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SPONTANEOUSLY BROKEN AND DYNAMICALLY ENHANCED GLOBAL AND LOCAL SYMMETRIES

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Abstract.

I re-examine the notions of spontaneously broken, global and local symmetries and discuss them in terms of some examples in quantum field theory or statistical mechanics. I then briefly recall some basic ideas and facts about the renormalization group. They are used to introduce and discuss the concept of dynamically enhanced (or "generated") asymptotic symmetries.

Contents.

Remark. Most of the ideas and results described in these notes have been worked out in collaboration with T. Spencer to whom I am deeply indebted for his clear insights and his generosity. I have also benefitted from collaborations with D. Brydges, G. Morchio, C. Pfister, E. Seiler, B. Simon and F. Strocchi to whom I extend my gratitude.

These notes are intended for light reading. The interested reader is advised to consult the following references for further study :

Sect. 1 : [1,2,3,4] .
Sect. 2 : [5,6,7] and the reviews [2,3,4,8,9] .
Sect. 3 : [2,10,11] and in particular [12] .
Sect. 4 : [13]
Sect. 5 : [14,15,16] (see also the contribution of H.P. Dürr).

1. Remarks on the historical development of symmetry concepts; basic definitions.

1.1.

We distinguish two aspects of symmetry

a) a geometrical, static aspect, and

b) a dynamical aspect.

Moreover, dynamical symmetries can either be

b1) global symmetries, or

b2) local "symmetries".

a) The physics of the antique and of the middle ages (Ptolemy, Kepler...), before Galilei and Newton, only knew the static, geometrical aspect of the concept of symmetries. (The orbits of the planets were believed to have high symmetry. An attempt was made by Kepler to explain the interplanetary distances in terms of the platonic solids. Matter was conceived as being built of highly symmetric "elementary bodies", etc.).

The geometrical aspect of symmetry is of course still extremely important in molecular physics, crystallography, condensed matter physics, biophysics. Historically, it has been an important root in the development of group theory. In a more algebraic outfit, it still appears in every branch of physics in discussions of the symmetries of invariant states (vacuum - or equilibrium states) of physical systems.

b) The idea that it is not the orbits of physical systems which necessarily exhibit high symmetries, but that it is the dynamical laws of physical systems which may admit invariance or symmetry groups could of course appear only after dynamics was introduced into physics, i.e. after the discovery of Newtonian mechanics.

I now briefly characterize static and global dynamical symmetries of physical systems abstractly : We consider a physical system , S . The family of all possible states of S is called X , its elements $x, y, \ldots$. We assume that X carries the action of a group G , i.e. with each $x \in X$ and each $g \in G$ we associate a state $x_g \in X$, the image of x under the transformation g .

a) A state x of a physical system has a static symmetry, described by a subgroup $H \subseteq G$, if

$$x_h = x \text{ , for all } h \in H \text{ .}$$

(H is the stability group of x . All states on the same G-orbit have conjugate stability groups). An orbit of a dynamical, physical system, S , is a mapping from the real line, the time axis, into the space of states, X , of S , i.e.

$$\mathbb{R} \ni t \longmapsto x(t) \in X.$$

The family of all possible orbits is denoted O.

b1) S has a global (dynamical) invariance - or symmetry group $H \subseteq G$ if with each $x(\cdot) \in O$ the trajectory

$$\{x(t)_h : t \in \mathbb{R}\}$$

is again an orbit of S, i.e. an element of O, for all $h \in H$. This means that

$$x(t)_h = x_h(t) ,$$

(where $x = x(0)$, x_h = image of x under h, $x_h(t)$ = orbit with initial condition x_h).

b2) Next, we give a somewhat misleading, abstract characterization of local (dynamical) symmetries : Suppose S consists of (spatially separated, but coupled) constituents, $S_1,\dots,S_n$, $n = 2,3,\dots,$ i.e. $S \cong S_1 \times S_2 \times \cdots \times S_n$.

An orbit, $x(t)$, of S then consists of an n-tuple $(x_1(t),\dots,x_n(t))$, where $x_i(t)$ is an orbit of the constituent system S_i, $i = 1,\dots,n$. A subgroup $H \subseteq G$ is a "local symmetry group" of S if, for arbitrary $h_1,\dots,h_n$ in H and an arbitrary orbit, $x(t)$, of S,

$$x(t)_{\underset{\sim}{h}} = (x_1(t)_{h_1},\dots,x_n(t)_{h_n}) ,$$

$\underset{\sim}{h} = (h_1,\dots,h_n)$, is again a possible orbit of S. Usually, it is not assumed that $\underset{\sim}{h}$ be time-independent.

"Local symmetries" have little in common with global symmetries, since for physical systems admitting a local symmetry group (an invariance group of gauge transformations of the second kind) one cannot devise any experiment which would distinguish between the two orbits $x(t)$ and $x(t)_{\underset{\sim}{h}}$. Thus, $x(t)$ and $x(t)_{\underset{\sim}{h}}$ really correspond to the same, physical orbit of S, expressed in different "internal coordinates".

The idea of "local symmetries" has led to the development of gauge theories, (H. Weyl, O. Klein, W. Pauli, Yang and Mills), in physics, and in mathematics it gave rise to the theory of fibre - and principal bundles.

The present trend in particle physics is to eliminate global symmetries from the fundamental dynamical laws, but to find dynamical laws admitting a (usually non-abelian) "local symmetry group", although the spatial symmetries (Poincaré covariance) are still global symmetries, unless gravity is included. Such dynamical laws are expressed in the form of gauge theories. That trend poses the interesting, theoretical problem of deriving the global, internal symmetries of phenomenological models or macroscopic descriptions as dynamically generated, asymptotic symmetries; (see Sect. 5).

1.2.

We now turn to the discussion of symmetry breaking. We start with the breaking of global symmetries : We distinguish between

i) Explicit breaking.
ii) Spontaneous breaking.
iii) Dynamical breaking.

We speak of an explicitly broken symmetry if the dynamics of a physical system contains a "small" term which is not invariant under a symmetry group leaving invariant the other terms of the dynamics. This concept is of importance when one tries to understand small masses (like the pion mass). It often arises in quantum field theories which, in the classical limit (tree approximation), have a symmetry that does not survive on the quantum level, because of anomalies. (Examples may be chiral symmetries (PCAC), dilation or conformal invariance). A different example for i) is the eightfold way. This topic is not discussed in my notes, although it is very important.

We say that a global symmetry group, H , of a physical system, S , is broken spontaneously (or dynamically) if H is an invariance group of the dynamics, but it is impossible to transform an orbit, $x(t)$, of S into the orbit $x(t)_h$, for some $h \in H$, by a sequence of local operations (local = local in space) without passing through states (or "configurations") of the system which have infinite energy, (or infinite action).

By considering the example of an infinite, three-dimensional ferromagnet one convinces oneself that the definition sketched above is appropriate.

Spontaneously broken, global symmetries can in general be characterized by a local order parameter, (in the example of the ferromagnet by the spontaneous magnetization = expectation value of the spin observable associated with a point or some microscopically small region).

It is not easy to distinguish between the concepts of spontaneous and dynamical symmetry breaking on an abstract level. (One might say that dynamical symmetry breaking does not manifest itself on the tree level and is non-perturbative, while spontaneous breaking appears already on the tree level and may be discussed perturbatively).

It is much more difficult to give an abstract characterization of the "spontaneous or dynamical breaking of local symmetries". This notion cannot mean that the physics of a system admitting a local symmetry group, i.e. a general covariance under changes of the internal coordinates, depends on the choice of local, internal coordinates. Broken, local symmetries cannot be characterized by (gauge-invariant) local order parameters.

The concept of a broken, local symmetry is intrinsically dynamical and must, in the author's opinion, be discussed in the context of a specific formulation of specific theories. One might vaguely describe it as follows : A system with a local, internal symmetry group H (assumed to be a compact Lie group) has always dynamical degrees of freedom carried by gauge fields which are indexed by generators of the Lie algebra of H. One might say that H is "broken spontaneously (or dynamically)" down to a subgroup $H_1 \subset H$ if the degrees of freedom associated with the coset space H/H_1 are "frozen out at small energies", i.e. are invisible at large distances. (But see Sect. 3).

A discussion of the concept of symmetry enhancement is postponed to Sects. 4 and 5.

2. Spontaneous breaking of global symmetries.

General references are [2,3,4,5,6]. General, group theoretical discussions of the possible symmetry breaking patterns in physical systems can be found e.g. in [17]. The breaking of spatial symmetries is a general phenomenon in the physics of matter at positive density (and temperature), in the thermodynamic limit : The boosts are always broken, rotation invariance is often broken (directional long range order; e.g. in liquid crystals), translation invariance is broken in systems with a crystalline equilibrium state (translational long range order). The mathematical understanding of systems with broken rotation - or translation invariance is however rudimentary.

There is a prominent example of the breaking of Lorentz invariance in a system at zero temperature and density : The boost symmetries are broken on the charged sectors of quantum electrodynamics [18], although the term "broken" is used here in a slightly different (weaker) sense.

An important feature of the breaking of continuous, internal symmetries and translation invariance is that it is always accompanied by the appearence of zero mass excitations (the Goldstone bosons, the phonons, respectively) which manifest themselves in the slow decay of correlations of suitably chosen observables; [6,20]. (The breaking of rotation invariance also implies the existence of correlations with slow decay but not the existence of Goldstone excitations).

Starting from our definition in Sect. 1, one can easily convince oneself - at least heuristically - that, in one and two dimensions (space dimensions, in equilibrium statistical mechanics; space-time dimensions in quantum field theory), continuous internal symmetries or translation invariance cannot be broken spontaneously or dynamically [6,20,21,22] (I recall the droplet argument, made rigorous in [22]), except in systems with interactions of extremely long range [23,24].

It may not be generally appreciated that we do not know any example of a system with translation invariant dynamics for which we can prove mathematically that the breaking of translation invariance occurs, (except in the rather unphysical, one-dimensional jellium model which exhibits a Wigner lattice or in idealized models of systems at positive density, but at zero temperature). Moreover, there are no known continuum models of liquid crystals for which we can rigorously establish directional long range order. These are serious gaps in the mathematical foundations of condensed matter physics.

Up until 1976 there were no known examples of models with a continuous, internal symmetry group for which spontaneous symmetry breaking, and hence the existence of Goldstone excitations, was established rigorously (except for the somewhat trivial spherical model of Berlin and Kac). That situation changed with the appearence of

[7], where a class of three - or higher dimensional lattice models of statistical mechanics, including the classical Heisenberg model, and the well known $\lambda|\vec{\varphi}|^4$-quantum field model in three space-time dimensions were discussed which exhibit spontaneous breaking of internal $O(N)$ symmetries at suitably chosen values of the thermodynamic parameters, the coupling constant λ and the bare mass, respectively. The method in [7] involves a rigorous version of spin wave theory. It was extended in [25] to quantum-mechanical lattice models, including the quantum XY model and the Heisenberg anti-ferromagnet. The method was generalized considerably in [24]. However, the Heisenberg ferromagnet, for example, has so far resisted all attempts at a mathematically rigorous understanding. All these developments have been reviewed pedagogically e.g. in [3,8,9,26] , and I do not repeat that here.

A new method in the theory of phase transitions and continuous symmetry breaking has been introduced recently by Spencer and myself [27]. Unfortunately, it can only be applied to systems with an abelian symmetry group, so far, (although it is conceivable that one might recover the results on $O(N)$ lattice σ-models of [7]). The method has the advantage of extending to abelian lattice gauge theories, permitting us to give fairly simple, new proofs of the results in [28,29]. It relies on a representation of abelian lattice spin systems or - field theories as gases of defects of co-dimension 2 carrying integer flux numbers. In three dimensions those defects are closely related to the Abrikosov vortices in a super conductor. They interact through long range forces. The broken symmetry appears when the defects have low effective activity , z , and form a dilute gas. The symmetry is restored when the defects condense. A heuristic discussion of the transition is based on an energy-entropy argument : The entropy , S, of a defect line (or-network) of length L is bounded by

$$S \leq c \cdot L ,$$

where c is a geometric constant, the energy is bounded by

$$E \geq c' |\varphi| \cdot L ,$$

where φ is the flux carried by the defect, i.e. $|\varphi| \geq 1$, and $c' > 0$. The effective activity z is therefore bounded above by

$$z \leq e^{-(\beta c' - c)L} , \beta = 1/kT ,$$

and is exponentially small in L for large $\beta (> c/c')$. Thus, for small temperatures T , the defects have small sizes and form a dilute gas. Hence the medium is ordered, and the symmetry is spontaneously broken. This argument is clearly reminiscent of the Peierls argument, e.g. [26], where one considers a gas of Bloch walls. It might have interesting applications to the theory of melting of solids. For a rigorous version see [27].

3. "Spontaneous breaking" of local, internal symmetries.

This topic is huge and less well understood than the breaking of global symmetries. It would require a series of lectures of its own. I therefore concentrate my attention on the discussion of a few specific aspects which are probably not typical for the whole circle of problems.

Systems admitting a local, internal symmetry group (i.e. general covariance under changes of internal "coordinates") are always described in the form of gauge (Yang-Mills) theories. Presently, the most widely used, non-perturbative ultraviolet regularization of gauge theories consists of putting these theories on a lattice. This preserves gauge-invariance, translation invariance and positivity of the metric in the physical Hilbert space, [30,31,32,33].

We now consider an example : The gauge group G is chosen to be $SU(2)$. The gauge field is denoted by $\underset{\sim}{g} = \{g_{xy}\}$, where xy is an arbitrary pair of nearest neighbors in $\mathbb{Z}^d$, and g_{xy} is an element of G formally given by

$$g_{xy} = P(e^{\int_x^y A_\mu(\xi)d\xi^\mu}), \quad \text{for all } xy.$$

We also introduce a Higgs field ,

$$\phi : x \in \mathbb{Z}^d \to \vec{\phi}_x \in \mathbb{E}^3,$$

with isospin 1 . Let χ be the spin 1/2 character of $SU(2)$, and U the spin 1 representation. The Euclidean functional measure which determines the vacuum state is given by

$$d\mu(g,\phi) \equiv Z^{-1} \prod_{(xy)} e^{\zeta(\vec{\phi}_x \cdot U(g_{xy})\vec{\phi}_y)} \cdot \prod_p e^{\beta\chi(g_{\partial p})} \prod_{(xy)} dg_{xy} \prod_x d\lambda(\vec{\phi}_x) , \tag{1}$$

$$d\lambda(\vec{\phi}) \overset{\text{e.g.}}{=} \exp[-\frac{\lambda}{4}|\vec{\phi}|^4 + \frac{\mu^2}{2}|\vec{\phi}|^2]d^3\phi .$$

$g_{\partial p} = \prod_{xy\subset\partial p}^{\circlearrowright} g_{xy}$, (p a unit square = plaquette), β, ζ and λ are positive constants. The r.s. of (1) is defined as the thermodynamic limit of the measures associated with finite sublattices, with arbitrary <u>boundary conditions</u> (b.c.) imposed at the boundary of each sublattice.

Let $h : x \to h_x$ be an arbitrary function from $\mathbb{Z}^d$ into G with the property that $h_x = 1$, except for finitely many sites x . We make the change of variables

$$g_{xy} \to h_x g_{xy} h_y^{-1} \equiv (g^h)_{xy} \quad ,$$
$$\vec{\phi}_x \to U(h_x)\vec{\phi}_x \equiv (\vec{\phi}^h)_x \quad . \tag{2}$$

This change of variables leaves $d\mu(g,\phi)$ invariant, i.e. $d\mu(g,\phi) = d\mu(g^h,\phi^h)$, <u>no matter what b.c. have been used to construct</u> $d\mu$. Let

$$\langle\cdot\rangle \equiv \int(\cdot)d\mu(g,\phi) \quad .$$

It clearly follows that gauge-dependent observables, like g_{xy} or $\vec{\phi}_x$, have zero expectation, i.e.

$$\langle g_{xy}\rangle = \langle\vec{\phi}_x\rangle = 0 \quad , \tag{3}$$

independent of the b.c. used in the construction of $\langle\cdot\rangle$. This is an immediate consequence of a definition of $\langle\cdot\rangle$ by means of the Dobrushin-Lanford-Ruelle equations, for example. It really expresses the triviality that if one does not fix a gauge by hand, the gauge is not fixed, and therefore gauge-dependent variables are averaged out when one computes their expectation (an observation made e.g. in [10,11]) .

Let us now fix a gauge and see whether we obtain a useful notion of "breaking of local symmetries". Gauge fixing is achieved by multiplying $d\mu$ by a function $F(g,\phi)$ with the property

$$\int F(g^h,\phi^h)\prod_x dh_x = 1 \ . \tag{4}$$

Let $\langle\cdot\rangle_F$ be the expectation determined by $F(g,\phi)d\mu(g,\phi)$. For gauge-invariant observables, A ,

$$\langle A\rangle_F = \langle A\rangle \quad . \tag{5}$$

However, it is now possible, a priori, that

$$\langle\vec{\phi}_x\rangle_F \neq 0 \ ,$$

if suitable b.c. are imposed.

One possible, <u>partial</u> gauge fixing is to turn all Higgs variables, $\vec{\phi}_x$, parallel to the 3-axis, e_3 , of $\mathbb{E}^3$. (Choose F to be proportional to $\prod_x \delta(\vec{\phi}_x - |\vec{\phi}_x|e_3)$) . Then

$$\langle\vec{\phi}_x\rangle_F = \langle|\vec{\phi}_x|\rangle_F \cdot e_3 \ , \ \langle|\vec{\phi}_x|\rangle_F > 0 \ , \tag{6}$$

(no matter whether μ^2 is positive or negative). For this choice of F, $F(\phi)d\mu(g,\phi)$ has a residual, local invariance group : If in (2) every h_x leaves e_3 invariant $(U(h_x)e_3 = e_3, \forall x)$, hence belonging to a $U(1)$ subgroup of G, then

$$F(\phi^h)d\mu(g^h,\phi^h) = F(\phi)d\mu(g,\phi) . \tag{7}$$

Thus, in a sense the coupling of the gauge field to the Higgs field has broken the gauge group down to a residual $U(1)$, but since this happens no matter whether μ^2 is positive or negative, it does not provide a terribly useful notion of "spontaneous, local symmetry breaking". (This becomes particularly evident in a theory with a large gauge group G, e.g. $SU(3)$, and a Higgs field, ϕ, with the property that the action of G on the vector space, V, of possible values of $\vec{\phi}_x$ decomposes V into several inequivalent G-orbits). The above notion depends of course on our choice of F. It has been shown in [12] that, for some class of complete gauge fixings, F, including the temporal gauge, and arbitrary "symmetry breaking" b.c.

$$\langle\vec{\phi}_x\rangle_F = 0 . \tag{8}$$

(This follows either from a "spin-wave" argument related to the one in [22], or from the principle of "symmetry restoration via defects", such as instantons). In [12] Morchio, Strocchi and the author have therefore proposed a gauge-invariant description of the physics of a Higgs theory in the continuum limit, with the hope that this might lead to a useful notion of "spontaneous local symmetry breaking". In the example of the Georgi-Glashow model considered above, the appropriate, gauge invariant fields are

$$\vec{\phi}\cdot\vec{F}_{\mu\nu} \text{ (photon), and } \vec{\phi}\cdot\vec{\phi} \text{ (Higgs particle)} . \tag{9}$$

Moreover there is a gauge-covariant field

$$\pi_{\vec{\phi}}\, \vec{F}_{\mu\nu} \equiv N[|\vec{\phi}|^2\vec{F}_{\mu\nu} - \vec{\phi}(\vec{\phi}\cdot\vec{F}_{\mu\nu})] \ (W^+ \text{ and } W^-) , \tag{10}$$

where N indicates normal ordering. From this field one may formally construct fields localized on curves (γ_{xy}) with given endpoints (x,y) :

$$(\pi_{\vec{\phi}}\, \vec{F}_{\mu\nu})(x)P[\exp \int_{\gamma_{xy}} A_\rho(\xi)\cdot d\xi^\rho](\pi_{\vec{\phi}}\, \vec{F}_{\sigma\lambda})(y) . \tag{11}$$

The conventional, gauge-dependent picture is recovered if $\vec{\phi}_x = e_3$. Note that the physical fields introduced in (9), (10) and (11) do not form any $SU(2)$ multiplets, and there is no reason why the masses of the photon and the W^+ and W^- boson ought to be degenerate. In fact, this theory is expected to have a non-perturbative phase in which there is only one massive, neutral vector particle (a massive photon), a

"Higgson" and neutral W^+-W^- bound states. In that phase the electric charge would be confined, (region I of Fig.1). At large renormalized values of ζ, μ^2 and β, the theory should however have a QED phase with a massless photon (coupled to the vacuum by $\vec{\phi}\cdot\vec{F}_{\mu\nu}$), a massive (unstable) "Higgson", massive W^+ and W^- vector bosons and massive magnetic monopoles, (region II, Fig.1). When one speaks of "spontaneous breaking of SU(2)" in this theory one is thinking of phase II. The picture developped here can be tested in the lattice Georgi-Glashow model for which one expects the following phase diagram, ($\lambda,\mu^2 > 0$; $\lambda,\mu^2 \to \infty$, as $\zeta \to \infty$) :

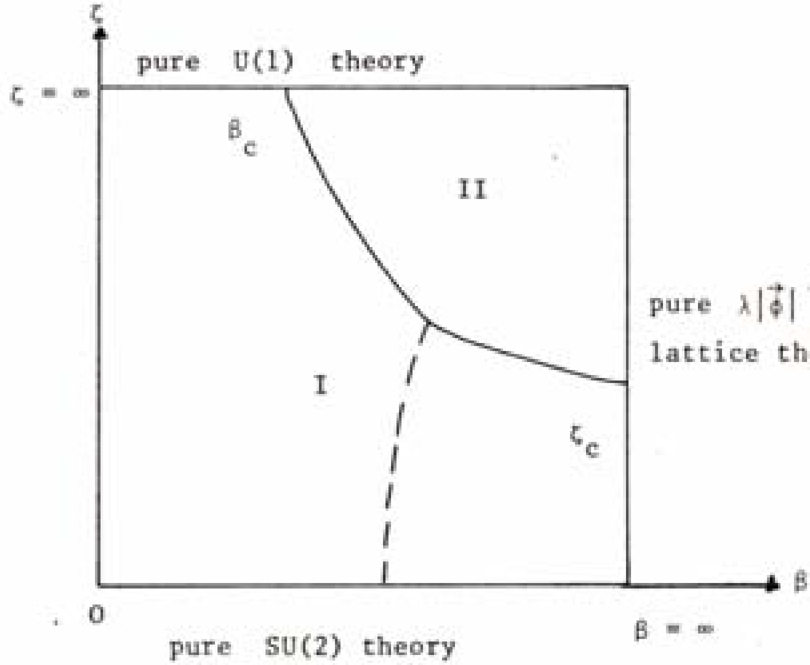

Fig. 1

I : confining phase/II : QED phase. The dashed line might correspond to a line of singularities of the electric string tension, [34]. The only rigorous results concern the existence and nature of the transitions on the lines $\beta = \infty$ and $\zeta = \infty$ (at ζ_c, β_c, resp.). See [7,28,27,33].

The above considerations extend to more general Higgs theories with the property that the stability group $H_{\vec{\phi}_x}$ of $\vec{\phi}_x \neq 0$ is conjugate to one subgroup H of the gauge group G, for all $\vec{\phi}_x$. The gauge fields corresponding to generators of the Lie algebra of $H_{\vec{\phi}_x}$, for all x, are the "electromagnetic and gluon fields", the remaining generators of the Lie algebra $\mathfrak{g}$ of G are the "broken generators" and should correspond to massive (H-neutral) bound states or massive vector bosons.

Things become problematic when there are several, inequivalent Higgs orbits, i.e. the abstract group corresponding to $H_{\vec{\phi}_x}$ depends non-trivially on $\vec{\phi}_x, \vec{\phi}_x \neq 0$. (Example : $G = SU(3)$, $\vec{\phi}$ in the adjoint representation, ...). Let $V_{eff}(\vec{\phi})$ be the effective Higgs potential, (including radiative corrections). Let $(\vec{\phi}_o)$ denote the orbit on which $V_{eff.}(\vec{\phi})$ takes its minimum, H_o the corresponding (abstract) stability group, and $\mathfrak{h}_o$ the Lie algebra of H_o. Let $\overline{\vec{\phi}_x}$ be the Higgs field averaged over a ball centered at x of radius $\approx M^{-1}$, where M is a typical mass scale (a fluctuation length scale of ϕ) of the theory. Then with high probability

$$\overline{\vec{\phi}_x} \in (\vec{\phi}_o) . \tag{12}$$

Perturbation theorists then say that G is broken down to H_o, that the gauge fields corresponding to generators of $\mathfrak{h}_o$ (in the sense indicated above and in [12]) remain massless, while the ones corresponding to $\mathfrak{g} \ominus \mathfrak{h}_o$ acquire masses.

On a non-perturbative level, this prediction is probably wrong, as argued by Morchio and Strocchi [35] on the basis of ideas and results in statistical mechanics [36,37,38] and of [12] : Suppose $V_{eff.}(\vec{\phi})$ has a <u>local minimum</u> on an orbit $(\vec{\phi}_1)$, $\vec{\phi}_1 = \vec{\phi}_o + \delta\vec{\phi}_1$, with stability group $H_1 \subset H_o$. (In general $H_1 \not\subset H_o$, but we make this assumption for clarity). Let M_o^2, M_1^2 be the curvatures of $V_{eff.}$ transversal to $(\vec{\phi}_o)$, $(\vec{\phi}_1)$ at $\vec{\phi}_o, \vec{\phi}_1$, respectively. If

$$\Delta\alpha \equiv V_{eff.}(\vec{\phi}_1) - V_{eff.}(\vec{\phi}_o) + a(M_1^2 - M_o^2)M^2 >> 0 \quad , \tag{13}$$

for some positive constant $a \approx (\zeta^{-1})_{ren.}$, then $\overline{\vec{\phi}_x}$ is close to $(\vec{\phi}_o)$ predominantly. (If (13) fails it may happen that $\vec{\phi}_x$ is close to $(\vec{\phi}_1)$, predominantly, <u>even</u> if $V_{eff.}(\vec{\phi}_1) > V_{eff.}(\vec{\phi}_o)$) . However, with some probability (vanishing in perturbation theory, but <u>positive</u> non-perturbatively) there appear "bubbles, B, of the false vacuum" such that $\overline{\vec{\phi}_x}$ is close to $(\vec{\phi}_1)$, for $x \in B$. The effect of these bubbles on the physics of such a theory can be estimated by a <u>Peierls (action-entropy) argument</u> [36] and a study of <u>mass generation</u> [38]. (I follow a presentation in [37]): First we must estimate the probability p of the event E_x that $\vec{\phi}_x$ is close to $(\vec{\phi}_1)$, i.e. <u>that</u> x (e.g. the origin) <u>belongs to a bubble</u> $B \supseteq \{y \,|\, |y-x| \lesssim M^{-1}\}$ of <u>false vacuum</u>. We choose b.c. such that $\overline{\vec{\phi}_z} \in (\vec{\phi}_o)$, as $|z| \to \infty$. A connected piece, Γ, of the boundary of a bubble is called a <u>contour</u> (or <u>phase boundary</u>). For the event E_x to occur it is necessary that there be a <u>contour</u> Γ separating $\{y \,|\, |y-x| \lesssim M^{-1}\}$ from ∞, as follows from our definitions of $\vec{\phi}_x$ and of contours. The action of a bubble B such that $\partial B \supset \Gamma$ is bounded below by $A(|\Gamma|)$, where

$$A(|\Gamma|) \gtrsim \sigma M^{-3}|\Gamma| + \Delta\alpha \cdot M^{-1}|\Gamma| \quad . \tag{14}$$

Here σ is a constant $\approx(\zeta\mu^2\lambda^{-1})_{ren.}$ and $|\Gamma|$ is the volume of Γ. The first term is a surface term, the second term a lower bound for a volume term. The precise dependence of M, a and σ on coupling constants is not known, presently. If

$$A(|\Gamma|)|\Gamma|^{-1} >> 1 \tag{15}$$

the statistical weight of a contour Γ is bounded above by $\exp[-A(|\Gamma|)]$. Therefore the probability p for E_x to occur is bounded by

$$p \leq \sum_{\Gamma}^{x} \exp[-A(|\Gamma|)] \quad ^{1)} \quad , \tag{16}$$

where $\sum\limits_{\Gamma}^{x}$ ranges over all contours Γ of volume $|\Gamma| = \text{const.} M^{-3} n$, $n = 1,2,3,\ldots$, surrounding $\{y \,|\, |y-x| \leq M^{-1}\}$. The number of such contours with given volume, $|\Gamma| = \text{const.} M^{-3} n$, is bounded by

$$e^{cn} \text{ , } (c \approx O(1) \text{ is a geometrical constant}) \ . \tag{17}$$

From (14), (16) and (17) we conclude that

$$p \ll 1 \quad \text{if (15) holds.} \tag{18}$$

By the results of [38,35] one then expects that gauge fields corresponding to generators in $\mathfrak{g} \ominus \mathfrak{h}_o$ (in the sense of [12]) acquire masses $\approx |\vec{\phi}_o|$, while gauge fields corresponding to the generators of $\mathfrak{h}_o \ominus \mathfrak{h}_1$ acquire masses

$$\approx \sqrt{p}\,|\delta\vec{\phi}_1| \, . \tag{19}$$

Similar considerations apply to Fermion masses. Thus - if there are no further local minima giving rise to other bubbles - G is really "broken down" to H_1 , rather than to the larger group H_o . Moreover, there is <u>no</u> elementary Higgs field causing the "breaking" from H_o down to H_1 , [35]. Finally, one does <u>not</u> expect any dynamical monopoles with charges labelled by $\pi_2(H_o/H_1)$, (π_k = k^{th} homotopy group), but only ones with charges labelled by $\pi_2(G/H_o) \simeq \pi_1(H_o)$, as argued in [37].

Finally, we should mention that the applicability of conventional perturbation theory, based on the (generally incorrect) assumption that $\langle\vec{\phi}_x\rangle_F \neq 0$, to Higgs gauge theories has been discussed in [12], with the result that the deviations can generally be expected to be entirely <u>non-perturbative</u>.

This ends our discussion of the notion of "spontaneous breaking of local (gauge) symmetries" : In abstracto, it is somewhat vague and misleading. It must be understood dynamically, and one should be aware of the fact that non-perturbative effects generally alter the conventional interpretation. Such effects, together with the requirements of renormalizability, some form of asymptotic freedom and the requirement that there exist an "unbroken" $SU(3)_c \times U(1)$ may be useful guides for model builders.

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1) A lower bound on p is more difficult to derive, see [37].

4. Renormalization group ideas.

We consider a class of physical systems which can be described by a family (algebra), $\mathfrak{A}$, of local "observables", e.g. Euclidean fields, in quantum field theory (QFT), or spin fields, in statistical mechanics (SM), and some space, X, of time-translation invariant states, e.g. Euclidean vacuum functionals in QFT, equilibrium states in SM. Let $A \in \mathfrak{A}$. By A_x we denote the translate of A by a vector x in space-imaginary time (QFT), or space (SM). Let $\rho \in X$ be a state characterizing a specific physical system. Question : How do correlations,

$$\rho(A_x \cdot B_y) \text{ , } A,B \text{ in } \mathfrak{A} \text{ ,}$$

behave, as $|x-y| \to \infty$, i.e. in the (infrared) scaling limit ? (In a continuum system one may also be interested in the behaviour of $\rho(A_x \cdot B_y)$, as $|x-y| \to 0$: the short distance, or ultraviolet limit. We focus our attention on the scaling limit). In order to answer that question, one tries to construct functions, $\alpha_A(\theta)$, depending on $A \in \mathfrak{A}$ and on a scale parameter θ, such that

$$G_{A,B}(x,y) \equiv \lim_{\theta\to\infty} \alpha_A(\theta)\alpha_B(\theta)\{\rho(A_{\theta x} \cdot B_{\theta y}) - \rho(A_{\theta x})\rho(B_{\theta y})\} \qquad (20)$$

exists. (My discussion is slightly oversimplified at this point, since one often chooses ρ on the r.s. of (20) to depend on θ, as well, such that ρ_θ approaches a critical state, as $\theta \to \infty$). In order to find $\alpha_A(\theta)$, $A \in \mathfrak{A}$, and other quantities of interest at large distances, one tries to determine the large scale effective dynamics, by integrating out fluctuations on a sequence of increasing length scales. One popular scheme to accomplish that is the Kadanoff "block spin transformations". Abstractly, they can be described as a non-linear transformation, τ, acting on $X \times \mathfrak{A}$:

$$\tau : (\rho, A) \to \tau(\rho, A) \equiv (\rho_\tau, A_\tau) \text{ ,} \qquad (21)$$

$$\text{with } \rho_\tau(A_\tau) = \rho(A) \text{ }^{2)} \text{ ,}$$

such that each application of τ increases the scale of effective fluctuations, i.e. transforms a dynamics on a given scale into an effective dynamics on the next larger scale. In order to answer the question raised at the beginning by means of such a scheme one must study the manifold M_m of fixed points of τ :

$$\rho^* \in M_m \subset X \text{ iff } \rho^*_\tau = \rho^* \text{ .} \qquad (22)$$

---

2) or $\tau = \tau_\theta$, with $\rho(A_{\theta x} \cdot B_{\theta y}) = \text{const.}\rho_{\tau_\theta}(A_x \cdot B_y)$. (21')

Under suitable hypotheses on the properties of τ , one can decompose X in the vicinity of some $\rho^* \in M_m$ into a stable manifold, $M_s(\rho^*)$, and an unstable manifold, $M_u(\rho^*)$:

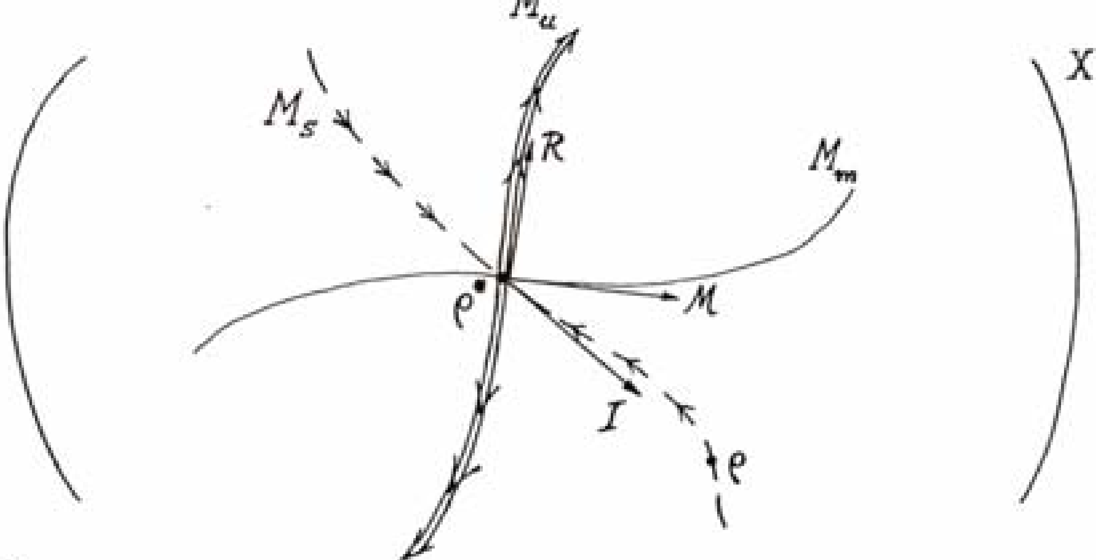

Fig. 2

States on $M_s(\rho^*)$ are driven towards ρ^* , states on $M_u(\rho^*)$ are driven away from ρ^* , under the action of τ . The tangent space, R , to $M_u(\rho^*)$ at ρ^* is the linear space spanned by eigenvectors of $D\tau_{\rho^*}$, the linearization of τ at ρ^* , corresponding to eigenvalues of modulus > 1 . R is called the space of "relevant perturbations". The space, I , of "irrelevant perturbations" is defined by replacing τ by τ^{-1} in the definition of R , and the space, M , of "marginal perturbations" is the tangent space to M_m at ρ^* . Let $\rho \in M_s(\rho^*)$. One argues that the functions $\alpha_A(\theta)$ are computable in terms of A and of the rate of approach of $\underbrace{(\cdots(\rho_\tau)_\tau)_\tau\cdots}_{n \text{ times}}$ to ρ^* . (See (21), (21')) .

The point of interest to us is now the following : It may happen that the fixed point ρ^* has a larger symmetry group than a state ρ on $M_s(\rho^*)$. This entails that the scaled correlations, $G_{A,B}(x,y)$, exhibit a larger symmetry than the original correlations $\rho(A_x \cdot B_y)$. If this happens we speak of asymptotic enhancement of symmetry, or of the (dynamical) generation of asymptotic symmetries. It is quite irrelevant in this general discussion, whether the symmetry in question is internal or spatial, global or local (i.e. gauged).

One might argue that the concept of symmetry enhancement is only interesting for physics if it has some stability properties. Let G be some (global or local, internal or spatial) symmetry group, and let H be a subgroup of G . Consider a G-invariant fixed point, ρ^* , of τ , (τ is assumed to have suitable smoothness properties). Suppose that the H-invariant subspace of M coincides with the G-invariant subspace of M . Then, in some vicinity, N , of ρ^* , every H-invariant fixed point of τ is also G-invariant. Thus, all states in $\underset{\tilde{\rho}/N}{\cup} M_s(\tilde{\rho})$, where $\underset{\tilde{\rho}/N}{\cup}$ ranges over all H-invariant fixed points, $\tilde{\rho}$, of τ in N, are driven towards G-invariant fixed points. Moreover if the H-invariant subspace of $\mathcal{M} \oplus \mathcal{R}$ coincides with the G-invariant subspace of $\mathcal{M} \oplus \mathcal{R}$, at ρ^*, then for some neighborhood N of ρ^*, the H-invariant subspace of marginal and relevant perturbations of a H-invariant fixed point $\tilde{\rho} \in N$ is also G-invariant, (i.e. H-invariant states near ρ^* tend to approach G-invariant states under the action of τ). This is the desired stability of our concept. The concept of symmetry enhancement has been described e.g. in [14], (see also Dürr's notes). The first rigorous study of models exhibiting this phenomenon (e.g. the $\mathbb{Z}_N$-models, see Sect. 5) probably appeared in [27,39]. An abstract discussion very similar to the one presented here appeared subsequently in [16] (which inspired the present section).

5. Symmetry enhancement : Generation of asymptotic, global and local symmetries.

In this final section we sketch very briefly some examples of the phenomena described at the end of Sect. 4 and point out why "symmetry enhancement" might be the right concept permitting us to decide whether a local (gauge) symmetry in some gauge theory is "spontaneously broken", or not.

First, we consider the $\mathbb{Z}_N$ spin models on the square lattice, $\mathbb{Z}^2$, $N = 5,6,7,\ldots$. The classical spin, $\vec{S}_x$, at a site $x \in \mathbb{Z}^2$ is given by

$$\vec{S}_x = (\cos\theta_x, \sin\theta_x)\ ,\ \theta_x = \frac{2\pi n_x}{N}\ ,\ n_x = 0,\ldots,N-1,\ \forall x\ .$$

The equilibrium state of the model at inverse temperature β is given by the measure

$$d\mu_\beta^{(N)}(\theta) = [Z_\beta^{(N)}]^{-1} \exp[\beta \sum_{xy} \cos(\theta_x - \theta_y)]\ , \tag{23}$$

where xy is an arbitrary pair of nearest neighbors. This measure is the limit of the measures

$$d\mu_{\beta,h}(\theta) = Z_{\beta,h}^{-1} \exp[\beta \sum_{xy} \cos(\theta_x - \theta_y)] \prod_x e^{h\cos(N\theta_x)} d\theta_x\ , \tag{24}$$

as $h \to \infty$; ($d\theta$ = Lebesgue measure on unit circle). The classical XY-, or rotator model corresponds to $h = 0$. For $h > 0$, the measure $d\mu_{\beta,h}$ and $d\mu_\beta^{(N)}$ have a discrete, global symmetry group (generated by $\mathbb{Z}_N$ and reflections) while the rotator $(h = 0)$ has a continuous, global symmetry group. In [39] Spencer and the author have shown that, for all $h \in [0,\infty]$ and $N \geq N_o$, where N_o is a suffiently large integer independent of h, there exists an interval $[\underline{\beta}(h,N)\ ,\ \overline{\beta}(h,N)]$ of values of β which are all critical points and at which the correlation length of the spin systems described by $d\mu_{\beta,h}$ is infinite. Moreover,

$$\underline{\beta}(h,N) \leq \underline{\beta}(0,N) \equiv \beta_c(\text{rotator}) < \infty\ ,$$

$$\overline{\beta}(h,N) \to \infty\ ,\ \text{as}\ \ h \to 0\ \ \text{or}\ \ N \to \infty\ .$$

We have constructed an infinite sequence of renormalization transformations which drive $d\mu_{\beta,h}$ towards a U(1)-invariant state $d\mu^*_{\beta,h}$, for all $h \in (0,\infty]$, $N \geq N_o$, and all $\beta \in (\beta_1, \beta_2)$, with $\underline{\beta}(h,N) \leq \beta_1 < \beta_2 \leq \overline{\beta}(h,N)$. Thus, asymptotically, the discrete symmetry of the $\mathbb{Z}_N$ models is enhanced to a continuous symmetry. We conjecture that for each $h \in (0,\infty]$ and each $\beta \in (\underline{\beta}(h,N), \overline{\beta}(h,N))$, $N \geq N_o$, there exists some $\beta' \equiv \beta'(\beta,h) \geq \beta_c$ (rotator) such that spin correlations in $d\mu_{\beta,h}$ and in $d\mu_{\beta'}$ have identical (long distance) scaling limits, (although this does not quite follow from our construction).

In [27] we have established similar results for the QED phases [29] of the $\mathbb{Z}_N$ lattice gauge theory in four dimensions : Local $\mathbb{Z}_N$-invariance is asymptotically enhanced to local U(1)-invariance.

Recently, we have also examined examples of non-abelian gauge theories coupled to some Higgs fields (not transforming under the fundamental representation) for which we argue that, for suitable choices of the coupling constants $\beta,\zeta,\lambda,\mu^2$, the theory is in the same (long distance) "<u>universality class</u>" as the corresponding pure Yang-Mills theory (for some $\beta' = \beta'(\beta,\zeta,\ldots)$, $\zeta = 0$, $\phi = 0$), if only gauge field expectations are considered. In such a case one could say that the matter fields leave the full gauge group "unbroken". (In the opposite case it would be appropriate to speak of "local symmetry breaking"). It would be interesting to study <u>symmetry enhancement at short distances</u> in continuum grand unified theories.

More standard examples of symmetry enhancement which are, however, not very well understood mathematically are :

- <u>Restoration of full Euclidean invariance of correlations of lattice theories in the scaling limit</u> (as $\beta \nearrow \beta_c$, where β_c is a critical point).

- <u>Restoration of translation invariance above the roughening temperature in the three-dimensional Ising model or in a lattice gauge theory</u>, [34].

Problems of symmetry enhancement are typically very involved, technically, so that we cannot present any details here.

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CONTINUUM (SCALING) LIMITS OF LATTICE FIELD THEORIES

(TRIVIALITY of $\lambda\varphi^4$ IN $d \underset{(=)}{>} 4$ DIMENSIONS)

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SUMMARY :

I describe some recent techniques for constructing the continuum (= scaling) limit of lattice field theories, including the one - and two - component $\lambda|\vec{\varphi}|^4$ theories and the Ising - and rotator models in a space (- imaginary time) of dimension $d \underset{(=)}{>} 4$. These techniques should have applications to other related models, like the self-avoiding random walk in five or more dimensions and bond percolation in seven or more dimensions. Some plausible conjectures concerning the Gaussian nature of the scaling limit of the $d \geq 2$ dimensional rotator model and the $d \geq 4$ dimensional $U(1)$ lattice gauge theory in the low temperature (weak coupling) phase are described.

1. INTRODUCTORY REMARKS, RESULTS AND CONJECTURES.

An important topic in Euclidean (quantum) field theory and statistical mechanics is the study of the scaling = continuum limit of lattice field theories. That limit corresponds to the large distance limit of rescaled lattice correlation functions at values of the inverse square coupling constant β (= field strength, or inverse temperature) approaching a critical value, as the distance scale tends to infinity. Given a family of lattice field theories, e.g.

lattice $\lambda\varphi^4$ theories or Ising models, indexed by the dimension d of the (space - imaginary time) lattice $\mathbb{Z}^d$, we define the upper critical dimension $\overline{d}_c$ by the property that, for $d > \overline{d}_c$, the scaling limit of the corresponding lattice field theory is trivial (Gaussian, in the case of a scalar field theory), and an appropriate version of mean field theory provides an exact description of the approach to the critical point.

The lower critical dimension $\underline{d}_c$ of those families of models is defined by the property that in dimension $d > \underline{d}_c$ there exists a critical point, $\beta_c < \infty$, of β at which some correlation length diverges.

It is often a subtle problem to determine the behaviour of a field theory when $d = \underline{d}_c$ or $d = \overline{d}_c$. For example, in the N-vector models (O(N) non-linear sigma models on the lattice) $\underline{d}_c = 2$ and $\overline{d}_c = 4$. For the $N = 2$ model, i.e. the rotator or classical XY model, it has recently been proven rigorously, by T. Spencer and the author[1], that there exists a Kosterlitz-Thouless transition, in particular that $\beta_c < \infty$ and that the susceptibility diverges as $\beta \nearrow \beta_c$, in dimension $d = \underline{d}_c = 2$. The proof is, however, fairly complicated. For $N \geq 3$ and $d = \underline{d}_c = 2$, it is conjectured that $\beta_c = \infty$ (asymptotic freedom), but no rigorous proof is known. Moreover the nature of the scaling limit, as $\beta \nearrow \beta_c$, is unknown, except in the two-dimensional Ising model $(N=1)$. For all these models, the analysis of the scaling limit in dimension $d = \overline{d}_c = 4$ is incomplete, although for $N = 1,2$ there are promising partial results. (When $N \geq 3$, not even the fact that $\overline{d}_c = 4$ has been proven rigorously, although that result appears to be within reach of present mathematical methods; $\underline{d}_c = 2$ follows from[2]). The analysis of these models in dimension $d = \underline{d}_c$ and $d = \overline{d}_c$ is important for the study of the scaling limit and the approach to the critical point (critical exponents) in dimension d, with $\underline{d}_c < d < \overline{d}_c$ by means of a $2+\varepsilon$ - or $4-\varepsilon$ expansion[3,4]. At present, not much is known about how to study the approach to the critical point directly when $d < \overline{d}_c$.

Fortunately, this is not always necessary for the construction of the continuum limit of lattice field theories in dimension $d < \bar{d}_c$. We may think of the massive, weakly coupled $\lambda\varphi^4$ theories in two or three dimensions the continuum limit of which is under rigorous mathematical and quantitative control (see e.g.[5-10]), although we are not able to calculate e.g. the critical exponents for these models.

In these notes we discuss the following rigorous results :

1) For one - and two - component $\lambda|\vec{\varphi}|^4$ theories, the Ising - and the rotator model

$\bar{d}_c = 4$, [11,12] .

2) For $d > \bar{d}_c$, the continuum limit in the single phase region of these models is Gaussian (i.e. a free or generalized free field), and the critical exponent γ of the susceptibility takes its mean field value, i.e. $\gamma = 1$, [11,12].

3) For $d = \bar{d}_c = 4$, the continuum limit in the single phase region is Gaussian if field strength renormalization is infinite, i.e. if the ultraviolet dimension of the fields is not canonical, [12].

4) If hyperscaling holds for $\underline{d}_c \leq d \leq \bar{d}_c$ then the critical exponent η of the two-point function at $\beta = \beta_c$ satisfies

$0 \leq \eta \leq 2 - \frac{d}{2}$, [12] .

For the one-component models most of these results were first obtained by Aizenman[11] who invented some very beautiful and clever inequalities. He also discovered a very simple proof of hyperscaling in two-dimensional Ising models[11]. Subsequently, the author found new proofs and extensions of some of Aizenman's results, in particular proofs of 1) - 4), [12], by adapting a technique developed with D. Brydges and T. Spencer which was inspired by ideas due to Symanzik[13]. See[14]; and[15,16] for some related results.

There are also partial results towards proving the following conjectures :

5) For the models introduced in 1) - 4)

$\eta = 0$, for $d > \overline{d}_c$.

(see[12] for a discussion).

6) When $d = \overline{d}_c = 4$, the continuum limit is Gaussian,[12,17].

7) For the self-avoiding random walk $\overline{d}_c = 4$; for bond percolation $\overline{d}_c = 6$,[18].

8) The scaling limit of the rotator model in the low temperature (multiple phase) region, i.e. $\beta > \beta_c$, is Gaussian in dimension $d \geq 2$ [19] . An analogous result is expected for the weakly coupled $U(1)$ lattice gauge theory in dimension $d \geq 4$.

An interesting open problem is to analyze the scaling limits of the $d = 3$ Ising - and the $d = 2,3$ rotator models and of the $U(1)$ lattice gauge theory in dimension $d = 4$, as $\beta \nearrow \beta_c$. We conjecture that these scaling limits are non-trivial (hyperscaling).

It might be mentioned that, in addition to 1) - 4) , there are rigorous results on the scaling limits of lattice field theories with long range ferromagnetic two-body interactions[12,18]. As an example we mention that the scaling limit of the one-dimensional Ising model with ferromagnetic $1/r^2$ interaction energy is Gaussian for $\beta < \beta_c$. (The existence of a phase transition, i.e. $\beta_c < \infty$, and spontaneous magentization at large values of β has recently been proven in[20]).

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2. GENERAL REMARKS ON SCALING LIMITS.

To be specific, we consider a real, scalar field φ ,

$$\varphi : j \in \mathbb{Z}^d \longrightarrow \varphi(j) \in I \; , \qquad (1)$$

with $I \subseteq \mathbb{R}$. The Ising model corresponds to $I = \{-1,1\}$, a lattice $\lambda\varphi^4$ theory to $I = \mathbb{R}$. We imagine the field $\varphi = \{\varphi(j)\}_{j \in \mathbb{Z}^d}$ is distributed according to some probability measure $d\mu_\beta(\varphi)$ (typically a Gibbs measure) depending on a real parameter β interpreted as the inverse temperature or field strength. We define $\varphi_x(j) = \varphi(j+x)$, $x \in \mathbb{Z}^d$, and assume that $d\mu_\beta(\varphi)$ is translation - invariant, i.e.

$$d\mu_\beta(\varphi_x) = d\mu_\beta(\varphi) \text{ , for all } x . \tag{2}$$

The correlation functions of this lattice theory are defined as the moments of $d\mu_\beta$, i.e.

$$< \varphi(x_1) \ldots \varphi(x_n) >_\beta = \int \prod_{k=1}^{n} \varphi(x_k) \, d\mu_\beta(\varphi) . \tag{3}$$

We may assume that $< \varphi(x) >_\beta = 0$. Of particular importance are the two-point function

$$< \varphi(x) \varphi(y) >_\beta = \int \varphi(x) \varphi(y) \, d\mu_\beta(\varphi) \tag{4}$$

and the susceptibility

$$\chi(\beta) = \sum_{x \in \mathbb{Z}^d} < \varphi(0) \varphi(x) >_\beta . \tag{5}$$

In the following we are interested in analyzing the long distance limit of the correlations defined in (3). We suppose there exists some value β_c of β such that for $\beta < \beta_c$ there exists a constant $m(\beta) > 0$, the inverse correlation length or mass, with the property that

$$< \varphi(x) \varphi(y) >_\beta \leq \text{const. } e^{-m(\beta)|x-y|} , \tag{6}$$

as $|x-y| \to \infty$, and

$$m(\beta) \searrow 0 \text{ , as } \beta \nearrow \beta_c . \tag{7}$$

These assumptions are known to hold for Ising models and lattice $\lambda\varphi^4$ fields[21]. We now define the scaled correlations

$$G_\theta(x_1,\ldots,x_n) = \alpha(\theta)^n < \varphi(\theta x_1) \ldots \varphi(\theta x_n) >_{\beta(\theta)} , \tag{8}$$

where $1 < \theta < \infty$, $x_j \in \mathbb{Z}^d_{\theta^{-1}} \equiv \{y : \theta y \in \mathbb{Z}^d\}$,

$j = 1,\ldots,n$, and $\beta(\theta) < \beta_c$, $\alpha(\theta)$ are functions of θ determined by the requirements that for $0 < |x-y| < \infty$

$$0 < \lim_{\theta\to\infty} G_\theta(x,y) \equiv G^*(x-y) < \infty . \tag{9}$$

It follows from (6), (7) and (9) that

$$\beta(\theta) \nearrow \beta_c , \quad \text{as} \quad \theta \to \infty \tag{10}$$

If we want to construct a massive continuum $(\theta \to \infty)$ limit we choose $\beta(\theta)$ such that

$$\theta m(\beta(\theta)) \longrightarrow m^* > 0 , \text{ as } \theta \longrightarrow \infty ; \tag{11}$$

if we try to construct a scale-invariant continuum limit we set $\beta(\theta) = \beta_c$ for all θ . (There are other possible conditions fixing the choice of $\beta(\theta)$. See e.g.[22]). Once $\beta(\theta)$ is chosen, $\alpha(\theta)$ is essentially determined by (9). The correlation functions in the continuum limit are then given by

$$G^*(x_1,\ldots,x_n) = \lim_{\theta\to\infty} G_\theta(x_1,\ldots,x_n) , \; n = 2,3,4,\ldots . \tag{12}$$

Thus, in order to construct the continuum limit, it is crucial to know the behaviour of the two-point function. In many models (e.g. the self-avoiding random walk or bond percolation) this turns out to be very hard. We now elaborate on this point. Suppose one can prove a power law a priori bound on $< \varphi(x)\varphi(y) >_\beta$, for $\beta < \beta_c$, e.g.

$$< \varphi(x)\varphi(y) >_\beta \; \leq c(\beta) \; |x-y|^{-(d-2+\eta)} , \tag{13}$$

with $\sup_{\beta<\beta_c} c(\beta) < \infty$. Then condition (9) imposes the following lower bound on $\alpha(\theta)$:

$$\alpha(\theta) \geq \text{const. } \theta^{(d-2+\eta)/2} . \tag{13'}$$

One may attempt to analyze the two-point function by studiing its operator inverse, the two-point vertex function $\Gamma_\beta(x-y)$ which, in perturbation theory, is expressed as a sum of one-particle irreducible diagrams. For this reason one may hope to estimate it by means of a convergent, infrared-finite expansion if the dimension d is large enough. This is a difficult analytical problem. Fortunately, for the models studied in the following and in[11,12,14], an a priori bound of the form (12), with $\eta \geq 0$, has been established in[2,22] (the infrared - or spin - wave bound), and this turns out to be sufficient to show that $\overline{d}_c = 4$ and that the continuum limit is Gaussian when $d > 4$, thanks to new correlation inequalities[11,12] discussed in the next section.

A more systematic procedure to determine $\beta(\theta)$ and $\alpha(\theta)$ relies on the renormalization group e.g. in the form of Kadanoff Block spin transformations, (calculation of large scale effective Hamiltonian or action) :

Let κ be a function on $\mathbb{R}^d$ defined by

$$\kappa(x) = \begin{cases} \varepsilon^{-d}, & -\frac{\varepsilon}{2} \leq x^\alpha \leq \frac{\varepsilon}{2}, \ \alpha = 1,\ldots,d \\ 0, & \text{otherwise}, \end{cases}$$

where x^α is the α^{th} component of $x \in \mathbb{R}^d$. Let $\kappa_x(y) = \kappa(y-x)$, $x \in \mathbb{Z}_\varepsilon^d$. We consider

$$G_\theta(\kappa_{x_1},\ldots,\kappa_{x_n}) = \sum_{y_1,\ldots,y_n \in \mathbb{Z}^d_{\theta^{-1}}} G_\theta(y_1,\ldots,y_n) \prod_{k=1}^n \theta^{-d} \kappa_{x_k}(y_k), \tag{14}$$

$x_j \in \mathbb{Z}_\varepsilon^d$, $j = 1,\ldots,n$. Now, note that $G_\theta(\kappa_{x_1},\ldots,\kappa_{x_n})$ depends only on the variables

$$\{\varphi_\theta(x_j) : x_j \in \mathbb{Z}_\varepsilon^d\}$$

where

$$\varphi_\theta(x) = \frac{\alpha(\theta)}{\theta^d \varepsilon^d} \sum_{\substack{y \in \mathbb{Z}^d \\ -\frac{\varepsilon}{2} \leq \theta^{-1} y^\alpha - x^\alpha \leq \frac{\varepsilon}{2}}} \varphi(y), \quad x \in \mathbb{Z}_\varepsilon^d, \tag{15}$$

and $\theta = \varepsilon^{-1} L^m$, L is some positive integer and $m = 1,2,3,\ldots$. Given $d\mu_{\beta(\theta)}(\varphi)$, let $d\mu_\theta$ be the unique measure on the configurations of the "block field" $\varphi_\theta = \{\varphi_\theta(x)\}_{x \in \mathbb{Z}_\varepsilon}$ with the property that

$$\int \prod_{k=1}^n \varphi_\theta(x_k) d\mu_\theta(\varphi_\theta) = \int \prod_{k=1}^n \varphi_\theta(x_k) d\mu_{\beta(\theta)}(\varphi), \tag{16}$$

for all $x_1,\dots,x_n$ in $\mathbb{Z}_\varepsilon^d$ and all n. If $d\mu_\beta(\varphi)$ is a Gibbs measure for all β one expects that $d\mu_\theta(\varphi_\theta)$ is again a Gibbs measure, i.e. $d\mu_\theta$ is given in terms of an effective Hamilton function, or action, on a scale of $\varepsilon\theta = L^m$. The calculation of the effective Hamilton function proceeds by a <u>succession of Block spin</u> (or - <u>field</u>) <u>transformations</u>, and each such transformation increases the scale by a factor of L. A mathematical description of the general features of that technique may be found in[23], explicit examples have been studied in[24].

In this approach the functions $\beta(\theta)$ and $\alpha(\theta)$ are determined by the requirements that

$$\theta m(\beta(\theta)) \equiv m^* = \text{const.} \geq 0 \text{ , and that} \tag{17}$$

$$\lim_{\theta\to\infty} d\mu_\theta(\varphi_\theta) \equiv d\mu^*(\varphi^*)$$

is a well-defined probability measure with moments $\neq 0,\infty$. It is hoped that $d\mu^*$ is again a Gibbs measure, but this does not always seem to be the case. Condition (17) and the functions $\beta(\theta)$ and $\alpha(\theta)$ determine the exponents ν and γ of the mass and the susceptibility, respectively. If $m^* = 0$ and $d\mu^*$ is scale-invariant then $d\mu^*$ is a fixed point of the Block spin transformations, and the fall-off of the two-point function is determined by $\alpha(\theta)$.

The smeared continuum correlation functions are given by

$$G^*(\kappa_{x_1},\dots,\kappa_{x_n}) = \int \prod_{k=1}^{n} \varphi^*(x_k)\, d\mu^*(\varphi^*) \text{ .}$$

In practice, it turns out that it is usually very hard to construct the limiting measures $d\mu^*$ and to show that they are Gibbs measures, but there are now some examples where the Block spin transformations can be made to work in a rigorous fashion[24]. However, in these examples $d\mu^*$ is a massless Gaussian.

In the following, we investigate the scaling limits of the Ising model and lattice $\lambda\varphi^4$ models in $d \geq 4$ dimensions using merely an a priori bound of the form (13) with $\eta \geq 0$ and new inequalities on the four-point Ursell function which permit us to avoid applying Block spin transformations.

3. THE CONTINUUM (= SCALING) LIMIT OF THE $\lambda\varphi_d^4$ LATTICE FIELD THEORY AND THE ISING MODEL IN $d \underset{(=)}{\geq} 4$ DIMENSIONS.

We now explain some basic ideas in the proofs of results 1) - 4) described in Sect. 1. Let φ be a real scalar lattice field (or a classical spin) with action (Hamilton function)

$$H(\varphi) = -\sum_{(jj')} \varphi(j)\,\varphi(j')\ , \tag{18}$$

where (jj') are nearest neighbor pairs in $\mathbb{Z}^d$. The measure $d\mu_\beta$, i.e. the Euclidean vacuum functional (or Gibbs state) is given by

$$d\mu_\beta(\varphi) = Z_\beta^{-1}\, e^{-\beta H(\varphi)} \prod_j d\lambda(\varphi(j))\ , \tag{19}$$

where

$$d\lambda(\varphi) = \exp[-\frac{\lambda}{4}\,\varphi^4 + \frac{\mu}{2}\,\varphi^2 - \varepsilon]d\varphi\ , \tag{20}$$

$0 < \beta < \infty$, $\lambda > 0$, μ and ε real, and Z_β is the partition function. Equation (19)' is to be understood as the thermodynamic limit of measures associated with finite sublattices. The limit exists for a large class of boundary conditions by correlation inequalities,[6,7]. If $\mu = \lambda$, $\varepsilon = \frac{\lambda}{4}$ and $\lambda \rightarrow \infty$ we obtain the Ising model. For all such models it is shown in[22], by using the infrared or spin wave bound of[2], that

$$< \varphi(x)\varphi(y) >_\beta \leq \frac{c_d}{\beta}\ |x-y|^{2-d}\ ,\ d \geq 3\ , \tag{21}$$

for some geometric constant c_d independent of β, λ and μ, as long as $\beta < \beta_c$. (It is shown e.g. in[2] that the critical inverse temperature, β_c, is finite, and properties (6) and (7) are proven in[21,25]). As shown in Sect. 2, $\alpha(\theta)$ must therefore satisfy the lower bound

$$\alpha(\theta) \geq \text{const.}\ (\beta(\theta)\theta^{d-2})^{1/2}\ , \tag{22}$$

or if $\beta_c < \infty$

$$\alpha(\theta) \geq (\beta_c \theta^{d-2})^{1/2} \quad . \tag{23}$$

The four-point Ursell function, u_4, is defined by

$$u_{4,\beta}(x_1 \dots, x_4) = < \varphi(x_1) \dots \varphi(x_4) >_\beta - \sum_p < \varphi(x_{p(1)}) \varphi(x_{p(2)}) >_\beta \cdot \\ \cdot < \varphi(x_{p(3)}) \varphi(x_{p(4)}) >_\beta \tag{24}$$

where $\sum_p$ ranges over all three pairings of $\{1,2,3,4\}$. It satisfies the following inequalities

$$0 \geq u_{4,\beta}(x_1, \dots, x_4) \geq -3\beta^2 \sum_{z_1 \dots z_4}{}' \prod_{k=1}^{4} < \varphi(x_k)\varphi(z) >_\beta, \tag{25}$$

where z_1 ranges over $\mathbb{Z}^d$, and $|z_\ell - z_1| = 1$, $\ell = 2,3,4$. The upper bound is the well-known Lebowitz inequality[26], the lower bound is the new inequality proven in[12] which is closely related to Aizenman's inequality[11] . We define

$$u_{4,\theta}(x_1, \dots, x_4) = \alpha(\theta)^4 \, u_{4,\beta(\theta)}(\theta x_1, \dots, \theta x_4) \; . \tag{26}$$

In order to satisfy as general a class of renormalization conditions as possible we permit λ and α to depend on θ , as well : A minimal condition on $\lambda = \lambda(\theta)$, $\mu = \mu(\theta)$ and on $\beta(\theta)$ and $\alpha(\theta)$ is that inequalities (9), Sect. 2, be satisfied. We now obtain from (25), (26)

$$0 \geq u_{4,\theta}(x_1, \dots, x_4) \geq \alpha(\theta)^{-4} \theta^d 3\beta(\theta)^2 \; \cdot \\ \cdot \sum_{z_1, \dots, z_4}{}' \theta^{-d} \prod_{k=1}^{4} G_\theta(x_k, \theta^{-1} z_k) \; . \tag{27}$$

The nice feature of (27) is that the upper and lower bounds are <u>independent</u> of $\lambda(\theta)$, $\mu(\theta)$. (In a sense, (27) says that the prediction of the linearized renormalization group provides a rigorous bound). Now, by (21) and (22)

$$\alpha(\theta)^{-4} \theta^d \beta(\theta)^2 \leq \text{const.}\ \theta^{4-d} \tag{28}$$

which tends to 0 , as $\theta \to \infty$, in dimension $d > 4$. One can use the infrared bound (21) to show that

$$\sum_{z_1 \ldots z_4}' \theta^{-d} \prod_{k=1}^{4} G_\theta(x_k, \theta^{-1} z_k) \leq \text{const.} \,, \tag{29}$$

uniformly in θ, provided $x_i \neq x_j$, for $i \neq j$, and $d > 4$. See[12,17]. By (27) - (29)

$$u_4^*(x_1,\ldots,x_4) = \lim_{\theta\to\infty} u_{4,\theta}(x_1,\ldots,x_4) = 0 \,, \tag{30}$$

at non-coinciding arguments, in $d > 4$ dimensions. Thus $\overline{d}_c = 4$, and the continuum limit is a free or generalized free field, provided $d > \overline{d}_c = 4$. For $d = 4$, the same result follows if either

(i) $\alpha(\theta)^2 / \beta(\theta)\theta^{d-2} \longrightarrow \infty$, or (31)

(ii) the limiting theory is scale-invariant. (Some uniformity in the limit of the two-point function is assumed; see[12]). In case (i), (27) and (31) yield (30), and the limiting theory has <u>non-canonical short distance behaviour</u>. In case (ii), triviality follows from a theorem of Pohlmeyer[27]. (Thus, in the language of the Callan-Symanzik equation, $\lambda\varphi_4^4$ is trivial, unless the β- and γ functions have a non-trivial common zero, and the corresponding theory is <u>not</u> scale-invariant). In[12] we have also proven a sharper form of (25) which suggests that the continuum limit in $d = 4$ is trivial, without any additional hypotheses, but we do not have a complete argument. Inequality (25) can also be used to calculate critical exponents. Combining it with an argument due to Glimm and Jaffe[21] one shows that the critical exponent γ of the susceptibility χ has its mean field value, $\gamma = 1$, in $d \geq 5$ dimensions[17,12]. It follows directly from (25) that if hyper scaling holds (i.e. the critical theory is non-trivial) then $\eta \leq 2 - \frac{d}{2}$, $d = 2,3,4$. See[12].

We conclude with some brief comments on the proof of the basic inequality (25). The proof in[12] relies on Symanzik's random walk -, or polymer representation of scalar Euclidean field theories[13,14]: One reexpresses a lattice field theory as a gas of random walks interacting through some soft core repulsion determined by $d\lambda(\varphi)$. Let $\omega_1,\ldots,\omega_n$ be n arbitrary random walks immersed in that gas and interacting with each other and with the (closed) random walks in the gas through the same soft core repulsion. Let $z(\omega_1,\ldots,\omega_n)$ be their joint correlation. Then

$$< \varphi(x)\varphi(y) >_\beta = \sum_{\omega:x\to y} z(\omega) \text{ , and}$$

$$u_{4,\beta}(x_1,\ldots,x_4) = \sum_p \sum_{\substack{\omega_1:x_{p(1)}\to x_{p(2)} \\ \omega_2:x_{p(3)}\to x_{p(4)}}} [z(\omega_1,\omega_2) - z(\omega_1)z(\omega_2)] \text{ .} \quad (32)$$

By a correlation inequality[12,14]

$$z(\omega_1,\omega_2) \geq z(\omega_1)z(\omega_2) \text{ ,} \quad (33)$$

<u>unless</u> ω_1 and ω_2 intersect. Thus

$$0 \geq u_{4,\beta}(x_1,\ldots,x_4) \geq - \sum_{p;\omega_1,\omega_2} z(\omega_1)z(\omega_2)\chi(\{\omega_1,\omega_2:\omega_1 \cap \omega_2 \neq \emptyset\}), \quad (34)$$

where $\sum_{p;\omega_1,\omega_2}$ is a short hand for the sum on the r.s. of (32). By requiring that $\omega_1 \cap \omega_2$ contains some lattice point z and then summing over all possible points z one can, after splitting ω_i into two walks, $i = 1,2$, and applying a Simon-type inequality[25,14], resum the r.s. of (34) to obtain (25). See[12], and[17] for an alternate, prior proof of a related inequality. Rather than discussing these technical aspects we emphasize that the r.s. of (34) should really vanish in the continuum limit, in dimension $d \geq 4$, because the random paths in the continuum are expected to have a Hausdorff dimension $D_H \leq 2$, so that, for $d \underset{(=)}{>} 4 \geq 2D_H$, two random paths do not intersect with probability 1. This is, in fact, a known theorem for Brownian paths in four or more dimensions[28]. Because of the repulsive character of the self-interaction, the field theoretic paths appear to have rather less tendency to intersect each other than Brownian paths and thus the r.s. of (34) is expected to vanish in the continuum limit in four dimensions. (I am indebted to T. Spencer for explaining such arguments to me).

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RESULTS AND PROBLEMS NEAR THE INTERFACE BETWEEN STATISTICAL MECHANICS AND QUANTUM FIELD THEORY

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Summary.

I present a brief survey of progress in the understanding of quantum field models, lattice gauge theories and spin systems that has been achieved during the past few years - roughly since the $M \cap \phi$ conference in Rome. Significant progress has occurred in the analysis of systems with abelian global or local symmetry groups, such as the Ising - and rotator models, one- and two-component $\lambda|\vec{\varphi}|^4$-theories and abelian lattice gauge theories, and in the mathematical foundations of renormalization group schemes for such systems. Some progress has been made with zero mass cluster expansions, with the construction of continuum gauge theories, and in surface physics.

A list of non-perturbative problems in non-abelian gauge theory and statistical physics concludes my notes.

Contents.

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I. Introduction.

A convenient point of orientation for these notes might be the author's contribution to the proceedings of the $M \cap \phi$ conference in Rome, in 1977, [1] . In that article I have emphasized the importance of non-perturbative methods in contemporary physics, proposed a list of typical, non-perturbative problems and then stated what was known, or could be hoped to be proved about them. For the convenience of the reader I state those problems, somewhat freely, once more :

A. Non-super-renormalizable field theories, infinite field strength - and charge renormalization.

B. Gauge theories in general (confinement, significance of topological field configurations,..., construction of super-renormalizable gauge theories, ...).

C. Super-selection sectors, topological charges, quantum solitons.

D. Critical phenomena, theory of critical points, mass generation, interactions of very long range.

E. Scattering of charged particles interacting with the radiation field.

What I knew (or, may be, was known in general) about either of those problems, in 1977, was certainly a bit disappointing, even to a mathematical physicist. (I did refer to or sketch a few perhaps somewhat interesting results which, however do not seem to have made much of an impression : I referred to work on quantum solitons in two space-time dimensions [2,3] where e.g. topological commutation relations between the fundamental field and the soliton field were derived and analyzed, and the mass of the soliton was estimated in terms of a surface tension, a quantity that can be expressed in terms of partition functions with "twisted boundary conditions". Although I was aware of the fact that such ideas could be applied to three space-time dimensional gauge theories, I did not write anything about this topic, and I did not understand the extension of those ideas to four dimensions and their implications for the confinement problem. As well known, these gaps have since been filled in; see [4,5] and refs. — I also explicitly mentioned arguments for the existence of a phase transition accompanied by the breaking of parity in two- and four-dimensional gauge theories with θ-vacua, at $\theta = \pi$. Those arguments are rigorous for some two-dimensional models [6] and were inspired by results of Coleman et al. [7] . In a somewhat different form they were finally rediscovered in 1980 [8,9].)

In the past few years, substantial progress has been made in solving the problems summarized above. Among the people who have posed and solved some of them are D. Brydges, E. Seiler and T. Spencer. I was fortunate to have had scientific contact with them. Thanks to their efforts and the efforts of M. Aizenman, T. Bałaban,

D. Buchholz and K. Fredenhagen, P. Federbush, G. Gallavotti et al., K. Gawedzki and A. Kupiainen, G. Mack et al., J. Magnen and R. Sénéor and others we now do have something to say about how to solve those problems. Some of the achievements have been described in Lausanne and at other places, like Cargèse, in 1979, see [10,11] and refs. to be found there.

As in Rome, I limit my review on the problems indicated under D , with some allusions to B . But what can now be said about D has interesting consequences for the problems mentioned under A. These notes are meant as comments on the lectures of M. Aizenman, K. Gawedzki, J. Imbrie and T. Spencer who have reported on specific, recent results concerning phase transitions, critical points or - intervals, scaling limits and other related topics in classical spin systems and quantum field models. Moreover, G. Mack has described recent results in lattice gauge theory. See also [12]. For sources on gauge theories other than his notes, see e.g. [13-16].

Much of the work referred to above has been inspired or is relying in a sense on work of J. Glimm and A. Jaffe - see [17,18] and refs. - Ja. G. Sinai - see [19] and refs. - and others, who were among the first to incorporate renormalization group ideas into mathematical physics.

There has been very impressive progress on the problems described under C and E above, as well. One should recall the work of Faddeev et al. [20] and Thacker et al. [21] on two-dimensional, completely integrable quantum field models, Jimbo, Miwa and Sato, see [22] and refs. given there, on the two-dimensional Ising model (they have understood how to make use of the topological commutation relations mentioned above in a very powerful way) and the work of Buchholz and Fredenhagen on the axiomatic theory of superselection sectors and scattering of charged particles. But these topics are not discussed in my notes.

2. Problems concerning phase transitions and critical phenomena and what we have learned about how to solve them.

As is well known, the mathematical structures of classical equilibrium statistical mechanics (ESM) and quantum field theory in the Euclidean description (EFT) - not involving Fermi fields - are identical. They are a branch of probability theory, namely the theory of random fields, in a wide sense. The following problems therefore come up in both, ESM and EFT.

2.1. Bulk problems.

Bulk problems concern the study of bulk thermodynamic functions and of the properties of equilibrium states (in ESM), or Euclidean vacuum functionals (in EFT).

Presently, there are good mathematical techniques to investigate those objects in three situations : Let $\langle(\cdot)\rangle_\beta$ denote an equilibrium state or a vacuum functional of some physical system. (In ESM, $\beta = (kT)^{-1}$ is the inverse temperature, in EFT $\beta = g^{-2}$ is the inverse square coupling or field strength).

(1) Small β : $\langle(\cdot)\rangle_\beta \approx$ uncorrelated ("ultralocal") state.

→ Convergent high temperature (strong coupling) expansions.

(2) Large β : $\langle(\cdot)\rangle_\beta \approx$ equilibrium state of ideal gas of defects [$\times$ Gaussian spin waves]

The defects may be Bloch walls, vortices... .

→ Combined low temperature - cluster expansions; rigorous spin wave theory.

As an asymptotic expansion one can often use (renormalization group improved) perturbation theory.

(3) Intermediate β , vicinity of critical points or - intervals : $\langle(\cdot)\rangle_\beta \approx$ perturbation of a critical state ("scaling distribution"), e.g. a Gaussian, by irrelevant and marginal operators.

→ Convergent renormalization group analysis; qualitative analysis of scaling limits, (based on group theoretical and bifurcation analysis, a priori bounds, correlation inequalities, etc.)

So far, successful, mathematically rigorous applications of (3) have been limited to the following situations : The scaling distribution is Gaussian (see [19,24] for a description of this class of critical states), and

(a) no relevant or marginal perturbations are present, scaling distribution is an attractive fixed point of renormalization group transformations; e.g. [25] , where a zero-mass cluster expansion is proposed;

(b) space of relevant perturbations of scaling distribution is empty, space of marginal distributions is one-dimensional (consisting of a quadratic polynomial in the field- or spin variables); [26,27,14,28]. While in [27,14] a combination of renormalization group techniques and a generalized Peierls argument has been applied to exhibit transitions in the two-dimensional rotator and related models and in higher dimensional, abelian lattice gauge theories, genuine real-space renormalization group transformations have been used in [28] to study charicatures of dipole gases.

(c) In [29] and subsequently in [30], it has been shown that the scaling limits of the correlation functions in the Ising- and rotator models and some class of lattice field theories are Gaussian in $d > 4$ dimensions, and some critical exponents have their mean field values. (Thus, in these examples, the fixed point of the renormalization group is Gaussian, the spaces of relevant and marginal perturbations are one-dimensional, and these perturbations preserve the Gaussian nature of the state).

In [29,30] qualitative methods, based on correlation inequalities and infrared bounds have been used. See also Sect. 3.

So far successful applications of renormalization group techniques have been limited to abelian models or models with linear fields. The reasons are two-fold : A technical reason is that duality transformations (linearizing the fields) and certain correlation inequalities, both only available in abelian models, have proved to be very useful and quite indispensable tools, [14,27,29,30]. A more fundamental reason is that for models with non-abelian symmetries and non-linear fields we do not know any scaling distributions (fixed points). Of course they would be non-Gaussian. Therefore one does not know any efficient renormalization group schemes, (except to estimate the ultraviolet cutoff dependence of unnormalized expectations [15]). It is conceivable, though perhaps not very likely, that Polyakov's solution of the string dynamics [31] may improve that situation for non-abelian gauge theories.

For a general description of real-space renormalization group transformations see e.g. [19] and for concrete, recent results the notes by Aizenman, Gawedzki and Spencer.

Impressive, new applications of combined low temperature - cluster expansions - see (2) - drawing some inspiration from the renormalization group are described in the notes of Imbrie (phase diagrams of $P(\phi)_2$ models) and Mack (confinement in the three-dimensional $U(1)$ model), and refs. given there.

2.2. Surface problems.

Typical examples of surface problems are :

- Rate of approach to the thermodynamic limit; (this is important e.g. for the theoretical evaluation of Monte Carlo calculations).
- Finite size scaling; see [31] and refs..
- Roughening transitions :
  In ESM, the instability of interfaces, e.g. in the three-dimensional Ising model.
  In EFT, the instability of electric or magnetic flux sheets in lattice gauge theory.
- Surface thermodynamics :
  In ESM, surface free energies and surface tensions,... .
  In EFT, the Casimir effect (see the notes by Symanzik), string tensions and ratios of partition functions with twisted boundary conditions in lattice gauge theory,...

- Physical effects of infinite surface layers inside infinite systems : Behaviour of state near such layers or near boundary; surface phase transitions.

For recent results see e.g. [33-36] and Sect. 7 of [27]. In [34] it is shown that in the three-dimensional Ising gauge theory a roughening transition would not coincide with the deconfining transition. In [36] it is proved - among other results - that if the surface tension in an abelian spin system vanishes then there is no interface; in particular, there are no interfaces in the rotator model in any dimension.

The methods described under (1) - (3) in Sect. 2.1 can also be applied to surface problems, but such applications tend to be much more complicated than the corresponding applications to bulk problems. In particular, I do not know of any mathematically rigorous renormalization group analysis of a genuine surface problem. Surface problems are likely to stay with us for quite some time.

3. The scaling limit of the $\lambda\varphi_d^4$ lattice field theory in $d \underset{(=)}{>} 4$ dimensions.

In this section I propose to explain some basic ideas behind the results (orginally due to Aizenman [29]) described in Sect. 2.1, (3), (c). I follow the approach taken in [30] which is based on very suggestive ideas of Symanzik [37].

Let φ denote a real scalar lattice field (or classical spin) with action (Hamilton function)

$$H(\varphi) = - \sum_{(jj')} \varphi_j \varphi_{j'} \quad , \tag{1}$$

where (jj') are nearest neighbors in $\mathbb{Z}^d$. The Euclidean vacuum functional (Gibbs state) of the system is given by the measure

$$d\mu_{\beta,\lambda}(\varphi) \equiv Z_{\beta,\lambda}^{-1} \, e^{-\beta H(\varphi)} \prod_j d\lambda(\varphi_j) \quad , \tag{2}$$

where $Z_{\beta,\lambda}$ is the partition function, and

$$d\lambda(\varphi_j) \equiv \exp[-\tfrac{\lambda}{4}\varphi_j^4 + \tfrac{\mu}{2}\varphi_j^2 - \varepsilon]d\varphi_j \ , \tag{3}$$

$0 < \beta < \infty$, $\lambda > 0$, μ and ε real. (The measure $d\mu_{\beta,\lambda}$ is to be understood as the thermodynamic limit of measures associated with finite sublattices and defined by (2). The limit exists by correlation inequalities). If $\mu = \lambda$, $\varepsilon = \frac{\lambda}{4}$ and $\lambda \to \infty$ we obtain the Ising model. The objects of our study are the scaling limits of the correlations

$$\langle \varphi_{x_1} \cdots \varphi_{x_{2m}} \rangle_{\beta,\lambda} \equiv \int \varphi_{x_1} \cdots \varphi_{x_{2m}} \, d\mu_{\beta,\lambda}(\varphi) \ , \tag{4}$$

defined by

$$G(x_1,\dots,x_{2m}) \equiv \lim_{\theta\to\infty} \alpha(\theta)^{2m} \langle\varphi_{\theta x_1}\dots\varphi_{\theta x_{2m}}\rangle_{\beta_\theta,\lambda_\theta} , \tag{5}$$

where β_θ and $d\lambda_\theta(\varphi)$ are chosen such that the system is in the single phase region, for all $\theta < \infty$, and a critical point is approached, as $\theta \to \infty$, (keeping e.g. the effective mass positive). Moreover, $\alpha(\theta)$ is determined by the condition that

$$0 < G(x,y) < \infty , \tag{6}$$

for $0 < |x-y| < \infty$. It follows from the infrared (spin wave) bound [38] that the lower bound in (6) can only hold if

$$\alpha(\theta) \geq \theta^{\frac{d}{2}-1} . \tag{7}$$

Let $u^{(4)}_{\beta,\lambda}$ be the connected (Ursell) four-point function. A new correlation inequality proven in [30] (see [29] for a related, prior result) says that if $x_i \neq x_j$, for $i \neq j$,

$$0 \geq u^{(4)}_{\beta,\lambda}(x_1,\dots,x_4) \geq -3\beta^2 \sum_{z_1,\dots z_4}{}' \prod_{k=1}^{4} \langle\varphi_{x_k}\varphi_{z_k}\rangle_{\beta,\lambda} , \tag{8}$$

where z_1 ranges over $\mathbb{Z}^d$, and $|z_\ell - z_1| \leq 1$, $\ell = 2,3,4$. (The upper bound is the well known Lebowitz inequality). Together with (5) and (7) inequality (8) implies that the scaled four-point Ursell function is bounded by

$$0 \geq \alpha(\theta)^4 u^{(4)}_{\beta_\theta,\lambda_\theta}(\theta x_1,\dots,\theta x_4) \geq -\theta^{4-d} 3\beta_\theta^2 \cdot$$

$$\cdot \Big(\sum_{z_1\dots z_4}{}' \theta^{-d} \prod_{k=1}^{4} \alpha(\theta)^2 \langle\varphi_{\theta x_k}\varphi_{z_k}\rangle_{\beta_\theta,\lambda_\theta} \Big)$$

From (6), the infrared bound of [38] and this inequality it follows easily that, for $d > 4$ and $x_i \neq x_j$, for $i \neq j$,

$$\lim_{\theta\to\infty} \alpha(\theta)^4 u^{(4)}_{\beta_\theta,\lambda_\theta}(\theta x_1,\dots,\theta x_4) = 0 . \tag{9}$$

See [29,30] for details. From (9) one may deduce that the scaling limits of all correlations are Gaussian (at non-coinciding arguments).

The basic idea behind the proof of inequality (8) and of (9) is to represent $u^{(4)}_{\beta,\lambda}$ as a sum over all pairs of random walks, ω_1,ω_2, connecting pairs $(x_{p(1)},x_{p(2)})$, $(x_{p(3)},x_{p(4)})$ of points. More precisely,

$$u^{(4)}_{\beta,\lambda}(x_1,\ldots,x_4) = \sum_{P} \sum_{\substack{\omega_1: x_{p(1)} \to x_{p(2)} \\ \omega_2: x_{p(3)} \to x_{p(4)}}} \{z(\omega_1,\omega_2) - z(\omega_1)z(\omega_2)\} \quad , \tag{10}$$

where $z(\omega_1,\ldots,\omega_k)$ is a correlation function of k open polymer chains, described by walks $\omega_1,\ldots\omega_k$, in a certain gas of closed polymers with soft core repulsive interactions [37,30] . Now, it is shown in [30] that all negative contributions to $u^{(4)}_{\beta,\lambda}$ come exclusively from those walks, ω_1 and ω_2 , which intersect each other. If the walks were ordinary random walks then the probability of intersection would approach 0 in the scaling ($\equiv$ continuum) limit in dimension $d \geq 4$, [39]. One would thus conclude that the scaling limit of $u^{(4)}_{\beta,\lambda}$ vanishes. This intuition is made precise in the form of inequality (8) which, however, only proves convergence to a Gaussian limit in dimension $d \geq 5$. One can improve inequality (8) in such a way that that result appears to extend to four dimensions, but there is only a partial result, so far, [30].

Next, suppose that

$$\langle \varphi_x \varphi_y \rangle_{\beta,\lambda} \lesssim \text{const.}\, |x-y|^{-(d-2+\eta)} \quad , \tag{11}$$

as $|x-y| \to \infty$, if β and λ are chosen so as to approach a critical point. We are interested in the nature of the scaling limit, assuming that (11) holds. Condition (6) and (11) now imply that

$$\alpha(\theta) \geq \theta^{(d-2+\eta)/2} \tag{12}$$

Using (8) and (12) it is then not hard to see that if $x_i \neq x_j$, for $i \neq j$,

$$\alpha(\theta)^4 u^{(4)}_{\beta_\theta,\lambda_\theta}(\theta x_1,\ldots,\theta x_4) \geq -\text{const.}\,\theta^{4-d-2\eta} \quad , \tag{13}$$

provided $d \geq 3$. Thus, the scaling limit of $u^{(4)}$ (at non-coinciding arguments) vanishes, unless

$$\eta \leq 2 - \frac{d}{2} \tag{14}$$

(For $d = 2$, (14) is always true, as shown by Simon). Thus if we can prove that the scaling limit is non-Gaussian (hyperscaling) then $\eta \leq 1$ $(d = 2)$, $\eta \leq 1/2$ $(d = 3)$, $\eta = 0$ $(d = 4)$. I recall that η is always non-negative in these models [38] and that $\eta = 0$ is only compatible with a non-Gaussian scaling limit if the latter is not scale-invariant. For further recent results on critical exponents ($\gamma = 1$, in $d \geq 5$) see [29,30]. A problem that has not been settled in [29,30] is to show that $\eta = 0$, in $d \geq 5$, or - more precisely - that the covariance of the Gaussian scaling limit of the $\lambda\varphi^4_{d \geq 5}$ theory is actually the standard free field two-point function, (as expected). The qualitative methods discussed here and the Block spin transformations used in [28] can presumably be extended to other systems : It seems likely that, using those techniques, one will be able to prove that the scaling limits of the $d \geq 2$ dimensional rotator and the $d \geq 4$ dimensional $U(1)$ lattice gauge theory are Gaussian in the low temperature region ($\beta > \beta_{crit.}$). Other applications may concern the self-avoiding random walk and percolation.

For further applications of Symanzik's polymer representation, (e.g. to a mass generation mechanism) see [30].

4. Open problems.

Here is a list of open problems which may keep us busy for the next several years.

1. Show that the physical mass of the two-dimensional N-vector models, with $N \geq 3$, is strictly positive, for all $\beta < \infty$.

2. Prove that pure, non-abelian lattice gauge theories (with Wilson- or Villain action) have deconfining transitions in dimension ≥ 5. (Is there a non-perturbative, rigorous form of spin- , or "glue wave" theory for such theories, analogous to the one for spin systems [38]?)

3. Exhibit permanent confinement in these models in dimension ≤ 4.

4. Prove the existence of a QED phase in the four-dimensional lattice Georgi-Glashow and Weinberg-Salam models without Fermions, at weak coupling.

5. Find efficient real-space renormalization group transformations and some of their fixed points for some non-trivial models with non-linear fields and/or non-abelian symmetries.

6. Develop concrete stochastic-geometric methods useful in statistical physics and Euclidean field theory. (Examples : Develop the statistical mechanics of defect gases. Prove convergence of the Regge-calculus (simplicial) approximation of the Euclidean string model to Polyakov's solution of that model [31], etc.)

7. Exhibit directional long range order in the two-dimensional jellium model at low temperature. Exhibit crystalline (translational) ordering in three-dimensional, (classical or quantum) particle systems at low temperature. Discuss the nature of the melting transition in three dimensions.

The problems described here are non-perturbative equilibrium problems.

However, the action may be in dynamical problems and the study of disordered and chaotic systems, during the coming years. (Would this not correspond to the state of the world ?) Disordered and dynamical systems theory, non-equilibrium statistical mechanics, fluid dynamics and turbulence are very active fields of research, and, quite generally, macroscopic physics seems to celebrate a comeback.

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