POLYMERS AND $g|\vec{\phi}|^4$ THEORY IN FOUR DIMENSIONS

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Abstract :

We investigate the approach to the critical point and the scaling limit of a variety of models on a four-dimensional lattice, including $g|\vec{\phi}|^4_4$-theory and the self-avoiding random walk. Our results, both theoretical and numerical, provide strong evidence for the triviality of the scaling limit and for logarithmic corrections to mean field scaling laws, as predicted by the perturbative renormalization group. We relate logarithmic corrections to scaling to the triviality of the scaling limit. Our numerical analysis is based on a novel, high-precision Monte-Carlo technique.

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1. Introduction

This paper is motivated by three different but physically related problems :

1) We want to sharpen the theoretical and numerical evidence that the continuum (= scaling) limit of $N = 1$, or 2 component lattice $g|\vec{\phi}|^4$-theory, the corresponding σ-models and the self-avoiding random walk model (the $N \to 0$ limit of the N-component σ-models) in four dimensions are trivial (i.e. Gaussian), for arbitrary mass-, charge- and field strength renormalizations compatible with the convergence of the two-point function (renormalized propagator). This continues the work of Aizenman [1] and of Brydges et. al [2] and extends subsequent results on four-dimensional theories in [3] ; see also [4,5,6].

2) We want to supply non-perturbative and numerical tests for the logarithmic violations of mean field theory in the models mentioned in (1) on a four-dimensional, simple hypercubical lattice and discuss the relevance of such violations for the problem described in (1). (In short : Logarithmic violations of any of the mean field scaling laws appear to imply triviality.)

3) We want to develop a high-precision Monte-Carlo method for the numerical calculation of critical exponents and log exponents in the self-avoiding random walk model - which is relevant in polymer physics [7,8] - and potentially in $g|\vec{\phi}|^4$-theory on three-, and four-dimensional lattices.

We feel that this paper reports on satisfactory progress in all three problems. In most respects, this paper contains few original ideas or innovations. It might show, however, that some progress can be achieved by systematic and careful use of existing ideas and techniques.

We now briefly describe the contents of the various sections of our paper.

In Section 2, we introduce several models of Euclidean lattice field theory which are studied in subsequent sections.

In Section 3, the random walk representation of scalar Euclidean lattice fields is reviewed. This representation is one of the main tools of our analysis.

In Section 4, we reconsider the limit when the number, N , of components of the lattice field tends to 0. It is well known that the self-avoiding random walk model, one of the standard models in polymer physics, is the $N \to 0$ limit of some lattice field theory, the N-component, non-linear σ-model. The random walk representation described in Section 3 is a very convenient tool in the study of the $N \to 0$ limit.

In Section 5, we describe, in general terms, how to construct the continuum (≡ scaling) limit of lattice models exhibiting continuous transitions, and we explain how the construction of the continuum limit can be reduced to analyzing the approach to the critical point of the lattice model.

In Section 6, we use the tools prepared in previous sections to study the continuum limit of four-dimensional lattice theories, in particular of $g|\vec{\phi}|^4$-lattice theories with $N = 0,1,$ or 2 component fields and the corresponding $g \to \infty$ limits, the self-avoiding random walk, the Ising - and the classical rotor model, respectively. We describe results suggesting that all continuum limits of that class of lattice theories are trivial (Gaussian), and we explain in which way triviality of the continuum limit is related to logarithmic violations of scaling laws which are predicted by the renormalization group equations in the perturbative regime. We prove partial results supporting the claim that all continuum limits of our lattice field theories are trivial unless mean field theory is exact.

The proofs of the main results in Section 6 are given in Section 7.

In Section 8, we report on a detailed, numerical analysis of the approach to the critical point in the self-avoiding random walk model, using a Monte-Carlo technique which is accurate enough to detect logarithmic violations of scaling laws which, in view of the results of Section 6, are so important for the sutdy of continuum limits. Our numerical data are in excellent agreement with the predictions of the perturbative renormalization group.

We have briefly reviewed, in appendices, results that the reader will find useful. In particular, Appendix A explains how to derive the analytic continuation in the number N of components of the field. In Appendix B, we recall relations among critical exponents and illustrate their connection with exponents of quantities like the entropy factor, the mean end-to-end distance, the Hausdorff dimension of random walks, etc. more commonly used in the context of polymer physics. Finally, in Appendix C, the fundamental correlation inequalities for measures on random walks, on which the results in Section 6 are based are briefly rederived.

We should stress that much of the material in Sections 2 through 5 has appeared already in the literature and is included here for reasons of a clear, self-contained exposition. In particular, Section 3 recall results which have been obtained in [2], Section 4 and Appendix A contain ideas which have also appeared in [9,10] and in Section 5 we draw on a very clear and pedagogical discussion of the continuum limit that has appeared in [4].

2. Definition of models

As announced, we shall analyze $g|\vec{\phi}|^4$ theory, non-linear σ-models (the Ising- and classical rotor model) and the self-avoiding random walk on the four-dimensional, simple hypercubical lattice $\mathbb{Z}^4$. On a lattice of arbitrary dimension d, $d = 1,2,3,4,\ldots$, these models are defined as follows : With each site $j \in \mathbb{Z}^d$ we associate a classical spin - or field variable, $\vec{\phi}(j) \in \mathbb{R}^N$. The a priori distribution of $\vec{\phi}(j)$ is given by a measure $d\lambda(\vec{\phi}(j))$, independent of j, which is invariant under rotations and reflections of $\vec{\phi}(j)$. The Hamilton function, or Euclidean lattice action, of this theory is given by

$$H(\vec{\phi}) = - \sum_{(j,j')} \vec{\phi}(j)\cdot\vec{\phi}(j')$$

$$= - \frac{1}{2}(\vec{\phi},\Delta\vec{\phi}) - d(\vec{\phi},\vec{\phi}) \quad , \tag{2.1}$$

where (j,j') denotes a nearest neighbor pair $(|j-j'| = 1)$, Δ is the finite difference Laplacian, and $(.,.)$ is the scalar product on $\ell_2(\mathbb{Z}^d)$.

The equilibrium state, or Euclidean vacuum functional, of this lattice theory is given by the measure

$$d\mu_{\beta,\lambda}(\vec{\phi}) = Z_{\beta,\lambda}^{-1} e^{-\beta H(\vec{\phi})} \prod_j d\lambda(\vec{\phi}(j)) \quad ; \tag{2.2}$$

the partition function, $Z_{\beta,\lambda}$, is chosen such that

$$\int d\mu_{\beta,\lambda}(\vec{\phi}) = 1 \quad ;$$

the parameter β is interpreted as inverse temperature or field strength.

Mathematically, the measure $d\mu_{\beta,\lambda}(\vec{\phi})$ must be defined as a limit of measures $d\mu^{\Lambda}_{\beta,\lambda}$, defined on configurations, $\{\vec{\phi}(j)\}_{j\in\Lambda}$, of spins in finite sublattices $\Lambda \subset \mathbb{Z}^d$, as Λ increases to $\mathbb{Z}^d$. Some limit can always be constructed by compactness. In the cases of interest to us $(N = 0,1,2)$ the limit $\Lambda \nearrow \mathbb{Z}^d$ is known to exist [11] .

The objects of interest to us are the correlation (or Euclidean Green) functions which are defined as the moments of the measure $d\mu_{\beta,\lambda}$, i.e.

$$<\prod_{j=1}^{n} \phi^{\alpha_j}(x_j)>_{\beta,\lambda} = \int \prod_{j=1}^{n} \phi^{\alpha_j}(x_j) d\mu_{\beta,\lambda}(\vec{\phi}) \quad , \tag{2.3}$$

where ϕ^α is the α-th component of $\vec{\phi}$.

In this paper we shall, as announced, investigate the scaling limits = continuum limits of these correlation functions, or, equivalently, their behavior for β in the vicinity of a critical point, in four dimensions. We study the approach to the critical point within the symmetric phase.

We now mention two examples of such lattice field theories :

(1) the $g|\vec{\phi}|^4$ theory

In this example we set

$$d\lambda(\vec{\phi}) = f(|\vec{\phi}|^2) d^N\phi \quad , \text{ with}$$

$$f(|\vec{\phi}|^2) = \exp[- \frac{g}{4}|\vec{\phi}|^4 + \frac{\mu^2}{2}|\vec{\phi}|^2 + c] \quad , \tag{2.4}$$

where $g > 0$, μ^2 real and c is some constant ; $d^N\phi \equiv \prod_{\alpha=1}^{N} d\phi^\alpha$.

By a trivial rescaling, $\vec{\phi}(j) \longrightarrow \text{const.}\vec{\phi}(j)$, it is seen that the expectation $<(.)>_{\beta,\lambda}$ really only depends on two-parameters, β and g , say : (μ^2 could be set = 1, for example). We change our notation to

$$< (\cdot) >_{\beta,g} \equiv < (\cdot) >_{\beta,\lambda} \quad ,$$

and we shall usually drop the subscript g .

(2) <u>non-linear</u> σ-<u>models on the lattice</u>

If in formula (2.4) we set

$$\mu^2 = g \cdot N \quad ,$$

let the constant c depend on g and N in a suitable way and then take $g \to \infty$ we obtain

$$d\lambda_{\infty}(\vec{\phi}) = \delta(|\vec{\phi}|^2 - N) d^N\phi \quad . \tag{2.5}$$

If we replace $d\lambda$ by $d\lambda_{\infty}$ in the preceeding formulas we obtain the vacuum functional and the Green functions of the non-linear σ-model on the lattice, also called N-vector model. This model has only one free parameter, β . We use the notation

$$< (.) >_{\beta} \equiv < (.) >_{\beta,\lambda_{\infty}} \quad .$$

Next we introduce <u>the self-avoiding random walk model</u>. This model arises from example (2) by analytic continuation in N to $N = 0$.

A random walk ω on the lattice $\mathbb{Z}^d$ is specified by an ordered sequence of nearest-neighbor jumps, $(\omega(s),\omega(s+1))$, $s = 1,\ldots,|\omega|$, where $|\omega|$ denotes the total number of such jumps. To each random walk ω we assign a statistical weight, $z^o_{\xi}(\omega)$, given by

$$z^o_{\xi}(\omega) = \xi^{|\omega|} \quad . \tag{2.6}$$

A random walk ω is said to be self-avoiding if no site of the lattice is visited by ω more than once. The notation

$$\omega : x \longrightarrow y$$

indicates that the walk ω starts at the site x and ends at y. The sites visited by a random walk ω really form an ordered set, ordered by the parameter s introduced above.

Next, we introduce Green functions for the self-avoiding random walk model : We choose n pairs of sites, $x_1, y_1, \ldots, x_n, y_n$, and define

$$F^o_\xi(x_1, y_1, \ldots, x_n, y_n) = \sum_{\substack{\omega_j : x_j \to y_j \\ j=1,\ldots,n}}' \prod_{i=1}^{n} z^o_\xi(\omega_i) \quad , \tag{2.7}$$

where Σ' ranges over all self-avoiding random walks, $\omega_1, \ldots, \omega_n$, which avoid each other, i.e.

$$\omega_\ell \cap \omega_k = \emptyset \quad ,$$

for $\ell \neq k$. We now define Green functions

$$G_\xi(x_1, \ldots, x_{2n}) = \sum_p F^o_\xi(x_{p(1)}, x_{p(2)}, \ldots, x_{p(2n-1)}, x_{p(2n)}), \tag{2.8}$$

where $\sum_p$ ranges over all pairings of $\{x_1, \ldots, x_{2n}\}$ into pairs $x_{p(2k-1)}$, $x_{p(2k)}$. Note that the functions $G_\xi(x_1, \ldots, x_{2n})$ are symmetric in their arguments, just like the correlation functions

$$< \phi^\alpha(x_1), \ldots, \phi^\alpha(x_n) >_\beta \quad , \quad \alpha \in \{1, \ldots, N\} \quad . \tag{2.9}$$

As is known, the functions $G_\xi(x_1, \ldots, x_{2n})$ are actually equal to the limits of the correlation functions (2.9) of the non-linear σ-model as $N \to 0$ if we set

$$\xi = \beta \qquad . \tag{2.10}$$

This is further discussed in § 4, and the proof is outlined in Appendix 1.

3. The random walk representation

In this section we briefly recall a random walk represenation of classical spin systems, originally introduced in [10,12,2] . It exhibits a basic mathematical relationship between lattice field theory and random walk models. The following calculations are formal ; the rigorous justification has been given in [2] . We assume that

$$d\lambda(\vec{\phi}) = f(|\vec{\phi}|^2)d^N\phi \qquad , \tag{3.1}$$

where f is a smooth function on the real line decreasing stronger than exponentially at infinity. A general class of single spin distributions can be obtained from distributions satisfying (3.1) by limiting arguments. Let

$$f(|\vec{\phi}|^2) = \int \hat{f}(a)e^{-ia|\vec{\phi}|^2} \, da \tag{3.2}$$

be the Fourier decomposition of f .

Let $F(\vec{\phi})$ be a function depending smoothly on a finite number of spin variables, $\vec{\phi}(j)$. We consider the correlation function

$$< \phi^1(x)F(\vec{\phi}) >_{\beta,\lambda} \qquad .$$

Inserting (3.2) into (2.2),(2.3) and interchanging integration over the $\vec{\phi}$- and over the a-variables we obtain

$$< \phi^1(x)F(\vec{\phi}) >_{\beta,\lambda} =$$

$$Z^{-1}_{\beta,\lambda} \int \prod_j \hat{f}(a(j))da(j) \cdot \int \phi^1(x)F(\vec{\phi})e^{-\frac{1}{2}(\vec{\phi},(\beta P+2ia)\vec{\phi})} \prod_j d^N\phi(j), \tag{3.3}$$

where

$$(Ph)(j) = - \sum_{j':|j-j'|=1} h(j') \qquad ,$$

and a denotes the function which takes the value $a(j)$ at site j . The $\vec{\phi}$-integral on the r.s. of (3.3) is Gaussian and can be evaluated explicitly. This yields

$$\begin{aligned} &< \phi^1(x)F(\vec{\phi}) >_{\beta,\lambda} = \\ &= Z^{-1}_{\beta,\lambda} \sum_y \int \prod_j \hat{f}(a(j))da(j)(\beta P+2ia)^{-1}_{xy} \cdot \\ &\int \frac{\partial F(\vec{\phi})}{\partial \phi^1(y)} e^{-\frac{1}{2}(\vec{\phi},(\beta P+2ia)\vec{\phi})} \prod_j d^N\phi(j) \quad . \end{aligned} \tag{3.4}$$

Following [12,2] we expand $(\beta P+2ia)^{-1}_{xy}$ in a Neumann series in βP . [Under our assumptions on f , $\hat{f}(a)$ is an entire function of a . We may therefore shift the contours of the $a(j)$-integrals in (3.4) in such a way that on the shifted integration contours the Neumann expansion of $(\beta P+2ia)^{-1}$ in βP converges.] Each term in that series is labelled by a random walk, ω , starting at x and ending at y . Let $n_j(\omega)$ denote the number of visits of the walk ω at the site j . Then

$$(\beta P+2ia)^{-1}_{xy} = \sum_{\omega: x\to y} \beta^{|\omega|} \prod_j (2ia(j))^{-n_j(\omega)} \quad , \tag{3.5}$$

as one easily verifies ; see also [2]. We define

$$d\nu_n(t) = \begin{cases} \delta(t)dt & , \quad \text{if } n = 0 \\ \\ \Gamma(n)^{-1}t^{n-1}\theta(t)dt \ , & n = 1,2,3,\dots \ , \end{cases} \tag{3.6}$$

where $\Gamma(n) = (n-1)!$, and θ is the Heaviside step function. Inserting (3.5) and the identity

$$(2ia)^{-n} = \int e^{-2iat} d\nu_n(t) \tag{3.7}$$

into (3.4) and carrying out the $a(j)$-integrals, we obtain

$$
\begin{aligned}
&< \phi^1(x) F(\vec{\phi}) >_{\beta,\lambda} = \\
&\sum_y \sum_{\omega: x \to y} \beta^{|\omega|} Z_{\beta,\lambda}^{-1} \int \prod_j d\nu_{n_j(\omega)}(t(j)) \cdot \\
&\cdot \int e^{-\beta H(\vec{\phi})} \frac{\partial F(\vec{\phi})}{\partial \phi^1(y)} \prod_j f(|\vec{\phi}(j)|^2 + 2t(j)) d^N\phi(j) \quad . \qquad (3.8)
\end{aligned}
$$

The variables $t(j)$ have the interpretation of <u>local times</u> : $t(j)$ is the total time the walk ω spends at site j . By (3.6), these local times have a Poissonian a priori distribution.

Identity (3.8) is the basic formula relating lattice field theories to random walks. It can be iterated by writing

$$
\frac{\partial F(\vec{\phi})}{\partial \phi^1(y)} = \begin{cases} \text{const., or} \\ \\ \phi^\alpha(y') \ F'(\vec{\phi}) \end{cases} ,
$$

for some $y' \in \mathbb{Z}^d$, $\alpha = 1, \ldots, N$, and some new function F' . We now define

$$
\begin{aligned}
d\nu_\omega(t) &= \prod_j d\nu_{n_j(\omega)}(t(j)) \\
z_{\beta,\lambda}^N(t) &= Z_{\beta,\lambda}^{-1} \int e^{-\beta H(\vec{\phi})} \prod_j f(|\vec{\phi}(j)|^2 + 2t(j)) d^N\phi(j)
\end{aligned} \qquad (3.9)
$$

and

$$
z_{\beta,\lambda}^N(\omega_1, \ldots, \omega_n) = \int \prod_{k=1}^n \beta^{|\omega_k|} d\nu_{\omega_k}(t_k) z_{\beta,\lambda}^N(t_1 + \ldots + t_n) \qquad (3.10)
$$

Finally, we define

$$F^N_{\beta,\lambda}(x_1,y_1,\ldots,x_n,y_n) = \sum_{\substack{\omega_k : x_k \to y_k \\ k=1,..,n}} z^N_{\beta,\lambda}(\omega_1,\ldots,\omega_n) \quad . \qquad (3.11)$$

Then

$$< \prod_{j=1}^{2n} \phi^\alpha(x_j) >_{\beta,\lambda} = \sum_p F^N_{\beta,\lambda}(x_{p(1)},\ldots,x_{p(2n-1)},x_{p(2n)}) \ , (3.12)$$

and

$$< \prod_{j=1}^{2n} \phi^\alpha(x_j) \prod_{k=1}^{2m} \phi^{\alpha'}(y_k) >_{\beta,\lambda}$$

$$= \sum_p \sum_{p'} F^N_{\beta,\lambda}(x_{p(1)},\ldots,x_{p(2n)},y_{p'(1)},\ldots,y_{p'(2m)}), \quad (3.12')$$

for $\alpha \neq \alpha'$; etc.

We want to note the following identity : Let $\omega_1,\ldots,\omega_m$ be m given random walks. Then $\omega_1 \cdot \ldots \cdot \omega_m$ denotes the union of the m ordered sequences corresponding to $\omega_1,\ldots,\omega_m$, respectively, so that

$$n_j(\omega_1 \cdot \ldots \cdot \omega_m) = \sum_{k=1}^{m} n_j(\omega_k) \quad . \qquad (3.13)$$

By (3.6)

$$d(\nu_{n_1} * \ldots * \nu_{n_m})(t) = d\nu_{n_1+\ldots+n_m}(t) \quad ,$$

where $*$ indicates convolution, so that

$$d(\nu_{\omega_1} * \ldots * \nu_{\omega_m})(t) = d\nu_{\omega_1 \cdot \ldots \cdot \omega_m}(t) \quad . \qquad (3.14)$$

It then follows from (3.10) that

$$z^N_{\beta,\lambda}(\omega_1,\ldots,\omega_n) = z^N_{\beta,\lambda}(\omega_1 \cdot \ldots \cdot \omega_n)$$
$$= \int d\nu_{\omega_1\cdot\ldots\cdot\omega_n}(t) z^N_{\beta,\lambda}(t) \quad . \tag{3.15}$$

In order to familiarize ourselves with these notions we consider a simple example :

$$d\lambda_o(\vec{\phi}) = \exp[-\tfrac{1}{2}(2d\beta+m_o^2)|\vec{\phi}|^2]d^N\phi \quad . \tag{3.16}$$

In this case

$$f(|\vec{\phi}|^2+2t) = f(|\vec{\phi}|^2)e^{-(2d\beta+m_o^2)t} \quad ,$$

hence

$$z^N_{\beta,\lambda_o}(t) = \prod_j e^{-(2d\beta+m_o^2)t_j}$$

Note that this quantity is N-independent. Thus,

$$dP(\omega,t) = \beta^{|\omega|} z^N_{\beta,\lambda_o}(t) d\nu_\omega(t) \quad . \tag{3.17}$$

is the usual lattice approximation to the Wiener measure, expressed in terms of local times $\{t(j)\}_{j\in\mathbb{Z}^d}$. In this simple case, the $t(j)$-integrals can be computed easily, and we obtain

$$Z^N_{\beta,\lambda_o}(\omega) = \beta^{|\omega|} \prod_j (2d\beta+m_o^2)^{-n_j(\omega)} \quad . \tag{3.18}$$

By (3.13), (3.15) and (3.18)

$$z^N_{\beta,\lambda_o}(\omega_1,\ldots,\omega_n) = z^N_{\beta,\lambda_o}(\omega_1 \cdot \ldots \cdot \omega_n)$$

$$= \prod_{k=1}^{n} z^{N}_{\beta,\lambda_o}(\omega_k) \qquad . \tag{3.19}$$

Now note that by (3.11), (3.12) and (3.18)

$$\begin{aligned} < \phi^{\alpha}(x)\phi^{\alpha}(y)>_{\beta,\lambda_o} &= F^{N}_{\beta,\lambda_o}(x,y) \\ &= \sum_{\omega:x\to y} z^{N}_{\beta,\lambda_o}(\omega) \\ &= (-\beta\Delta+m_o^2)^{-1}_{xy} \qquad , \end{aligned} \tag{3.20}$$

and, using also (3.19),

$$F^{N}_{\beta,\lambda_o}(x_1,y_1,\ldots,x_n,y_n) = \prod_{k=1}^{n} (-\beta\Delta+m_o^2)^{-1}_{x_k y_k} \qquad . \tag{3.21}$$

Thus, this example is nothing but the well known <u>Brownian motion</u> (<u>proper time</u>) <u>representation</u> of the free (Gaussian) lattice field of mass m_o .

4. The limit $N \to 0$

In this section we review the well known results [7,8] that

(a) the $N = 0$ limit of $g|\vec{\phi}|^4$ theory is the Edwards model [13] of self-suppressing random walks ;

(b) the $N = 0$ limit of the non-linear σ-model is the self-avoiding random walk model. Our arguments are based on the results of Section 3. See also [9].

By (3.11), (3.12) and (3.15), it is enough to identify the $N = 0$ limit of the weights $z^N_{\beta,\lambda}(\Omega)$, where $\Omega \equiv \omega_1 \cdot \ldots \cdot \omega_n$, for some random walks $\omega_1,\ldots,\omega_n$. We recall that

$$z^N_{\beta,\lambda}(\Omega) = \beta^{|\Omega|} \int \prod_{j\in\mathbb{Z}^d} d\nu_{n_j(\Omega)}(t_j) z^N_{\beta,\lambda}(t) \quad . \qquad (4.1)$$

In Appendix 1 we sketch the proofs of the following facts :

If

$$d\lambda(\vec{\phi}) \equiv d\lambda^N(\vec{\phi}) \equiv f^N(|\vec{\phi}|^2)d^N\phi$$

is chosen appropriately, e.g. as in $g|\vec{\phi}|^4$ theory or in the non-linear σ-model with N-component fields - see (2.4) and (2.5) - $z^N_{\beta,\lambda}(\Omega)$ extends to a function of N which is analytic in a neighborhood of $N = 0$, (provided we work in a finite volume, or keep β sufficiently small ; see e.g. [9].) Moreover

$$z^o_{\beta,\lambda}(\Omega) \equiv \lim_{N\to 0} z^N_{\beta,\lambda}(\Omega) = \beta^{|\Omega|} \prod_{j\in\mathbb{Z}^d} p(n_j(\Omega)) \quad , \qquad (4.2)$$

where

$$|\Omega| = \sum_{k=1}^{n} |\omega_k| ,$$

$$p(n) = \lim_{N \to 0} \frac{\int d\nu_{n+(N/2)}(t) f^N(2t)}{\int d\nu_{N/2}(t) f^N(2t)} \quad , \tag{4.3}$$

and

$$d\nu_x(t) = \Gamma(x)^{-1} t^{x-1} \theta(t) dt \quad , \tag{4.4}$$

for $x \neq 0,-1,-2,\ldots$.

In $g|\vec{\phi}|^4$-theory we choose $f^N(|\vec{\phi}|^2) = f(|\vec{\phi}|^2)$ as in (2.4) , independently of N with

$$\mu^2 \equiv 2dg + m_o^2 \quad , \tag{4.5}$$

and obtain

$$p(n) = \int d\nu_n(t) e^{-(gt^2+(2dg+m_o^2)t)} \quad , \tag{4.6}$$

$n = 1,2,3,\ldots$.

Note that, in this case, $z^o_{\beta,\lambda}(\Omega)$ converges to the statistical weight of ordinary random walks, defined in (3.18), (3.19), as $g \searrow 0$.

In the σ-model, we choose $d\lambda^N(\vec{\phi})$ as in (2.5) and obtain

$$p(n) = \delta_{n1} + \delta_{no} \tag{4.7}$$

We conclude that, in this case,

$$\lim_{N \to 0} \langle \phi^{\alpha}(x_1) \ldots \phi^{\alpha}(x_n) \rangle^N_{\beta} = G_{\xi=\beta}(x_1,\ldots,x_n) \quad , \tag{4.8}$$

where $\langle (.) \rangle^N_{\beta} \equiv \langle (.) \rangle_{\vec{\beta}}$ is the expectation of the N-component σ-model defined in Section 2, and G_{ξ} is the Green function of the self-avoiding random walk model introduced in (2.8). There are further relations, similar to (4.8), which follow from (4.2), (4.6), (4.7) and the results in Sections 2 and 3 and which we refrain from writing out explicitly. For details see Appendix A .

§5. Scaling limit and critical exponents.

In order to simplify our notations we only consider $N = 1$ component fields and the self-avoiding random walk model throughout this section. Moreover, we first imagine that the $|\vec{\phi}|^4$-coupling constant, g, is fixed, (e.g. at $g = +\infty$, corresponding to the Ising model limit).

By $G_\beta(x_1,\ldots,x_n)$ we denote the correlation functions, $< \phi(x_1) \ldots \phi(x_n) >_{\beta,\lambda}$, of the lattice field theories defined in section 2, and the Green functions of the self-avoiding random walk model, respectively. We want to study the behavior of the functions $G_\beta(x_1 \ldots, x_n)$ at large distances, or, equivalently, in the continuum limit.

In $d \geq 3$ dimensions, all models studied in our paper are known to have a phase transition, from a high temperature (small β) disordered phase with exponentially decaying connected correlations to a large β phase with long range order. Thus, for β small enough,

$$G_\beta(x,y) \leq \text{const. } e^{-m(\beta)|x-y|}, \tag{5.1}$$

for some positive constant $m(\beta)$, (the inverse correlation length), and the susceptibility

$$\chi(\beta) \equiv \sum_{y \in \mathbb{Z}^d} G_\beta(x,y) \tag{5.2}$$

is finite, while for sufficiently large β, $\chi(\beta)$ diverges. See [14] and refs. given there.

In the $N = 1, 2$ $(3,4)$ component field theory models it is known that there exists a critical point β_c (which in the $g|\vec{\phi}|^4$-theory depends, of course, on the coupling constant g) with the property that $m(\beta)$ and $\chi(\beta)^{-1}$ are positive, continuous functions of β, for $\beta < \beta_c$, and

$$\lim_{\beta \nearrow \beta_c} m(\beta) = \lim_{\beta \nearrow \beta_c} \chi(\beta)^{-1} = 0 \tag{5.3}$$

This result can be found in [14] and is conjectured to hold for all N with

$$-2 \leq N \leq \infty \quad ,$$

in particular for $N = 0$.

In view of result (5.3) one must ask how $m(\beta)$ tends to 0 and how $\chi(\beta)$ diverges, as $\beta \nearrow \beta_c$. In the class of models studied in this paper it is expected that, for $d \neq 4$, a scaling law holds, i.e.

$$\begin{aligned} m(\beta) &\sim \tau^{\nu} \\ \chi(\beta) &\sim \tau^{-\gamma} \end{aligned} \tag{5.4}$$

where $\tau \equiv \frac{\beta_c - \beta}{\beta_c}$. It is known that the critical exponents ν and γ satisfy the inequalities

$$\nu \geq 1/2 \quad \text{and} \quad \gamma \geq 1 \tag{5.5}$$

see [14]. Moreover, for $d \geq 5$ and ($N = 1$, or 2)-component fields,

$$\gamma = 1 \ , \tag{5.6}$$

see [1,3], which is the value predicted by mean field theory. It has also been shown [15] that the critical exponent, α , of the specific heat vanishes in $d \geq 5$. There are strong indications that in five or more dimensions and for an arbitrary number $N = 0,1,2,3, \ldots$ of components mean field theory provides an exact description of the approach to the critical point; in particular

$$\nu = 1/2 \ , \gamma = 1 \ , \alpha = 0 \ .$$

In four dimensions, one expects that there are logarithmic violations of the meanfield theory scaling laws. Such violations are predicted by the renormalization group. See [16,17]. For example

$$\begin{aligned} m(\beta) &\sim \tau^{1/2} |\ell n \tau|^{-N} \\ \chi(\beta) &\sim \tau^{-1} |\ell n \tau|^{G} \end{aligned} \tag{5.7}$$

Next, we introduce scaled correlations. Let θ be a parameter varying between 1 and ∞ . We define

$$G^{(\theta)}(x_1, \ldots, x_n) \equiv \alpha(\theta)^n \, G_{\beta(\theta)} \, (\theta x_1, \ldots, \theta x_n) \ , \tag{5.8}$$

where

$$x_j \in \mathbb{Z}^d \ , \text{ i.e. } \ x_j \in \mathbb{Z}^d_{\theta^{-1}} \ ,$$

and $\mathbb{Z}_a^d$ is the lattice with lattice spacing a ,

$$\alpha(\theta) > 0 \quad \text{and} \quad \beta(\theta) \leq \beta_c$$

are functions which one tries to choose in such a way that the two-point correlation function, $G^{(\theta)}(x,y)$, has a non-trivial limit, as $\theta \to \infty$, i.e. for $0 < |x-y| < \infty$,

$$0 < G^{(\infty)}(x,y) \equiv \lim_{\theta \to \infty} G^{(\theta)}(x,y) < \infty \tag{5.9}$$

From (5.1) and (5.3) it follows that

$$\beta(\theta) \to \beta_c \quad , \quad \text{as} \quad \theta \to \infty \tag{5.10}$$

more precisely, one may choose $\beta(\theta)$ such that the physical mass, i.e. the inverse correlation length measured in physical units,

$$m^* \equiv \theta m(\beta(\theta)) \tag{5.11}$$

is kept fixed. By (5.3) this is guaranteed if

$$\beta(\theta) = m^{-1} \left(\frac{m^*}{\theta}\right) \tag{5.12}$$

and the inverse function, m^{-1} , of m exists for $\frac{m^*}{\theta}$ sufficiently small, [14].

Next, we try to choose $\alpha(\theta)$ in such a way that our <u>renormalization condition</u> (5.9) is fulfilled. Not much is known about whether this is actually possible. (The only examples where rigorous results are known are the two-dimensional Ising model and super-renormalizable $\lambda|\vec{\phi}|^4$ models in two and three dimensions.) By scaling at large distances we mean the property that

$$\alpha(\theta)^2 \sim \theta^{d-2+\eta} , \qquad (5.13)$$

where η is a critical exponent. It is expected that in $d \neq 4$ dimensions condition (5.9) can be fulfilled with functions $\beta(\theta)$ and $\alpha(\theta)$ satisfying (5.12) and (5.13), respectively. From this it would follow that

$$\gamma = \nu(2-\eta) , \qquad (5.14)$$

a well-known scaling relation due to Fisher.

In four dimensions, there might be logarithmic violations of (5.13), i.e.

$$\alpha(\theta)^2 \sim \theta^2 |\ln\theta|^E . \qquad (5.15)$$

If (5.9) can be imposed, the relation

$$G = 2N - E \qquad (5.16)$$

(corresponding to (5.14)) holds. These matters and further relations between critical exponents are reviewed in Appendix B. Here, we draw the reader's attention to the fact that, for all $N = 1,2,3,\ldots$,

$$\eta \geq 0 , \quad E \geq 0 , \qquad (5.17)$$

a consequence of the infrared bound

$$0 \leq G_\beta(x,y) \leq \beta^{-1} c_d \, |x-y|^{2-d} \quad , \tag{5.18}$$

where c_d is a geometrical constant; see [18,4] .

Next, we define the connected four-point (Ursell) function, $\mu_{4,\beta}$:

$$\mu_{4,\beta}(x_1,x_2,x_3,x_4) \equiv G_\beta(x_1,x_2,x_3,x_4) - $$

$$- \sum_p G_\beta(x_{p(1)} x_{p(2)}) \, G_\beta(x_{p(3)},x_{p(4)}) \; . \tag{5.19}$$

The scaled Ursell function is defined by

$$\mu_4^{(\theta)}(x_1,x_2,x_3,x_4) = \alpha(\theta)^4 \mu_{4,\beta(\theta)}(\theta x_1,\theta x_2,\theta x_3,\theta x_4) \; , \tag{5.20}$$

and

$$\mu_4^{(\infty)}(x_1,x_2,x_3,x_4) = \lim_{\theta\to\infty} \mu_4^{(\theta)} \; (x_1,x_2,x_3,x_4) \; .$$

For $\beta < \beta_c$, $m(\beta) > 0$, and one can evaluate the Fourier transform of $\mu_{4,\beta}$ at 0 momentum :

$$\overline{\mu}_{4,\beta} \equiv \sum_{\substack{x_j \in \mathbb{Z}^d \\ j=2,3,4}} \mu_{4,\beta}(0,x_2,x_3,x_4) \tag{5.21}$$

It is believed that in $d \neq 4$ dimension

$$\overline{\mu}_{4,\beta} \sim \tau^{-(\gamma+2\Delta)} \tag{5.22}$$

for some critical exponent $\Delta > 0$, (and $\tau \equiv \beta_c^{-1}(\beta_c-\beta)$ small).

Let

$$\overline{\mu}_4^{(\theta)} \equiv \sum_{\substack{x_j \in \mathbb{Z}^d_{\theta^{-1}} \\ j=2,3,4}} \theta^{-3d} \mu_4^{(\theta)}(0,x_2,x_3,x_4) \tag{5.23}$$

$$\chi^{(\theta)} \equiv \sum_{x \in \mathbb{Z}^d_{\theta^{-1}}} \theta^{-d} G^{(\theta)} (0,x) \tag{5.24}$$

$$m^{(\theta)} \equiv \theta m(\beta(\theta)) = m^* = \text{const.} \tag{5.25}$$

We now define a dimensionless coupling constant $g_r^{(\theta)}$ as follows :

$$g_r^{(\theta)} \equiv -\overline{\mu}_4^{(\theta)} (\chi^{(\theta)})^{-2} (m^{(\theta)})^d \quad . \tag{5.26}$$

By a trivial change of variables, one sees that

$$g_r^{(\theta)} = g_r(\beta(\theta)) \ , \tag{5.27}$$

where

$$g_r(\beta) \equiv -\overline{\mu}_{4,\beta} \ \chi(\beta)^{-2} m(\beta)^d \quad . \tag{5.28}$$

By (5.22), the critical exponent of $g(\beta)$ is

$$d\nu + \gamma - 2\Delta \ . \tag{5.29}$$

For small values of the critical exponent η and positive physical mass,

$$g_r^{(\infty)} \equiv \lim_{\theta\to\infty} g_r^{(\theta)} = \lim_{\beta\nearrow\beta_c} g_r(\beta) \tag{5.30}$$

measures the deviation of the theory in the scaling (= continuum) limit from a free, i.e. Gaussian theory, for which $\mu_4 \equiv 0$. Indeed, under our assumption on η and m^* ,

$$g_r^{(\infty)} \propto - \int d^dx\, d^dy\, d^dz\, \mu_4^{(\infty)}(0,x,y,z) \; .$$

For $N = 0,1,2,$

$$\mu_4^{(\infty)}(x_1,x_2,x_3,x_4) \leq 0 \; , \text{ for all } \theta \; , \tag{5.31}$$

the Lebowitz inequality [19] . Hence,

$$g_r^{(\infty)} = \lim_{\beta\nearrow\beta_c} g_r(\beta) = 0 \leftrightarrow \mu_4^{(\infty)} \equiv 0 \; . \tag{5.32}$$

For $N = 1,2$, Newman has shown that the theory is trivial, i.e. Gaussian if and only if

$$\mu_4^{(\infty)} = 0 \quad ; [20]$$

An earlier result of this sort was proven, in the context of axiomatic field theory by Borchers [21].

If $\mu_4^{(\infty)} \neq 0$ then $g_r^{(\infty)} \neq 0$ which implies

$$d\nu + \gamma - 2\Delta = 0 \quad (\underline{\text{hyperscaling}}) \tag{5.33}$$

if scaling holds.

Finally, we wish to emphasize that the $g|\vec{\phi}|^4$ lattice theories have really two independent coupling constants, g and β. In the construction of the scaling limit, not only β but also g may be chosen to depend on the scale parameter θ in such a way that $(\beta(\theta), g(\theta))$ converge to a critical point $(\beta_c, g(\beta_c))$, as $\theta \to \infty$. The critical points of this theory are expected to form a curve of the type shown in Fig. 1.
If one passes to the scaling limit without (infinite) field strength - and charge renormalization - the procedure adopted in constructive quantum field theory in two and three dimensions - one chooses (for $d < 4$)

$$\beta(\theta) \xrightarrow[\theta\to\infty]{} \beta_o \, , \; \alpha(\theta)^2 \sim \theta^{d-2} \, , \;\; g(\theta) \sim \theta^{d-4} \quad . \tag{5.34}$$

Such theories lie in the vicinity of the Gaussian fixed point and have canonical (free field) ultraviolet behaviour.

It is one of the basic problems of two - and three dimensional stastistical mechanics to construct scaling limits associated with nontrivial critical points $(\beta_c, g(\beta_c))$, where $\beta_c > \beta_o$ and $g(\beta_c) > 0$, in particular with $\beta_c = \beta_c$ (Ising), $g = +\infty$.

The scaling limit of the two-dimensional Ising model has been constructed by means of an explicit solution; see [22]. Aizenman has recently found a very simple and elegant proof of hyperscaling in a class of two-dimensional Ising models [1]. The only general result valid in arbitrary dimension is an absolute upperbound on $g_r(\beta)$, due to Glimm and Jaffe [14]. See also [4] for a general discussion of these matters.

In the following we study "all possible" scaling limits in <u>four</u> dimensions, a problem which should be easier than its lower-dimensional analogue. Our results, partly theoretical, partly numerical, suggest logarithmic violations of hyperscaling and triviality of "all" scaling limits.

§6. Estimates on $|\mu_4^{(\infty)}|$ and $\lim_{\beta\nearrow\beta_c} g_r(\beta)$.

The purpose of this section is to review inequalities on the four-point Ursell function, $\mu_{4,\beta}(x_1,x_2,x_3,x_4)$, and the dimensionless coupling constant, which permit us to analyze the scaling limit of $g|\vec{\phi}|^4$ theory, $0 \leq g \leq \infty$, and the self-avoiding random walk model $(N = 0,1,2)$ in four or more dimensions. Proofs are given in section 7. In four dimensions, we obtain results which sharpen earlier results in [3,6]. Our methods are based on the random walk representation of [2] which we have reviewed in sections 3 and 4.

Let

$$z_\beta(\omega) = z^N_{\beta,\lambda}(\omega) \quad , \quad N = 0,1 \ , \quad \text{or} \quad 2 \ . \tag{6.1}$$

For $N = 0$ this is the statistical weight of a self-suppressing $(g < \infty)$ or self-avoiding random walk; see section 4. For $g = 0$, it is the weight of the standard (non-interacting) random walk, and for $N = 1,2$, the weight which appears in the random walk representation of $g|\vec{\phi}|^4$ theory, or the non-linear σ-models; see section 3.

We now introduce a quantity which plays a basic role in our approach :

$$Q(\beta) \equiv \chi(\beta)^{-2} \sum_{y_1,y_2 \text{ in } \mathbb{Z}^d} \ \sum_{\substack{\omega_1:0\to y_1 \\ \omega_2:j\to j_2}} \chi_\phi(\omega_1,\omega_2)\, z_\beta(\omega_1) z_\beta(\omega_2) \tag{6.2}$$

where j is some fixed nearest neighbor of 0 , and

$$\chi_\phi(\omega_1,\omega_2) = \begin{cases} 1 & \text{if } \omega_1 \text{ and } \omega_2 \text{ do not intersect, } (\omega_1 \cap \omega_2 = \emptyset). \\ 0 \ , & \text{otherwise.} \end{cases} \tag{6.3}$$

By definition (5.2) of $\chi(\beta)$ and the random walk representation (3.9) - (3.12) of $G_\beta(x,y)$, we have

$$\chi(\beta) = \sum_y \sum_{w:x\to y} z_\beta(\omega) \quad , \tag{6.4}$$

for arbitrary x.

Therefore $Q(\beta)$ <u>is the probability</u> (with respect ot the statistical weight $z_\beta(\omega_1)\, z_\beta(\omega_2)$) <u>that two random walks</u>, ω_1 <u>and</u> ω_2, <u>starting at neighboring lattice sites and ending at arbitrary sites do nowhere intersect</u>.

Triviality of the continuum limit of $g|\vec{\phi}|^4$-theory or the self-avoiding random walk in four dimensions is intimately connected with the behaviour of $Q(\beta)$, as $\beta \nearrow \beta_c$. We shall see that, for the $N = 1$, or 2 component lattice theories,

$$\lim_{\beta\nearrow\beta_c} Q(\beta) = 0 \tag{6.5}$$

implies triviality of the continuum limit.

Next, we summarize our main inequalities on the four-point function.

It has been shown in [3] (following a very similar result in [1]) that in $N = 0,1,2$ component theories, and for $x_i \neq x_j$, $i \neq j$,

$$0 \geq u_{4,\beta}(x_1,x_2,x_3,x_4) \geq -3\,\beta^2 \sum_{z_1;z_2,z_3,z_4} \prod_{j=1}^{4} G_\beta\,(x_j,z_j) \ , \tag{6.6}$$

up to a term which vanishes in the continuum limit in $d \geq 2$ dimensions.

$\sum\limits_{z_1;z_2,z_3,z_4}$ extends over all $z_1 \in \mathbb{Z}^d$, and points z_2, z_3, z_4 with

$$|z_j - z_1| = 0 \text{ , or } 1 \text{ , } j = 2,3,4 \text{ .}$$

By scaling distances, see (5.8) and (5.20), we obtain from (6.6)

$$0 \geq \mu_4^{(\theta)}(x_1,x_2,x_3,x_4) \geq -3\beta(\theta)^2 \alpha(\theta)^{-4}\, \theta^d \cdot$$

$$\cdot \{ \sum_{z_1;z_2,z_3,z_4} \theta^{-d} \prod_{j=1}^{4} G^{(\theta)}(x_j,z_j)\} \text{ ,} \tag{6.7}$$

where now z_1 ranges over $\mathbb{Z}^d_{\theta^{-1}}$, and $|z_j - z_1| \leq \theta^{-1}$, $j = 2,3,4$. At non-coinciding arguments, i.e. $x_i \neq x_j$, for $i \neq j$, and under mild uniformity assumptions on $G^{(\theta)}(x,y)$ which were discussed in [3] (in particular if $m^* > 0$)

$$\beta(\theta)^2 (\theta^{d-2}/\alpha(\theta)^2) \{ \sum \theta^{-d} \prod_{j=1}^{4} G^{(\theta)}(x_j,z_j) \tag{6.8}$$

is bounded uniformly in θ. Thus

$$|\mu_4^{(\theta)}(x_1,x_2,x_3,x_4)| \leq k_\varepsilon\, \theta^{4-d}(\theta^{d-2}/\alpha(\theta)^2) \tag{6.9}$$

where k_ε is a constant which is finite provided $|x_i - x_j| \geq \varepsilon$, $i \neq j$, for arbitrarily small, positive values of ε.

For $N = 1,2$,

$$\theta^{d-2}/\alpha(\theta)^2 \text{ is bounded ,} \tag{6.10}$$

as $\theta \to \infty$. See (5.17).

We conclude that, in four dimensions,

$$\lim \mu_4^{(\theta)} (x_1, x_2, x_3, x_4) = 0 \tag{6.11}$$

unless

$$\alpha(\theta) \sim \theta, \quad \text{hence} \quad \eta = E = 0 , \tag{6.12}$$

i.e. the short distance behavior of the theory is canonical.

In order to complete a proof of triviality of the continuum limit of the four-dimensional $N = 1$, or 2 component lattice field theories, we may henceforth assume that (6.12) holds and try to sharpen inequality (6.6). As recalled in section 5, it suffices, in this case, to analyze the behaviour of the dimensionless coupling constant,

$$g_r(\beta) = |\bar{\mu}_{4,\beta}| \, \chi(\beta)^{-2} \, m(\beta)^d ,$$

as $\beta \nearrow \beta_c$. More precisely, if η is small the continuum limit is trivial if

$$\lim_{\beta \nearrow \beta_c} g_r(\beta) = 0 . \tag{6.13}$$

Our new inequality for $g_r(\beta)$ is

$$0 \leq g_r(\beta) \leq \text{const.} \, \chi(\beta)^2 \, m(\beta)^d \, Q(\beta) , \tag{6.14}$$

(up to a term vanishing as $\beta \nearrow \beta_c$. This inequality sharpens one in [3]). Now

$$\chi(\beta)^2 \ m(\beta)^4$$

is bounded uniformly in $\beta \leq \beta_c$; (for $N = 1,2$) . Assuming a scaling law, or a behaviour as in (5.7) and (5.15),

$$\chi(\beta)^2 m(\beta)^4 \sim \begin{cases} \tau^{2\nu\eta} \\ |\ell n \tau|^{-2E} \end{cases} \tag{6.15}$$

Thus if $\eta = E = 0$ (6.13) follows if

$$\lim_{\beta \nearrow \beta_c} Q(\beta) = 0 \tag{6.16}$$

Our results for the behaviour of $Q(\beta)$ near the critical point are as follows :

(1) If $z(\omega)$ is the statistical weight of standard (non-interacting) random walk

$$Q(\beta) \sim |\ell n \tau|^{-\kappa} \quad , \tag{6.17}$$

where

$$1/2 \leq \kappa \leq 1 \ . \tag{6.17}$$

This is an immediate consequence of recent results by Lawler [23] who conjectures that $\kappa = 1/2$. This is in agreement with "renormalization group" calculations which we have performed (which are non-rigorous). It follows easily from (6.17) that, in the continuum limit, two Brownian paths starting at different points do not intersect. This is a well known result [24].

The advantate of the proof based on (6.17) is that it shows in which way intersection probabilities vanish, as the continuum limit is approached. See section 7.

(2) If $z_\beta(\omega) = z^o_{\beta,\lambda}(\omega)$ is the statistical weight of the self-avoiding random walk - or the Edwards model then

$$Q(\beta) \leq -(2d)^{-1} \frac{\partial}{\partial\beta} \chi^{-1}(\beta) \quad ; \tag{6.18}$$

with <u>equality</u> in the case of the self-avoiding random walk model. Now note that for $d = 4$,

$$(6.18) \text{ implies } \lim_{\beta\to\beta_c} Q(\beta) = 0 \ , \tag{6.19}$$

<u>unless mean field theory provides an exact description of the behaviour of $\chi(\beta)$ near β_c</u> .

For the self-avoiding random walk model, the equation

$$Q(\beta) = -(2d)^{-1} \frac{\partial}{\partial\beta} \chi^{-1}(\beta)$$

provides us with the possibility of determining the critical exponent of the susceptibility χ via measuring the exponent of $Q(\beta)$. For $d = 4$, our numerical data indicate

$$Q(\beta) \sim \tau\chi(\beta) \sim |\ell n\tau|^{-G} \quad \text{with} \quad G = \cdot 24 \pm \cdot 02 \tag{6.20}$$

see section 8.

(3) Consider the $N = 2$ lattice field theory. In (2.1) we have defined the lattice action of this theory as

$$H(\vec{\phi}) \equiv - \sum_{(j,j')} \vec{\phi}(j) \cdot \vec{\phi}(j') .$$

One can introduce an additional coupling term yielding an anisotropic iteraction :

$$H(\vec{\phi}) \to H_\varepsilon(\phi^1, \phi^2) = H(\phi) - \varepsilon \sum_{(j,j')} \phi^1(j)\phi^1(j') .$$

We define

$$\chi(\beta,\varepsilon) = \sum_x < \phi_o^1 \phi_x^1 >_{\beta,\varepsilon} \tag{6.21}$$

where $<(\cdot)>_{\beta,\varepsilon}$ is the equilibrium expectation corresponding to the action H_ε .

By arguments similar to the one given in section 7 for the self-avoiding random walk model and correlation inequalities one can show that

$$Q(\beta) \leq (2d)^{-1} \, 3\chi(\beta,0)^{-2} \beta^{-1} \frac{\partial\chi(\beta,\varepsilon)}{\partial\varepsilon} \Big|_{\varepsilon=0} \tag{6.22}$$

Heuristically,

$$\beta^{-1} \frac{\partial\chi(\beta,\varepsilon)}{\partial\varepsilon}\Big|_{\varepsilon=0} \sim \frac{\partial\chi(\beta)}{\partial\beta} \quad , \text{ for } d \geq 4 \quad ,$$

But the proof is incomplete in $d = 4$.

Results (1) through (3) lead us to the following conjectures :

(4) In $N = 1$, or 2 component field theories, for all $0 \leq g \leq \infty$ and $d \geq 4$,

$$Q(\beta) \leq -\text{const.} \frac{\partial}{\partial \beta} \chi^{-1}(\beta) \tag{6.23}$$

$$\lim_{\beta \nearrow \beta_c} Q(\beta) = 0 \text{ , when } d = 4 \tag{6.24}$$

Inequality (6.23) would imply that, in four dimensions,

$$\lim_{\beta \nearrow \beta_c} Q(\beta) = 0 \text{ , unless } -\frac{\partial}{\partial \beta} \chi^{-1}(\beta) \geq \text{const} > 0 \text{ ,} \tag{6.25}$$

for β near β_c . Since by [19]

$$-\frac{\partial}{\partial \beta} \chi^{-1}(\beta) \leq 4d \text{ ,}$$

(6.25) would give

$$\chi(\beta) \sim \tau^{-1} \text{ .} \tag{6.26}$$

Combining (6.12), (6.13), (6.16) and (6.26), we conclude, using Fisher's relations (5.14) and (5.16) that, <u>in four dimensions</u>,

$$u_4^{(\theta)}(x_1,x_2,x_3,x_4) = 0 \text{ , (for } x_i \neq x_j \text{ , } i \neq j) \text{ ,}$$

i.e. the continuum limit of the $N = 1$, or 2 component field theories is <u>trivial</u> (Gaussian), <u>unless</u>

(a) mean field theory is an exact description of the approach to the critical point, i.e. $\eta = 0$, $\nu = 1/2$, $\eta = 1$, $E = N = G = 0$; and (b) the probability that two (field-theoretic) random walks starting at neighboring points do not intersect is positive. Statement (a) would, of course, violate the predictions of the renormalization group, [16,17].

§7. Proofs of inequalities (6.6), (6.14) and (6.18).

Our proofs rely on the random walk representation reviewed in section 3. By the definition of $\mu_{4,\beta}$, see (5.19), definition (3.11) and the random walk formula (3.12), we have

$$\mu_{4,\beta}(x_1,x_2,x_3,x_4) = \sum_p F^c_\beta(x_{p(1)},x_{p(2)},x_{p(3)},x_{p(4)}) , \tag{7.1}$$

where

$$F^c_\beta(x_1,y_1,x_2,y_2) = \sum_{\substack{\omega_1:x_1\to y_1 \\ \omega_2:x_2\to y_2}} [z^N_{\beta,\lambda}(\omega_1,\omega_2) - z^N_{\beta,\lambda}(\omega_1)z^N_{\beta,\lambda}(\omega_2)] \tag{7.2}$$

For the standard, non-interacting random walk on the lattice, we set

$$z^N_{\beta,\lambda}(\omega_1,\omega_2) \quad \equiv \quad z^N_{\beta,\lambda}(\omega_1)z^N_{\beta,\lambda}(\omega_2)\chi_\phi(\omega_1,\omega_2) , \tag{7.3}$$

where χ_ϕ has been defined in (6.3), and

$$z^N_{\beta,\lambda}(\omega) = z^N_{\beta,\lambda_o}(\omega) ,$$

with $z^n_{\beta,\lambda_o}(\omega)$ given by (3.16). Thus, in the standard random walk model, $F^c_\beta(x_1,y_1,x_2,y_2)$ is proportional to the probability that a random walk going from x_1 to y_1 and a walk from x_2 to y_2 do not intersect, just as in the self-avoiding random walk model. In all cases we simplify our notation to

$$z_\beta(\omega) \ , \ z_\beta(\omega_1,\omega_2) .$$

Our results in section 6 are consequences of the following inequalities for the weights, z_β :

(A) If $\omega_1 \cap \omega_2 = \emptyset$ (i.e. ω_1 and ω_2 do not intersect) then

$$z_\beta(\omega_1,\omega_2) \geq z_\beta(\omega_1)z_\beta(\omega_2) \; ; \tag{7.4}$$

see [3].

(B) $$\sum_{\omega_2} z_\beta(\omega_1,\omega_2) \leq z_\beta(\omega_1) \sum_{\omega_2} z_\beta(\omega_2) \; , \tag{7.5}$$

see [2,3].

For the standard and the self-avoiding random walk models, the proofs of (7.4) and (7.5) are trivial. For $N = 1$, or 2 component field theories, the proofs are sketched in Appendix C. Note that the Lebowitz inequality,

$$\mu_{4,\beta} \leq 0 \; ,$$

follows directly from (7.1), (7.2) and (B). By (A),

$$F^c_\beta(x_1,y_1,x_2,y_2) \geq \sum_{\substack{\omega_1:x_1\to y_1\\ \omega_2:x_2\to y_2\\ \omega_1\cap\omega_2\neq\phi}} [z_\beta(\omega_1,\omega_2)-z_\beta(\omega_1)z_\beta(\omega_2)]$$

$$\geq - \sum_{\substack{\omega_1:x_1\to y_1\\ \omega_2:x_2\to y_2\\ \omega_1\cap\omega_2\neq\phi}} z_\beta(\omega_1)z_\beta(\omega_2) \quad ,$$

(and we have used that $z_\beta(\omega_1,\omega_2) \geq 0$, for all ω_1 and ω_2 .) Given ω_1 and ω_2 , let $z(\omega_1,\omega_2)$ be the last site on ω_1 , with respect to the natural ordering of the jumps made by ω_1 and thus of the sites visited by ω , where ω_1 intersects ω_2 . By making an error, E_1 , which vanishes in the continum limit, for $d \geq 2$, we can ignore walks ω_1 and ω_2 for which $z(\omega_1,\omega_2) \in \{x_1,y_1,x_2,y_2\}$. Thus

$$|F^c_\beta(x_1,y,x_2,y_2)| \leq \sum_{z \notin \{x_1,y_1,x_2,y_2\}} \sum_{\substack{\omega_1 : x_1 \to y_1 \\ \omega_2 : x_2 \to y_2 \\ z(\omega_1,\omega_2)=z}} z_\beta(\omega_1) z_\beta(\omega_2) + E_1$$

$$\leq \sum_{\substack{z \notin \{x_1,y_1,x_2,y_2\} \\ |z'-z| = 1}} \beta \sum_{\substack{\omega_1' : x_1 \to z \\ \omega_1'' : z' \to y_1 \\ \omega_2 : x_2 \to y_2 \\ \omega_2 \ni z}} z_\beta(\omega_1' \cdot \omega_1'') z_\beta(\omega_2) \chi_\phi(\omega_1'',\omega_2) + E_1 \quad (7.6)$$

where

$$|E_1| \leq \text{const.}\ \beta \{ G_\beta(x_1,x_2) G_\beta(x_2',y_1) G_\beta(x_2,y_2) + \ldots \} \quad (7.7)$$

with $|x_2 - x_2'| = 1$.

We denote by ω_2'' a part of ω_2 from z to y_2 and by ω_2' a part of ω_2 from x_2 to z'' , where z'' is a nearest neighbor of z , in such a way that

$$\omega_z' \circ (z'',z) \circ \omega_2'' = \omega_2 ,$$

$$\omega_2' \cdot \omega_2'' = \omega_2 ,$$

where $\omega_1 \circ \omega_2$ denotes the composition of the sequence of jumps corresponding to ω_1 with the sequence of jumps corresponding to ω_2, (providing the end-point of ω, agrees with the starting point of ω_2.) Clearly

$$\chi_\phi(\omega_1'',\omega_2) \leq \chi_\phi(\omega_1'',\omega_2'') \quad . \tag{7.8}$$

By summing independently over ω_2' and ω_2'' and using (7.7) and (7.8), we obtain

$$|F_\beta^c(x_1,y_1,x_2,y_2)| \leq \beta^2 \sum_{\substack{z\in\mathbb{Z}^d \\ |z'-z|=1 \\ |z''-z|=1}} \sum_{\substack{\omega_1':x_1\to z \\ \omega_1:z'\to y_1}} \sum_{\substack{\omega_2':x_2\to z'' \\ \omega_2'':z\to y}} z_\beta(\omega_1' \cdot \omega_1'') \cdot$$

$$\cdot\, z_\beta(\omega_2' \cdot \omega_2'')\chi_\phi(\omega_1'',\omega_2'') + E \tag{7.9}$$

where $E \equiv E_\beta\ (x_1,y_1,x_2,y_2)$ satisfies

$$|E_\beta(x_1,y_1,x_2,y_2)| \leq K_\beta \cdot \beta\{G_\beta(x_1,x_2)G_\beta(x_2',y_1)G_\beta(x_2,y_2)$$

$$+ (x_2 \to y_2) + (x_2 \to x_1) + (x_2 \to y_1)\}\ , \tag{7.10}$$

where K_β is a finite constant, and $|x_2-x_2'| = 1$. The error E_β vanishes in the continuum limit for $d \geq 2$; (to see this, one just applies the scale transformations of section 5).

For standard random walks,

$$z_\beta(\omega_1 \cdot \omega_2) = z_\beta(\omega_1) z_\beta(\omega_2) \ , \tag{7.11}$$

see (3.19), while for the lattice field theories and the self-avoiding random walk

$$z_\beta(\omega_1 \cdot \omega_2) = z_\beta(\omega_1, \omega_2) \ ,$$

by (3.15). Thus, applying (7.11), inequality (7.5) - see (B) - respectively, we obtain from (7.9)

$$|F^c_\beta(x_1,y_1,x_2,y_2)| \le \beta^2 \sum_{\substack{z \in \mathbb{Z}^d \\ |z'-z|=1 \\ |z''-z|=1}} G_\beta(x_1,z) G_\beta(x_2,z) \ \cdot$$

$$\cdot \ \{ \sum_{\substack{\omega: z' \to y_1 \\ \omega': z \to y_2}} z_\beta(\omega) z_\beta(\omega') \chi_\phi(\omega,\omega') \} + E \ , \tag{7.12}$$

and we have resummed over ω_1', ω_2', using

$$G_\beta(x,y) = \sum_{\omega: x \to y} z_\beta(\omega) \ .$$

The error $E = E_\beta(x_1,y_1,x_2,y_2)$ is given by (7.10). From that estimate it follows immediately that

$$m(\beta)^d \chi(\beta)^{-2} \sum_{y_1,x_2,y_2} E_\beta(x_1,y_1,x_2,y_2) \longrightarrow 0 \ , \tag{7.13}$$

as $\beta \nearrow \beta_c$, in dimension $d \geq 3$; see also [3,5]. We shall therefore omit the error term henceforth. If, on the r.s. of (7.12), we insert the trivial inequality

$$\chi_\phi(\omega,\omega') \leq 1$$

and resume over ω and ω' we obtain inequality (6.6).

Next, we sum both sides of (7.12) over y_1,x_2,y_2 , keeping x_1 fixed. Using the definition (6.2) of $Q(\beta)$ and (6.4) , we obtain

$$m(\beta)^d \chi(\beta)^{-2} \sum_{y_1,x_2,y_2} |F^c(x_1,y_1,x_2,y_2)|$$

$$\leq \beta^2 \, m(\beta)^d \chi(\beta)^2 \, Q(\beta) \ , \qquad (7.14)$$

up to a term which, by (7.13), tends to 0 as $\beta \nearrow \beta_c$, in $d \geq 3$. Applying (7.14) to all three terms in equation (7.1) for $\mu_{4,\beta}$, we obtain

$$g_r(\beta) \equiv |\overline{\mu}_{4,\beta}| \chi(\beta)^{-2} \, m(\beta)^d$$

$$\leq 3\beta^2 \, m(\beta)^d \, \chi(\beta)^2 \, Q(\beta) \qquad (7.15)$$

where $$\overline{u}_{4,\beta} \equiv \sum_{\substack{x_j \in \mathbb{Z}^d \\ j=2,3,4}} u_{4,\beta}(0,x_2,x_3,x_4) \ ,$$

(up to a term vanishing, as $\beta \nearrow \beta_c$, in three or more dimensions.) We note that (7.15) is the desired inequality (6.14).

Finally, we turn to our proof of (6.18), i.e. for $N = 0$,

$$Q(\beta) \leq (2d)^{-1} \frac{\partial}{\partial\beta} \chi^{-1}(\beta) . \tag{7.16}$$

By (4.1), (4.2) and (4.3), we have

$$\chi(\beta) \equiv \sum_{x \in \mathbb{Z}^d} G_\beta(0,x)$$

$$= \sum_{x \in \mathbb{Z}^d} \sum_{\omega:0\to x} \beta^{|\omega|} \prod_{j \in \mathbb{Z}^d} p(n_j(\omega)) ,$$

where $p(n)$ is independent of β. Thus, for $\beta < \beta_c$

$$\frac{\partial}{\partial\beta} \chi(\beta) = \sum_{x \in \mathbb{Z}^d} \sum_{\substack{z \in \mathbb{Z}^d \\ z':|z'-z|=1}} \sum_{\substack{\omega=\omega_1 \circ (z,z') \circ \omega_2 \\ \omega_1:0\to z \\ \omega_2:z'\to x}} \beta^{|\omega_1|+|\omega_2|} \cdot$$

$$\cdot \prod_{j\in\mathbb{Z}^d} p(n_j(\omega_1 \cdot \omega_2)) .$$

If $\omega_1 \cap \omega_2 = \emptyset$

$$p(n_j(\omega_1 \cdot \omega_2)) = p(n_j(\omega_1))p(n_j(\omega_2)) .$$

Thus,

$$\frac{\partial}{\partial\beta} \chi(\beta) \geq \sum_{\substack{x,z \\ z':|z'-z|=1}} \sum_{\substack{\omega_1:0\to z \\ \omega_2:z'\to x}} \beta^{|\omega_1|} \prod_{j \in \mathbb{Z}^d} p(n_j(\omega_1)) \cdot$$

$$\cdot\ \beta^{|\omega_1|} \prod_{j\in\mathbb{Z}^d} p(n_j(\omega_2))\chi_o(\omega_1,\omega_2) \quad .$$

But

$$\beta^{|\omega|} \prod_{j\in\mathbb{Z}^d} p(n_j(\omega)) = z^o_{\beta,\lambda}(\omega) \quad ,$$

hence, using the definition (6.2) of $Q(\beta)$, we obtain

$$\frac{\partial}{\partial\beta}\,\chi(\beta) \geq (2d)\ \chi(\beta)^2\ Q(\beta) \ ,$$

whence (7.16) .

§8. The numerical data.

Our numerical results were obtained by means of a Monte-Carlo simulation which used the Metropolis algorithm [25]. We generated a sequence of self-avoiding random walks by repeatedly applying a set of elementary local deformations (an idea originating in our study of lattice string theories) and checking, at every step, that the nonlocal constraint that the walk be self-avoiding was respected. The transition probability T between consecutive members ω , ω' of the sequence was taken to be :

$$T(\omega \to \omega') = \frac{1}{|\omega|} \; P(\Delta|\omega|)\chi_{sAw}(\omega') \tag{8.1}$$

where $\Delta|\omega| = |\omega'| - |\omega|$ and $\chi_{sAw}(\omega')$ is unity if ω' is self-avoiding, and zero otherwise. The probability $P(\Delta|\omega|)$ of each deformation was taken to depend only on the change of length of the walks and assigned as in ref [26], where such a procedure was first tested numerically. Thus (see Figure 2)

$$\begin{aligned} P(+2) &= \frac{\xi^2}{[1+(2d-3)\xi^2]} \\ P(0) &= \frac{(1+\xi^2)}{2[1+(2d-3)\xi^2]} \\ P(-2) &= \frac{1}{[1+(2d-3)\xi^2]} \end{aligned} \tag{8.2}$$

The factor $\frac{1}{|\omega|}$ in (8.1) corresponds to the arbitrary choice of the link where the deformation should act.

In more conventional approaches to lattice field theories, one assigns the degrees of freedon of the field in question to every lattice point (link). Each configuration of the system is specified by a number of variables proportional to the size of the lattice. Computer memory limits one to lattices of the order of 10^4 points. On the other hand, the method we used only stores points along the walk. We could thus code the four components of each point into a single-precision word (32 binary digits), which allows each component to be as large as 2^8. The length of the walks is then limited by the number of (single-precision) words in the available memory, which means that we may have lengths of thousands of steps. Even in regions which were sufficiently close to the critical point for scaling behavior to be observed, typical walks (a few hundred steps) never touched the boundaries of our lattice. As we plunged deeper and deeper into the critical region, the only consideration that we had to keep in mind was the time of approach to equilibrium. Within very reasonable computer times we were able to achieve very high precision, an essential requirement, as our aim consisted of measuring logarithmic deviations from mean field behavior. It is clear that the $N = 0$ case introduces an enormous simplification in the calculation. Had we treated the field theory case using a random walk description, the calculation of the weights carried by each point of the curve (which here is just a constant) would have to involve a determinant viewed as an expansion in terms of closed walks (see Appendix A).

In four dimensions, the upper critical dimensionality for the ferromagnetic vector model, logarithmic corrections to scaling are expected from the perturbative solutions of the Callan-Symanzik renormalization group equations [16, 17]. However, they were not yet tested in a nonperturbative context; we are able to provide such a test with the help of our numerical simulations. Thus, we measured, for a different values of the parameter ξ (see §2), the mean length of curves with fixed end points :

$$< |\omega| > = \frac{\sum_{\omega:0\to x} |\omega|\xi^{|\omega|}}{\sum_{\omega:0\to x} \xi^{|\omega|}} = \frac{d}{d\ln\xi} \ln\Big(\sum_{\omega:0\to x} \xi^{|\omega|}\Big) \underset{|x|\to\infty}{\sim} -\text{const.}\ \frac{dm}{d\ln\xi}\,, \quad (8.3)$$

where m is the inverse correlation length, see (5.1), and the last equality follows from the identification :

$$G_\xi(0,x) = \sum_{\omega:0\to x} \xi^{|\omega|} \ . \qquad (8.4)$$

Letting $\xi = \beta \equiv e^{-b}$ and $\tau_b \equiv (b/b_c)-1$, an ansatz of the form :

$$m(\tau_b) \sim \tau_b^{1/2}\ |\ln\tau_b|^{-N} \qquad (8.5)$$

together with (8.3) allowed us to determine from the data listed in Table I that $N = \cdot 14 \pm \cdot 03$. This value is to be compared with the renormalization group prediction of (1/8). Figures 3 and 4 show a linear and a logarithmic plot of the data and best fits obtained from mean field theory $(m(\tau_b) \sim \tau_b^{1/2})$ and from (8.5).

Encouraged by these results, we set out to gather numerical data on the behavior of $Q(\beta)$, defined in (6.2), as $\beta \nearrow \beta_c$. In fact, we actually measured a related quantity $\tilde{Q}$; defined as :

$$\tilde{Q}(\beta;y_1,y_2) = \frac{\displaystyle\sum_{|j|=1} \sum_{\substack{\omega_1:0\to y_1 \\ \omega_2:j\to y_2}} \chi_\phi(\omega_1,\omega_2) z_\beta(\omega_1) z_\beta(\omega_2)}{\displaystyle\sum_{|j|=1} \sum_{\substack{\omega_1:0\to y_1 \\ \omega_2:j\to y_2}} z_\beta(\omega_1) z_\beta(\omega_2)} \qquad (8.6)$$

To compute $\tilde{Q}$ numerically, we generated sequences of two self-avoiding walks that started from the origin and ended at two distinct points, y_1 and y_2. We then recorded the fraction of the total number of iterations corresponding to configurations in which the two curves did not intersect. The results of our measurements, for two different choices of the endpoints, are shown on Table II. Assuming a scaling law for $\tilde{Q}(\beta;y_1,y_2)$ of the type :

$$\tilde{Q}(\beta;y_1,y_2) \overset{\beta\nearrow\beta_c}{\sim} C(y_1,y_2) \quad |\ln\tau|^{-\tilde{G}} \tag{8.7}$$

where $\tilde{G}$ is independent of the endpoints, and using the relation :

$$\chi^2(\beta)Q(\beta) = \sum_{y_1,y_2} \sum_{j=1} \tilde{Q}(\beta;y_1,y_2)\, G_\beta(0,y_1)G_\beta(j,y_2) \tag{8.8}$$

which follows from (8.6), we can easily obtain :

$$\tilde{Q}(\beta) \overset{\beta\nearrow\beta_c}{\sim} \{\chi^{-2}(\beta) \sum_{y_1,y_2} \sum_{|j|=1} C(y_1,y_2)G_\beta(0,y_1)G_\beta(j,y_2)\}|\ln\tau|^{-\tilde{G}} . \tag{8.9}$$

The expression in the curly brackets tends to some constant as $\beta\nearrow\beta_c$, which permits us to identify, using (6.20) :

$$G = \tilde{G} . \tag{8.10}$$

For the case of the self-avoiding random walk we may, as shown in sections 6, 7, relate $Q(\beta)$ to the susceptibility $\chi(\beta)$ via (6.18), where the equality holds. Thus for $\tilde{Q}$, in view of (8.9); we have :

$$\tilde{Q}(\beta) \sim Q(\beta) = -(2d)^{-1} \frac{\partial}{\partial\beta} \chi^{-1}(\beta) \quad . \tag{8.11}$$

Resorting again to the renormalization group equations one can try to fit the data with an expression such as :

$$\tilde{Q}(\tau;y_1,y_2) = C(y_1;y_2) \, \{|\ell n\tau| - G\ell n|\ell n\tau|\}^{-G} \quad , \tag{8.12}$$

which results from the relation $-\frac{\partial\chi^{-1}}{\partial\tau} \sim |\ell n\chi^{-1}|^{-G}$, whose solution yields $\chi^{-1} \sim \tau|\ell n\tau|^{-G}$ (perturbatively one finds $E = 0$, thus $G = 2N$) . From the data in Table II, we obtain (instead of β and τ , our fits were made for b and τ_b , as in (8.5)) :

$$C(y_1,y_2) = \cdot 89 \pm\cdot 02 \quad G = \cdot 24 \pm\cdot 02 \qquad \begin{array}{l} y_1 = (5,5,0,0) \\ \\ y_2 = (5,-5,0,0) \end{array} \quad . \tag{8.13}$$

The perturbative prediction of the renormalization group in $G = 1/4$.

The dependence of $\tilde{Q}$ on the choice of endpoints helps considerably. A mere inspection of the data could not rule out the possiblity that $\tilde{Q}$ could tend to some constant $C' \in (0,1]$. One could try a fit of the form :

$$\tilde{Q}(\tau;y_1,y_2) = C(y_1,y_2) \, \{|\ell n\tau| - G\ell n|\ell n\tau|\}^{-G} + C' \quad . \tag{8.14}$$

If G is not too large, this would be numerically indistinguishable from :

$$\tilde{Q} \simeq (C+C') - (GC)\ell n\{|\ell n\tau| - G\ell n|\ell n\tau|\} \approx (C+C') - (GC)\ell n|\ell n\tau| \quad . \tag{8.15}$$

The best one can make is to determine the combinations $(C+C')$ and (GC), which leaves us with one free parameter, and allows for a number of equally good fits. To cope with this problem, one can then use the dependence of $\tilde{Q}$ on the choice of endpoints. In (8.14) we have assumed the scaling law (8.7), and the independence of C' on the endpoints. Should $\tilde{Q}$ tend to a nonzero value as $\tau \to 0$, such a value would be connected to the behavior at the critical point, where curves are infinitely long and thus, insensitive to the endpoints. Taking the ratio :

$$R(\tau;\text{ endpoints}) \equiv \frac{\tilde{Q}(\tau;y_1,y_2)}{\tilde{Q}(\tau;y_1',y_2')} \qquad (8.16)$$

a value of $C' \neq 0$ would imply that R should depend on τ. Our data, however, show this ratio to be independent of τ, to a very good accuracy as $\tau \to 0$. Using $y_1' = (10,1,0,0)$, $y_2' = (10,-1,0,0)$ as endpoints, we computed (8.16) for three different values of τ and found it to be $1.57 \pm .03$. This tends to support our contention that $\tilde{Q} \to 0$, thus $Q \to 0$, as $\tau \to 0$ [27].

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TABLE I

(Values of $|<\omega>|$ versus b).

b	$<|\omega|>^{(1)}$	$<|\omega|>^{(2)}$
2.100	19.99	19.78
2.060	21.81	21.66
2.021	24.19	24.38
2.002	26.01	26.27
1.984	28.23	28.65
1.966	32.61	32.00
1.948	37.61	37.23
1.931	46.21	46.35

(1) Values obtained from the Monte-Carlo similation.

(2) Best fit to the data assuming

i) $m(\tau_b) = C\,\tau_b^{1/2}|\ln\tau_b|^{-N}$

ii) $\tau = \dfrac{b-b_c}{b_c}$

$C = 12.69 \pm .13$; $b_c = 1.904 \pm .002$; $N = .14 \pm .03$

TABLE II

Values of $\widetilde{Q}(\tau_b;y_1,y_2)$ versus b .

b	$\widetilde{Q}^{(1)}$	$\widetilde{Q}^{(2)}$
2.021	·681	·681
2.002	·671	·671
1.984	·663	·661
1.966	·645	·648
1.948	·635	·634

(1) Values obtained from the Monte-Carlo simulation.

(2) Best fit to the data assuming $b_c = 1.904$ and

$$\widetilde{Q}(\tau_b;y_1,y_2) = C(y_1,y_2)(|\ell n\tau_b| - G\ell n|\ell n\tau|)^{-G}$$

$C(y_1,y_2) = {\cdot}89 \pm {\cdot}02$ $\qquad$ $G = 24 \pm {\cdot}02$.

Appendix A : The $N \to 0$ limit.

Let us consider the random walk representation of paragraph 3. The expressions (3.9) and (3.10) for $z^N_{\beta,\lambda}$ may be rewritten as :

$$z^N_{\beta,\lambda}(t) = Z^{-1}_{\beta,\lambda} \int \prod_j da(j)\ \hat{f}(a(j)) e^{-2ia(j)t(j)}$$

$$\int e^{-1/2(\vec{\phi},(\beta P+2ia)\vec{\phi})} \prod_j d^N\phi(j) \tag{A.1}$$

$$z^N_{\beta,\lambda}(\omega) = Z^{-1}_{\beta,\lambda}\ \beta^{|\omega|} \int \prod da(j) \hat{f}(a(j))(2ia(j))^{-n_j(\omega)} \det{}^{-N/2}(\beta P+2ia) \tag{A.2}$$

In (A.1), we have just reintroduced the Fourier decomposition of f , whereas in (A.2), we have made use of (3.7) and identified the Gaussian integral of (A.1) with the determinant. We can once more appeal to the random walk representation to express this determinant as [2] :

$$\det{}^{-N/2}(\beta P+2ia) = (\prod_j 2ia(j))^{-N/2} \exp[\sum_{z\in\mathbb{Z}^d} \sum_{\omega':z\to z} \frac{N}{2|\omega'|} \prod_\ell (2ia(\ell))^{-n_\ell(\omega')}] \tag{A.3}$$

We may then insert (A.3) into (A.2) . The resulting expression displays N as a parameter, and it is not hard to show that $z^N_{\beta,\lambda}(\omega)$ is analytic around $N = 0$, (provided we work in finite volume, or keep β small; see e.g. [9]). We shall examine the $N \to 0$ limit of $z^N_{\beta,\lambda}$ and for that purpose we expand the exponential to obtain :

$$z^N_{\beta,\lambda}(\omega) = Z^{-1}_{\beta,\lambda} \beta^{|\omega|} \int \prod_j da(j) \hat{f}(a(j))(2ia(j))^{-n_j(\omega)-N/2} \quad .$$

$$\cdot \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{N}{2}\right)^k \sum'_{\omega_1 \ldots \omega_k} \frac{1}{\omega_1 \ldots \omega_k} \prod_{\ell} (2ia(\ell))^{-[n_\ell(\omega_1)+\ldots+n_\ell(\omega_k)]} \tag{A.4}$$

where $\sum'_{\omega} \equiv \sum_{z\in\mathbb{Z}^d} \sum_{\omega:z\to z}$. Each term in the expansion corresponds to a sum over k closed random walks, i.e., a polymer gas. However, in the $N \to 0$ limit only the $k = 0$ term will contribute, yielding :

$$z^{o}_{\beta,\lambda}(\omega) = \lim_{N\to 0} z^{N}_{\beta,\lambda}(\omega) = \beta^{|\omega|} \prod \lim_{N\to 0} \frac{\int \hat{f}(a(j))(2ia(j))^{-n_j(\omega)-N/2} da(j)}{\int \hat{f}(a(j))(2ia(j))^{-N/2} da(j)} =$$

$$= \beta^{|\omega|} \prod \lim_{N\to 0} \frac{\int d\nu_{n_j(\omega)+N/2}(t)\ f(2t)}{\int d\nu_{N/2}(t)\ f(2t)} \tag{A.5}$$

where the last equality follows from (3.7). The expression for the two-point correlation in the $N \to 0$ limit becomes :

$$\lim_{N\to 0} < \phi^1(x)\phi^1(y) > = \sum_{\omega:x\to y} z^{o}_{\beta,\lambda}(\omega) \tag{A.6}$$

Using the notation introduced in (4.2),

$$z^{o}_{\beta,\lambda}(\omega) = \beta^{|\omega|} \prod_{j\in\omega} p(n_j(\omega)) \tag{A.7}$$

we shall now compute $p(n)$ for $g|\vec{\phi}|^4$-theory and the nonlinear σ-model.

a) $g|\vec{\phi}|^4$-theory.

$$f(|\phi|^2) = \exp[-\frac{g}{4}|\vec{\phi}|^4 + \frac{(2d\beta+m_o^2)}{2}|\vec{\phi}|^2 + C]$$

$$p(n) = \lim_{N\to 0} \frac{\int d\nu_{n+N/2}(t)\ e^{-(gt^2+[2d\beta+m_o^2]t)}}{\int d\nu_{N/2}(t)\ e^{-(gt^2+[2d\beta+m_o^2]t)}} \quad . \tag{A.8}$$

Recalling from (3.6) that $d\nu_o(t) = \delta(t)dt$ we immediately obtain :

$$p(n) = \int d\nu_n(t)\, e^{-(gt^2+[2d\beta+m_o^2]t)} \quad . \tag{A.9}$$

Inserting this into (A.6) , (A.7), we obtain the propagator for the Edwards model of self-suppressing random walks. Letting $g = 0$ and using (3.7) we obtain, for the simple random walk :

$$p(n) = \frac{1}{(2d\beta+m_o^2)^n} \quad . \tag{A.10}$$

b) The nonlinear σ-model.

$$f(|\vec{\phi}|^2) = \delta(|\vec{\phi}|^2-N)$$

$$p\ (n) = \lim_{N\to 0} \frac{\int d\nu_{n+N/2}(t)\ \delta(2t-N)}{\int d\nu_n(t)\ \delta(2t-N)} \quad . \tag{A.11}$$

Using (3.6), we integrate numerator and denominator by means of :

$$\int d\nu_k(t)\ \delta(2t-N) = (\tfrac{1}{2})^k\ \frac{N^{k-1}}{\Gamma(k)} \tag{A.12}$$

after cancellations we arrive at :

$$p(n) = \lim_{N\to 0} \frac{(N/2)^n\ \Gamma(N/2)}{\Gamma(n+N/2)}\quad . \tag{A.13}$$

This vanishes unless $n=0,1$, in which case we may use the identity $x\Gamma(x) = \Gamma(x+1)$ to establish :

$$p(n) = \delta_{n1} + \delta_{no}$$

Therefore in (A.7), (A.8), we have the restriction that each site of the walk ω cannot be visited more than once. This corresponds to the self-avoiding random walk.

Appendix B : relations among critical exponents.

Let us first consider the susceptibility of the continuum theory. By definition :

$$\chi^{(\infty)} = \lim_{\theta\to\infty} \chi^{(\theta)} \equiv \lim_{\theta\to\infty} \sum_{x\in \mathbb{Z}^d_{\theta^{-1}}} \theta^{-d} G^{(\theta)}(0,x) =$$

$$= \lim_{\theta\nearrow\infty} \theta^{-d}\, \alpha^2(\theta) \sum_{x\in\mathbb{Z}^d} G_{\beta(\theta)}(0,x) \tag{B.1}$$

where $\sum_{x\in\mathbb{Z}^d} G_{\beta(\theta)}(0,x) \equiv \chi(\beta)$ is the susceptibility of the lattice theory at inverse temperature $\beta(\theta)$.

The expected behavior of such a quantity near the critical temperature is the power law (5.4) for $d \neq 4$. However, for $d = 4$ one might have expression (5.7), which admits a logarithmic deviation in the form :

$$\chi(\beta) \sim \tau^{-\gamma} |\ell n\tau|^{G} \tag{B.2}$$

analogously we may write the solution of eq. (5.12) near the critical point as

$$\tau(\theta) \sim \theta^{-1/\gamma}\ |\ell n\theta|^{N/\nu} \tag{B.3}$$

with $N \neq 0$ only for $d = 4$, where $\nu = 1/2$. In this way, requiring that the limit in (B.1) exist

$$\chi^{(\infty)} \sim \alpha^2(\theta)\theta^{-d}\ \theta^{\gamma/\nu} |\ell n\theta|^{G-\gamma N/\nu}$$

$$\sim \theta^{-2+\eta+\gamma/\nu} \, |\ell n\theta|^{E+G-\gamma N/\nu}$$

implies, at the same time, the relations

$$\gamma = \nu(2-\eta) \tag{B.4}$$

and its extension for the logarithmic deviations in $d = 4$

$$G = 2N - E \tag{B.5}$$

Furthemore, one can relate the exponents of the susceptibility for the S.A.W. to the exponents which appear in the counting problem for chains with L steps : if $N(L)$ is the number of such chains, asymptotically for L large

$$N(L) \sim \mu^L \, L^\rho \, |\ell nL|^R \, . \tag{B.6}$$

Then

$$\chi(\beta) = \sum_{x\in\mathbb{Z}^d} \; \sum_{\omega:0\to x} \xi^{|\omega|} = \sum_{L=0}^{\infty} N(L) \, \xi^L \, .$$

From this identity, we obtain the asymptotic relations

$$\mu = \xi_c^{-1} \tag{B.7}$$

$$\rho+1 = \gamma \tag{B.8}$$

$$R = G \tag{B.9}$$

A quantity often studied in polymer physics is the so called mean end-to-end distance, which is the square root of the mean square distance for paths of length L . Once again, let us introduce indices for the asymptotic behavior of this quantity, which is expected to diverge as $L^{1/\delta}|\ln L|^{\mathcal{D}}$, where δ is in fact the Hausdorff dimension of these chains, and $\mathcal{D}$ measures violations to the self-similar behavior on different scales of the chain.

Let us also consider another quantity. If $r^2 = \sum_{i=1}^{d} x_i^2$

$$\sum_{x\in\mathbb{Z}^d} r^2 G_\beta(0,x) = \sum_{x\in\mathbb{Z}^d} \sum_{L=0}^{\infty} \sum_{\substack{\omega:0\to x\\ |\omega|=L}} r^2 \xi^L \sim$$

$$\sim \sum_L N(L)\xi^L \; L^{2/\delta} \; |\ln L|^{2\mathcal{D}} \quad \sim$$

$$\sim \sum_L L^{\rho+2/\delta}|\ln L|^{\mathcal{R}+2\mathcal{D}}\xi^L \tag{B.10}$$

On the other hand

$$\sum_{x\in\mathbb{Z}^d} r^2 \; G_\beta(0,x) \sim \sum_r r^{d-1} \; r^2 \; \frac{e^{-m(\beta)r}}{r^{d+2+\eta}|\ln r|^{\mathcal{E}}} \sim$$

$$\sim m(\beta)^{\eta-4}|\ln \, m(\beta)|^{-\mathcal{E}} \sim [\tau^{\nu}|\ln\tau|^{-\mathcal{N}}]^{\eta-4} \; |\ln\tau|^{-\mathcal{E}} \tag{B.11}$$

so that, by comparing (B.10) and (B.11), we obtain

$$\rho+1+2/\delta = \nu(4-\eta) \tag{B.12}$$

and for $d = 4$

$$\mathcal{R} + 2\mathcal{D} = 4\,\mathcal{N} - \mathcal{E} \tag{B.13}$$

which by (B.4), (B.8) and (B.5), (B.9) reduce to

$$\delta = 1/\gamma \tag{B.14}$$

$$\mathcal{D} = \mathcal{N} \quad . \tag{B.15}$$

It is worthwhile noting that the exponents of the mean end-to-end distance are functions only of the exponents of the correlation length.

Appendix C : Inequalities for the z's

We shall rederive here [3] the inequalities (7.4) and (7.5) which are the fundamental ingredients for the estimates contained in section 7.

Let us first consider the case (A), in which the two walks ω_1 and ω_2 do not intersect. We shall show

$$z(\omega_1,\omega_2) \geq z(\omega_1)z(\omega_2) \quad . \tag{C.1}$$

(In this appendix, we shall omit the super (sub)-scripts for the z's) .
By (3.10) this follows from

$$z(t_1+t_2) \geq z(t_1)z(t_2) \quad . \tag{C.2}$$

If we define

$$\mathring{z}(t) = z(t) \prod_j e^{gt^2(j)-\mu^2 t(j)} \tag{C.3}$$

and
$$F(x,y) = \ell n\, \mathring{z}\,(xt+ys) \tag{C.4}$$

then
$$F(1,1) = \int_0^1 dx \frac{\partial}{\partial x} F(x,1) + F(0,1) =$$

$$= \int_0^1 dx \frac{1}{\mathring{z}(xt+s)} \frac{\partial}{\partial x} \mathring{z}(xt+s) + \ell n\, \mathring{z}(s) \quad . \tag{C.5}$$

Thus we have

$$\frac{1}{\mathring{Z}(xt+s)} \frac{\partial}{\partial x} \mathring{z}(xt+s) = -g \sum_j t^2(j) < |\vec{\phi}(j)|^2 >_{(tx+s)} \tag{C.6}$$

with $< \cdot >_{(t)}$ denoting an average value in the measure

$$\frac{1}{Z(t)} e^{-\beta H(\vec{\phi})} \prod_j f(|\vec{\phi}(j)|^2) e^{-g|\vec{\phi}(j)|^2 t(j)} d^N\phi(j) \tag{C.7}$$

where $Z(t)$ is, as usual, a normalization factor. However,

$$\frac{\partial}{\partial y} < |\vec{\phi}(j)|^2 >_{(xt+ys)} = -g \sum_k s(k) < |\vec{\phi}(j)|^2 ; |\vec{\phi}(k)|^2 >_{(xt+ys)} \leq 0 \tag{C.8}$$

for all values of y, as

$$< |\vec{\phi}(j)|^2 ; |\vec{\phi}(k)|^2 > \geq 0 \tag{C.9}$$

is one of the Griffiths inequalities. So, we arrive at a lower bound if we set $s = 0$ in (C.6). Thus, using (C.5), we obtain

$$\ln \mathring{z}(t+s) \geq \ln \mathring{z}(t) + \ln \mathring{z}(s) \tag{C.10}$$

which through exponentiation and the definition of $\mathring{z}$ implies

$$z(t+s)e^{-2g\sum_j t(j)s(j)} \geq z(t)z(s) \tag{C.11}$$

when the two curves do not intersect $\sum_j t(j)s(j) = 0$ and (C.1) follows.

For the inequality (B) let us first observe that the expectation $< \phi^1(x)\phi^1(y) >_{(t)}$ has a random walk representation :

$$< \phi^1(x)\, \phi^1(y) >_{(t)} = \frac{1}{z(t)} \sum_{\omega : x \to y} \int d\nu_\omega(s)\, z(t+s) \quad . \tag{C.12}$$

So that

$$\sum_{\omega_2 : x \to y} z(\omega_1, \omega_2) = \sum_{\omega_2 : x \to y} \int d\nu_{\omega_1}(t_1) d\nu_{\omega_2}(t_2) z(t_1 + t_2) =$$

$$= \int d\nu_{\omega_1}(t_1)\, z(t_1) < \phi^1(x)\phi^1(y) >_{(t_1)} \tag{C.13}$$

but
$$< \phi^1(x)\phi^1(y) >_{(t)} \leq < \phi^1(x)\phi^1(y) > \tag{C.14}$$

by Ginibre's inequality and as

$$z(\omega_1) = \int d\nu_{\omega_1}(t_1)\, z(t_1)$$

$$\sum_{\omega_2 : x \to y} z(\omega_2) = < \phi^1(x)\phi^1(y) >$$

inequality (B) follows, that is

$$\sum_{\omega_2} z(\omega_1, \omega_2) \leq z(\omega_1) \sum_{\omega_2} z(\omega_2) \ . \tag{C.15}$$

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FIGURE CAPTIONS

Figure 1 - Critical line for $g|\vec{\phi}|^4$ theories.

Figure 2 - Elementary deformations used in the Monte-Carlo procedure.

Figure 3 - Linear plot of $< |\omega| >$ vs b . The full curve is our best-fit, whereas the dotted one corresponds to a mean field fit (no log corrections) with $b_c = 1.904$

Figure 4 - Log plot of $< |\omega| >$ vs b .

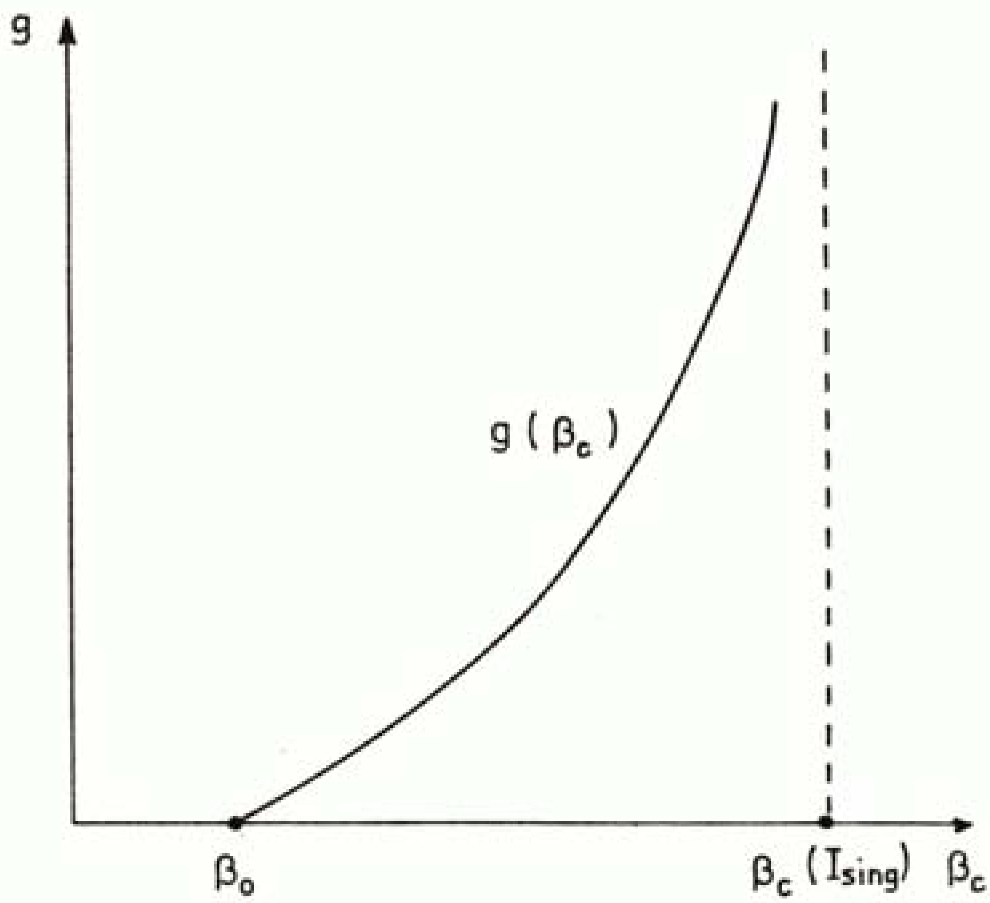

Fig. 1

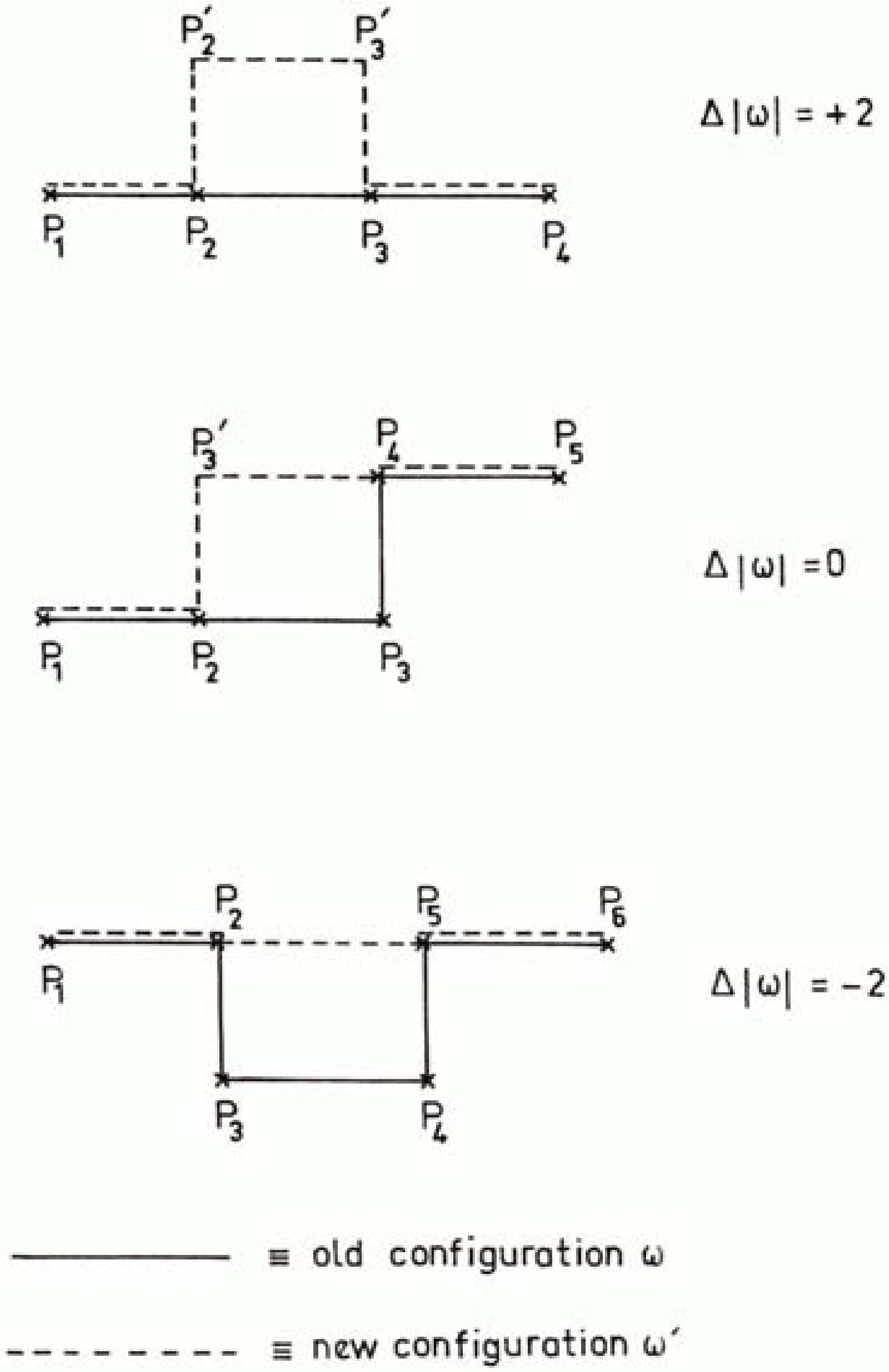
P'2
P'3
P1
P2
P3
P4
Δ|ω| = +2
P'3
P4
P5
P1
P2
P3
Δ|ω| = 0
P2
P5
P6
P1
P3
P4
Δ|ω| = −2
≡ old configuration ω
≡ new configuration ω'

Fig. 2

⟨|ω|⟩ vs b

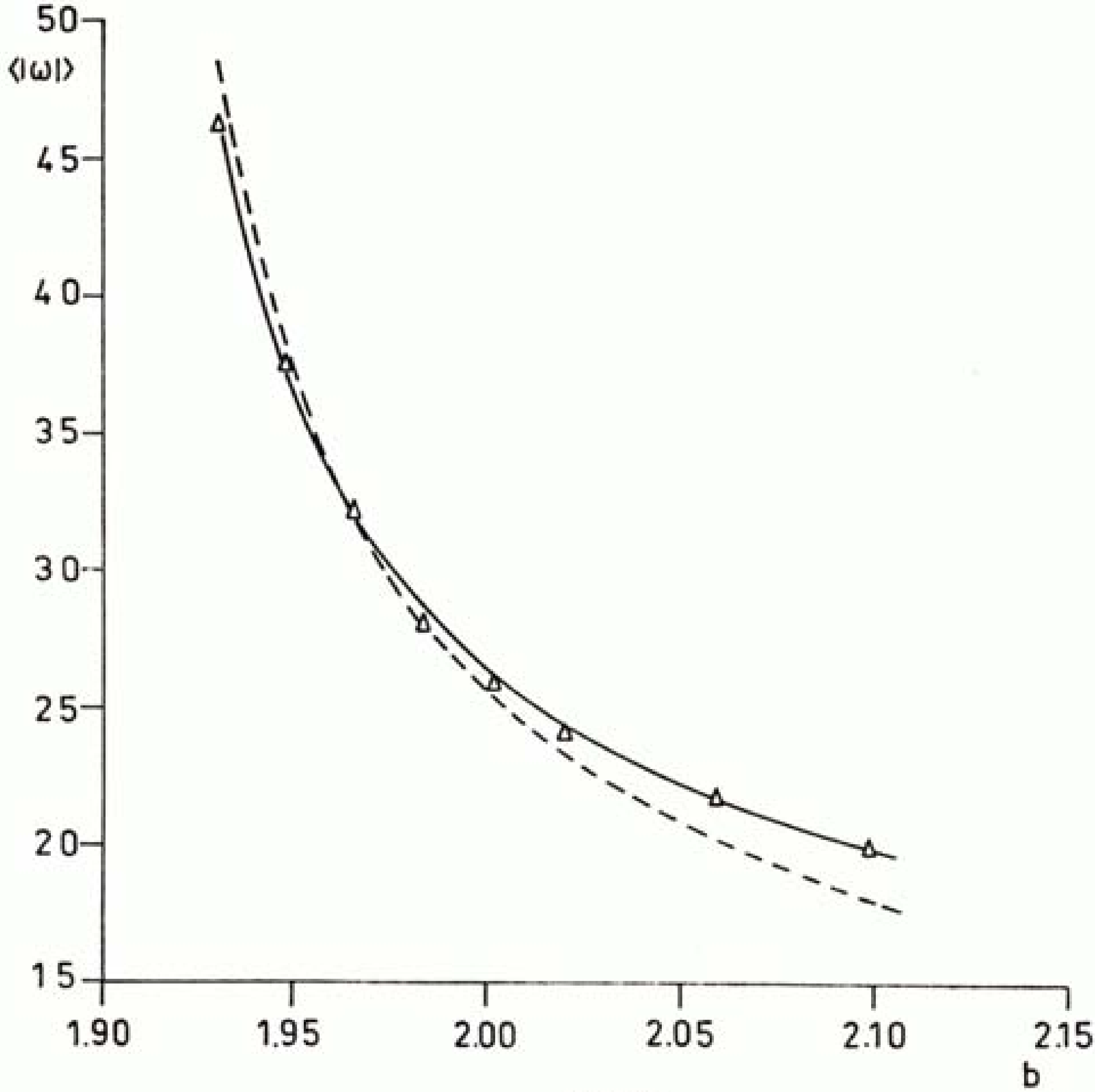

Fig.3

ln ⟨|ω|⟩ vs ln τ_b

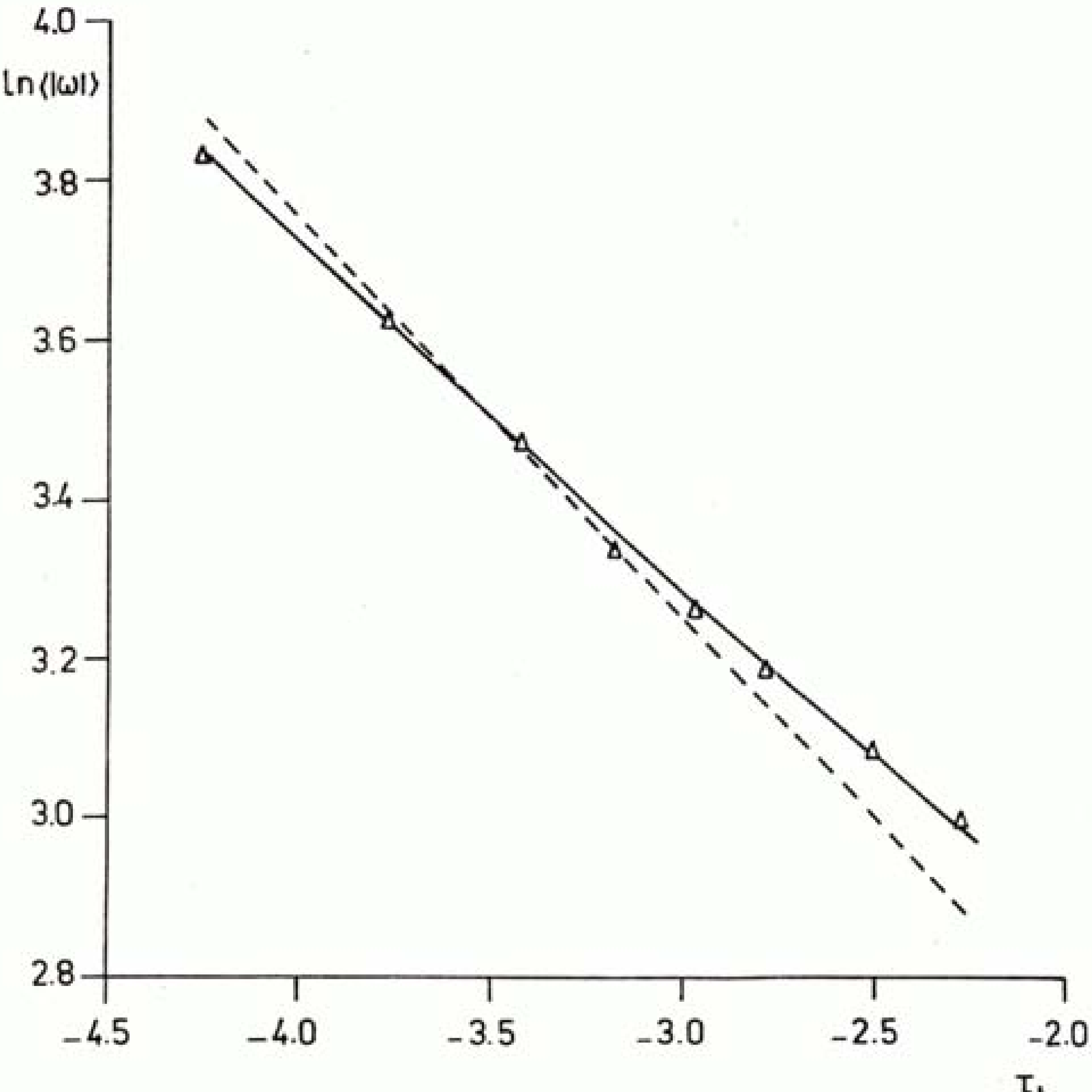

Fig. 4

to those we had in the (quasi-) linear case. We have to ensure that the data g_{AB} are of class $W^{2m+2}(H)$ in order to obtain a $W^m(M)$ solution. But the transformation to harmonic coordinates $x^{\alpha'}$ requires a $\partial_a g_{ab}$ of class $W^{2m-1}(H_1)$, $\partial_a \partial_\beta x^{\alpha'} \in W^{2m-1}(H_1)$, and the $g_{ab,1}$ needs a $g_{ab,\alpha\beta}$. As usually in our propagation equations on H_i the calculation of $\omega \in W^{2m+1}$ requires a $\partial^2 h_{\alpha\beta} \in W^{2m}$, so we finally have the following:

In order to obtain a $g \in W^m(M)$ we have to require

$$h_{\alpha\beta} \in W^{2m+2}(H), \omega \in W^{2m+1}(H_\cap), \partial\omega \in W^{2m}(H_\cap)$$

A beautiful argument of Choquet-Bruhat [20] on maximal developments (as described also in [12; p. 251]) applies to the characteristic as well as to the Cauchy problem. It shows that for data in W^{m+r}_{data} $(r > 0)$ the maximal W^m and W^{m+r} developments coincide; hence for C^∞ data the solution is C^∞ in the whole W^4 development.

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Note Added in Proof:

In a forthcoming paper in which we shall extend our results to the perfect fluid case we shall use coordinate conditions which are less stringent than those in section 6.3.

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