_2 \end{cases} .$$

Because of the position of the factor $\Omega^1_D(\mathcal{A})$, $\widetilde{\nabla}$ is not quite a connection in the usual sense. In the classical case, the last ingredient needed for the definition of the two Dirac operators of an $N = (1,1)$ Dirac bundle are the two anti-commuting Clifford actions on $\widetilde{\mathcal{H}}$. Their obvious generalizations to the non-commutative case are the $\mathbb{C}$-linear maps

$$\mathsf{c} : \begin{cases} \mathcal{H} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathcal{A}} \mathcal{H} \\ \xi_1 \otimes \omega \otimes \xi_2 \longmapsto \xi_1 \otimes \omega\, \xi_2 \end{cases}$$

and

$$\overline{\mathsf{c}} : \begin{cases} \mathcal{H} \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \mathcal{H} \longrightarrow \mathcal{H} \otimes_{\mathcal{A}} \mathcal{H} \\ \qquad\qquad \xi_1 \otimes \omega \otimes \xi_2 \longmapsto \xi_1 \, \omega \otimes \gamma \xi_2 \end{cases} .$$

With these, we may introduce two operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ on $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{H}$ in analogy to the classical case:

$$\mathcal{D} := \mathsf{c} \circ \widetilde{\nabla} , \qquad \overline{\mathcal{D}} := \overline{\mathsf{c}} \circ \widetilde{\nabla} .$$

In order to obtain a set of $N = (1,1)$ spectral data, one has to find a connection $\nabla$ on $\mathcal{H}$ which makes the operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ self-adjoint and ensures that the relations $\mathcal{D}^2 = \overline{\mathcal{D}}^2$ and $\{ \mathcal{D}, \overline{\mathcal{D}} \} = 0$ of Definition 2.20 are satisfied. The $\mathbb{Z}_2$-grading on $\mathcal{H} \otimes_{\mathcal{A}} \mathcal{H}$ is simply the tensor product grading, and the Hodge operator can be taken to be $* = \gamma \otimes \mathbf{1}$.

In section 4 below, we will verify these conditions in the example of the non-commutative torus. In the general case, we have, up to now, not been able to prove the existence of a connection $\nabla$ on $\mathcal{H}$ which supplies $\mathcal{D}$ and $\overline{\mathcal{D}}$ with the correct algebraic properties, but the naturality of the construction presented above as well as the similarity with the procedure of section I 2.2.2 lead us to expect that this problem can be solved in many cases of interest.
More precisely, we expect that the relation $\{ \mathcal{D}, \overline{\mathcal{D}} \} = 0$ can be satisfied under rather general assumptions, whereas it may often be appropriate to deal with a non-vanishing operator $\mathcal{D}^2 - \overline{\mathcal{D}}^2$ that generates an $S^1$-action.

### 2.2.6 Riemannian and Spin$^c$ "manifolds" in non-commutative geometry

In this section, we address the following question: What is the additional structure that makes an $N = (1,1)$ non-commutative space into a *non-commutative "manifold"*, into a *Spin$^c$ "manifold"*, or into a *quantized phase space*? There is a definition of non-commutative manifolds in terms of $K$-homology, see e.g. [*Co1* ]. In our search for the characteristic features of non-commutative manifolds we will, as before, be guided by the classical case and by the principle that they should be natural from the point of view of quantum physics.

Extrapolating from classical geometry, we are e.g. led to the following requirement an $N = (1,1)$ space $(\mathcal{A}, \mathcal{H}, \mathrm{d}, \gamma, *)$ should satisfy in order to describe a "manifold": The data must extend to a set of $N = 2$ *spectral data* $(\mathcal{A}, \mathcal{H}, \mathrm{d}, T, *)$ where $T$ is a self-adjoint operator on $\mathcal{H}$ such that

*i)* $[\, T, a \,] = 0$ for all $a \in \mathcal{A}$;

*ii)* $[\, T, \mathrm{d} \,] = \mathrm{d}$;

*iii)* $T$ has integral spectrum, and $\gamma$ is the mod 2 reduction of $T$, i.e. $\gamma = \pm 1$ on $\mathcal{H}_\pm$, where

$$\mathcal{H}_\pm = \mathrm{span} \{ \, \xi \in \mathcal{H} \, | \, T\xi = n \, \xi \text{ for some } n \in \mathbb{Z}, (-1)^n = \pm 1 \, \} .$$

Such $N = 2$ spectral data have been used in section I 1.2 already, and have also been briefly discussed in section I 3.

Before we can formulate further properties that we suppose to characterize non-commutative manifolds, we recall some basic facts about *Tomita-Takesaki theory*. Let $\mathcal{M}$ be a von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, and assume that $\xi_0 \in \mathcal{H}$ is a cyclic and separating vector for $\mathcal{M}$, i.e.

$$\overline{\mathcal{M}\,\xi_0} = \mathcal{H}$$

and

$$a\,\xi_0 = 0 \quad \Longrightarrow \quad a = 0$$

for any $a \in \mathcal{M}$, respectively. Then we may define an anti-linear operator $S_0$ on $\mathcal{H}$ by setting

$$S_0\,a\,\xi_0 = a^*\xi_0$$

for all $a \in \mathcal{M}$. One can show that $S_0$ is closable, and we denote its closure by $S$. The polar decomposition of $S$ is written as

$$S = J\Delta^{\frac{1}{2}}$$

where $J$ is an anti-unitary involutive operator, referred to as *modular conjugation*, and the so-called *modular operator* $\Delta$ is a positive self-adjoint operator on $\mathcal{H}$. The fundamental result of Tomita-Takesaki theory is the following theorem:

$$J\mathcal{M}J = \mathcal{M}' \;, \quad \Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$$

for all $t \in \mathbb{R}$; here, $\mathcal{M}'$ denotes the commutant of $\mathcal{M}$ on $\mathcal{H}$. Furthermore, the vector state $\omega_0(\cdot) := (\xi_0, \cdot\,\xi_0)$ is a *KMS-state* for the automorphism $\sigma_t := \mathrm{Ad}_{\Delta^{it}}$ of $\mathcal{M}$, i.e.

$$\omega_0(\sigma_t(a)\,b) = \omega_0(b\,\sigma_{t-i}(a))$$

for all $a, b \in \mathcal{M}$ and all real $t$.

Let $(\mathcal{A}, \mathcal{H}, \mathtt{d}, T, *)$ be a set of $N = 2$ spectral data coming from an $N = (1,1)$ space as above. We define the analogue $Cl_{\mathcal{D}}(\mathcal{A})$ of the space of sections of the Clifford bundle,

$$Cl_{\mathcal{D}}(\mathcal{A}) = \big\{\, a_0\,[\,\mathcal{D}, a_1\,] \ldots [\,\mathcal{D}, a_k\,]\,|\, k \in \mathbb{Z}_+,\; a_i \in \mathcal{A}\,\big\} \;,$$

where $\mathcal{D} = \mathtt{d} + \mathtt{d}^*$, and, corresponding to the second generalized Dirac operator $\overline{\mathcal{D}} = i(\mathtt{d} - \mathtt{d}^*)$,

$$Cl_{\overline{\mathcal{D}}}(\mathcal{A}) = \big\{\, a_0\,[\,\overline{\mathcal{D}}, a_1\,] \ldots [\,\overline{\mathcal{D}}, a_k\,]\,|\, k \in \mathbb{Z}_+,\; a_i \in \mathcal{A}\,\big\} \;.$$

In the classical setting, the sections $Cl_{\mathcal{D}}(\mathcal{A})$ and $Cl_{\overline{\mathcal{D}}}(\mathcal{A})$ operate on $\mathcal{H}$ by the two actions $c$ and $\overline{c}$, respectively, see Definition I 2.6. In the general case, we notice that, in contrast to the algebra $\Omega^\bullet_{\mathtt{d}}(\mathcal{A})$ introduced before, $Cl_{\mathcal{D}}(\mathcal{A})$ and $Cl_{\overline{\mathcal{D}}}(\mathcal{A})$ form $^*$-algebras of operators on $\mathcal{H}$, but are neither $\mathbb{Z}$-graded nor differential.

We want to apply Tomita-Takesaki theory to the von Neumann algebra $\mathcal{M} := \big(Cl_{\mathcal{D}}(\mathcal{A})\big)''$. Suppose there exists a vector $\xi_0 \in \mathcal{H}$ which is cyclic and separating for $\mathcal{M}$, and let $J$ be the anti-unitary conjugation associated to $\mathcal{M}$ and $\xi_0$. Suppose, moreover, that for all $a \in {}^J\!\mathcal{A} := J\mathcal{A}J$ the operator $[\,\overline{\mathcal{D}}, a\,]$ uniquely extends to a bounded operator on $\mathcal{H}$. Then we can form the algebra of bounded operators $Cl_{\overline{\mathcal{D}}}({}^J\!\mathcal{A})$ on $\mathcal{H}$ as above. The properties $J\mathcal{A}J \subset \mathcal{A}'$ and $\{\,\mathcal{D}, \overline{\mathcal{D}}\,\} = 0$ imply that $Cl_{\mathcal{D}}(\mathcal{A})$ and $Cl_{\overline{\mathcal{D}}}({}^J\!\mathcal{A})$ commute in the graded sense; to arrive at truly commuting algebras, we first decompose $Cl_{\overline{\mathcal{D}}}({}^J\!\mathcal{A})$ into a direct sum

$$Cl_{\overline{\mathcal{D}}}({}^J\!\mathcal{A}) = Cl^{+}_{\overline{\mathcal{D}}}({}^J\!\mathcal{A}) \oplus Cl^{-}_{\overline{\mathcal{D}}}({}^J\!\mathcal{A})$$

with

$$Cl^{\pm}_{\overline{\mathcal{D}}}({}^J\!\mathcal{A}) = \left\{\, \omega \in Cl_{\overline{\mathcal{D}}}({}^J\!\mathcal{A}) \,|\, \gamma\,\omega = \pm\omega\,\gamma \right\} .$$

Then we define the "twisted algebra" $\widetilde{Cl}_{\overline{\mathcal{D}}}({}^J\!\mathcal{A}) := Cl^{+}_{\overline{\mathcal{D}}}({}^J\!\mathcal{A}) \oplus \gamma\, Cl^{-}_{\overline{\mathcal{D}}}({}^J\!\mathcal{A})$. This algebra commutes with $Cl_{\mathcal{D}}(\mathcal{A})$.

We propose the following definitions: The $N = 2$ spectral data $(\mathcal{A}, \mathcal{H}, \mathrm{d}, T, *)$ describe a *non-commutative manifold* if

$$\widetilde{Cl}_{\overline{\mathcal{D}}}({}^J\!\mathcal{A}) = J\, Cl_{\mathcal{D}}(\mathcal{A})\, J \ .$$

Furthermore, inspired by classical geometry, we say that a non-commutative manifold $(\mathcal{A}, \mathcal{H}, \mathrm{d}, T, *, \xi_0)$ is *spin*$^c$ if the Hilbert space factorizes as a $Cl_{\mathcal{D}}(\mathcal{A}) \otimes \widetilde{Cl}_{\overline{\mathcal{D}}}({}^J\!\mathcal{A})$ module in the form

$$\mathcal{H} = \mathcal{H}_{\mathcal{D}} \otimes_{\mathcal{Z}} \mathcal{H}_{\overline{\mathcal{D}}}$$

where $\mathcal{Z}$ denotes the center of $\mathcal{M}$.

Next, we introduce a notion of "quantized phase space". We consider a set of $N = (1,1)$ spectral data $(\mathcal{A}, \mathcal{H}, \mathrm{d}, \gamma, *)$, where we now think of $\mathcal{A}$ as the algebra of phase space "functions" (i.e. of pseudo-differential operators, in the Schrödinger picture of quantum mechanics) rather than functions over configuration space. We are, therefore, not postulating the existence of a cyclic and separating vector for the algebra $Cl_{\mathcal{D}}(\mathcal{A})$. Instead, we define for each $\beta > 0$ the *temperature* or *KMS state*

$$⨍_{\beta} : \left\{ \begin{array}{rcl} Cl_{\mathcal{D}}(\mathcal{A}) & \longrightarrow & \mathbb{C} \\ \omega & \longmapsto & ⨍_{\beta} \omega := \dfrac{\mathrm{Tr}_{\mathcal{H}}\left(\omega e^{-\beta\mathcal{D}^2}\right)}{\mathrm{Tr}_{\mathcal{H}}\left(e^{-\beta\mathcal{D}^2}\right)} \end{array} \right. ,$$

with no limit $\beta \to 0$ taken, in contrast to Definition 2.3. The $\beta$-integral $⨍_{\beta}$ clearly is a faithful state, and through the GNS-construction we obtain a faithful representation of $Cl_{\mathcal{D}}(\mathcal{A})$ on a Hilbert space $\mathcal{H}_\beta$ with a cyclic and separating vector $\xi_\beta \in \mathcal{H}_\beta$ for $\mathcal{M}$. Each bounded operator $A \in \mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ induces a bounded operator $A_\beta$ on $\mathcal{H}_\beta$; this is easily seen by computing matrix elements of $A_\beta$,

$$\langle\, A_\beta x, y \,\rangle = ⨍_{\beta} A x y^*$$

for all $x, y \in \mathcal{M} \subset \mathcal{H}_\beta$, and using the explicit form of the $\beta$-integral. We denote the modular conjugation and the modular operator on $\mathcal{H}_\beta$ by $J_\beta$ and $\triangle_\beta$, respectively, and we assume that for each $a \in \mathcal{M}$ the commutator

$$[\,\overline{\mathcal{D}}, J_\beta a J_\beta\,] = \frac{1}{i}\,\frac{d}{dt}\,\left(\left(e^{it\overline{\mathcal{D}}}\right)_\beta J_\beta a J_\beta \left(e^{-it\overline{\mathcal{D}}}\right)_\beta\right)\Big|_{t=0}$$

defines a bounded operator on $\mathcal{H}_\beta$.

Then we can define an algebra of bounded operators $\widetilde{Cl}_{\overline{\mathcal{D}}}\,({}^{J_\beta}\!\mathcal{A})$ on $\mathcal{H}_\beta$, which is contained in the commutant of $Cl_{\mathcal{D}}\,(\mathcal{A})$, and we say that the $N = (1,1)$ spectral data $(\mathcal{A}, \mathcal{H}, \mathrm{d}, \gamma, *)$ describe a *quantized phase space* if the following equation holds:

$$J_\beta\, Cl_{\mathcal{D}}\,(\mathcal{A})\, J_\beta = \widetilde{Cl}_{\overline{\mathcal{D}}}\,({}^{J_\beta}\!\mathcal{A})$$

### 2.3 Hermitian and Kähler non-commutative geometry

In this section, we introduce the spectral data describing complex non-commutative spaces, more specifically spaces that carry a Hermitian or a Kähler structure; the terminology is of course carried over from the classical case, see part I. Since these structures are more restrictive than the data of Riemannian non-commutative geometry, we will be able to derive some appealing properties of the space of differential forms. We also find a necessary condition for a set of $N = (1,1)$ spectral data to extend to Hermitian data. A different approach to complex non-commutative geometry has been proposed in [$BC$].

#### 2.3.1 Hermitian and $N = (2,2)$ spectral data

**Definition 2.26** A set of data $(\mathcal{A}, \mathcal{H}, \partial, \overline{\partial}, T, \overline{T}, \gamma, *)$ is called a set of *Hermitian spectral data* if

1) the quintuple $(\mathcal{A}, \mathcal{H}, \partial + \overline{\partial}, \gamma, *)$ forms a set of $N = (1,1)$ spectral data;
2) $T$ and $\overline{T}$ are self-adjoint bounded operators on $\mathcal{H}$, $\partial$ and $\overline{\partial}$ are densely defined, closed operators on $\mathcal{H}$ such that the following (anti-)commutation relations hold:

$$\begin{aligned} &\partial^2 = \overline{\partial}^2 = 0\;, && \{\,\partial, \overline{\partial}\,\} = 0\;, \\ &[\,T, \partial\,] = \partial\;, && [\,T, \overline{\partial}\,] = 0\;, \\ &[\,\overline{T}, \partial\,] = 0\;, && [\,\overline{T}, \overline{\partial}\,] = \overline{\partial}\;, \\ &[\,T, \overline{T}\,] = 0\;; && \end{aligned}$$

3) for any $a \in \mathcal{A}$, $[\,T, a\,] = [\,\overline{T}, a\,] = 0$ and each of the operators $[\,\partial, a\,]$, $[\,\overline{\partial}, a\,]$ and $\{\,\partial, [\,\overline{\partial}, a\,]\,\}$ extends uniquely to a bounded operator on $\mathcal{H}$;

4) the $\mathbb{Z}_2$-grading $\gamma$ satisfies

$$\{\gamma, \partial\} = \{\gamma, \overline{\partial}\} = 0 ,$$
$$[\gamma, T] = [\gamma, \overline{T}] = 0 ;$$

5) the Hodge $*$-operator satisfies

$$* \partial = \zeta \, \overline{\partial}^* * , \qquad * \overline{\partial} = \zeta \, \partial^* *$$

for some phase $\zeta \in \mathbb{C}$.

Some remarks on this definition may be useful: The Jacobi identity and the equation $\{\partial, \overline{\partial}\} = 0$ show that condition 3 above is in fact symmetric in $\partial$ and $\overline{\partial}$.

As in section 2.2.1, a set $(\mathcal{A}, \mathcal{H}, \partial, \overline{\partial}, T, \overline{T})$ that satisfies the first three conditions but does not involve $\gamma$ or $*$, can be made into a complete set of Hermitian spectral data.

In classical Hermitian geometry, the $*$-operator can always be taken to be the usual Hodge $*$-operator – up to a multiplicative redefinition in each degree – since complex manifolds are orientable.

Next, we describe conditions sufficient to equip a set of $N = (1,1)$ spectral data with a Hermitian structure. In subsection 2.3.2, Corollary 2.34, a necessary criterion is given as well.

**Proposition 2.27** Let $(\mathcal{A}, \mathcal{H}, \mathrm{d}, \gamma, *)$ be a set of $N = (1,1)$ spectral data with $[\gamma, *] = 0$, and let $T$ be a self-adjoint bounded operator on $\mathcal{H}$ such that

a) the operator $\partial := [T, \mathrm{d}]$ is nilpotent: $\partial^2 = 0$;

b) $[T, \partial] = \partial$;

c) $[T, a] = 0$ for all $a \in \mathcal{A}$;

d) $[T, \omega] \in \pi(\Omega^1(\mathcal{A}))$ for all $\omega \in \pi(\Omega^1(\mathcal{A}))$;

e) the operator $\overline{\partial} := \mathrm{d} - \partial$ satisfies $* \partial = \zeta \, \overline{\partial}^* *$, where $\zeta$ is the phase appearing in the relations of $*$ in the $N = (1,1)$ data;

f) $[T, \gamma] = 0$ and $[T, \overline{T}] = 0$, where $\overline{T} := - * T *^{-1}$.

Then $(\mathcal{A}, \mathcal{H}, \partial, \overline{\partial}, T, \overline{T}, \gamma, *)$ forms a set of Hermitian spectral data.

Notice that the conditions a - d) are identical to those in Definition I 2.20 of section I 2.4.1. Requirement e) will turn out to correspond to part e) of that definition. The relations in f) ensure compatibility of the operators $T$, $\gamma$ and $*$ and were not needed in the classical setting.

Proof: We check each of the conditions in Definition 2.26: The first one is satisfied by assumption, since $\mathrm{d} = \partial + \overline{\partial}$ is the differential of $N = (1,1)$ spectral data.

The equalities $\partial^2 = \overline{\partial}^2 = \{\partial, \overline{\partial}\} = [T, \overline{\partial}] = 0$ follow from a) and b), as in the proof of Lemma I 2.21. With this, we compute

$$[\overline{T}, \overline{\partial}] = -[* T *^{-1}, \overline{\partial}] = -\zeta * [T, \partial^*] *^{-1} = \overline{\partial} ,$$

and since

$$[\overline{T},\mathtt{d}] = [\,*\,T\,*^{-1},\mathtt{d}^*\,]^* = \zeta\,*\,[\,T,\mathtt{d}\,]^*\,*^{-1} = \overline{\partial}\ ,$$

we obtain $[\,\overline{T},\partial\,] = 0$. The relation $[\,T,\overline{T}\,] = 0$ and self-adjointness of $T$ were part of the assumptions, and $\overline{T}^* = \overline{T}$ is clear from the unitarity of the Hodge $*$-operator.
That $[\,\partial,a\,]$ and $[\,\overline{\partial},a\,]$ are bounded for all $a\in\mathcal{A}$ follows from the corresponding property of $\mathtt{d}$ and from the assumption that $T$ is bounded. As in the proof of Proposition I 2.22, one shows that $\{\,\partial,[\,\overline{\partial},a\,]\} \in \pi(\Omega^2_{\mathtt{d}}(\mathcal{A}))$, and therefore $\{\,\partial,[\,\overline{\partial},a\,]\}$ is a bounded operator. $T$ and $*$ commute with all $a\in\mathcal{A}$ by assumption, and thus the same is true for $\overline{T}$.
Using f) and the Jacobi identity, we get

$$\{\,\gamma,\partial\,\} = \{\,\gamma,[\,T,\mathtt{d}\,]\,\} = [\,T,\{\,\mathtt{d},\gamma\,\}\,] + \{\,\mathtt{d},[\,\gamma,T\,]\,\} = 0$$

and

$$\{\,\gamma,\overline{\partial}\,\} = \{\,\gamma,\mathtt{d}-\partial\,\} = 0\ .$$

By assumption, $\gamma$ commutes with $T$ and $*$, therefore also with $\overline{T}$.
Finally, the relations of condition 5 in Definition 2.26 between the $*$-operator and $\partial$, $\overline{\partial}$ follow directly from e) and $*\,\mathtt{d} = \zeta\,\mathtt{d}^*\,*$. ∎

As in classical differential geometry, Kähler spaces arise as a special case of Hermitian geometry. In particular, Kähler spectral data provide a realization of the $N = (2,2)$ supersymmetry algebra:

**Definition 2.28** Hermitian spectral data $(\mathcal{A},\mathcal{H},\partial,\overline{\partial},T,\overline{T},\gamma,*)$ are called $N=(2,2)$ or *Kähler spectral data* if

$$\{\,\partial,\overline{\partial}^*\,\} = \{\,\overline{\partial},\partial^*\,\} = 0\ ,$$
$$\{\,\partial,\partial^*\,\} = \{\,\overline{\partial},\overline{\partial}^*\,\}\ .$$

Note that the first line is a consequence of the second one in classical complex geometry, but has to be imposed as a separate condition in the non-commutative setting.

One can also define Kähler spectral data, as in section I 1.2, as containing a nilpotent differential $\mathtt{d}$ – together with its adjoint $\mathtt{d}^*$ – and two commuting U(1) generators $L^3$ and $J_0$, say, which satisfy the relations (I 1.49-51). This approach has the virtue that the complex structure familiar from classical differential geometry is already present in the algebraic formulation; see eq. (I 1.54) for the precise relationship with $J_0$. Moreover, this way of introducing non-commutative complex geometry makes the role of Lie group symmetries of the spectral data explicit, which is somewhat hidden in the formulation of Definitions 2.26 and 2.28 and in Proposition 2.27: The presence of the $\mathrm{U}(1)\times\mathrm{U}(1)$ symmetry, acting in an appropriate way, ensures that a set of $N=(1,1)$ spectral data acquires an $N=(2,2)$ structure.
Because of the advantages in the treatment of differential forms, we will stick to the setting using $\partial$ and $\overline{\partial}$ for the time being, but the data with generators $L^3$ and $J_0$ will appear naturally in the context of symplectic geometry in section 2.5.

### 2.3.2 Differential forms

In the context of Hermitian non-commutative geometry, we have two differential operators $\partial$ and $\overline{\partial}$ at our disposal. We begin this section with the definition of an abstract algebra of universal forms which is appropriate for this situation.

**Definition 2.29** A *bi-differential algebra* $\mathcal{B}$ is a unital algebra together with two *anti-commuting* nilpotent derivations $\delta, \overline{\delta}\ :\ \mathcal{B} \longrightarrow \mathcal{B}$.
A *homomorphism of bi-differential algebras* $\varphi : \mathcal{B} \longrightarrow \mathcal{B}'$ is a unital algebra homomorphism which intertwines the derivations.

**Definition 2.30** The *algebra of complex universal forms* $\Omega^{\bullet,\bullet}(\mathcal{A})$ over a unital algebra $\mathcal{A}$ is the (up to isomorphism) unique pair $(\iota, \Omega^{\bullet,\bullet}(\mathcal{A}))$ consisting of a unital bi-differential algebra $\Omega^{\bullet,\bullet}(\mathcal{A})$ and an injective unital algebra homomorphism $\iota : \mathcal{A} \longrightarrow \Omega^{\bullet,\bullet}(\mathcal{A})$ such that the following universal property holds: For any bi-differential algebra $\mathcal{B}$ and any unital algebra homomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{B}$, there is a unique homomorphism $\widetilde{\varphi} : \Omega^{\bullet,\bullet}(\mathcal{A}) \longrightarrow \mathcal{B}$ of bi-differential algebras such that $\varphi = \widetilde{\varphi} \circ \iota$.

The description of $\Omega^{\bullet,\bullet}(\mathcal{A})$ in terms of generators and relations is analogous to the case of $\Omega^{\bullet}(\mathcal{A})$, and it shows that $\Omega^{\bullet,\bullet}(\mathcal{A})$ is a *bi-graded bi-differential algebra*

$$\Omega^{\bullet,\bullet}(\mathcal{A}) = \bigoplus_{r,s=0}^{\infty} \Omega^{r,s}(\mathcal{A}) \tag{2.39}$$

by declaring the generators $a, \delta a, \overline{\delta} a$ and $\delta\overline{\delta} a$, $a \in \mathcal{A}$, to have bi-degrees (0,0), (1,0), (0,1) and (1,1), respectively.
As in the $N = (1,1)$ framework, we introduce an involution $\natural$, called *complex conjugation*, on the algebra of complex universal forms, provided $\mathcal{A}$ is a ${}^*$-algebra:

$$\natural\ :\ \Omega^{\bullet,\bullet}(\mathcal{A}) \longrightarrow \Omega^{\bullet,\bullet}(\mathcal{A})$$

is the unique anti-linear anti-automorphism acting on generators by

$$\begin{aligned} &\natural(a) \equiv a^{\natural} := a^* \ , \\ &\natural(\delta a) \equiv (\delta a)^{\natural} := \overline{\delta}(a^*\ ) \ , \qquad\qquad \natural(\overline{\delta} a) \equiv (\overline{\delta} a)^{\natural} := \delta(a^*\ ) \ , \\ &\natural(\delta\overline{\delta} a) \equiv (\delta\overline{\delta} a)^{\natural} := \delta\overline{\delta}(a^*\ ) \ . \end{aligned} \tag{2.40}$$

Let $\tilde{\gamma}$ be the $\mathbb{Z}_2$-reduction of the total grading on $\Omega^{\bullet,\bullet}(\mathcal{A})$, i.e., $\tilde{\gamma} = (-1)^{r+s}$ on $\Omega^{r,s}(\mathcal{A})$. Then it is easy to verify that

$$\overline{\delta}\natural\tilde{\gamma} = \natural\delta \ . \tag{2.41}$$

This makes $\Omega^{\bullet,\bullet}(\mathcal{A})$ into a unital bi-graded bi-differential $\natural$-algebra.

Let $(\mathcal{A}, \mathcal{H}, \partial, \overline{\partial}, T, \overline{T}, \gamma, *)$ be a set of Hermitian spectral data. Then we define a $\mathbb{Z}_2$-graded representation $\pi$ of $\Omega^{\bullet,\bullet}(\mathcal{A})$ as a unital bi-differential algebra on $\mathcal{H}$ by setting

$$\begin{aligned} &\pi(a) = a \ , \\ &\pi(\delta a) = [\,\partial, a\,] \ , \qquad\qquad \pi(\overline{\delta} a) = [\,\overline{\partial}, a\,] \ , \\ &\pi(\delta\overline{\delta} a) = \{\,\partial, [\,\overline{\partial}, a\,]\,\} \ . \end{aligned} \tag{2.42}$$

Note that, by the Jacobi identity, the last equation is compatible with the anti-commutativity of $\delta$ and $\overline{\delta}$.
As in the case of $N=(1,1)$ geometry, we have that

$$\pi(\delta\omega) = [\,\partial, \pi(\omega)\,]_g \ , \qquad\qquad \pi(\overline{\delta}\omega) = [\,\overline{\partial}, \pi(\omega)\,]_g \ , \tag{2.43}$$

for any $\omega \in \Omega^{\bullet,\bullet}(\mathcal{A})$, and therefore the graded kernel of the representation $\pi$ has good properties: We define

$$J^{\bullet,\bullet} := \bigoplus_{r,s=0}^{\infty} J^{r,s} \ , \qquad J^{r,s} := \{\,\omega \in \Omega^{r,s}(\mathcal{A}) \,|\, \pi(\omega) = 0\,\} \ , \tag{2.44}$$

and we prove the following statement in the same way as Proposition 2.21:

**Proposition 2.31** The set $J$ is a two-sided, bi-graded, bi-differential $\natural$-ideal of $\Omega^{\bullet,\bullet}(\mathcal{A})$.

We introduce the space of complex differential forms as

$$\Omega^{\bullet,\bullet}_{\partial,\bar{\partial}}(\mathcal{A}) := \bigoplus_{r,s=0}^{\infty} \Omega^{r,s}_{\partial,\bar{\partial}}(\mathcal{A}) \ , \qquad \Omega^{r,s}_{\partial,\bar{\partial}}(\mathcal{A}) := \Omega^{r,s}(\mathcal{A})/J^{r,s} \ . \tag{2.45}$$

The algebra $\Omega^{\bullet,\bullet}_{\partial,\bar{\partial}}(\mathcal{A})$ is a unital bi-graded bi-differential $\natural$-algebra, too, and the representation $\pi$ determines a representation, still denoted $\pi$, of this algebra on $\mathcal{H}$.

Due to the presence of the operators $T$ and $\overline{T}$ among the Hermitian spectral data, the image of $\Omega^{\bullet,\bullet}_{\partial,\bar{\partial}}(\mathcal{A})$ under $\pi$ enjoys a property not present in the $N=(1,1)$ case:

**Proposition 2.32** The representation of the algebra of complex differential forms satisfies

$$\pi\big(\Omega^{\bullet,\bullet}_{\partial,\bar{\partial}}(\mathcal{A})\big) = \bigoplus_{r,s=0}^{\infty} \pi\big(\Omega^{r,s}_{\partial,\bar{\partial}}(\mathcal{A})\big) \ . \tag{2.46}$$

In particular, $\pi$ is a representation of $\Omega^{\bullet,\bullet}_{\partial,\bar{\partial}}(\mathcal{A})$ as a unital, bi-graded, bi-differential $\natural$-algebra. The $\natural$-operation is implemented on $\pi\big(\Omega^{\bullet,\bullet}_{\partial,\bar{\partial}}(\mathcal{A})\big)$ with the help of the Hodge $*$-operator and the $^*$-operation on $\mathcal{B}(\mathcal{H})$:

$$\natural : \left\{ \begin{array}{rcl} \pi\big(\Omega^{r,s}_{\partial,\bar{\partial}}(\mathcal{A})\big) & \longrightarrow & \pi\big(\Omega^{r,s}_{\partial,\bar{\partial}}(\mathcal{A})\big) \\ \omega & \longmapsto & \omega^{\natural} := (-\zeta)^{r+s} * \omega^* *^{-1} \end{array} \right. .$$

**Proof**: Let $\omega \in \pi\big(\Omega^{r,s}_{\partial,\bar{\partial}}(\mathcal{A})\big)$. Then part 2) of Definition 2.26 implies that

$$[\,T, \omega\,] = r\,\omega \ , \qquad [\,\overline{T}, \omega\,] = s\,\omega \ ,$$

which gives the direct sum decomposition (2.46). It remains to show that the $\natural$-operation is implemented on the space $\pi\big(\Omega^{\bullet,\bullet}_{\partial,\bar\partial}(\mathcal{A})\big)$: For $a \in \mathcal{A}$, we have that

$$\begin{aligned}\pi\big((\delta a)^\natural\big) &= \pi\big(\overline{\delta}(a^*)\big) = [\,\overline{\partial}, a^*\,] = -[\overline{\partial}^*, a]^* = -[\,\bar\zeta * \partial *^{-1}, a\,]^* = -\zeta * [\,\partial, a\,]^* *^{-1} \\ &= -\zeta * \pi(\delta a)^* *^{-1} ,\end{aligned}$$

and, similarly, using (2.40) and the properties of the Hodge $*$-operator,

$$\pi\big((\overline{\delta} a)^\natural\big) = -\zeta * \pi(\overline{\delta} a)^* *^{-1} , \qquad \pi\big((\delta\overline{\delta} a)^\natural\big) = \zeta^2 * \pi(\delta\overline{\delta} a)^* *^{-1} .$$

This proves that $\pi(\omega^\natural) = \pi(\omega)^\natural$. ∎

As an aside, we mention that the implementation of $\natural$ on $\pi\big(\Omega^{\bullet,\bullet}_{\partial,\bar\partial}(\mathcal{A})\big)$ via the Hodge $*$-operator shows that the conditions e) of the "classical" Definition I 2.20 and of Proposition 2.27 are related; more precisely, the former is a consequence of the latter.

Hermitian spectral data carry, in particular, an $N = (1,1)$ structure, and thus we have two notions of differential forms available. Their relation is described in our next proposition.

**Proposition 2.33** The space of $N = (1,1)$ differential forms is included in the space of Hermitian forms, i.e.,

$$\pi\big(\Omega^p_{\mathrm{d}}(\mathcal{A})\big) \subset \bigoplus_{r+s=p} \pi\big(\Omega^{r,s}_{\partial,\bar\partial}(\mathcal{A})\big) , \tag{2.47}$$

and the spaces coincide if and only if

$$[\,T, \omega\,] \in \pi\big(\Omega^1_{\mathrm{d}}(\mathcal{A})\big) \quad \text{for all} \quad \omega \in \Omega^1_{\mathrm{d}}(\mathcal{A}) . \tag{2.48}$$

PROOF: The inclusion (2.47) follows simply from $\mathrm{d} = \partial + \overline{\partial}$. If the spaces are equal then the equation

$$[\,T, \omega\,] = r\,\omega ,$$

for all $\omega \in \pi\big(\Omega^{r,s}_{\partial,\bar\partial}(\mathcal{A})\big)$ , implies (2.48). The converse is shown as in the proof of Proposition I 2.22 in section 2.4.1 of part I, concerning classical Hermitian geometry. ∎

Note that even if the spaces of differential forms do not coincide, the algebra of complex forms contains a graded differential algebra $\big(\Omega^{\ \bullet}_{\partial,\bar\partial}(\mathcal{A}), \mathrm{d}\big)$ with $\mathrm{d} = \partial + \overline{\partial}$ and

$$\Omega^{\ \bullet}_{\partial,\bar\partial}(\mathcal{A}) := \bigoplus_p \Omega^{\ p}_{\partial,\bar\partial}(\mathcal{A}) , \qquad \Omega^{\ p}_{\partial,\bar\partial}(\mathcal{A}) := \bigoplus_{r+s=p} \Omega^{r,s}_{\partial,\bar\partial}(\mathcal{A}) . \tag{2.49}$$

By Proposition 2.32, we know that

$$\pi\big(\Omega^{\bullet,\bullet}_{\partial,\bar\partial}(\mathcal{A})\big) = \bigoplus_p \pi\big(\Omega^{\ p}_{\partial,\bar\partial}(\mathcal{A})\big) ,$$

and hence we obtain a necessary condition for $N = (1,1)$ spectral data to extend to Hermitian spectral data:

**Corollary 2.34** If a set of $N = (1,1)$ spectral data extends to a set of Hermitian spectral data then

$$\pi\big(\Omega_{\mathrm{d}}^{\bullet}(\mathcal{A})\big) = \bigoplus_{p} \pi\big(\Omega_{\mathrm{d}}^{p}(\mathcal{A})\big) .$$

This condition is clearly not sufficient since it is always satisfied in classical differential geometry.

Beyond the complexes (2.45) and (2.49), one can of course also consider the analogue of the *Dolbeault complex* using only the differential $\overline{\partial}$ acting on $\Omega_{\partial,\bar{\partial}}^{\bullet,\bullet}(\mathcal{A})$. The details are straightforward.

We conclude this subsection with some remarks concerning possible variations of our Definition 2.26 of Hermitian spectral data. For example, one may wish to drop the boundedness condition on the operators $T$ and $\overline{T}$, in order to include infinite-dimensional spaces into the theory. This is possible, but then one has to make some stronger assumptions in Proposition 2.27.

Another relaxation of the requirements in Hermitian spectral data is to avoid introducing $T$ and $\overline{T}$ altogether, and to replace them by a decomposition of the $\mathbb{Z}_2$-grading

$$\gamma = \gamma_{\partial} + \gamma_{\bar{\partial}}$$

such that

$$\begin{aligned} \{\,\gamma_{\partial}, \partial\,\} &= 0 \ , \qquad [\,\gamma_{\partial}, \overline{\partial}\,] = 0 \ , \\ \{\,\gamma_{\bar{\partial}}, \overline{\partial}\,\} &= 0 \ , \qquad [\,\gamma_{\bar{\partial}}, \partial\,] = 0 \ . \end{aligned}$$

Then the space of differential forms may be defined as above, but Propositions 2.32 and 2.33, as well as the good properties of the integral established in the next subsection, will not hold in general.

### 2.3.3 Integration in complex non-commutative geometry

The definition of the integral is completely analogous to the $N = (1,1)$ setting: Again we use the operator $\triangle = \mathtt{d}\,\mathtt{d}^* + \mathtt{d}^*\,\mathtt{d}$, where now $\mathtt{d} = \partial + \overline{\partial}$. Due to the larger set of data, the space of square-integrable, complex differential forms, now obtained after quotienting by the two-sided *bi-graded* $\natural$-ideal $K$, has better properties than the corresponding space of forms in Riemannian non-commutative geometry. There, two elements $\omega \in \Omega_{\mathrm{d}}^{p}(\mathcal{A})$ and $\eta \in \Omega_{\mathrm{d}}^{q}(\mathcal{A})$ with $p \neq q$ were not necessarily orthogonal with respect to the sesqui-linear form $(\cdot,\cdot)$ induced by the integral. For Hermitian and Kähler non-commutative geometry, however, we can prove the following orthogonality statements:

**Proposition 2.35** Let $\omega_i \in \pi\big(\Omega_{\partial,\bar{\partial}}^{r_i,s_i}(\mathcal{A})\big)$, $i = 1,2$. Then

$$(\omega_1, \omega_2) = 0 \tag{2.50}$$

if $r_1 + s_1 \neq r_2 + s_2$ in the Hermitian case; if the spectral data also carry an $N = (2,2)$ structure, then eq. (2.50) holds as soon as $r_1 \neq r_2$ or $s_1 \neq s_2$.

PROOF: In the case of Hermitian spectral data, the assertion follows immediately from cyclicity of the trace, from the commutation relations

$$[\,T, \omega_i\,] = r_i\,\omega_i\ , \qquad [\,\overline{T}, \omega_i\,] = s_i\,\omega_i\ ,$$

which means that $T+\overline{T}$ counts the total degree of a differential form, and from the equation

$$[\,T + \overline{T}, \triangle\,] = 0\ .$$

In the Kähler case, Definition 2.28 implies the stronger relations

$$[\,T, \triangle\,] = [\,\overline{T}, \triangle\,] = 0\ . \qquad \blacksquare$$

### 2.3.4 Generalized metric on $\widetilde{\Omega}^{\,1}_{\partial,\bar\partial}(\mathcal{A})$

The notions of vector bundles, Hermitian structure, torsion, etc. are defined just as for $N = (1,1)$ spectral data in section 2.2. The definitions of holomorphic vector bundles and connections can be carried over from the classical case; see section I 2.4.4. Again, we pass from $\Omega^{\,1}_{\partial,\bar\partial}$, see eq. (2.49), to the space of all square-integrable 1-forms $\widetilde{\Omega}^{\,1}_{\partial,\bar\partial}$, which is equipped with a generalized Hermitian structure $\langle\cdot,\cdot\rangle_{\partial\bar\partial}$ according to the construction in Theorem 2.9. Starting from here, we can define an analogue

$$\langle\!\langle\cdot,\cdot\rangle\!\rangle\ :\ \widetilde{\Omega}^{\,1}_{\partial,\bar\partial}(\mathcal{A}) \times \widetilde{\Omega}^{\,1}_{\partial,\bar\partial}(\mathcal{A}) \longrightarrow \mathbb{C}$$

of the $\mathbb{C}$-bi-linear metric in classical complex geometry by

$$\langle\!\langle\,\omega, \eta\,\rangle\!\rangle := \langle\,\omega, \eta^{\natural}\,\rangle_{\partial\bar\partial}\ .$$

**Proposition 2.36** The generalized metric $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ on $\widetilde{\Omega}^{\,1}_{\partial,\bar\partial}(\mathcal{A})$ has the following properties:

1) $\langle\!\langle\,a\omega, \eta\,b\,\rangle\!\rangle = a\,\langle\!\langle\,\omega, \eta\,\rangle\!\rangle\,b$;
2) $\langle\!\langle\,\omega\,a, \eta\,\rangle\!\rangle = \langle\!\langle\,\omega, a\,\eta\,\rangle\!\rangle$;
3) $\langle\!\langle\,\omega, \omega^{\natural}\,\rangle\!\rangle \geq 0$ ;

here $\omega,\,\eta \in \widetilde{\Omega}^{\,1}_{\partial,\bar\partial}(\mathcal{A})$ and $a, b \in \mathcal{A}$. If the underlying spectral data are Kählerian, one has that

$$\langle\!\langle\,\omega, \eta\,\rangle\!\rangle = 0$$

if $\omega, \eta \in \widetilde{\Omega}^{0,1}_{\partial,\bar\partial}(\mathcal{A})$ or $\omega, \eta \in \widetilde{\Omega}^{1,0}_{\partial,\bar\partial}(\mathcal{A})$.

PROOF: The first three statements follow directly from the definition of $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ and the corresponding properties of $\langle\cdot,\cdot\rangle_{\partial\bar\partial}$ listed in Theorem 2.9. The last assertion is a consequence of Proposition 2.35, using the fact that the spaces $\widetilde{\Omega}^{r,s}_{\partial,\bar\partial}(\mathcal{A})$ are $\mathcal{A}$-bi-modules. Note that this property of the metric $\langle\!\langle\cdot,\cdot\rangle\!\rangle$ corresponds to the property $g_{\mu\nu}=g_{\bar\mu\bar\nu}=0$ (in complex coordinates) in the classical case. ∎

### 2.4 The $N=(4,4)$ spectral data

We just present the definition of spectral data describing non-commutative Hyperkähler spaces. Obviously, it is chosen in analogy to the discussion of the classical case in section 2.5 of part I.

**Definition 2.37** A set of data $(\mathcal{A},\mathcal{H},G^{a\pm},\overline{G}^{a\pm},T^i,\overline{T}^i,\gamma,*)$ with $a=1,2$, $i=1,2,3$, is called a set of $N=(4,4)$ or *Hyperkähler spectral data* if

1) the subset $(\mathcal{A},\mathcal{H},G^{1+},\overline{G}^{1+},T^3,\overline{T}^3,\gamma,*)$ forms a set of $N=(2,2)$ spectral data;
2) $G^{a\pm}$, $a=1,2$ are closed, densely defined operators on $\mathcal{H}$, and $T^i$, $i=1,2,3$, are bounded operators on $\mathcal{H}$ which satisfy $\left(G^{a\pm}\right)^*=G^{a\mp}$, $\left(T^i\right)^*=T^i$ and the following (anti-)commutation relations ($a,b=1,2$, $i,j=1,2,3$, and $\tau^i$ are the Pauli matrices):

$$\begin{aligned}
&\{\,G^{a+},G^{b+}\}=0\ , && \{\,G^{a-},G^{b+}\}=\delta^{ab}\,\Box\ ,\\
&[\,\Box,G^{a+}\,]=0\ , && [\,\Box,T^i\,]=0\ ,\\
&[\,T^i,T^j\,]=i\epsilon^{ijk}\,T^k\ , && [\,T^i,G^{a+}\,]=\frac{1}{2}\,\overline{\tau^i_{ab}}\,G^{b+}\ ,
\end{aligned}$$

   for some self-adjoint operator $\Box$ on $\mathcal{H}$, which, in the classical case, is the holomorphic part of the Laplace operator;
3) the operators $\overline{G}^{a\pm}$, $a=1,2$, and $\overline{T}^i$, $i=1,2,3$, also satisfy the conditions in 2) and (anti-)commute with $G^{a\pm}$ and $T^i$.

The construction of non-commutative differential forms and the integration theory is precisely the same as for $N=(2,2)$ spectral data. We therefore refrain from giving more details. It might, however, be interesting to see whether the additional information encoded in $N=(4,4)$ spectral data gives rise to special properties, beyond the ones found for Kähler data in subsection 2.3.3.

### 2.5 Symplectic non-commutative geometry

Once more, our description in the non-commutative context follows the algebraic characterization of classical symplectic manifolds given in section 2.6 of part I. The difference between our approaches to the classical and to the non-commutative case is that, in the former, we could derive most of the algebraic relations – including the SU(2) structure showing up on symplectic manifolds – from the specific properties of the symplectic 2-form, whereas now we will instead include those relations into the defining data, as a "substitute" for the symplectic form.

**Definition 2.38** The set of data $(\mathcal{A}, \mathcal{H}, \mathtt{d}, L^3, L^+, L^-, \gamma, *)$ is called a set of *symplectic spectral data* if

1) $(\mathcal{A}, \mathcal{H}, \mathtt{d}, \gamma, *)$ is a set of $N = (1,1)$ spectral data;
2) $L^3$, $L^+$ and $L^-$ are bounded operators on $\mathcal{H}$ which commute with all $a \in \mathcal{A}$ and satisfy the $\mathrm{sl}_2$ commutation relations

$$[\,L^3, L^{\pm}\,] = \pm 2L^{\pm}\ , \quad [\,L^+, L^-\,] = L^3$$

as well as the Hermiticity properties $(L^3)^* = L^3$, $(L^{\pm})^* = L^{\mp}$; furthermore, they commute with the grading $\gamma$ on $\mathcal{H}$;
3) the operator $\widetilde{\mathtt{d}}^* := [\,L^-, \mathtt{d}\,]$ is densely defined and closed, and together with $\mathtt{d}$ it forms an SU(2) doublet, i.e., the following commutation relations hold:

$$\begin{aligned} [\,L^3, \mathtt{d}\,] &= \mathtt{d}\ , & [\,L^3, \widetilde{\mathtt{d}}^*\,] &= -\widetilde{\mathtt{d}}^*\ , \\ [\,L^+, \mathtt{d}\,] &= 0\ , & [\,L^+, \widetilde{\mathtt{d}}^*\,] &= \mathtt{d}\ , \\ [\,L^-, \mathtt{d}\,] &= \widetilde{\mathtt{d}}^*\ , & [\,L^-, \widetilde{\mathtt{d}}^*\,] &= 0\ . \end{aligned}$$

As in the classical case, there is a second SU(2) doublet spanned by the adjoints $\mathtt{d}^*$ and $\widetilde{\mathtt{d}}$. The Jacobi identity shows that $\widetilde{\mathtt{d}}^*$ is nilpotent and that it anti-commutes with $\mathtt{d}$.

Differential forms and integration theory are formulated just as for $N = (1,1)$ spectral data, but the presence of SU(2) generators among the symplectic spectral data leads to additional interesting features, such as the following: Let $\omega \in \Omega^k_{\mathtt{d}}(\mathcal{A})$ and $\eta \in \Omega^l_{\mathtt{d}}(\mathcal{A})$ be two differential forms. Then their scalar product, see eq. (2.38), vanishes unless $k = l$:

$$(\omega, \eta) = 0 \quad \text{if } k \neq l\ . \tag{2.51}$$

This is true because, by the SU(2) commutation relations listed above, the operator $L^3$ induces a $\mathbb{Z}$-grading on differential forms, and because $L^3$ commutes with the Laplacian $\triangle = \mathtt{d}^*\mathtt{d} + \mathtt{d}\mathtt{d}^*$. One consequence of (2.51) is that the reality property of $(\cdot,\cdot)$ stated in Proposition 2.22 is valid independently of the phase occurring in the Hodge relations.

The following proposition shows that we can introduce an $N = (2,2)$ structure on a set of symplectic spectral data if certain additional properties are satisfied. As was the case for Definition 2.38, the extra requirements are slightly stronger than in the classical situation, where some structural elements like the almost-complex structure are given automatically. In the Kähler case, the latter allows for a separate counting of holomorphic resp. anti-holomorphic degrees of differential forms, which in turn ensures that the symmetry group of the symplectic data associated to a classical Kähler manifold is in fact $\mathrm{SU}(2) \times \mathrm{U}(1)$ – see also section 3 of part I. Without this enlarged symmetry group, it is impossible to re-interpret the $N = 4$ as an $N = (2,2)$ supersymmetry algebra. Therefore, we explicitly postulate the existence of an additional U(1) generator in the non-commutative context – which coincides with the U(1) generator $J_0$ in eq. (I 1.49) of section I 1.2 and is intimately related to the complex structure.

**Proposition 2.39** Suppose that the SU(2) generators of a set of symplectic spectral data satisfy the following relations with the Hodge operator:

$$* L^3 = -L^3 * , \qquad * L^+ = -\zeta^2 L^- * ,$$

where $\zeta$ is the phase appearing in the Hodge relations of the $N = (1,1)$ subset of the symplectic data. Assume, furthermore, that there exists a bounded self-adjoint operator $J_0$ on $\mathcal{H}$ which commutes with all $a \in \mathcal{A}$, with the grading $\gamma$, and with $L^3$, whereas it acts like

$$[\, J_0, \mathtt{d}\,] = -i\,\widetilde{\mathtt{d}} , \qquad [\, J_0, \widetilde{\mathtt{d}}\,] = i\,\mathtt{d}$$

between the SU(2) doublets. Then the set of symplectic data carries an $N = (2,2)$ Kähler structure with

$$\partial = \frac{1}{2}\,(\mathtt{d} - i\,\widetilde{\mathtt{d}}) , \qquad \overline{\partial} = \frac{1}{2}\,(\mathtt{d} + i\,\widetilde{\mathtt{d}}) ,$$
$$T = \frac{1}{2}\,(L^3 + J_0) , \qquad \overline{T} = \frac{1}{2}\,(L^3 - J_0) .$$

PROOF: All the conditions listed in Definition 2.26 of Hermitian spectral data can be verified easily: Nilpotency of $\partial$ and $\overline{\partial}$ follows from $\mathtt{d}^2 = \widetilde{\mathtt{d}}^2 = 0$ and

$$\{\, \mathtt{d}, \widetilde{\mathtt{d}}\,\} = 0 , \tag{2.52}$$

and the action of the Hodge operator on the SU(2) generators ensures that $*$ intertwines $\partial$ and $\overline{\partial}$ in the right way. As for the extra conditions in Definition 2.28 of Kähler spectral data, one sees that the first one is always true for symplectic spectral data, whereas the second one, namely the equality of the "holomorphic" and "anti-holomorphic" Laplacians, is again a consequence of relation (2.52). ∎

## 3. The non-commutative 3-sphere

Here and in the next section, we present two examples of non-commutative spaces and show how the general methods developed above can be applied. We first discuss the "quantized" or "fuzzy" 3-sphere. We draw some inspiration from the conformal field theory associated to a non-linear $\sigma$-model with target being a 3-sphere, the so-called SU(2)-WZW model, see [*Wi3*] and also [*FGK, PS*]. But while the ideas on a non-commutative interpretation of conformal field theory models proposed in [*FG*] are essential for placing non-commutative geometry into a string theory context, the following calculations are self-contained; the results of subsections 3.2 and 3.3 are taken from [*Gr*]. Although there is no doubt that the methods used in [*Gr*] and below can be extended to arbitrary compact, connected and simply connected Lie groups, we will, for simplicity, restrict ourselves to the case of SU(2).

We first introduce a set of $N = 1$ spectral data describing the non-commutative 3-sphere, then discuss the de Rham complex and its cohomology, and finally turn towards geometrical aspects of this non-commutative space. Subsection 3.4 briefly describes the $N = (1,1)$ formalism.

### 3.1 The $N = 1$ data associated to the 3-sphere

In this subsection, we introduce $N = 1$ data describing the non-commutative 3-sphere.

Since the 3-sphere is diffeomorphic to the Lie group $G = \mathrm{SU}(2)$, we are looking for data describing a Lie group $G$. Let $\{T_A\}$ be a basis of $\mathbf{g} = T_eG$, the Lie algebra of $G$. By $\vartheta_A$ and $\overline{\vartheta}_A$ we denote the left- and right-invariant vector fields associated to the basis elements $T_A$, and by $\theta^A$ and $\overline{\theta}^A$ the corresponding dual basis of 1-forms. The structure constants $f_{AB}^C$ are defined, as usual, by

$$[\vartheta_A, \vartheta_B] = f_{AB}^C \vartheta_C \ . \tag{3.1}$$

The Killing form on $\mathbf{g}$ induces a canonical Riemannian metric on $TG$ given by

$$g_{AB} \equiv g(\vartheta_A, \vartheta_B) = -\mathrm{Tr}\,(\mathrm{ad}_{T_A} \circ \mathrm{ad}_{T_B}) = -f_{AC}^D f_{BD}^C \ , \tag{3.2}$$

and the Levi-Civita connection reads

$$\nabla_A \vartheta_B \equiv \nabla_{\vartheta_A} \vartheta_B = \frac{1}{2} f_{AB}^C \vartheta_C \ . \tag{3.3}$$

The left-invariant vector fields $\vartheta_A$ define a trivialization of the (co-)tangent bundle. We denote by $\nabla^L$ the flat connection associated to that trivialization,

$$\nabla^L \theta^A = 0$$

for all $A$. We introduce the operators

$$a^{A*} = \theta^A \wedge \ , \quad a^A = g^{AB}\, \vartheta_B \llcorner$$

on the space of differential forms, as well as the usual gamma matrices

$$\gamma^A = a^{A\,*} - a^A \;, \quad \overline{\gamma}^A = i\,(\,a^{A\,*} + a^A\,)\;. \tag{3.4}$$

It is easy to verify that $\gamma^A$ and $\overline{\gamma}^A$ generate two anti-commuting copies of the Clifford algebra,

$$\{\,\gamma^A,\gamma^B\} = \{\,\overline{\gamma}^A,\overline{\gamma}^B\} = -2g^{AB}\;, \quad \{\,\gamma^A,\overline{\gamma}^B\} = 0\;. \tag{3.5}$$

Following the notations of section I2.2, we shall denote by $\mathcal{S}$ the bundle of differential forms endowed with the above structures.
We define two connections $\nabla^{\mathcal{S}}$ and $\overline{\nabla}^{\mathcal{S}}$ on $\mathcal{S}$ by setting

$$\begin{aligned}\nabla^{\mathcal{S}} &= \theta^A \otimes (\nabla^L_{\vartheta_A} + \frac{1}{12}\,f_{ABC}\gamma^B\gamma^C)\;,\\ \overline{\nabla}^{\mathcal{S}} &= \overline{\theta}^A \otimes (\nabla^L_{\overline{\vartheta}_A} - \frac{1}{12}\,f_{ABC}\overline{\gamma}^B\overline{\gamma}^C)\;,\end{aligned} \tag{3.6}$$

where $f_{ABC} = f^D_{AB}g_{DC}$ , and we put

$$J_A := i\,\nabla^L_{\vartheta_A}\;, \quad \psi^A := -i\,\gamma^A\;, \quad \overline{J}_A := -i\,\nabla^L_{\overline{\vartheta}_A}\;, \quad \overline{\psi}^A := i\,\overline{\gamma}^A\;. \tag{3.7}$$

These objects satisfy the commutation relations

$$[\,J_A, J_B\,] = i f^C_{AB}\,J_C\;, \quad \{\,\psi^A,\psi^B\,\} = 2g^{AB}\;, \tag{3.8}$$

with analogous relations for $\overline{J}_A$ and $\overline{\psi}^A$; barred and unbarred operators (anti-)commute.
The two anti-commuting Dirac operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ on $\mathcal{S}$ read $[FG]$

$$\begin{aligned}\mathcal{D} &= \psi^A J_A - \frac{i}{12}\,f_{ABC}\psi^A\psi^B\psi^C\\ \overline{\mathcal{D}} &= \overline{\psi}^A\overline{J}_A - \frac{i}{12}\,f_{ABC}\overline{\psi}^A\overline{\psi}^B\overline{\psi}^C\end{aligned} \tag{3.9}$$

where $\mathbf{c}$ and $\overline{\mathbf{c}}$ are the Clifford actions defined by the gamma matrices of eq. (3.4). The $\mathbb{Z}_2$-grading operator $\gamma$ on $\mathcal{S}$ , anti-commuting with $\mathcal{D}$ and $\overline{\mathcal{D}}$, is given by

$$\gamma = \frac{1}{i\,(3!)^2}\,g\,\varepsilon_{ABC}\,\varepsilon_{DEF}\;\psi^A\psi^B\psi^C\,\overline{\psi}^D\overline{\psi}^E\overline{\psi}^F\;, \tag{3.10}$$

where $g = \det g_{AB}$ . By $L^2(\mathcal{S}) \simeq L^2(G)\otimes W$, where $W$ is the irreducible representation of the Clifford algebra of eqs. (3.4,5), we denote the Hilbert space of square integrable sections of the bundle $\mathcal{S}$ , with respect to the normalized Haar measure on $G$ . In the language of Connes' spectral triples, the classical 3-sphere is described by the $N=1$ data $(L^2(\mathcal{S}),\, C^\infty(G),\, D,\, \gamma)$ , with $D \equiv \mathcal{D}$.
The Hilbert space $L^2(\mathcal{S})$ carries a unitary representation $\pi$ of $G\times G$ given by

$$\big(\pi(g_1,\,g_2)f\big)(h) = f(g_1^{-1}hg_2)\;, \tag{3.11}$$

for all $g_i$, $h \in G$ and $f \in L^2(G)$. For each $j \in \frac{1}{2}\mathbb{Z}_+$ we denote by $(\pi^j, V_j)$ the irreducible unitary $(2j+1)$-dimensional (spin $j$) representation of $G$, and to each vector $\xi^* \otimes \eta \in V_j^* \otimes V_j$ we associate a smooth function $f_{\xi^* \otimes \eta} \in C^\infty(G)$ by setting

$$f_{\xi^* \otimes \eta}(g) = \frac{1}{\sqrt{2j+1}} \langle \xi^*, \pi^j(g)\eta \rangle . \tag{3.12}$$

This defines a linear isometry

$$\varphi : \bigoplus_{j \in \frac{1}{2}\mathbb{Z}_+} V_j^* \otimes V_j \longrightarrow L^2(G) , \tag{3.13}$$

and the Peter-Weyl theorem states that the image of $\varphi$ is dense in $L^2(G)$ and also in $C(G)$ in the supremum norm topology. It is easy to verify that the operators $J_A$ and $\overline{J}_A$ act on $\bigoplus_{j \in \frac{1}{2}\mathbb{Z}_+} V_j^* \otimes V_j$ as $d\pi(T_A, 1)$ and $d\pi(1, T_A)$, respectively. For each positive integer $k$, we denote by $P_{(k)}$ the orthogonal projection

$$P_{(k)} : L^2(G) \longrightarrow \mathcal{H}_0 := \bigoplus_{j=0, \frac{1}{2}, \ldots}^{k/2} V_j^* \otimes V_j . \tag{3.14}$$

The Dirac operator $D$ and the $\mathbb{Z}_2$-grading $\gamma$ clearly leave the *finite-dimensional* Hilbert space $\mathcal{H}_0 \otimes W$ invariant. We define $\mathcal{A}_0$ to be the unital subalgebra of $\mathrm{End}(\mathcal{H}_0)$ generated by operators of the form $P_{(k)} f_{\xi^* \otimes \eta}$, where $\xi^* \otimes \eta \in \mathcal{H}_0$. The following theorem is proven in [*Gr*]:

**Theorem 3.1** The algebra $\mathcal{A}_0$ coincides with the algebra of endomorphisms of $\mathcal{H}_0$, i.e.,

$$\mathcal{A}_0 = \mathrm{End}(\mathcal{H}_0) .$$

The proof in [*Gr*] shows that $\mathcal{A}_0$ is a full matrix algebra for any compact, connected and simply connected group. That $\mathcal{A}_0$ equals the endomorphism ring of $\mathcal{H}_0$ was only proved for SU(2), but a slight generalization of the proof for SU(2) should yield the result for all groups of the above type.

We define the non-commutative 3-sphere by the $N = 1$ data $(\mathcal{A}_0, \mathcal{H}_0 \otimes W, D, \gamma)$. Notice that this definition of the non-commutative 3-sphere is very close to that of the non-commutative 2-sphere [*Ber*, *Ho*, *Ma*, *GKP*]. For an alternative derivation of this definition, the reader is referred to [*FG*] where it is shown how this space arises as the quantum target of the WZW model based on $SU(2)$.

We note that $1/k$ plays the role of Planck's constant $\hbar$ in the quantization of symplectic manifolds, i.e., it is a deformation parameter. Formally, the classical 3-sphere emerges as the limit of non-commutative 3-spheres as the deformation parameter $1/k$ tends to zero.

### 3.2 The topology of the non-commutative 3-sphere

In this subsection, we shall apply the tools of subsection 2.1 to the non-commutative space $(\mathcal{A}_0, \mathcal{H}_0 \otimes W, D, \gamma)$ describing the non-commutative 3-sphere; we follow the presentation in [Gr]. For convenience, we shall choose the basis $\{T_A\}$ of $T_eG$ in such a way that $g_{AB} = 2\delta_{AB}$. The structure constants are then given by the Levi-Civita tensoı, $f_{AB}^C = \varepsilon_{ABC}$.

#### 3.2.1 The de Rham complex

First, we determine the structure of the spaces of differential forms $\Omega_D^n(\mathcal{A}_0)$ and the action of the exterior differentiation $\delta : \Omega_D^\bullet(\mathcal{A}_0) \longrightarrow \Omega_D^\bullet(\mathcal{A}_0)$. We use the same notations as in subsection 2.1.2.

The space of 1-forms is

$$\Omega_D^1(\mathcal{A}_0) \simeq \pi(\Omega^1(\mathcal{A}_0)) = \left\{ \sum_i a_0^i \, [\, J^A, a_1^i \,] \otimes \psi^A \;\middle|\; a_j^i \in \mathcal{A}_0 \right\} . \tag{3.15}$$

Since $\mathcal{A}_0$ is a full matrix algebra, see Theorem 3.1, it follows that

$$\Omega_D^1(\mathcal{A}_0) \simeq \{\, a_A \otimes \psi^A \,|\, a_A \in \mathcal{A}_0 \,\} . \tag{3.16}$$

Using the fact that any element of $\pi(\Omega^2(\mathcal{A}_0))$ can be written as a linear combination of products of pairs of elements in $\pi(\Omega^1(\mathcal{A}_0))$, we get

$$\pi(\Omega^2(\mathcal{A}_0)) = \{\, a_{AB} \otimes \psi^A \psi^B \,|\, a_{AB} \in \mathcal{A}_0 \,\} . \tag{3.17}$$

Our next task is to determine the space $\pi(\delta J^1)$ of so-called "auxiliary 2-forms", see eq. (2.2). To this end, let $\omega = \sum_i a_i \delta b_i \in \Omega^1(\mathcal{A}_0)$ be such that

$$\pi(\omega) = \sum_i a_i [D\,,\, b_i] = 0 . \tag{3.18}$$

Using eqs. (3.8) and (3.18), we see that the coefficient of $[\,\psi^A, \psi^B\,]$ in $\pi(\delta\omega)$ is proportional to

$$\begin{aligned} \varepsilon^{AB} \sum_i [\, J^A, a_i \,][\, J^B, b_i \,] &= -\varepsilon^{AB} \sum_i a_i [\, J^A, [\, J^B, b_i \,]] \\ &= -\frac{1}{2}\varepsilon^{AB} \sum_i a_i [\, [\, J^A, J^B \,], b_i \,] = -\frac{i}{2}\varepsilon^{AB}\varepsilon^{ABC} \sum_i a_i [\, J^C, b_i \,] = 0 \, , \end{aligned}$$

where $\varepsilon^{AB}$ denotes the Levi-Civita antisymmetric tensor. This shows that $\pi(\delta J^1)$ is included in $\mathcal{A}_0$, and since $\mathcal{A}_0$ is a full matrix algebra, this implies that $\pi(\delta J^1)$ is either 0 or equal to $\mathcal{A}_0$. We construct a non-vanishing element of $\pi(\delta J^1)$ explicitly. Let $P_j$ be the orthogonal projection onto $V_j^* \otimes V_j$. We define $a\,,\, b \in \mathcal{A}_0$ by

$$a = P_0 \, a \, P_{1/2} \ , \quad b = P_{1/2} \, b \, P_0$$

and

$$a\,:\,V_{1/2}^*\otimes V_{1/2}\ni\begin{pmatrix}\alpha\\ \beta\end{pmatrix}\otimes\begin{pmatrix}\gamma\\ \delta\end{pmatrix}\;\longmapsto\;\alpha-2\beta+2\gamma+\delta\;\in\;V_0^*\otimes V_0$$

$$b\,:\,V_0^*\otimes V_0\;\ni\;\alpha\;\longmapsto\;\alpha\cdot\begin{pmatrix}1\\ -2\end{pmatrix}\otimes\begin{pmatrix}2\\ 1\end{pmatrix}\in\;V_{1/2}^*\otimes V_{1/2}\,.$$

It is straightforward to verify that $\omega := a\delta b$ satisfies $\pi(\omega)=0$ and $\pi(\delta\omega)\neq 0$. This proves that $\pi(\delta J^1)=\mathcal{A}_0$, and we get

$$\Omega_D^2(\mathcal{A}_0)\simeq\{\,a_{AB}\otimes\psi^A\psi^B\,|\,a_{AB}=-a_{BA}\in\mathcal{A}_0\,\}\,. \tag{3.19}$$

In order to determine the space of 3-forms, we first notice that

$$\pi(\Omega^3(\mathcal{A}_0))=\{\,a_{ABC}\otimes\psi^A\psi^B\psi^C\,\}\,, \tag{3.20}$$

and we compute the space $\pi(\delta J^2)$. Let $a_i,b_i,c_i\in\mathcal{A}_0$ be such that $\omega=\sum_i a_i\delta b_i\delta c_i$ satisfies

$$\pi(\omega)=\sum_i a_i[\,D,b_i\,][\,D,c_i\,]=0\,. \tag{3.21}$$

The coefficient of $\psi^1\psi^2\psi^3$ in $\pi(\delta\omega)$ is proportional to

$$\begin{aligned}\varepsilon^{ABC}\sum_i\,[\,J^A,a_i\,][\,J^B,b_i\,][\,J^C,c_i\,]&=-\varepsilon^{ABC}\sum_i\,a_i\,\left[\,J^A,[\,J^B,b_i\,][\,J^C,c_i\,]\right]\\&=-\varepsilon^{ABC}\sum_i\,a_i\big([\,[J^A,[\,J^B,b_i\,]]\;[\,J^C,c_i\,]+[\,J^B,b_i\,]\;[\,J^A,[\,J^C,c_i\,]]\big)\\&=-\frac{i}{2}\,\varepsilon^{ABC}\sum_i\,a_i\big(\varepsilon^{ABD}[\,J^D,b_i\,][\,J^C,c_i\,]+\varepsilon^{ACD}[\,J^B,b_i\,][\,J^D,c_i\,]\big)=0\end{aligned}$$

where we have used eq. (3.21) and the Jacobi identity. Thus, $\pi(\delta J^2)$ is included in $\pi(\Omega^1(\mathcal{A}_0))$, and since $\mathcal{A}_0$ is a full matrix algebra, it is either 0 or equal to $\pi(\Omega^1(\mathcal{A}_0))$. Let $\omega,\eta\in\Omega^1(\mathcal{A}_0)$ be such that $\pi(\omega)=-\mathbf{1}\otimes\psi^A$, $\pi(\eta)=0$ and $\pi(\delta\eta)=\mathbf{1}\otimes\mathbf{1}$. The existence of $\omega$ and $\eta$ is ensured by eqs. (3.16) and the fact that $\pi(\delta J^1)=\mathcal{A}_0$. We have $\omega\eta\in\Omega^2(\mathcal{A}_0)$, $\pi(\omega\eta)=0$ and $\pi(\delta(\omega\eta))=\mathbf{1}\otimes\psi^A$ as $\pi(\delta\omega)=0$. This proves that $\pi(\delta J^2)=\pi(\Omega^1(\mathcal{A}_0))$, and we get

$$\Omega_D^3(\mathcal{A}_0)\simeq\{\,a\otimes\psi^1\psi^2\psi^3\,|\,a\in\mathcal{A}_0\,\}\,. \tag{3.22}$$

We proceed with the space of 4-forms. First, we notice that due to the Clifford algebra relations, eqs. (3.4,5,8), we have

$$\pi(\Omega^4(\mathcal{A}_0))=\{\,a_{AB}\otimes\psi^A\psi^B\,|\,a_{AB}\in\mathcal{A}_0\,\}\,. \tag{3.23}$$

Let $\omega\in\Omega^1(\mathcal{A}_0)$ and $\eta\in\Omega^2(\mathcal{A}_0)$ be such that $\pi(\omega)=0$, $\pi(\delta\omega)=\mathbf{1}\otimes\mathbf{1}$, and $\pi(\eta)=\mathbf{1}\otimes\psi^A\psi^B$. The existence of $\omega$ and $\eta$ is ensured by the fact that $\pi(\delta J^1)=\mathcal{A}_0$ and by eq. (3.17). We have $\omega\eta \in \Omega^3(\mathcal{A}_0)$, $\pi(\omega\eta) = 0$ and $\pi(\delta(\omega\eta)) = \mathbf{1}\otimes\psi^A\psi^B$ as $\pi(\omega) = 0$. Since $\mathcal{A}_0$ is a full matrix algebra, this proves that $\pi(\Omega^4(\mathcal{A}_0)) = \pi(\delta J^3)$, and we get $\Omega^4_D(\mathcal{A}_0) = 0$. Using the fact that the product of differential forms induces a surjective map

$$\Omega^n_D(\mathcal{A}_0) \otimes \Omega^m_D(\mathcal{A}_0) \longrightarrow \Omega^{n+m}_D(\mathcal{A}_0)$$

we obtain

$$\Omega^n_D(\mathcal{A}_0) = 0 \quad \forall n > 3 . \tag{3.24}$$

Collecting eqs. (3.16), (3.22) and (3.24), we arrive at the following theorem on the structure of differential forms over the non-commutative space $(\mathcal{A}_0, \mathcal{H}_0 \otimes W, D, \gamma)$:

**Theorem 3.2** The left $\mathcal{A}_0$-modules $\Omega^n_D(\mathcal{A}_0)$ are all free and given as follows:

0) $\Omega^0_D(\mathcal{A}_0) = \mathcal{A}_0$ is one-dimensional with basis $\{\mathbf{1}\}$;

1) $\Omega^1_D(\mathcal{A}_0)$ is three-dimensional with basis $\{\mathbf{1} \otimes \psi^A\}$;

2) $\Omega^2_D(\mathcal{A}_0)$ is three-dimensional with basis $\{\mathbf{1} \otimes \psi^A\psi^{A+1}\}$ (where addition is taken modulo 3);

3) $\Omega^3_D(\mathcal{A}_0)$ is one-dimensional with basis $\{\mathbf{1} \otimes \psi^1\psi^2\psi^3\}$;

4) $\Omega^n_D(\mathcal{A}_0) = 0$ for all $n > 3$.

Notice that the structure of the modules $\Omega^n_D(\mathcal{A}_0)$ is the same as that of the spaces of differential forms on $\mathrm{SU}(2) \simeq S^3$.

In the following, we compute the action of the exterior differential

$$\delta \,:\, \Omega^n_D(\mathcal{A}_0) \longrightarrow \Omega^{n+1}_D(\mathcal{A}_0) .$$

We introduce the following bases of $\Omega^1_D(\mathcal{A}_0)$ and $\Omega^2_D(\mathcal{A}_0)$

$$e^A = \mathbf{1} \otimes \psi^A \quad \in \Omega^1_D(\mathcal{A}_0) , \tag{3.25}$$

$$f^A = \varepsilon^{ABC} \otimes \psi^B\psi^C \quad \in \Omega^2_D(\mathcal{A}_0) , \tag{3.26}$$

which allows us to identify $\Omega^1_D(\mathcal{A}_0)$ and $\Omega^2_D(\mathcal{A}_0)$ with the standard free module $\mathcal{A}_0^3$, and we decompose their elements with respect to these bases,

$$\omega = \omega_A\, e^A \quad \text{for } \omega \in \Omega^1_D(\mathcal{A}_0) , \tag{3.27}$$

$$\omega = \omega_A\, f^A \quad \text{for } \omega \in \Omega^2_D(\mathcal{A}_0) . \tag{3.28}$$

It is easily verified that the product of 1-forms $\omega, \eta \in \Omega^1_D(\mathcal{A}_0)$ is given by

$$\omega \cdot \eta = \varepsilon^{ABC}\, \omega_B\, \eta_C\, f^A . \tag{3.29}$$

By the Leibniz rule for the exterior differential $\delta$, knowledge of the action of $\delta$ on the elements $a \in \mathcal{A}_0$, $e^A$ and $f^A$ fully determines the action of the differential on $\Omega^\bullet_D(\mathcal{A}_0)$. By definition, we have

$$\delta a = [\, J^A, a\,]\, e^A . \tag{3.30}$$

Using eq. (3.30) and the nilpotency of $\delta$ we get

$$0 = \delta^2 J^A = i\varepsilon^{ABC}\delta(J^C e^B) = -\varepsilon^{BAC}\varepsilon^{DCF} J^F e^D e^B + i\varepsilon^{BAC} J^C \delta e^B$$

from which we can successively conclude that

$$\varepsilon^{BAE}\varepsilon^{DEC} = i\varepsilon^{BAC}\delta e^B \ ,$$
$$e^A e^C = i\varepsilon^{BAC}\delta e^B \ .$$

With eq. (3.29), we finally get

$$\delta e^A = -if^A \ . \tag{3.31}$$

This equation, together with the nilpotency of $\delta$, furthermore implies that

$$\delta f^A = 0 \ . \tag{3.32}$$

We summarize these results in the following

**Theorem 3.3** Let $g = \frac{1}{3!}\,\varepsilon^{ABC}\psi^A\psi^B\psi^C$ be the basis element of $\Omega_D^3(\mathcal{A}_0)$, and $e^A$ and $f^A$ as in eqs. (3.25,26). Then the algebra structure of $\Omega_D^\bullet(\mathcal{A}_0)$ is given as follows:

a1) $$[\,a, e^A\,] = [\,a, f^A\,] = [\,a, g\,] = 0 \quad \text{for all } \ a \in \mathcal{A}_0 \tag{3.33}$$

a2) $$e^A e^B = \varepsilon^{ABC} f^C \ , \quad e^A e^B e^C = \varepsilon^{ABC}\, g \ , \tag{3.34}$$

$$e^A f^B = \delta^{AB}\, g \ . \tag{3.35}$$

The differential structure on $\Omega_D^\bullet(\mathcal{A}_0)$ is given by

b1) $$\delta a = [\,J^A, a\,]\, e^A \ , \tag{3.36}$$

b2) $$\delta e^A = -if^A \ , \quad \delta f^A = 0 \ . \tag{3.37}$$

### 3.2.2 Cohomology of the de Rham complex

Let us now compute the cohomology groups of the de Rham complex $(\Omega_D^\bullet(\mathcal{A}_0), \delta)$ of Theorems 3.2 and 3.3.

The zeroth cohomology group $H^0$ consists of those elements $a \in \mathcal{A}_0$ that are closed, i.e., satisfy $\delta a = 0$. We have

$$\begin{aligned} a \in H^0 \quad &\Longleftrightarrow \quad \delta a = [\,J^A, a\,]e^A = 0 \\ &\Longleftrightarrow \quad [\,J^A, a\,] = 0 \quad \text{for all } A \end{aligned}$$

and it follows that

$$H^0 = \mathcal{A}_R \equiv \bigoplus_{j=0}^{k/2} \mathbf{1}_{V_j^\bullet} \otimes \mathrm{End}\,(V_j) \ , \tag{3.38}$$

and

$$\dim_{\mathbb{C}} H^0 = \sum_{j=0}^{k/2} (2j+1)^2 = \frac{1}{6}(2k+3)(k+2)(k+1) \ . \tag{3.39}$$

In order to compute the first cohomology group, we first determine the closed 1-forms. For any 1-form $\omega = \omega_A e^A \in \Omega^1_D(\mathcal{A}_0)$, relation (3.37) implies that

$$\delta\omega = \left( [\, J^A, \omega_B \,]\varepsilon^{ABC} - i\omega_C \right) f^C \ ,$$

and thus $\delta\omega = 0$ is equivalent to

$$[\, J^A, \omega_B \,]\varepsilon^{ABC} = i\omega_C \ . \tag{3.40}$$

We show that all closed 1-forms are exact. First, notice that if we view $\mathcal{A}_0$ as a representation space of su(2), then, for a closed 1-form, eq. (3.40) must hold in all isotypic components. Therefore, there is no loss of generality in assuming that all coefficients $\omega_A$ transform under the spin $j$ representation, i.e.,

$$[\, J^A, [\, J^A, \omega_B \,]] = j(j+1)\,\omega_B \ . \tag{3.41}$$

Furthermore, we can assume that $j \neq 0$ since otherwise $\omega = 0$, as follows from eq. (3.40). We define $a(\omega) \in \mathcal{A}_0$ by

$$a(\omega) = \frac{1}{j(j+1)} [\, J^A, \omega_A \,]$$

and we compute $\delta a$. Using eqs. (3.40,41) and the Jacobi identity, we get

$$\begin{aligned} \delta a(\omega) &= \frac{1}{j(j+1)} [\, J^A, [\, J^B, \omega_B \,]] e^A \\ &= \frac{1}{j(j+1)} \left( i\varepsilon^{ABC}[\, J^C, \omega_B \,] + [\, J^B, [\, J^A, \omega_B \,]] \right) e^A \\ &= \frac{1}{j(j+1)} [\, J^B, [\, J^B, \omega_A \,]] e^A = \omega_A e^A \ . \end{aligned}$$

This proves that

$$H^1 = 0 \ . \tag{3.42}$$

We proceed towards the second cohomology group. The condition for a 2-form $\omega = \omega_A f^A$ to be closed reads

$$\delta\omega = 0 \iff [\, J^A, \omega_A \,] = 0 \ . \tag{3.43}$$

Again, we assume that the components $\omega_A$ belong to a spin $j$ representation of su(2). If $j = 0$, then setting $\eta_A = i\omega_A$ we get

$$\delta(\eta_A e^A) = \omega_A f^A \ ,$$

proving that $\omega$ is exact. If $j \neq 0$, we set

$$\eta_A = -\frac{1}{j(j+1)}\,\varepsilon^{ABC}[\,J^B, \omega_C\,]\ ,$$

and one easily verifies that $\delta(\eta_A e^A) = \omega_A f^A$. This proves that

$$H^2 = 0\ . \tag{3.44}$$

Finally, we compute the third cohomology group. Since all 3-forms are closed, we just have to compute the image of the exterior differential in $\Omega^3_D(\mathcal{A}_0)$. For any 2-form $\omega$ we have

$$\delta\omega = [\,J^A, \omega_A\,]\,g$$

with $g$ being the basis element of $\Omega^3_D(\mathcal{A}_0)$ as in Theorem 3.3. This means that the image of $\delta$ in $\Omega^3_D(\mathcal{A}_0)$ is given by

$$\operatorname{im}\delta\,\Big|_{\Omega^2_D(\mathcal{A}_0)} = \operatorname{span}\Big(\bigcup_{A=1}^{3}\operatorname{im}(\operatorname{ad}J^A)\Big)\cdot g\ ,$$

and this space consists of linear combinations of elements of $\mathcal{A}_0$ transforming under a spin $j$ representation for $j \neq 0$, multiplied by $g$. Thus, the quotient $\Omega^3_D(\mathcal{A}_0)/\operatorname{im}\delta$ is given by

$$H^3 \simeq \mathcal{A}_R \equiv \bigoplus_{j=0}^{k/2} \mathbf{1}_{V_j^*} \otimes \operatorname{End}(V_j)\ . \tag{3.45}$$

Collecting our results of eqs. (3.38,39,42,44) and (3.45), we get the following

**Theorem 3.4** The cohomology groups of the de Rham complex of Theorem 3.3 are

$$H^0 \simeq H^3 \simeq \mathcal{A}_R\ , \qquad H^1 = H^2 = 0$$

with dimensions

$$\dim_{\mathbb{C}} H^0 = \dim_{\mathbb{C}} H^3 = \frac{1}{6}(2k+3)(k+2)(k+1)\ .$$

This theorem shows that the cohomology groups of the fuzzy 3-sphere – which is the quantum target of the WZW model based on SU(2) [$FG$, $Gr$] – look very much like those of the classical SU(2) group manifold, except for the unexpected dimensions of the spaces $H^0$ and $H^3$.

We observe that in the classical setting, the cohomology groups are modules over the ring $H^0$ and that, for a connected space, the Betti numbers coincide with the dimensions of these modules. We are thus led to the idea that the dimensions of the cohomology groups over $\mathbb{C}$ may be less relevant than their dimensions as modules over $H^0$. Of course, it may happen in general that some $H^0$-module is not free, and we would, in that case, lose the notion of dimension. For the cohomology groups of the de Rham complex $(\Omega_D^\bullet(\mathcal{A}_0), \delta)$ we get

$$\dim_{H^0} H^0 = \dim_{H^0} H^3 = 1 \ ,$$
$$\dim_{H^0} H^1 = \dim_{H^0} H^2 = 0 \ ,$$

which fits perfectly with the classical result. The above proposal is obviously tailored to make sense of the cohomology groups of Theorem 3.4 and its general relevance remains to be decided by the study of other examples of non-commutative spaces.

## 3.3 The geometry of the non-commutative 3-sphere

The $N = 1$ spectral data $(\mathcal{A}_0, \mathcal{H}_0 \otimes W, D, \gamma)$ permit us to investigate not only topological but also geometrical aspects of the quantized 3-sphere, namely integration of differential forms and Hermitian structures, as well as connections and the associated Riemann, Ricci and scalar curvatures. For a more detailed account of the results of this section, the reader is referred to [$Gr$].

### 3.3.1 Integration and Hermitian structures

We start with the canonical scalar product and the Hermitian structures on the spaces of differential forms. We use the same notations as in subsections 2.1.3 – 2.1.5.
Any element $\omega \in \pi(\Omega^\bullet(\mathcal{A}_0))$ can be written uniquely as

$$\omega = \omega^0 + \omega_A^1 e^A + \omega_A^2 f^A + \omega^3 g \tag{3.46}$$

where $\omega^i, \omega_A^i \in \mathcal{A}_0$. The integral $⨍$ , as given in Definition 2.3, is just the normalized trace on $\mathcal{H}_0 \otimes W$, denoted by Tr. Thus, for any element $\omega$ as above, we have

$$⨍ \omega = \text{Tr}\, \omega^0 \tag{3.47}$$

It is easy to show that the sesqui-linear form $(\cdot,\cdot)$ associated to the integral is given by

$$(\omega, \eta) = \text{Tr}\left[\omega^0(\eta^0)^* + \omega_A^1(\eta_A^1)^* + \omega_A^2(\eta_A^2)^* + \omega^3(\eta^3)^*\right] . \tag{3.48}$$

This proves that the kernels $K^i$ of the sesqui-linear form $(\cdot,\cdot)$ equal the kernels $J^i$ of the representation $\pi$. Thus, in this example we also have the equality

$$\widetilde{\Omega}_D^p(\mathcal{A}_0) = \Omega_D^p(\mathcal{A}_0) \ .$$

Furthermore, since $\pi(\delta J^1) = \mathcal{A}_0$ and $\pi(\delta J^2) = \pi(\Omega_D^1(\mathcal{A}_0))$, we see that the decomposition (3.46) gives the canonical representative $\omega^\perp$ of an arbitrary differential form $\omega \in \Omega_D^\bullet(\mathcal{A}_0)$.
The Hermitian structure on $\Omega_D^p(\mathcal{A}_0)$ is readily seen to be

$$\langle \omega, \eta \rangle = \omega_A(\eta_A)^* \ , \quad \omega, \eta \in \Omega_D^p(\mathcal{A}_0) \ . \tag{3.49}$$

Notice that, in this example, we get a true Hermitian structure on $\Omega_D^p(\mathcal{A}_0)$ and not only a *generalized* Hermitian structure on $\widetilde{\Omega}^p(\mathcal{A}_0)$, cf. subsection 2.1.5.

### 3.3.2 Connections on $\Omega_D^1(\mathcal{A}_0)$

This last property makes it possible to regard $\Omega_D^1(\mathcal{A}_0)$ as the cotangent bundle of the non-commutative 3-sphere and to study connections on $\Omega_D^1(\mathcal{A}_0)$.
Since the space of 1-forms $\Omega_D^1(\mathcal{A}_0)$ is a trivial left $\mathcal{A}_0$-module, a connection $\nabla$ on $\Omega_D^1(\mathcal{A}_0)$ is uniquely determined by the images of the basis elements, i.e.,

$$\nabla e^A = -\omega^A_{BC}\, e^B \otimes e^C \tag{3.50}$$

where $\omega^A_{BC}$ are *arbitrary* elements of $\mathcal{A}_0$.

**Proposition 3.5** A connection $\nabla$ is unitary if and only if its coefficients satisfy the Hermiticity condition

$$\omega^{A*}_{BC} = \omega^C_{BA} \,. \tag{3.51}$$

PROOF: It follows from (3.49) that $\langle e^A, e^B\rangle = \delta^{AB}$. Then we have for a unitary connection (see Definition 2.12)

$$0 = \delta\langle e^A, e^B\rangle = -\omega^A_{CB}e^C + e^C\omega^{B*}_{CA} \qquad \blacksquare$$

**Proposition 3.6** The torsion of a connection is given by

$$\mathrm{T}^A = (-i\delta^{AD} + \omega^A_{BC}\varepsilon^{BCD})\, f^D \,.$$

PROOF: Using Definition 2.14 and eqs. (2.20), (3.34,37), we get

$$\mathrm{T}(\nabla)\, e^A = -if^A + \omega^A_{BC}\varepsilon^{BCD} f^D \,. \qquad \blacksquare$$

**Proposition 3.7** A connection is torsionless and unitary if and only if its coefficients satisfy the following conditions

$$\text{i)} \quad \omega^A_{BC} - \omega^{A*}_{BC} = i\varepsilon^{ABC} \,, \tag{3.52}$$

$$\text{ii)} \quad \omega^A_{BC} = \omega^{A*}_{CB} \,, \tag{3.53}$$

$$\text{iii)} \quad \omega^A_{BC} = \omega^C_{AB} \,. \tag{3.54}$$

In particular, such a connection is uniquely determined by the nine self-adjoint elements $\omega^A_{AB} \in \mathcal{A}_0$ and the self-adjoint part of $\omega^1_{23}$.
PROOF: The condition of vanishing torsion,

$$\omega^A_{BC}\,\varepsilon^{BCD} = i\delta^{AD} \,,$$

can equivalently be written as

$$\omega^A_{BC} - \omega^A_{CB} = i\varepsilon^{ABC} \ . \tag{3.55}$$

Using alternatively eqs. (3.55) and the unitarity condition eq. (3.51) we get

$$\omega^A_{BC} = i\varepsilon^{ABC} + \omega^A_{CB} = i\varepsilon^{ABC} + \omega^{B*}_{CA} = \omega^{B*}_{AC} = \omega^C_{AB} \ .$$

which proves the result. ∎

This proposition shows that, as in the classical case, there are many unitary and torsionless connections. There are two possibilities to reduce the space of "natural" connections further. First, we can consider *real* connections, i.e., connections whose associated parallel transport maps real forms to real forms. In the classical setting, a 1-form $\omega$ is real if $\omega^* = -\omega$ (the sign comes from the fact that the Clifford matrices are anti-Hermitian). Thus, we see that our basis of 1-forms consists of *imaginary* 1-forms, i.e., $e^{A*} = e^A$. If the covariant derivative of an imaginary 1-form is to be imaginary, then the connection coefficients $\omega^A_{BC}$ must be anti-Hermitian. We call such a connection a *real connection.*

**Corollary 3.8** There is a unique real unitary and torsionless connection on the cotangent bundle $\Omega^1_D(\mathcal{A}_0)$, and its coefficients are given by

$$\omega^A_{BC} = \frac{i}{2}\,\varepsilon^{ABC} \ .$$

There is another way of reducing the number of "natural" connections. If we look at a general unitary and torsionless connection, we see that it does not have any isotropy property. For example, the coefficients $\omega^A_{AA}$ are all independent of one another. We hope that if we require the connection to be invariant under all 1-parameter group of isometries (see [*CFG*, *Gr*]) we shall get relations among these coefficients. We shall not pursue this route here, but we refer the reader to [*Gr*] for a detailed analysis.

We proceed with the computation of the scalar curvature of the *real* connection $\widetilde{\nabla}$ of Corollary 3.8.
For any connection $\nabla$ with coefficients $\omega^A_{BC}$ as defined in eq. (3.50), the curvature tensor is given by (see Definition 2.11)

$$\begin{aligned} \mathrm{R}(\nabla)\, e^A &= -\nabla^2 e^A \\ &= \left\{ [\, J^D, \omega^A_{EC}]\, \varepsilon^{DEB} - i\omega^A_{BC} + \omega^A_{DE}\, \omega^E_{FC}\, \varepsilon^{DFB} \right\} f^B \otimes e^C \ . \end{aligned} \tag{3.56}$$

In particular, for the *real* connection $\widetilde{\nabla}$ of Corollary 3.8, the curvature tensor reads

$$\mathrm{R}(\widetilde{\nabla})\, e^A = \frac{1}{4}\,\varepsilon^{ABC} f^B \otimes e^C \ . \tag{3.57}$$

In order to compute the Ricci curvature, we use a dual basis to the generators $e^A$, as in subsection 2.1.7, before eq. (2.15). It is clear that the elements $\varepsilon_A \in \Omega^1_D(\mathcal{A}_0)^*$ defined by

$$\varepsilon_A(\omega) = \varepsilon_A(\omega_B e^B) := \omega_A \tag{3.58}$$

for all $\omega \in \Omega^1_D(\mathcal{A}_0)$, form a dual basis to $e^A$. Using eq. (3.49) it is then easy to verify that the *dual 1-forms* $e_A$ and their dual maps $e^{\mathrm{ad}}_A$, eq. (2.16), are given by

$$e_A = e^A \; , \quad e^{\mathrm{ad}}_A(f^B) = -\varepsilon^{ABC} e^C \; . \tag{3.59}$$

For the real connection $\widetilde{\nabla}$, we get from eq. (3.57)

$$\mathtt{Ric}(\widetilde{\nabla}) = -\frac{1}{2}\, e^A \otimes e^A \; . \tag{3.60}$$

We proceed with the computation of the scalar curvature. The *right dual maps* $(e^A_R)^{\mathrm{ad}}$ to the basis 1-forms $e^A$, eq. (2.17), act as

$$(e^A_R)^{\mathrm{ad}}(e^B) = \delta^{AB} \; . \tag{3.61}$$

The scalar curvature of the real connection $\widetilde{\nabla}$ follows from eq. (3.60) and is given by

$$\mathtt{r}(\widetilde{\nabla}) = -\frac{3}{2} \; . \tag{3.62}$$

It is the same as the scalar curvature of the *unique* real unitary and torsionless connection for the classical SU(2) – recall that the definition of the scalar curvature in the non-commutative setting differs from the classical one by a sign, see the remark in Definition 2.16.

This completes our study of the non-commutative 3-sphere in terms of $N=1$ spectral data. Our results show that the non-commutative 3-sphere has striking similarities with its classical counterpart. As we saw in subsections 3.2.1 and 3.2.2, the spaces of differential forms have the same structure as left-modules over the algebra of functions, and the cohomology groups have the same dimensions as modules over the zeroth cohomology group, $H^0$. Furthermore, geometric invariants like the scalar curvature, too, coincide for the classical and the quantized 3-sphere.

### 3.4 Remarks on $N=(1,1)$

In the following, we consider $N=(1,1)$ data for the algebra $\mathcal{A}_0$. The construction of the first subsection starts from the BRST operator of the group $G$ and leads to a deformation of the de Rham complex for the classical 3-sphere in the form of $N=(1,1)$ data for the non-commutative 3-sphere. In the second subsection, we return to the two generalized Dirac operators provided by superconformal field theory [$FG$], which lead to a different formulation of $N=(1,1)$ data, displaying "spontaneously broken supersymmetry".

### 3.4.1 $N = (1,1)$ data from BRST

One way to arrive at $N = (1,1)$ data for the algebra $\mathcal{A}_0$ and at the associated (non-commutative) de Rham complex for the quantized 3-sphere is to use the action of the group $G$ on the Hilbert space $\mathcal{H}_0$ for introducing a BRST operator (see also section I 2.3). Let $\{ J_A \}$ be the basis of the complexified Lie algebra $\mathrm{g}^{\mathbb{C}}$ of $G$ introduced in eq. (3.7). The BRST operator $Q$ for the group $G$ is defined as usual: We introduce ghosts $c^A$ and anti-ghosts $b_A$ satisfying the ghost algebra

$$\{ c^A, c^B \} = \{ b_A, b_B \} = 0 \ , \quad \{ c^A, b_B \} = \delta^A_B \ . \tag{3.63}$$

Then the BRST operator is given by

$$Q = c^A J_A - \frac{i}{2} f^C_{AB}\, c^A c^B b_C \ , \tag{3.64}$$

and the ghost number operator is

$$T = c^A b_A \ . \tag{3.65}$$

The Hilbert space of the $N = (1,1)$ data will be of the form $\mathcal{H}_0 \otimes W$ where $W$ is a representation space for the ghost algebra.
In order to obtain $N = (1,1)$ data, we require that the ghost algebra acts unitarily on $W$ with respect to the natural $^*$-operation, namely

$$c^{A\,*} = g^{AB} b_B \ . \tag{3.66}$$

This choice is compatible with positive definiteness of the scalar product on $W$, and it renders the ghost number operator $T$ self-adjoint.[1] Furthermore, this choice of $^*$-operation leads to identifying the ghost algebra with the CAR

$$\{ c^A, c^B \} = 0 \ , \quad \{ c^A, c^{B\,*} \} = g^{AB} \ , \tag{3.67}$$

and the BRST operator can be written

$$Q = c^A J_A - \frac{i}{2} f^C_{AB}\, c^A c^B c^*_C \ , \tag{3.68}$$

where indices are raised and lowered with the metric $g_{AB}$ as usual. Under the identifications

$$c^A \sim a^{A\,*} := -i\, \theta^A \wedge$$

[1] In the context of gauge theories, one considers representations such that $c^{A*}=c^A$ , $b^*_A=b_A$ . These Hermiticity conditions together with the defining relations (3.63) imply that the inner product of the representation space is not positive definite – which is why $c^A$ and $b_A$ are called ghosts.

where $\{ \theta^A \}$ is a basis of 1-forms dual to $\{ \vartheta_A \}$, eq. (3.68) for the BRST operator formally coincides with the exterior derivative on $G$. This fact was already mentioned in section I2.3.

In order to complete our construction of $N = (1,1)$ data, we introduce the Hodge $*$-operator

$$* = \frac{1}{n!} \sqrt{g}\, \varepsilon_{A_1 \dots A_n} ( c^{A_1} + c^{A_1 *} ) \cdots ( c^{A_n} + c^{A_n *} ) \tag{3.69}$$

where $n = \dim G$. This operator clearly commutes with the algebra $\mathcal{A}_0$ of Theorem 3.1. Moreover, it is easy to verify that $*$ is unitary and satisfies

$$*^2 = (-1)^{\frac{n(n-1)}{2}} \tag{3.70}$$

as well as

$$*Q = (-1)^{n-1} Q * \ . \tag{3.71}$$

It follows that $(\mathcal{A}, \mathcal{H}, d, \gamma, *)$ with $\mathcal{A} = \mathcal{A}_0$, $\mathcal{H} = \mathcal{H}_0 \otimes W$, $d = Q$ and where $\gamma$ is the modulo 2 reduction of the $\mathbb{Z}$-grading $T$, form a set of $N = (1,1)$ data in the sense of Definition 2.20.

We refrain from presenting the details of the construction of differential forms and of the other geometrical quantities, since the computations are fairly straightforward. For example, the space of $k$-forms is given by

$$\Omega_d^k(\mathcal{A}_0) = \{ a_{A_1 \dots A_k} c^{A_1} \cdots c^{A_k} \,|\, a_{A_1 \dots A_k} \in \mathcal{A}_0 \} \ . \tag{3.72}$$

For $G = \mathrm{SU}(2)$, we see that these spaces are isomorphic to $\Omega_D^k(\mathcal{A}_0)$ as left $\mathcal{A}_0$-modules. Furthermore, it is easy to see that $\Omega_d^\bullet(\mathcal{A}_0)$ and $\Omega_D^\bullet(\mathcal{A}_0)$ are isomorphic as complexes, which proves that, in particular, their cohomologies coincide.

Of course, the same constructions and results apply to the BRST operator associated with the right-action of $G$ on $\mathcal{H}_0$ given by the generators $\overline{J}_A$ of eq. (3.7).

The Hilbert space $\mathcal{H} = \mathcal{H}_0 \otimes W$ can be decomposed into a direct sum of eigenspaces of the $\mathbb{Z}$-grading operator $T$,

$$\mathcal{H} = \bigoplus_{k=0}^{n} \mathcal{H}^{(k)}$$

where $\mathcal{H}^{(0)} = \mathcal{H}_0$, $n = \dim G$ ($= 3$ for $\mathrm{G} = \mathrm{SU}(2)$). The subspaces $\mathcal{H}^{(k)}$ are left-modules for $\mathcal{A}_0$. Furthermore, it follows from eqs. (3.65) and (3.68) that $d := Q$ maps $\mathcal{H}^{(k)}$ into $\mathcal{H}^{(k+1)}$ for $k = 0, \dots, n$ (with $\mathcal{H}^{(n+1)} := \{0\}$). Since $d^2 = 0$, $\mathcal{H}$ is a complex. Viewed as linear spaces, the cohomology groups of $\Omega_d^\bullet(\mathcal{A}_0)$ and $(\mathcal{H}, Q)$ are isomorphic, although the latter do not carry a ring structure.

As a side remark, consider an odd operator $H$ on $\mathcal{H}$. Then $\tilde{d} := d + H$ is nilpotent if and only if $\{ d, H \} + H^2 = 0$. If $H$ commutes with $\mathcal{A}_0$, then $\Omega_{\tilde{d}}^\bullet(\mathcal{A}_0)$ and $\Omega_d^\bullet(\mathcal{A}_0)$ are identical complexes. In the next subsection, we will meet a conformal field theory motivated example for $\tilde{d} = d + H$ which is nilpotent but for which $H$ does not commute with $\mathcal{A}_0$.

### 3.4.2 Spontaneously broken supersymmetry

In section 3.1 we introduced two connections $\nabla^S$ and $\overline{\nabla}^S$ and their associated Dirac operators $\mathcal{D}$ and $\overline{\mathcal{D}}$, see eqs. (3.6-9). Since these two Dirac operators correspond to *different* connections, they are not Dirac operators on an $N=(1,1)$ Dirac bundle in the sense of Definition I 2.6. It is interesting to notice that $\mathcal{D}$ and $\overline{\mathcal{D}}$ nevertheless satisfy the $N=(1,1)$ algebra [FG]

$$\mathcal{D}^2 = \overline{\mathcal{D}}^2 \ , \quad \{\, \mathcal{D}, \overline{\mathcal{D}} \,\} = 0 \ . \tag{3.73}$$

The easiest way to prove (3.73) is to verify that the generalized exterior derivative

$$\tilde{d} := \frac{1}{2}\,(\mathcal{D} + i\overline{\mathcal{D}}) \tag{3.74}$$

is nilpotent. Let $\{\, \vartheta_A \,\}$ and $\{\, \theta^A \,\}$ denote a basis of the Lie algebra and the dual basis of 1-forms, respectively, as before. We define the operators

$$a^{A\,*} = \theta^A \wedge \ , \quad a_A = \vartheta_A \llcorner$$

as usual, and we can express the fermionic operators $\psi^A$ and $\overline{\psi}^A$ as

$$\psi^A = -i(a^{A\,*} - a^A) \ , \quad \overline{\psi}^A = -(a^{A\,*} + a^A) \ , \tag{3.75}$$

where indices are raised and lowered with the metric $g_{AB}$. Using eqs. (3.9) and (3.74), we can rewrite the operator $\tilde{d}$ as a sum of terms of degree $1, -1$ and $-3$,

$$\tilde{d} = \tilde{d}_1 + \tilde{d}_{-1} + \tilde{d}_{-3} \tag{3.76}$$

where

$$\begin{aligned} \tilde{d}_1 \ &= a^{A\,*} J_A^+ - \frac{1}{4}\, f_{ABC} a^{A\,*} a^{B\,*} a^C \\ \tilde{d}_{-1} &= -a^A J_A^- \\ \tilde{d}_{-3} &= -\frac{1}{12}\, f_{ABC} a^A a^B a^C \end{aligned} \tag{3.77}$$

with

$$J_A^{\pm} = -\frac{i}{2}\,(J_A \pm \overline{J}_A) \ . \tag{3.78}$$

It is then straightforward to show that $\tilde{d}$ given by eqs. (3.76,77) satisfies $\tilde{d}^2 = 0$ and that the associated Laplacian $\triangle = \{\, \tilde{d}, \tilde{d}^* \,\}$ is given by

$$\triangle = g^{AB} J_A J_B + \frac{\dim G}{24} = g^{AB} \overline{J}_A \overline{J}_B + \frac{\dim G}{24} \ . \tag{3.79}$$

Thus, $\triangle$ is a strictly positive operator – corresponding to what one calls *spontaneously broken supersymmetry* in the context of field theory. This implies that the cohomology of the complex $(\mathcal{H}, \tilde{d})$ is trivial. However, the cohomology of the complex $\Omega^\bullet_{\tilde{d}}(\mathcal{A}_0)$, as introduced in Sect. 2.2, is not trivial. Notice that $\tilde{d}_1$ is the BRST operator associated to the generators $J_A^+$ and hence nilpotent. This implies that the BRST cohomology of the fuzzy 3-sphere can be extracted from $\Omega^\bullet_{\tilde{d}}(\mathcal{A}_0)$.

## 4. The non-commutative torus

As a second, "classic" example of non-commutative spaces, we discuss the geometry of the non-commutative 2-torus [$Ri, Co1, Co5$]. After a short review of the classical torus in subsection 4.1, we analyze the spin geometry ($N = 1$) of the non-commutative torus in subsection 4.2 along the lines of [$FGR2, Gr$]. In subsections 4.3 and 4.4, we successively extend the $N = 1$ data to $N = (1,1)$ and $N = (2,2)$ data – according to the general procedure proposed in subsection 2.2.5 above. In these two last subsections, we do not give detailed proofs, but merely state the results since the computations, although straightforward, are tedious and not very illuminating.

### 4.1 The classical torus

To begin with, we describe the $N = 1$ data associated to the classical 2-torus $\mathbb{T}_0^2$. By Fourier transformation, the algebra of smooth functions over $\mathbb{T}_0^2$ is isomorphic to the Schwarz space $\mathcal{A}_0 := \mathcal{S}(\mathbb{Z}^2)$ over $\mathbb{Z}^2$, endowed with the (commutative) convolution product:

$$(a \bullet b)(p) \;=\; \sum_{q\in\mathbb{Z}^2} a(q)\, b(p-q) \tag{4.1}$$

where $a, b \in \mathcal{A}_0$ and $p \in \mathbb{Z}^2$. Complex conjugation of functions translates into a $*$-operation:

$$a^*(p) \;=\; \overline{a(-p)}\,, \qquad a \in \mathcal{A}_0\,. \tag{4.2}$$

If we choose a spin structure over $\mathbb{T}_0^2$ in such a way that the spinors are periodic along the elements of a homology basis, then the associated spinor bundle is a trivial rank 2 vector bundle. With this choice, the space of square integrable spinors is given by the direct sum

$$\mathcal{H} \;=\; l^2(\mathbb{Z}^2) \,\oplus\, l^2(\mathbb{Z}^2) \tag{4.3}$$

where $l^2(\mathbb{Z}^2)$ denotes the space of square summable functions over $\mathbb{Z}^2$. The algebra $\mathcal{A}_0$ acts diagonally on $\mathcal{H}$ by the convolution product. We choose a flat metric $(g_{\mu\nu})$ on $\mathbb{T}_0^2$ and we introduce the corresponding 2-dimensional gamma matrices

$$\{\gamma^\mu, \gamma^\nu\} \;=\; -\,2\,g^{\mu\nu}\,, \qquad \gamma^{\mu *} \;=\; -\,\gamma^\mu\,. \tag{4.4}$$

Then, the Dirac operator $D$ on $\mathcal{H}$ is given by

$$(D\,\xi)(p) \;=\; i\,p_\mu\,\gamma^\mu\,\xi(p)\,, \qquad \xi \in \mathcal{H}\,. \tag{4.5}$$

Finally, the $\mathbb{Z}_2$-grading on $\mathcal{H}$, denoted by $\sigma$, can be written as

$$\sigma \;=\; \frac{i}{2}\,\sqrt{g}\,\varepsilon_{\mu\nu}\,\gamma^\mu\,\gamma^\nu \tag{4.6}$$

where $\varepsilon_{\mu\nu}$ is the Levi-Civita tensor. The data $(\mathcal{A}_0, \mathcal{H}, D, \sigma)$ are the canonical $N = 1$ data associated to the compact spin manifold $\mathbb{T}_0^2$, and it is thus clear that they satisfy all the properties of Definition 2.1.

### 4.2 Spin geometry ($N = 1$)

The non-commutative torus is obtained by deforming the product of the algebra $\mathcal{A}_0$. For each $\alpha \in \mathbb{R}$ , we define the algebra $\mathcal{A}_\alpha := \mathcal{S}(\mathbb{Z}^2)$ with the product

$$(a \bullet_\alpha b)\,(p) \;=\; \sum_{q\,\in\mathbb{Z}^2} a(q)\,b(p-q)\;e^{i\pi\alpha\omega(p,q)} \tag{4.7}$$

where $\omega$ is the integer-valued anti-symmetric bilinear form on $\mathbb{Z}^2 \times \mathbb{Z}^2$

$$\omega(p,q) \;=\; p_1 q_2 \;-\; p_2 q_1 \;, \qquad p,q \;\in\; \mathbb{Z}^2 \;. \tag{4.8}$$

The $^*$-operation is defined as before. Alternatively, we could introduce the algebra $\mathcal{A}_\alpha$ as the unital $^*$-algebra generated by the elements $U$ and $V$ subject to the relations

$$UU^* \;=\; U^*U \;=\; VV^* \;=\; V^*V \;=\; \mathbf{1}\,, \qquad UV \;=\; e^{-2\pi i\alpha}\,VU \;. \tag{4.9}$$

Having chosen an appropriate closure, the equivalence of the two descriptions is easily seen if one makes the following identifications:

$$U(p) \;=\; \delta_{p_1,1}\,\delta_{p_2,0}\;,\;\; V(p) \;=\; \delta_{p_1,0}\,\delta_{p_2,1}\;. \tag{4.10}$$

If $\alpha$ is a rational number, $\alpha = \frac{M}{N}$, where $M$ and $N$ are co-prime integers, then the centre $Z(\mathcal{A}_\alpha)$ of $\mathcal{A}_\alpha$ is infinite-dimensional:

$$Z(\mathcal{A}_\alpha) \;=\; \mathrm{span}\left\{U^{mN}V^{nN} \;\middle|\; m,n \in \mathbb{Z}\right\}\;. \tag{4.11}$$

Let $\mathrm{I}_\alpha$ denote the ideal of $\mathcal{A}_\alpha$ generated by $Z(\mathcal{A}_\alpha) - \mathbf{1}$. Then it is easy to see that the quotient $\mathcal{A}_\alpha/\mathrm{I}_\alpha$ is isomorphic, as a unital $^*$-algebra, to the full matrix algebra $M_N(\mathbb{C})$.
If $\alpha$ is irrational, then the centre of $\mathcal{A}_\alpha$ is trivial and $\mathcal{A}_\alpha$ is of type $\mathrm{II}_1$, the trace being given by the evaluation at $p = 0$. Unless stated differently, we shall only study the case of irrational $\alpha$.
We define the non-commutative 2-torus $\mathbb{T}^2_\alpha$ by its $N = 1$ data $(\mathcal{A}_\alpha, \mathcal{H}, D, \sigma)$ where $\mathcal{H}$, $D$ and $\sigma$ are as in eqs. (4.3), (4.5) and (4.6), and $\mathcal{A}_\alpha$ acts diagonally on $\mathcal{H}$ by the deformed product, eq. (4.7).
When $\alpha = \frac{M}{N}$ is rational, one may work with the data $(\mathcal{A}_\alpha/\mathrm{I}_\alpha, M_N(\mathbb{C})\otimes\mathbb{C}^2, D_\alpha, \sigma)$, where the Dirac operator $D_\alpha$ is given by

$$D_\alpha \;=\; i\,\gamma^\mu\;\frac{\sin\left(\frac{\pi}{N}\,p_\mu\right)}{\frac{\pi}{N}}\;. \tag{4.12}$$

### 4.2.1 Differential forms

Recall that there is a representation $\pi$ of the algebra of universal forms $\Omega^\bullet(\mathcal{A}_\alpha)$ on $\mathcal{H}$ (see subsection 2.1.2). The images of the homogeneous subspaces of $\Omega^\bullet(\mathcal{A}_\alpha)$ under $\pi$ are given by

$$\pi\left(\Omega^0\left(\mathcal{A}_\alpha\right)\right) = \mathcal{A}_\alpha \quad \text{(by definition)} \tag{4.13}$$

$$\pi\left(\Omega^{2k-1}\left(\mathcal{A}_\alpha\right)\right) = \left\{a_\mu\, \gamma^\mu \;\middle|\; a_\mu \in \mathcal{A}_\alpha\right\} \tag{4.14}$$

$$\pi\left(\Omega^{2k}\left(\mathcal{A}_\alpha\right)\right) = \left\{a + b\sigma \;\middle|\; a, b \in \mathcal{A}_\alpha\right\} \tag{4.15}$$

for all $k \in \mathbb{Z}_+$. In principle, one should then compute the kernels $J^n$ of $\pi$ (see eq. (2.2)), but these are generally huge and difficult to describe explicitly. To determine the space of $n$-forms, it is simpler to use the isomorphism

$$\Omega^n_D(\mathcal{A}_\alpha) \;=\; \Omega^n(\mathcal{A}_\alpha)/\left(J^n + \delta J^{n-1}\right) \;\simeq\; \pi\left(\Omega^n\left(\mathcal{A}_\alpha\right)\right)/\pi(\delta J^{n-1})\;. \tag{4.16}$$

First, we have to compute the spaces of "auxiliary forms" $\pi(\delta J^{n-1})$.

**Lemma 4.1** The spaces $\pi(\delta J^{n-1})$ of auxiliary forms are given by

$$\pi\left(\delta J^1\right) = \mathcal{A}_\alpha \tag{4.17}$$

$$\pi\left(\delta J^{2k}\right) = \pi\left(\Omega^{2k+1}\left(\mathcal{A}_\alpha\right)\right) \tag{4.18}$$

$$\pi\left(\delta J^{2k+1}\right) = \pi\left(\Omega^{2k+2}\left(\mathcal{A}_\alpha\right)\right) \tag{4.19}$$

for all $k \geq 1$.

PROOF: Let $a_i, b_i \in \mathcal{A}_\alpha$ be such that the universal 1-form $\eta = \sum_i a_i \delta b_i \in \Omega^1(\mathcal{A}_\alpha)$ satisfies $\pi(\eta) = 0$. This means that

$$i \sum_{j,q} (p-q)_\mu \gamma^\mu\, a_j(q) b_j(p-q)\, e^{i\pi\alpha\omega(p,q)} = 0 \tag{4.20}$$

for all $p \in \mathbb{Z}^2$. Using eq. (4.20), we have

$$\begin{aligned}\pi(\delta\eta) &= -\sum_{j,q} q_\mu (p-q)_\nu \gamma^\mu\gamma^\nu\, a_j(q) b_j(p-q)\, e^{i\pi\alpha\omega(p,q)} \\ &= -\sum_{j,q} (q^2 - p^2) a_j(q) b_j(p-q)\, e^{i\pi\alpha\omega(p,q)} \in \mathcal{A}_\alpha\;.\end{aligned} \tag{4.21}$$

This proves that $\pi(\delta J^1) \subset \mathcal{A}_\alpha$. Then, we construct an explicit non-vanishing element of $\pi(\delta J^1)$. We set

$$a_1(p) = b_2(p) = \delta_{p_1,-1}\delta_{p_2,0}\;,$$
$$a_2(p) = b_2(p) = \delta_{p_1,1}\delta_{p_2,0}\;,$$

and an easy computation shows that the element $\eta = \sum_{i=1}^{2} a_i \delta b_i$ satisfies

$$\pi(\eta) = 0 \,, \quad \pi(\delta\eta) = -g^{11} \,.$$

Since $\pi(\delta J^1)$ is an $\mathcal{A}_\alpha$-bimodule, eq. (4.17) follows.
Let $k \geq 3$ and $\eta \in \Omega^k(\mathcal{A}_\alpha)$. Then, using eqs. (4.14) and (4.15), we see that there exists an element $\psi \in \Omega^{k-2}(\mathcal{A}_\alpha)$ with $\pi(\eta) = \pi(\psi)$. The first part of the proof ensures the existence of an element $\phi \in \Omega^1(\mathcal{A}_\alpha)$ with $\pi(\phi) = 0$ and $\pi(\delta\phi) = \mathbf{1}$. Then we have $\phi\psi \in J^{k-1}$, and $\pi(\delta(\phi\psi)) = \pi(\psi) = \pi(\eta)$, proving that $\eta \in \delta J^{k-1}$, and therefore eqs. (4.18) and (4.19). ∎

As a corollary to this lemma, we obtain the following

**Proposition 4.2** Up to isomorphism, the spaces of differential forms are given by

$$\Omega_D^0(\mathcal{A}_\alpha) = \mathcal{A}_\alpha \,, \tag{4.22}$$
$$\Omega_D^1(\mathcal{A}_\alpha) \cong \{a_\mu \gamma^\mu \,|\, a_\mu \in \mathcal{A}_\alpha\} \,, \tag{4.23}$$
$$\Omega_D^2(\mathcal{A}_\alpha) \cong \{a\,\sigma \,|\, a \in \mathcal{A}_\alpha\} \,, \tag{4.24}$$
$$\Omega_D^n(\mathcal{A}_\alpha) = 0 \qquad \text{for } n \geq 3 \,, \tag{4.25}$$

where we have chosen special representatives on the right hand side.

Notice that $\Omega_D^1(\mathcal{A}_\alpha)$ and $\Omega_D^2(\mathcal{A}_\alpha)$ are free left $\mathcal{A}_\alpha$-modules of rank 2 and 1, respectively. This reflects the fact the bundles of 1- and 2-forms over the 2-torus are trivial and of rank 2 and 1, respectively.

### 4.2.2 Integration and Hermitian structure over $\Omega_D^1(\mathcal{A}_\alpha)$

It follows from eqs. (4.13-15) that there is an isomorphism $\pi(\Omega^\bullet(\mathcal{A}_\alpha)) \simeq \mathcal{A}_\alpha \otimes M_2(\mathbb{C})$. Applying the general definition of the integral – see subsection 2.1.3 – to the non-commutative torus, one finds for an arbitrary element $\omega \in \pi(\Omega^\bullet(\mathcal{A}_\alpha))$,

$$\int\!\!\!\!\!\!- \; \omega \;=\; \mathrm{Tr}_{\mathbb{C}^2}\,(\omega\,(0)) \tag{4.26}$$

where $\mathrm{Tr}_{\mathbb{C}^2}$ denotes the normalized trace on $\mathbb{C}^2$. The cyclicity property, Assumption 2.4 in subsection 2.1.3, follows directly from the definition of the product in $\mathcal{A}_\alpha$ and the cyclicity of the trace on $M_2(\mathbb{C})$. The kernels $K^n$ of the canonical sesqui-linear form on $\pi(\Omega^\bullet(\mathcal{A}_\alpha))$ – see eq. (2.5) – coincide with the kernels $J^n$ of $\pi$, and we get for all $n \in \mathbb{Z}_n$:

$$\widetilde{\Omega}^n(\mathcal{A}_\alpha) \;=\; \Omega^n(\mathcal{A}_\alpha)\,, \quad \widetilde{\Omega}_D^n(\mathcal{A}_\alpha) \;=\; \Omega_D^n(\mathcal{A}_\alpha)\,. \tag{4.27}$$

Note that the equality $K^n = J^n$ holds in all explicit examples of non-commutative $N = 1$ spaces studied so far. It is easy to see that the canonical representatives $\omega^\perp$ on $\mathcal{H}$ of differential forms $[\omega] \in \Omega^n_D(\mathcal{A}_\alpha)$, see eq. (2.10), coincide with the choices already made in eqs. (4.22-25). The canonical Hermitian structure on $\Omega^1_D(\mathcal{A}_\alpha)$ is simply given by

$$\langle \omega, \eta \rangle_D \; = \; \omega_\mu \, g^{\mu\nu} \, \eta^*_\nu \in \mathcal{A}_\alpha \tag{4.28}$$

for all $\omega, \eta \in \Omega^1_D(\mathcal{A}_\alpha)$. Note that this is a true Hermitian metric, i.e., it takes values in $\mathcal{A}_\alpha$ and not in the weak closure $\mathcal{A}''_\alpha$. Again, this is also the typical situation in other examples.

### 4.2.3 Connections on $\Omega^1_D(\mathcal{A}_\alpha)$ , and cohomology

Since $\Omega^1_D(\mathcal{A}_\alpha)$ is a free left $\mathcal{A}_\alpha$-module, it admits a basis which we can choose to be $E^\mu := \gamma^\mu$. A connection $\nabla$ on $\Omega^1_D(\mathcal{A}_\alpha)$ is uniquely specified by its coefficients $\Gamma^\lambda_{\mu\nu} \in \mathcal{A}_\alpha$,

$$\nabla E^\mu \; = \; -\Gamma^\mu_{\nu\lambda} \, E^\nu \otimes E^\lambda \;\; \in \; \Omega^1_D(\mathcal{A}_\alpha) \otimes_{\mathcal{A}_\alpha} \Omega^1_D(\mathcal{A}_\alpha) \; , \tag{4.29}$$

and these coefficients can be chosen arbitrarily. Note that in the classical case ($\alpha = 0$) the basis $E^\mu$ consists of real 1-forms. Accordingly, we say that the connection $\nabla$ is real if its coefficients in the basis $E^\mu$ are self-adjoint elements of $\mathcal{A}_\alpha$. A simple computation shows that there is a unique real, unitary, torsionless connection $\nabla^{L.C.}$ on $\Omega^1_D(\mathcal{A}_\alpha)$ given by

$$\nabla^{L.C.} \, E^\mu \; = \; 0 \; . \tag{4.30}$$

In the remainder of this subsection, we determine the de Rham complex and its cohomology. Let $U$ and $V$ be the elements of $\mathcal{A}_\alpha$ defined in eq. (4.10), then it is easy to verify that the elements $E^\mu$ of $\Omega^1_D(\mathcal{A}_\alpha)$ given by

$$E^1 = U^* \delta U \; , \qquad E^2 = V^* \delta V \; , \tag{4.31}$$

form a basis of $\Omega^1_D(\mathcal{A}_\alpha)$ and that they are closed,

$$\delta E^1 = \delta E^2 = 0 \; . \tag{4.32}$$

A word of caution is in order here: Eq. (4.32) does *not* mean that $\delta E^\mu$ is zero as an element of $\Omega^2(\mathcal{A}_\alpha)$, but that $\delta E^\mu \in \delta J^1$ which is zero in the quotient space $\Omega^2_D(\mathcal{A}_\alpha)$. As a basis of $\Omega^2_D(\mathcal{A}_\alpha)$ we choose

$$F = \frac{1}{2} \varepsilon_{\mu\nu} \gamma^\mu \gamma^\nu \tag{4.33}$$

and we get for the product of basis 1-forms,

$$E^\mu E^\nu = \varepsilon^{\mu\nu} F \; . \tag{4.34}$$

This completely specifies the de Rham complex, and we can now compute of the cohomology groups $H^p$. For $a \in \mathcal{A}_\alpha$, we have the equivalences

$$\begin{aligned} [D, a] = 0 \;\; &\Longleftrightarrow \;\; i p_\mu \gamma^\mu a(p) = 0 \quad \forall p \\ &\Longleftrightarrow \;\; a(p) = \delta_{p,0} \, \tilde{a} \end{aligned} \tag{4.35}$$

for some $\tilde{a} \in \mathbb{C}$. This shows that $H^0 \simeq \mathbb{C}$. Let $a_\mu E^\mu$ be a 1-form, then we obtain with eqs. (4.32) and (4.34) that

$$\delta(a_\mu E^\mu) = 0 \iff i\widehat{p}_\mu \, a_\nu \, \varepsilon^{\mu\nu} F = 0 \tag{4.36}$$

where $\widehat{p}_\mu$ denotes the multiplication operator by $p_\mu$, i.e., $(\widehat{p}_\mu a)(q) = q_\mu a(q)$. Suppose that the 1-form $a_\mu E^\mu$ is closed and satisfies $a_\mu(0) = 0$, and define the algebra element $b$ by $b = \left(2i\widehat{p}_\mu\right)^{-1} a_\mu$. Using eq. (4.36), we see that

$$\delta b = a_\mu E^\mu \ . \tag{4.37}$$

This proves that any closed 1-form is cohomologous to a "constant" 1-form $c_\mu E^\mu$ with $c_\mu \in \mathbb{C}$. On the other hand, a non-vanishing constant 1-form $c_\mu E^\mu$ cannot be exact since the equation

$$(\delta a)(p) = ip_\mu \, E^\mu \, a(p) = \delta_{p,0} \, c_\mu \, E^\mu \tag{4.38}$$

has no solution. Thus, we have $H^1 \simeq \mathbb{C}^2$. The same argument shows that a constant 2-form $cF$, with $c \in \mathbb{C}$, is not exact. If a 2-form $aF$ satisfies $a(0) = 0$, then it is the coboundary of the 1-form $i\,\varepsilon_{\mu\nu}\left(\widehat{p}_\nu\right)^{-1} aE^\mu$ and this proves the following

**Proposition 4.3** In the basis $\{\, \mathbf{1},\ E^\mu = \gamma^\mu,\ F = \frac{1}{2}\varepsilon_{\mu\nu}\gamma^\mu\gamma^\nu \,\}$ of $\Omega_D^\bullet(\mathcal{A}_\alpha)$, the de Rham differential algebra is specified by the following relations:

$$\begin{aligned} &E^\mu E^\nu = \varepsilon^{\mu\nu} F \ , \\ &\delta E^\mu = \delta F = 0 \ , \qquad \delta a = i\widehat{p}_\mu E^\mu \quad \forall a \in \mathcal{A}_\alpha \ . \end{aligned}$$

Furthermore, the cohomology of the de Rham complex is given by

$$H^0 \simeq H^2 \simeq \mathbb{C} \ , \qquad H^1 \simeq \mathbb{C}^2 \ . \tag{4.39}$$

This completes our study of the $N = 1$ data describing the non-commutative 2-torus at irrational deformation parameter.

### 4.3 Riemannian geometry ( $N = (1,1)$ )

At the end of our discussion of the non-commutative 3-sphere, in subsection 3.4, we have briefly outlined a description in terms of "Riemannian" $N = (1,1)$ data – with the two generalized Dirac operators borrowed from conformal field theory, see [*FG*]. In the following, we will treat the non-commutative torus (at irrational deformation parameter) as a Riemannian space. Here we can, moreover, *construct* a set of $N = (1,1)$ data from the Connes spectral triple along the general lines of subsection 2.2.5.

Our first task is to find a real structure $J$ on the $N = 1$ data $(\mathcal{A}_\alpha, \mathcal{H}, \underline{D, \sigma})$. To this end, we introduce the complex conjugation $\kappa : \mathcal{H} \to \mathcal{H}$, $(\kappa x)(p) := \overline{x}(p) := \overline{x(p)}$, as well as the charge conjugation matrix $C : \mathcal{H} \to \mathcal{H}$ as the unique (up to a sign) constant matrix such that

$$C\,\gamma^\mu = -\,\overline{\gamma}^\mu\, C \tag{4.40}$$

$$C \;=\; C^* = C^{-1}\;. \tag{4.41}$$

Then it is easy to verify that $J = C\kappa$ is a real structure.
The right actions of $\mathcal{A}_\alpha$ and $\Omega^1_D(\mathcal{A}_\alpha)$ on $\mathcal{H}$ (see subsection 2.2.5) are given as follows

$$\xi \bullet a \equiv Ja^*\, J^*\xi \;=\; \xi \bullet_\alpha a^\vee \tag{4.42}$$

$$\xi \bullet \omega \equiv J\omega^* J^*\xi \;=\; \gamma^\mu\xi \bullet_\alpha \omega^\vee_\mu \tag{4.43}$$

where $\xi \in \mathcal{H},\ a \in \mathcal{A}_\alpha, \omega \in \Omega^1_D(\mathcal{A}_\alpha)$, $\xi \bullet_\alpha a$ denotes the diagonal right action of $a$ on $\xi$ by the deformed product, and

$$a^\vee(p) := a(-p)\;.$$

Notice that $(a \bullet_\alpha b)^\vee = a^\vee \bullet_\alpha b^\vee$. We denote by $\mathring{\mathcal{H}}$ the dense subspace $\mathcal{S}(\mathbb{Z}^2) \oplus \mathcal{S}(\mathbb{Z}^2) \subset \mathcal{H}$ of smooth spinors. The space $\mathring{\mathcal{H}}$ is a two-dimensional free left $\mathcal{A}_\alpha$-module with canonical basis $\{e_1, e_2\}$. Then, any connection $\nabla$ on $\mathring{\mathcal{H}}$ is uniquely determined by its coefficients ${\omega_j}^i \in \Omega^1_D(\mathcal{A}_\alpha)$:

$$\nabla e_i \;=\; {\omega_i}^j \otimes e_j \;=\; {\omega_{\mu i}}^j\,\gamma^\mu \otimes e_j \in \Omega^1_D(\mathcal{A}_\alpha) \otimes_{\mathcal{A}_\alpha} \mathring{\mathcal{H}}\;. \tag{4.44}$$

The "associated right connection" $\overline{\nabla}$ is then given by

$$\overline{\nabla}\, e_i = e_j \otimes {\overline{\omega}_i}^j \in\; \mathring{\mathcal{H}} \otimes_{\mathcal{A}_\alpha} \Omega^1_D(\mathcal{A}_\alpha) \tag{4.45}$$

where

$${\overline{\omega}_j}^i \;=\; -\,C^i_k({\omega_l}^k)^*\, C^l_j \;=\; C^i_k({\omega_{\mu\, l}}^k)^*\, C^l_j\,\gamma^\mu\;. \tag{4.46}$$

An arbitrary element in $\mathring{\mathcal{H}} \otimes_{\mathcal{A}_\alpha} \mathring{\mathcal{H}}$ can be written as $e_i \otimes a^{ij} e_j$ where $a^{ij} \in \mathcal{A}_\alpha$. As in subsection 2.2.5, the "Dirac operators" $\mathcal{D}$ and $\overline{\mathcal{D}}$ on $\mathring{\mathcal{H}} \otimes_{\mathcal{A}_\alpha} \mathring{\mathcal{H}}$ associated to the connection $\nabla$ are given by

$$\mathcal{D}\left(e_i \otimes a^{ij}\, e_j\right) = e_i \otimes \left(\delta\, a^{ij} + {\overline{\omega}_k}^i\, a^{kj} + a^{ik}\, {\omega_k}^j\right) \bullet e_j \tag{4.47}$$

$$\overline{\mathcal{D}}\left(e_i \otimes a^{ij}\, e_j\right) = e_i \bullet \left(\delta\, a^{ij} + {\overline{\omega}_k}^i\, a^{kj} + a^{ik}\, {\omega_k}^j\right) \otimes \sigma\, e_j\;. \tag{4.48}$$

In order to be able to define a scalar product on $\mathring{\mathcal{H}} \otimes_{\mathcal{A}_\alpha} \mathring{\mathcal{H}}$, we need a Hermitian structure on the *right* module $\mathring{\mathcal{H}}$, denoted by $\langle\cdot,\cdot\rangle$, with values in $\mathcal{A}_\alpha$. It is defined by

$$⨍\ \langle \xi, \zeta\rangle\, a \;=\; (\xi, \zeta\, a) \quad \forall\, \xi, \zeta \in \mathring{\mathcal{H}}\;,\ \forall\, a \in \mathcal{A}_\alpha\;. \tag{4.49}$$

This Hermitian structure can be written explicitly as

$$\langle \xi, \zeta\rangle \;=\; \sum_{i=1}^{2} \overline{\xi^i} \bullet_\alpha \zeta^{i\vee}\;, \tag{4.50}$$

and it satisfies

$$\langle \xi\, a, \zeta\, b\rangle \;=\; a^*\,\langle \xi,\zeta\rangle\, b \tag{4.51}$$

for all $\xi,\zeta \in \mathring{\mathcal{H}}$ and $a,b\in\mathcal{A}_\alpha$. Then we define the scalar product on $\mathring{\mathcal{H}} \otimes_{\mathcal{A}_\alpha} \mathring{\mathcal{H}}$ as (see [*Co1* ])

$$(\xi_1\otimes\xi_2,\ \zeta_1\otimes\zeta_2) \;=\; (\xi_2, \langle\xi_1,\zeta_1\,\rangle\,\zeta_2)\ . \tag{4.52}$$

This expression can be written in a more suggestive way if one introduces a Hermitian structure, denoted $\langle\cdot,\cdot\rangle_L$, on the *left* module $\mathring{\mathcal{H}}$:

$$\langle \xi,\zeta\rangle_L \;:=\; \langle J\,\xi, J\,\zeta\rangle\ .$$

This Hermitian structure satisfies

$$\langle a\,\xi, b\,\zeta\rangle_L \;=\; a\,\langle\xi,\zeta\rangle_L\, b^*$$

for all $a,b\in\mathcal{A}_\alpha$ and $\xi,\zeta\in\mathring{\mathcal{H}}$, and the scalar product on $\mathring{\mathcal{H}}\otimes_{\mathcal{A}_\alpha}\mathring{\mathcal{H}}$ can be written as follows

$$(\xi_1\otimes\xi_2,\zeta_1\otimes\zeta_2) \;=\; ⨍\ \langle\xi_1,\zeta_1\rangle\,\langle\zeta_2,\xi_2\rangle_L\ .$$

A tedious computation shows that the relations

$$\mathcal{D}^* = \mathcal{D}\,,\qquad \overline{\mathcal{D}}^* = \overline{\mathcal{D}}\,,\qquad \{\,\mathcal{D},\overline{\mathcal{D}}\,\} = 0\,,\qquad \mathcal{D}^2 = \overline{\mathcal{D}}^2 \tag{4.53}$$

are equivalent to

$$\widetilde{\nabla}\, e_i\otimes e_j \;=\; 0\quad \forall i,\, j\ . \tag{4.54}$$

In particular, we see that the original $N=1$ data uniquely determine the operators $\mathcal{D}$ and $\overline{\mathcal{D}}$ satisfying the $N=(1,1)$ algebra – cf. Definition 2.20 –

$$\mathcal{D}^2 = \overline{\mathcal{D}}^2\ ,\qquad \{\,\mathcal{D},\overline{\mathcal{D}}\,\} = 0\ .$$

One can prove that there are unique $\mathbb{Z}_2$-grading operators

$$\gamma = \mathbf{1}\otimes\sigma\ ,\qquad \overline{\gamma} = \sigma\otimes\mathbf{1} \tag{4.55}$$

commuting with $\mathcal{A}_\alpha$ and such that

$$\begin{aligned}\{\,\mathcal{D},\gamma\,\} &= \{\,\overline{\mathcal{D}},\overline{\gamma}\,\} = 0\\ [\,\mathcal{D},\overline{\gamma}\,] &= [\,\overline{\mathcal{D}},\gamma\,] = 0\ .\end{aligned}$$

The combined $\mathbb{Z}_2$-grading

$$\Gamma = \gamma\overline{\gamma}$$

together with the Hodge operator

$$* = \overline{\gamma}$$

complete our data to a set of $N=1$ data $\left(\mathcal{A}_\alpha, \mathcal{H}\otimes_{\mathcal{A}_\alpha}\mathcal{H}, \mathcal{D}, \overline{\mathcal{D}}, \Gamma, *\right)$. Furthermore, these data admit a unique $\mathbb{Z}$-grading

$$\mathcal{T} = \frac{1}{2i}\, g_{\mu\nu}\gamma^\mu \otimes \gamma^\nu \sigma$$

commuting with $\mathcal{A}_\alpha$, whose mod 2 reduction equals $\Gamma$, and such that

$$[\,\mathcal{T}, d\,] = d\ .$$

### 4.4 Kähler geometry ( $N=(2,2)$)

The classical torus can be regarded as a complex Kähler manifold, and thus it is natural to ask whether we can extend the $N=(1,1)$ spectral data to $N=(2,2)$ data in the non-commutative case, too. The simplest way to determine such an extension is to look for an anti-selfadjoint operator $I$ commuting with $\mathcal{A}_\alpha$, $\Gamma$, $*$, and $\mathcal{T}$, and then to define a new differential by

$$d_I = [\,I, d\,]\ . \tag{4.56}$$

The nilpotency of $d_I$ implies further constraints on the operator $I$. The idea behind this construction is to identify $I$ with $i(T-\overline{T})$, where $T$ and $\overline{T}$ are as in Definition 2.26. In the classical setting, this operator has clearly the above properties.

The most general operator $I$ on $\mathcal{H}\otimes_{\mathcal{A}_\alpha}\mathcal{H}$ that commutes with all elements of $\mathcal{A}_\alpha$ is of the form

$$I \;=\; \sum_{\mu,\nu=0}^{3} \gamma^\mu \otimes \gamma^\nu\, I^R_{\mu\nu} \tag{4.57}$$

where $I^R_{\mu\nu}$ are elements of $\mathcal{A}_\alpha$ acting on $\mathcal{H}\otimes_{\mathcal{A}_\alpha}\mathcal{H}$ from the right, and where we have set

$$\gamma^0 \;=\; \mathbf{1}\ , \quad \gamma^3 \;=\; \sigma\ . \tag{4.58}$$

The vanishing of the commutators of $I$ with $\Gamma$ and $*$ implies that $I^R_{\mu\nu}=0$ unless $\mu,\nu\in\{0,3\}$. The equation $[\,I,\mathcal{T}\,]=0$ requires $I^R_{03}=I^R_{30}$ and leaves the coefficients $I^R_{00}$ and $I^R_{33}$ undetermined. Since the operator $I$ appears only through commutators, its trace part is irrelevant and we can set $I^R_{00}=0$. All constraints together give

$$I \;=\; (\sigma\otimes\mathbf{1}\;+\;\mathbf{1}\otimes\sigma)\; I^R_{03} \;+\; (\sigma\otimes\sigma)\; I^R_{33} \tag{4.59}$$

where $I^R_{03}$ and $I^R_{33}$ are anti-selfadjoint elements of $\mathcal{A}_\alpha$. We decompose $I$ into two parts

$$I_1 = (\sigma\otimes\mathbf{1}\;+\;\mathbf{1}\otimes\sigma)\; I^R_{03} \tag{4.60}$$

$$I_2 = (\sigma\otimes\sigma)\; I^R_{33} \tag{4.61}$$

and we introduce the new differentials according to eq. (4.56)

$$d_1 = d \;=\; \frac{1}{2}(\mathcal{D} \;-\; i\,\overline{\mathcal{D}}) \tag{4.62}$$

$$d_2 = [\,I_1, d\,] \tag{4.63}$$

$$d_3 = [\,I_2, d\,]\ . \tag{4.64}$$

The nilpotency of $d_2$ and $d_3$ implies that $I_{03}$ and $I_{33}$ are multiples of the identity, and we normalize them as follows

$$I_1 = \frac{i}{2}\ (\sigma \otimes \mathbf{1} \;+\; \mathbf{1} \otimes \sigma) \tag{4.65}$$

$$I_2 = i\ (\sigma \otimes \sigma)\ . \tag{4.66}$$

Comparing eqs. (4.66) and (4.55), we see that

$$I_2 \;=\; i\,\gamma\,\overline{\gamma} \tag{4.67}$$

and it follows, using eqs. (4.62) and (4.64), that

$$d_3 \;=\; [\,I_2, d\,] \;=\; 2\,i\,d\,\gamma\,\overline{\gamma}\,. \tag{4.68}$$

Thus, the differential $d_3$ is a trivial modification of $d$, and we discard it. It is then easy to verify that $(\mathcal{A}_\alpha, \mathcal{H} \otimes_{\mathcal{A}_\alpha} \mathcal{H}, d_1, d_2, \Gamma, *, \mathcal{T})$ form a set of $N=(2,2)$ spectral data together with a $\mathbb{Z}$-grading. Furthermore, they are, as we have shown, canonically determined by the original $N=(1,1)$ data. Therefore, a Riemannian non-commutative torus (at irrational deformation parameter $\alpha$) admits a canonical Kähler structure. Notice that if we choose the metric $g^{\mu\nu} = \delta^{\mu\nu}$ in eq. (4.4), then $\partial = -\frac{1}{2}(d_1 + i d_2)$ coincides with the holomorphic differential obtained in [*Co1*] from cyclic cohomology and using the equivalence of conformal and complex structures in two dimensions.

We have only given the definitions of the spectral data in the $N=(1,1)$ and the $N=(2,2)$ setting. As a straightforward application of the general methods described in section 2, we could compute the associated de Rham resp. Dolbeault complexes, or geometrical quantities like curvature. We do not carry out these calculations.

Instead, let us emphasize the following feature: In section 3, we already say that the topology of "the" non-commutative 3-sphere depends on the spectral data other than the algebra. Now, we learn once again that, for rational deformation parameter $\alpha = \frac{M}{N}$, the algebra $\mathcal{A}_\alpha$ does not specify the geometry of the underlying non-commutative space. It is only the selection of a specific $K$-cycle $(\mathcal{H}, D)$ that allows us to identify this space as a deformed torus. By choosing different K-cycles $(\mathcal{H}, D)$ for the same algebra $\mathcal{A} = M_N(\mathbb{C})$ (with $N = \sum_{j=1}^{l} j^2$) we are able to describe *either* a fuzzy three-sphere, as discussed in Sect. 3, *or* a non-commutative torus. In other words, choosing *different* spectral data, but keeping the algebra $\mathcal{A}$ *fixed*, may lead to *different* non-commutative geometries.

Yet, it is plausible that the sequence $\mathcal{A}_N := M_N(\mathbb{C})$, $N = 1, 2, 3, \ldots$, of algebras may be associated uniquely with non-commutative tori, while the sequence $\mathcal{A}_N := M_N(\mathbb{C})$, $N = \sum_{j=1}^{l} j^2$, $l = 1, 2, 3, \ldots$, may be associated uniquely with fuzzy three-spheres.

## 5. Directions for future work

In this work and in part I, we have presented an approach to (non-commutative) geometry rooted in supersymmetric quantum theory. We have classified the various types of classical and of non-commutative geometries according to the symmetries, or to the "supersymmetry content", of their associated spectral data. Obviously, many natural and important questions remain to be studied. In this concluding section, we describe a few of these open problems and sketch, once more, some of the physical motivations underlying our work.

**(1)** An obvious question is whether one can give a complete classification of the possible types of spectral data in terms of graded Lie algebras (and, perhaps, $q$-deformed graded Lie algebras). As an example, we recall the structure of $N = 4^+$ spectral data , describing an extension of Kähler geometry (see sections 1.2 and 3 of part I). The spectral data involve the operators $\mathtt{d}$, $\mathtt{d}^*$, $\widetilde{\mathtt{d}}$, $\widetilde{\mathtt{d}}^*$, $L^3$, $L^+$, $L^-$, $J_0$ and $\triangle$, which close under taking (anti-)commutators: They generate a graded Lie algebra defined by

$$
\begin{aligned}
&[\,L^3, L^\pm\,] = \pm 2L^\pm \;, \quad [\,L^+, L^-\,] = L^3 \;, \quad [\,J_0, L^3\,] = [\,J_0, L^+\,] = 0 \;, \\
&[\,L^3, \mathtt{d}\,] = \mathtt{d} \;, \quad [\,L^+, \mathtt{d}\,] = 0 \;, \quad [\,L^-, \mathtt{d}\,] = \widetilde{\mathtt{d}}^* \;, \qquad [\,J_0, \mathtt{d}\,] = -i\,\widetilde{\mathtt{d}} \;, \\
&[\,L^3, \widetilde{\mathtt{d}}\,] = \widetilde{\mathtt{d}} \;, \quad [\,L^+, \widetilde{\mathtt{d}}\,] = 0 \;, \quad [\,L^-, \widetilde{\mathtt{d}}\,] = -\mathtt{d}^* \;, \quad [\,J_0, \widetilde{\mathtt{d}}\,] = i\,\mathtt{d} \;, \\
&\{\,\mathtt{d}, \mathtt{d}\,\} = \{\,\widetilde{\mathtt{d}}, \widetilde{\mathtt{d}}\,\} = \{\,\mathtt{d}, \widetilde{\mathtt{d}}\,\} = \{\mathtt{d}, \widetilde{\mathtt{d}}^*\,\} = 0 \;, \\
&\{\,\mathtt{d}, \mathtt{d}^*\,\} = \{\,\widetilde{\mathtt{d}}, \widetilde{\mathtt{d}}^*\,\} = \triangle \;,
\end{aligned}
$$

where $\triangle$, the Laplacian, is a central element. The remaining (anti-)commutation relations follow by taking adjoints, with the rules that $\triangle$, $J_0$ and $L^3$ are self-adjoint, and $(L^-)^* = L^+$.

It would be interesting to determine all graded Lie algebras (and their representations) occurring in spectral data of a (non-commutative) space. In the case of classical geometry, we have given a classification up to $N = (4,4)$ spectral data, and there appears to be enough information in the literature to settle the problem completely; see [*Bes, HKLR, Joy*]. In the non-commutative setting, however, further algebraic structures might occur, including $q$-deformations of graded Lie algebras.

To give a list of all graded Lie algebras that are, in principle, admissible, appears possible; see [*FGR2*] for additional discussion. However, in view of the classical case, where we only found the groups U(1), SU(2), Sp(4) and direct products thereof (see part I, section 3) we expect that not all Lie group symmetries that may arise in principle are actually realized in (non-commutative) geometry.

Determining the graded Lie algebras that actually occur in the spectral data of geometric spaces is clearly just the first step towards a classification of non-commutative spaces. A more difficult problem will be to characterize the class of all $^*$-algebras $\mathcal{A}$ that admit a given type of spectral data, i.e. the class of algebras that possess a $K$-cycle $(\mathcal{H}, \mathtt{d}_i)$ with a collection of differentials $\mathtt{d}_i$ generating a certain graded Lie algebra such that the ordinary Lie group generators $X_j$ contained in the graded Lie algebra commute with the elements of $\mathcal{A}$.

**(2)** Given some non-commutative geometry defined in terms of spectral data, it is natural to investigate its symmetries, i.e. to introduce a notion of diffeomorphisms. For definiteness, we start from a set of data $(\mathcal{A}, \mathcal{H}, \mathrm{d}, \mathrm{d}^*, T, *)$ with an $N = 2$ structure, cf. section 2.2.6. To study notions of diffeomorphisms, it is useful to introduce an algebra $\Phi_{\mathrm{d}}^{\bullet}(\mathcal{A})$ defined as the smallest $^*$-algebra of (unbounded) operators containing $\mathcal{B} := \pi(\Omega^{\bullet}(\mathcal{A})) \vee \pi(\Omega^{\bullet}(\mathcal{A}))^*$ and arbitrary graded commutators of $\mathrm{d}$ and $\mathrm{d}^*$ with elements of $\mathcal{B}$. Due to the existence of the $\mathbb{Z}$-grading $T$, $\Phi_{\mathrm{d}}^{\bullet}(\mathcal{A})$ decomposes into a direct sum

$$\Phi_{\mathrm{d}}^{\bullet}(\mathcal{A}) := \bigoplus_{n\in\mathbb{Z}} \Phi_{\mathrm{d}}^{n}(\mathcal{A}) \ , \qquad \Phi_{\mathrm{d}}^{n}(\mathcal{A}) := \left\{ \phi \in \Phi_{\mathrm{d}}^{\bullet}(\mathcal{A}) \,\middle|\, [T, \phi]_g = n\, \phi \right\} .$$

Note that both positive and negative degrees occur. Thus, $\Phi_{\mathrm{d}}^{\bullet}(\mathcal{A})$ is a graded $^*$-algebra. This algebra is quite a natural object to introduce when dealing with $N = 2$ spectral data, as the algebra $\Omega_{\mathrm{d}}^{\bullet}(\mathcal{A})$ of differential forms does not have a $^*$-representation on $\mathcal{H}$, because $\mathrm{d}$ is not self-adjoint.

Ignoring operator domain problems arising because the (anti-)commutator of $\mathrm{d}$ with the adjoint of a differential form is unbounded, in general, we observe that $\Phi_{\mathrm{d}}^{\bullet}(\mathcal{A})$ has the interesting property that it forms a complex with respect to the action of $\mathrm{d}$ by graded commutation, and, in view of examples from quantum field theory, we call it the *field complex* in the following.

For $N = (2, 2)$ non-commutative Kähler data with holomorphic and anti-holomorphic gradings $T$ and $\overline{T}$, see Definition 2.26, one may introduce a *bi-graded complex* $\Phi_{\partial,\overline{\partial}}^{\bullet,\bullet}(\mathcal{A})$ in a similar way. A slight generalization of such bi-graded field complexes containing operators $\phi$ of degree $(n, m)$ with $n$ and $m$ *real*, but $n + m \in \mathbb{Z}$, naturally occurs in $N = (2, 2)$ superconformal field theory, see e.g. [*FG, FGR2*] and references given there.

Next, we show how the field complex appears when we attempt to introduce a notion of *diffeomorphisms* of a (non-commutative) geometric space described in terms of $N = 2$ spectral data: One possible generalization of the notion of diffeomorphisms to non-commutative geometry is to identify them with $^*$-automorphisms of the algebra $\mathcal{A}$ of "smooth functions". It may be advantageous, though, to follow concepts from classical geometry more closely: An infinitesimal diffeomorphism is then given by a derivation $\delta(\cdot) := [L, \cdot]$ of $\mathcal{A}$ where $L$ is an element of $\Phi_{\mathrm{d}}^{0}$ such that $\delta$ commutes with $\mathrm{d}$, i.e.

$$[\mathrm{d}, L] = 0 \ .$$

The derivation $\delta$ can then be extended to all of $\pi(\Omega_{\mathrm{d}}^{\bullet}(\mathcal{A}))$, and $\delta$ preserves the degree of differential forms iff $L$ commutes with $T$, i.e. iff $L \in \Phi_{\mathrm{d}}^{0}$.

For a classical manifold $M$, it turns out that each $L$ with the above properties can be written as

$$L = \{\mathrm{d}, X\}$$

for some vector field $X \in \Phi_{\mathrm{d}}^{-1}$, i.e. $L$ is the Lie derivative in the direction of this vector field. In the non-commutative situation, however, it might happen that the cohomology of the field complex at the zeroth position is non-trivial. In this case, the study of diffeomorphisms of the non-commutative space necessitates studying the cohomology of the field complex $\Phi_{\mathrm{d}}^{\bullet}(\mathcal{A})$ in degree zero.

As in classical differential geometry, it is interesting to investigate *special* diffeomorphisms, i.e. ones that preserve additional structure in the spectral data. As an example, consider derivations $\delta(\cdot) = [\, L, \cdot \,]$ such that $L$ commutes with $\mathtt{d}$ *and* $\mathtt{d}^*$ : They generate *isometries* of the non-commutative space. For complex spectral data, we may consider derivation not only commuting with $\mathtt{d}$ but also with $\partial$: They generate one-parameter groups of holomorphic diffeomorphisms. In the example of symplectic spectral data, we are interested in diffeomorphisms preserving the symplectic forms, i.e., in symplectomorphisms. One-parameter groups of symplectomorphisms are generated by derivations commuting with $\mathtt{d}$ and $\widetilde{\mathtt{d}}^*$.

**(3)** Another important topic in non-commutative geometry is *deformation theory*. Given spectral data specified in terms of generators $\{\, X_j,\ \mathtt{d},\ \mathtt{d}_\alpha,\ \triangle \,\}$ of a graded Lie algebra as in remark (1), we may study one-parameter families $\{\, X_j^{(t)},\ \mathtt{d},\ \mathtt{d}_\alpha^{(t)},\ \triangle^{(t)} \,\}_{t\in\mathbb{R}}$ of deformations. Here, we choose to keep one generator, $\mathtt{d}$, fixed, and we require that the graded Lie algebras are isomorphic to one another for all $t$. This means that we study deformations of the (non-commutative) complex or symplectic structure of a given space $\mathcal{A}$ while preserving the differential and the de Rham complex. Only those deformations of spectral data are of interest which cannot be obtained from the original ones by $^*$-automorphisms of the algebra $\mathcal{A}$ commuting with $\mathtt{d}$ (i.e. by "diffeomorphisms"). In classical geometry, the deformation theory of complex structures is well-developed (Kodaira-Spencer theory), and there are non-trivial results in the deformation theory of symplectic structures (e.g. Moser's theorem); but this last topic is still a subject of active research.
Next, we consider deformations $\mathtt{d}'$ of the differential $\mathtt{d}$ of a given set of $N = 2$ spectral data $(\mathcal{A}, \mathcal{H}, \mathtt{d}, \mathtt{d}^*, T, *)$ which are of the form

$$\mathtt{d}' := \mathtt{d} + \omega \ ,$$

for some operator $\omega \in \Phi_{\mathtt{d}}^{\bullet}(\mathcal{A})$ of odd degree. We require that $\mathtt{d}'$ again squares to zero, which implies that $\omega$ has to satisfy a zero curvature condition

$$\omega^2 + \{\, \mathtt{d}, \omega \,\} = 0 \ . \tag{5.1}$$

We distinguish between several possibilities: First, we require that the deformed data still carry an $N = 2$ structure with the same $\mathbb{Z}$-grading $T$ as before. Then $\omega$ must be an element of $\Phi_{\mathtt{d}}^{1}(\mathcal{A})$ satisfying (5.1), and we can identify it with the connection 1-form of a flat connection on some vector bundle; for an example, see the discussion of the structure of classical $N = (1,1)$ Dirac bundles in section 2.2.3 of part I.
More generally, we only require the deformed data to be of $N = (1,1)$ type, with a $\mathbb{Z}_2$-grading $\gamma$ given by the mod 2 reduction of $T$. As a simple example, consider an operator $\omega$ in $\Phi_{\mathtt{d}}^{\bullet}(\mathcal{A})$ of degree $2n+1$, with $n \neq 0$. Then condition (5.1) implies that

$$\omega^2 = 0 \quad \text{and} \quad \{\, \mathtt{d}, \omega \,\} = 0 \ .$$

If $\omega = [\, \mathtt{d}, \beta \,]$ and $[\, \beta, \omega \,] = 0$ then

$$\mathtt{d}' = e^{-\beta}\, \mathtt{d}\, e^{\beta} \ .$$

We then say that $\mathrm{d}$ and $\mathrm{d}'$ are equivalent. If $\omega$ represents a non-trivial cohomology class of the field complex $\Phi^\bullet_{\mathrm{d}}(\mathcal{A})$ then $\mathrm{d}$ and $\mathrm{d}'$ are inequivalent.

**(4)** In the introduction to paper I and in [*FGR2*] we have remarked that, from the point of view of physics, it is quite unnatural to attribute special importance to the algebra of functions over configuration space. The natural algebra in Hamiltonian mechanics is the algebra of functions over phase space, and, in quantum mechanics, it is a non-commutative deformation thereof, denoted $\mathcal{F}_\hbar$ (where $\hbar$ is Planck's constant), which is the natural algebra to study. In examples where phase space is given as the cotangent bundle $T^*M$ of a smooth manifold $M$, the configuration space, one may ask whether there are natural mathematical relations between spectral data involving the algebra $\mathcal{A} = C^\infty(M)$ and ones involving the algebra $\mathcal{F}_\hbar$. For example, it may be possible to represent $\mathcal{A}$ and $\mathcal{F}_\hbar$ on the same Hilbert space $\mathcal{H}$ and consider spectral data $(\mathcal{A}, \mathcal{H}, \mathrm{d}, T, *)$ and $(\mathcal{F}_\hbar, \mathcal{H}, \mathrm{d}, T, *)$ with the *same* choice of operators $\mathrm{d}, T$ and $*$ on $\mathcal{H}$. It is well known that from $(\mathcal{A}, \mathcal{H}, \mathrm{d}, T, *)$ configuration space $M$ can be reconstructed (Gelfand's theorem and extensions thereof). This leads to the natural question whether $M$ can also be reconstructed from $(\mathcal{F}_\hbar, \mathcal{H}, \mathrm{d}, T, *)$, or whether at least some of the topological properties of $M$, e.g. its Betti numbers, can be determined from these data.
It is known that, in string theory, spectral data generalizing $(\mathcal{F}_\hbar, \mathcal{H}, \mathrm{d}, T, *)$ do not determine configuration space uniquely; this is related to the subject of stringy dualities and symmetries, more precisely to $T$ dualities, see e.g. [*GPR*] and also [*KS, FG*]. The distinction between "algebras of functions on configuration space" $\mathcal{A}$ and "algebras of functions on phase space" $\mathcal{F}$ remains meaningful in many examples of non-commutative spaces. Typically, $\mathcal{F}$ arises as a crossed product of $\mathcal{A}$ by some group $G$ of "diffeomorphisms". Under what conditions properties of the algebra $\mathcal{A}$ can be inferred from spectral data $(\mathcal{F}_\hbar, \mathcal{H}, \mathrm{d}, T, *)$ *without* knowing explicitly how the group $G$ acts on $\mathcal{F}$ represents a problem of considerable interest in quantum theory.
For another perspective concerning the distinction between "algebras of functions on configuration space" and "algebras of functions on phase space" see section 2.2.6.

It is worth emphasizing that in quantum field theory and string theory, where $M$ is an infinite-dimensional space, the analogue of the "algebra of functions on $M$", i.e. of $\mathcal{A}$, does not exist, while the analogue of the "algebra of functions on phase space $T^*M$", i.e. of $\mathcal{F}$, still makes sense. For additional discussion of these matters see also [*FGR2*].

**(5)** A topic in the theory of complex manifold that has attracted a lot of interest, recently, is *mirror symmetry*. For a definition of mirrors of classical Calabi-Yau manifolds, see e.g. [*Y*] and references given there, and cf. the remarks at the end of section I2.4.3. It is natural to ask whether one can define mirrors of non-commutative spaces, and whether some classical manifolds may have non-commutative mirrors. Superconformal field theory with $N = (2, 2)$ supersymmetry suggests how one might define a mirror map in the context of non-commutative geometry (see [*FG, FGR2*]): Assume that two sets of $N = (2, 2)$ spectral data $(\mathcal{A}_i, \mathcal{H}, \partial_i, \overline{\partial}_i, T_i, \overline{T}_i, *_i)$, $i = 1, 2$, are given, where the algebras $\mathcal{A}_i$ act on the Hilbert spaces $\mathcal{H}_i$ which are subspaces of a single Hilbert space $\mathcal{H}$ on which the operators $\partial_i, \overline{\partial}_i, T_i, \overline{T}_i$ and $*_i$ are defined. We say that the space $\mathcal{A}_2$ is the *mirror* of $\mathcal{A}_1$ if

$$\partial_2 = \partial_1 \ , \quad \overline{\partial}_2 = \overline{\partial}_1^* \ , \quad T_2 = T_1 \ , \quad \overline{T}_2 = -\overline{T}_1 \ ,$$

and if the dimensions $b_i^{p,q}$ of the cohomology of the Dolbeault complexes (2.45) satisfy $b_2^{p,q} = b_1^{n-p,q}$, where $n$ is the top dimension of differential forms (recall that in Definition 2.26 we required $T$ and $\overline{T}$ to be bounded operators).
Let $\mathcal{A}$ be a non-commutative Kähler space with mirror $\widetilde{\mathcal{A}}$. Within superconformal field theory, there is the following additional relation between the two algebras: Viewing $\mathcal{A}$ as the algebra of functions over a (non-commutative) target $M$, and analogously for $\widetilde{\mathcal{A}}$ and $\widetilde{M}$, the phase spaces over the loop spaces over $M$ and $\widetilde{M}$ coincide.

**(6)** The success of the theory presented in this paper will ultimately be measured in terms of the applications it has to concrete problems of geometry and physics. In particular, one should try to apply the notions developed here to further examples of truly non-commutative spaces such as quantum groups, or the non-commutative complex projective spaces (see e.g. [*Ber*, *Ho*, *Ma*, *GKP*]), non-commutative Riemann surfaces [*KL*], and non-commutative symmetric spaces [*BLU*, *BLR*, *GP*, *BBEW*]. In most of these cases, it is natural to ask whether the "deformed" spaces carry a complex or Kähler structure in the sense of section 2.3 above.

From our point of view, however, the most interesting examples for the general theory and the strongest motivation to study spectral data with supersymmetry come from string theory: The "ground states" of string theory are described by certain $N = (2,2)$ superconformal field theories. They provide the spectral data of the loop space over a target which is a "quantization" of classical space – or rather of an internal compact manifold. It may happen that the conformal field theory is the quantization of a $\sigma$-model of maps from a parameter space into a classical target manifold. In general, the target space reconstructed from the spectral data of the conformal field theory then turns out to be a (non-commutative) deformation of the target space of the classical $\sigma$-model. The example of the superconformal SU(2) Wess-Zumino-Witten model, which is the quantization of a $\sigma$-model with target SU(2), has been studied in some detail in [*FG*, *Gr*, *FGR2*] and has motivated the results presented in section 3. A more interesting class of examples would consist of $N = (2,2)$ superconformal field theories which are quantizations of $\sigma$-models whose target spaces are given by three-dimensional Calabi-Yau manifolds. But one may also apply the methods developed in this paper to superconformal field theories which, at the outset, are *not* quantizations of some classical $\sigma$-models. They may enable us to reconstruct (typically non-commutative) geometric spaces from the supersymmetric spectral data of such conformal field theories. This leads to the idea that, quite generally, superconformal field theories are (quantum) $\sigma$-models, but with target spaces that tend to be non-commutative spaces. An interesting family of examples of this kind consists of the Gepner models, which are expected to give rise to non-commutative deformations of certain Calabi-Yau three-folds. For further discussion of these ideas see also [*FG*, *FGR2*].

## References


[*AG*] L. Alvarez-Gaumé, *Supersymmetry and the Atiyah-Singer index theorem*, Commun. Math. Phys. **90** (1983) 161-173

[*AGF*] L. Alvarez-Gaumé, D.Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric $\sigma$-model*, Commun. Math. Phys. **80** (1981) 443-451

[*BC*] P. Beazley Cohen, *Structure complexe non commutative et superconnexions*, preprint MPI für Mathematik, Bonn, MPI/92-19

[*Ber*] F.A. Berezin, *General concept of quantization*, Commun. Math. Phys. **40** (1975) 153-174

[*Bes*] A.L. Besse, *Einstein Manifolds*, Springer Verlag 1987

[*BLR*] D. Borthwick, A. Lesniewski, M. Rinaldi, *Hermitian symmetric superspaces of type IV*, J. Math. Phys. **343** (1993) 4817-4833

[*BLU*] D. Borthwick, A. Lesniewski, H. Upmeier, *Non-perturbative quantization of Cartan domains*, J. Funct. Anal. **113** (1993) 153-176

[*BBEW*] M. Bordemann, M. Brischle, C. Emmrich, S. Waldmann, *Phase space reduction for star-products: an explicit construction for* $\mathbb{C}\mathrm{P}^n$, Lett. Math. Phys. **36** (1996) 357-371

[*CF*] A.H. Chamseddine, J. Fröhlich, *Some elements of Connes' non-commutative geometry, and space-time geometry*, in: Chen Ning Yang, a Great Physicist of the Twentieth Century, C.S. Liu and S.-T. Yau (eds.), International Press 1995, pp. 10-34

[*CFF*] A.H. Chamseddine, G. Felder, J. Fröhlich, *Gravity in non-commutative geometry*, Commun. Math. Phys. **155** (1993) 205-217

[*CFG*] A.H. Chamseddine, J. Fröhlich, O. Grandjean, *The gravitational sector in the Connes-Lott formulation of the standard model*, J. Math. Phys. **36** (1995) 6255-6275

[*Co1*] A. Connes, *Noncommutative Geometry*, Academic Press 1994

[*Co2*] A. Connes, *Noncommutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. **62** (1985) 257-360

[*Co3*] A. Connes, *The action functional in noncommutative geometry*, Commun. Math. Phys. **117** (1988) 673-683

[*Co4*] A. Connes, *Reality and noncommutative geometry*, J. Math. Phys. **36** (1995) 6194-6231

[*Co5*] A. Connes, *$C^*$-algèbres et géométrie différentielle*, C.R. Acad. Sci. Paris Sér. A-B **290** (1980) 599-604

[*CoK*] A. Connes, M. Karoubi, *Caractère multiplicatif d'un module de Fredholm*, *K*-Theory **2** (1988) 431-463

[*DFR*] S. Doplicher, K. Fredenhagen, J.E. Roberts, *The quantum structure of space-time at the Planck scale and quantum fields*, Commun. Math. Phys. **172** (1995) 187-220

[*FG*] J. Fröhlich, K. Gawędzki, *Conformal field theory and the geometry of strings*, CRM Proceedings and Lecture Notes Vol. **7** (1994), 57-97

[*FGR1*] J. Fröhlich, O. Grandjean, A. Recknagel, *Supersymmetric quantum theory and differential geometry*, Commun. Math. Phys. **193** (1998) 527-594

[*FGR2*] J. Fröhlich, O. Grandjean, A. Recknagel, *Supersymmetric quantum theory, non-commutative geometry, and gravitation*, Lecture notes for the Les Houches 1995 summer school on "Quantum Symmetries", A. Connes, K. Gawędzki (eds.)

[*FGK*] G. Felder, K. Gawędzki, A. Kupiainen, *Spectra of Wess-Zumino-Witten models with arbitrary simple groups*, Commun. Math. Phys. **117** (1988) 127-158

[*FW*] D. Friedan, P. Windey, *Supersymmetric derivation of the Atiyah-Singer index theorem and the chiral anomaly*, Nucl. Phys. **B235** (1984) 395-416

[*GKP*] H. Grosse, C. Klimčik, P. Prešnajder, *Towards finite quantum field theory in non-commutative geometry*, Int. J. Theor. Phys. **35** (1996) 231-244

[*GP*] H. Grosse, P. Prešnajder, *The construction of non-commutative manifolds using coherent states*, Lett. Math. Phys. **28** (1993) 239-250

[*GPR*] A. Giveon, M. Porrati, E. Rabinovici, *Target space duality in string theory*, Phys. Rep. **244** (1994) 77-202

[*Gr*] O. Grandjean, *Non-commutative differential geometry*, Ph.D. Thesis, ETH Zürich, July 1997

[*GSW*] M.B. Green, J.H. Schwarz, E. Witten, *Superstring Theory I,II*, Cambridge University Press 1987

[*HKLR*] N.J. Hitchin, A. Karlhede, U. Lindstrom, M. Rocek, *Hyperkähler metrics and supersymmetry*, Commun. Math. Phys. **108** (1987) 535-589

[*Ho*] J. Hoppe, *Quantum theory of a massless relativistic surface and a two-dimensional boundstate problem*, Ph.D. Thesis, MIT 1982; *Quantum theory of a relativistic surface*, in: Constraint's theory and relativistic dynamics, G. Longhi, L. Lusanna (eds.), Proceedings Florence 1986, World Scientific

[*Ja1*] A. Jaffe, A. Lesniewski, K. Osterwalder, *Quantum K-theory I: The Chern character*, Commun. Math. Phys. **118** (1988) 1-14

[*Ja2*] A. Jaffe, A. Lesniewski, K. Osterwalder, *On super-KMS functionals and entire cyclic cohomology*, $K$-theory **2** (1989) 675-682

[*Ja3*] A. Jaffe, K. Osterwalder, *Ward identities for non-commutative geometry*, Commun. Math. Phys. **132** (1990) 119-130

[*Jac*] N. Jacobson, *Basic Algebra II*, W.H. Freeman and Company 1985

[*Joy*] D.D. Joyce, *Compact hypercomplex and quaternionic manifolds*, J. Differ. Geom. **35** (1992) 743-762; *Manifolds with many complex structures*, Q. J. Math. Oxf. II. Ser. **46** (1995) 169-184

[*Kar*] M. Karoubi, *Homologie cyclique et K-théorie*, Société Mathématique de France, Astérisque **149** (1987)

[*KL*] S. Klimek, A. Lesniewski, *Quantum Riemann surfaces I. The unit disc*, Commun. Math. Phys. **146** (1992) 103-122; *Quantum Riemann surfaces II. The discrete series*, Lett. Math. Phys. **24** (1992) 125-139

[*KS*] C. Klimčik, P. Ševera, *Dual non-abelian duality and the Drinfeld double*, Phys. Lett. B **351** (1995) 455-462

[*Ma*] J. Madore, *The commutative limit of a matrix geometry*, J. Math. Phys **32** (1991) 332-335

[*Pol*] J. Polchinski, *Dirichlet branes and Ramond-Ramond charges*, Phys. Rev. Lett. **75** (1995) 4724-4727; *TASI lectures on D-branes*, hep-th/9611050.

[*PS*] A. Pressley, G. Segal, *Loop groups*, Clarendon Press 1986

[*Ri*] M. Rieffel, *Non-commutative tori – a case study of non-commutative differentiable manifolds*, Contemp. Math. **105** (1990) 191-211

[*Sw*] R.G. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. **105** (1962) 264-277

[*Wi1*] E. Witten, *Constraints on supersymmetry breaking*, Nucl. Phys. **B202** (1982) 253-316

[*Wi2*] E. Witten, *Supersymmetry and Morse theory*, J. Diff. Geom. **17** (1982) 661-692

[*Wi3*] E. Witten, *Non-abelian bosonization in two dimensions*, Commun. Math. Phys. **92** (1984) 455-472

[*Wi4*] E. Witten, *Bound states of strings and D-branes*, Nucl. Phys. **B460** (1996) 335–350

[*Y*] *Essays on mirror symmetry*, S.T. Yau (ed.), International Press 1992