ON THE TRIVIALITY OF $\lambda\varphi_d^4$ THEORIES AND THE APPROACH

TO THE CRITICAL POINT IN $d \underset{(=)}{>} 4$ DIMENSIONS

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Abstract :

It is shown that one-and two-component $\lambda|\varphi|^4$ theories and non-linear σ-models in five or more dimensions approach free, or generalized free fields in the continuum (scaling) limit, and that in four dimensions there is no family of $\lambda|\varphi|^4$ theories to which renormalized perturbation theory is asymptotic. Some critical exponents for the lattice theories in five or more dimensions are shown to be mean field. The main tools are Symanzik's polymer representation of scalar field theories and correlation inequalities.

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1. In this note we sketch some ideas in the proofs of the results described in the abstract : triviality and approach to the critical point of $\lambda|\varphi|^4$ theories and non-linear σ-models in dimension $d \underset{(=)}{\geq} 4$. The one-component non-linear σ-model on a lattice is the usual Ising model, the two-component model is the classical rotor.

Our main results are related to some prior results of Aizenman [1] ; (see also [2,3] for some previous ideas on triviality). Our methods of proof, based on refs. [4,5] are, however, different from and complementary to his and yield some complementary information. They involve combining

i) Symanzik's representation of scalar field theories as models of interacting random walks or "polymer chains" [4] ; see [5,6] for further developments ;

ii) spin wave theory, in the form of infrared bounds [7] ;

iii) Ginibre's correlation inequalities [8] .

In one-and two-component $\lambda|\varphi|^4$ theories and σ-models on the lattice we establish an infinite family of new inequalities for each n-point correlation - (or Euclidean Green's) function, $n = 4,6,8,\ldots$, which express it in terms of the two-point function, i.e. the renormalized propagator, and the inverse temperature (field strength), β , but there is <u>no explicit dependence</u> on bare masses and charges. Those inequalities imply that, in the continuum limit, each n-point function approaches the n-point function of a (generalized) free field with the same two-point function, provided $d \underset{(=)}{\geq} 4$.

For simplicity, we only consider the four-point function of the one-component model in zero magnetic field. Details of our arguments and extensions of our results to more general models (including ones in a magnetic field) will be presented elsewhere.

2. Let $\mathbb{Z}_a^d$ denote the d-dimensional, simple hypercubic lattice with lattice spacing a measured in physical units, e.g. cm, and ranging over the interval $(0,1]$. At each site $j \in \mathbb{Z}_a^d$ there is attached a real-valued spin-or lattice field variable, φ_j, with a priori distribution

$$d\lambda(\varphi_j) = \exp[-\frac{\lambda}{4}\varphi_j^4 + \frac{\mu}{2}\varphi_j^2 - \varepsilon]\, d\varphi_j \quad , \tag{1}$$

where $d\varphi_j$ is Lebesgue measure, and $\lambda = \lambda(a)$, $\mu = \mu(a)$ and $\varepsilon = \varepsilon(a)$ are arbitrary functions of a with $\lambda(a) > 0$. If we set

$$\mu = \lambda R^2 \quad , \quad \varepsilon = \frac{\lambda}{4} R^4 \quad , \quad R > 0 \quad , \tag{2}$$

and let λ tend to $+\infty$ we obtain an Ising spin of length R.

The Hamilton function, or lattice action, of the models is defined by

$$H(\varphi) = - \sum_{(jj')} \varphi_j \varphi_{j'} \quad , \tag{3}$$

where $\sum_{(jj')}$ ranges over all pairs, (jj'), of nearest neighbors in $\mathbb{Z}_a^d$. The equilibrium state, or Euclidean vacuum functional, of the system at inverse temperature β is given by the measure

$$d\mu_{\beta,a}(\varphi) = Z^{-1} e^{-\beta H(\varphi)} \prod_j d\lambda(\varphi_j) \quad ,^{1)} \tag{4}$$

where Z is the usual partition function, and $\beta = \beta(a)$ is some positive function of a to be specified later. The correlation - , or Euclidean Green's functions are the moments of $d\mu_{\beta,a}$, i.e.

$$< \varphi_{x_1} \ldots \varphi_{x_n} >_{\beta,a} = \int \varphi_{x_1} \ldots \varphi_{x_n} d\mu_{\beta,a}(\varphi) \ .^{1)} \tag{5}$$

The distance, $|x-y|$, between two sites, x and y, of $\mathbb{Z}_a^d$ is measured in physical units, i.e. <u>lattice units multiplied by</u> a. We may choose

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1) Mathematically, these quantities are defined as limits of corresponding quantities in a finite region, Λ, of $\mathbb{Z}_a^d$, as $\Lambda \nearrow \mathbb{Z}_a^d$. The existence of the limit follows from correlation inequalities 8, 12.

$$a = a_n \equiv 2^{-n}\ [\mathrm{cm}] \quad , \quad n = 0,1,2,\ldots \quad ,$$

so that $\mathbb{Z}^d_{a_m} \subset \mathbb{Z}^d_{a_n}$, for $m < n$. If $a = 1$ we shall drop the subscript "a" . The two-point function in momentum space is given by

$$< |\hat{\varphi}(k)|^2>_{\beta,a} \equiv \sum_j a^d e^{ik\cdot j} <\varphi_o\varphi_j>_{\beta,a} \quad , \qquad (6)$$

where $k = (k_1,\ldots,k_d)$, $-\pi/a \leqq k_\alpha \leqq \pi/a$, $\alpha = 1,\ldots,d$. It is shown in [7] that

$$0 < <|\hat{\varphi}(k)|^2>_{\beta,a} \leqq M^2_{\beta,a}\delta(k) + \frac{\pi^2}{4}[\zeta(a)k^2]^{-1} \quad , \qquad (7)$$

where $\zeta(a) \equiv \beta(a)\, a^{2-d}$. Here $M^2_{\beta,a}$ is the long range order (spontaneous magnetization squared), and the second term on the r.h.s. of (7) is the spin wave contribution. From (7) and correlation inequalities one may derive

$$0 < <\varphi_o\varphi_x>_{\beta,a} \leqq M^2_{\beta,a} + c_d\, \zeta(a)^{-1}\, |x|^{2-d} \quad , \ d \geqq 3 \ , \qquad (8)$$

for some finite constant, c_d , independent of β and a ; see [9] . From (7) or (8) we conclude that

$$< \varphi_o\varphi_x >_{\beta,a} \rightarrow \text{const., as} \quad a \to 0 \ , \quad \text{for all} \quad x \ ,$$

<u>unless</u> $\quad 0 < \zeta(a) \leqq \zeta \overset{\text{e.g.}}{=} 1$, <u>for all</u> $a > 0$. (9)

In a theory with infinite field strength renormalization, i.e.

$$\dim\,[\varphi] \neq \frac{d-2}{2} \qquad (a \to 0) \quad ,$$

$$\lim_{a\to 0} \zeta(a) = 0 \quad , \qquad (10)$$

as follows from (8) .

The connected four-point, or Ursell function is defined by

$$u^{(4)}_{\beta,a}(x_1,..,x_4) \equiv \langle \varphi_{x_1}\varphi_{x_2}\varphi_{x_3}\varphi_{x_4}\rangle_{\beta,a}$$
$$- \sum_P \langle\varphi_{x_{P(1)}}\varphi_{x_{P(2)}}\rangle_{\beta,a}\langle\varphi_{x_{P(3)}}\varphi_{x_{P(4)}}\rangle_{\beta,a} \tag{11}$$

where $\sum_P$ ranges over all pairings, P, of $\{1,2,3,4\}$.

We can now state one of our main, new inequalities. Suppose that

$$|x_i - x_j| \geqq \delta \quad , \quad \text{for } i \neq j \quad ,$$

for some arbitrarily small, but positive δ. Then

$$0 \geqq u^{(4)}_{\beta,a}(x_1,\ldots,x_4) \geqq -\beta(a)^2 \sum_{z;z',z''} \langle\varphi_{x_{P(1)}}\varphi_z\rangle_{\beta,a} \cdot$$
$$\cdot \langle\varphi_{z'}\varphi_{x_{P(2)}}\rangle_{\beta,a}\langle\varphi_{x_{P(3)}}\varphi_z\rangle_{\beta,a}\langle\varphi_{z''}\varphi_{x_{P(4)}}\rangle_{\beta,a} + E(\beta,a) \tag{12}$$

where z ranges over $\mathbb{Z}^d_a$, $|z-z'| = |z-z''| = a$, and $E(\beta,a) \leqq \text{const. } \beta(a)^2\, a^{2-d}$.

The upper bound in (12) is the Lebowitz inequality [10], the lower bound is our new inequality. It is only useful if $M_{\beta,a} \equiv 0$; (if $M_{\beta,a} \neq 0$ other inequalities, discussed elsewhere, must be used).

3. We now show that, in $d \underset{(=)}{>} 4$ dimensions,

$$\lim_{a\to 0} u^{(4)}_{\beta,a}(x_1,\ldots,x_4) = 0 \quad , \tag{13}$$

provided $|x_i - x_j| \geqq \delta > 0$, for $i \neq j$, (i.e. at non-coinciding arguments), and $M_{\beta,a} \equiv 0$.

We recall that some limit of all correlation functions of the $\lambda\varphi^4$ lattice theories, as $a \to 0$, can always be constructed by using correlation inequalities and a compactness argument ; see [9] and references given there. The limiting correlation functions can be analytically continued in the time variables from the Euclidean region to the Minkowski region. It then follows from (13) that the connected four-point Wightman distribution in the limit $a = 0$ vanishes. (In fact, our inequalities can be used to show that all connected 2n-point Wightman distributions vanish, for $n = 2,3,\ldots$) Thus the theory is a free, or generalized free field.

Next, we state the <u>renormalization conditions</u>, which permit us to prove (13) in $d \geqq 5$ dimensions[2)]. We choose $\lambda(a)$, $\mu(a)$ and $\beta(a)$ such that

$$\left.\begin{aligned} &0 < \zeta(a) \equiv \beta(a)a^{2-d} \leqq 1 \text{ ; see (9) ,} \\ &M_{\beta,a} \equiv 0 \quad , \qquad \text{and} \\ &0 < \overline{\lim_{a\to 0}} <\varphi_0\varphi_x>_{\beta,a} \leqq \sup_a <\varphi_o\varphi_x>_{\beta,a} < \infty \quad , \end{aligned}\right\} \tag{14}$$

for $x = ze_1$, $0 < z < 1$, where e_1 is the unit vector in the 1-direction of $\mathbb{E}^d$. (If $\overline{\lim_{a\to 0}} <\varphi_o\varphi_x>_{\beta,a} = 0$, for all choices of $\lambda(a)$, $\mu(a)$ and $\beta(a)$ compatible with (9) and with $M_{\beta,a} \equiv 0$ then the field of the limiting theory vanishes identically, an uninteresting case). It is known that, in two and three dimensions, $\lambda(a)$, $\mu(a)$ and $\beta(a)$ can be chosen such that condition (14) holds, and, for sufficiently small, bare coupling, $\lim_{a\to 0} u^{(4)}_{\beta,a}(x_1,\ldots,x_4) \neq 0$. See [12,13] , (also [9,14] for some recent results). It appears that when $d \geqq 4$ there is no complete proof of the compatibility of condition (14), although there are some partial results [2,14] and, for $d \geqq 5$, a proof seems accessible, due to the absence of infrared divergences.

By (12) and the definition of $\zeta(a)$

$$u^{(4)}_{\beta,a}(x_1,\ldots,x_4) \geqq -\zeta(a)^2\, a^{d-4} \sum_{z;z',z''} a^d <\varphi_{x_{P(1)}}\varphi_z>_{\beta,a} \cdot$$

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2) We work with renormalized, rather than bare fields; see Sect. 7 .

$$\cdot \langle \varphi_{z'} \varphi_{x_{P(2)}} \rangle_{\beta,a} \langle \varphi_{x_{P(3)}} \varphi_z \rangle_{\beta,a} \langle \varphi_{z''} \varphi_{x_{P(4)}} \rangle_{\beta,a} + E(\beta,a) \tag{15}$$

Using (8) and (14) one sees that

$$\zeta(a) \sum_{z;z',z''}^{<} a^d \langle \varphi_{x_{P(1)}} \varphi_z \rangle_{\beta,a} \langle \varphi_{z'} \varphi_{x_{P(2)}} \rangle_{\beta,a} \cdot$$
$$\cdot \langle \varphi_{x_{P(3)}} \varphi_z \rangle_{\beta,a} \langle \varphi_{z''} \varphi_{x_{P(4)}} \rangle_{\beta,a} \leqq C_\delta < \infty \quad , \tag{16}$$

where $\Sigma^<$ ranges over all z with

$$\mathrm{dist}_{[cm]}(z,\{x_1,\ldots,x_4\}) \leqq 1 \quad ,$$

and C_δ is a finite constant (for $\delta > 0$) .

Moreover, one can show that

$$E(\beta,a) \leqq C'_\delta \ \zeta(a)\ a^{d-2} \quad . \tag{17}$$

Finally

$$\zeta(a)^2 \sum_{z;z',z''}^{>} a^d \langle \varphi_{x_{P(1)}} \varphi_z \rangle_{\beta,a} \langle \varphi_{z'} \varphi_{x_{P(2)}} \rangle_{\beta,a} \cdot$$
$$\cdot \langle \varphi_{x_{P(3)}} \varphi_z \rangle_{\beta,a} \langle \varphi_{z''} \varphi_{x_{P(4)}} \rangle_{\beta,a} \tag{18}$$
$$\leqq C'' \sum_z a^d \, |z-x_{P(1)}|^{2-d} \, |z-x_{P(3)}|^{2-d} \leqq C''' \quad ,$$

where $\Sigma^>$ ranges over all z with

$$\mathrm{dist}_{[cm]}(z,\{x_1,\ldots,x_4\}) \geqq 1 \quad ,$$

and C'' , C''' are uniform constants. Inequality (18) follows from (8), (14) (combined with physical positivity) and the fact that for $d \geqq 5$

$$\sum_z^{>} |z-x|^{2-d} \, |z-y|^{2-d} \leqq C^{IV} < \infty \quad .$$

Thus, for $d \geqq 5$ and arbitrary $\delta > 0$, (12) and (16)-(18) yield

$$0 \geqq u^{(4)}_{\beta,a}(x_1,\dots,x_4) \geqq - a^{d-4}(k_\delta\ \zeta(a) + k'),$$

for finite constants k_δ and k', which, together with (9), proves (13).

In four dimensions, the bounds (16) and (17) are still useful, but (18) must be improved. This requires a condition on the propagator $< \varphi_x\varphi_y >_{\beta,a}$ which is stronger than (14), namely, for $|x-y| > 1$

$$< \varphi_x\varphi_y >_{\beta,a} \leqq k\,|x-y|^{-\varepsilon} \quad , \tag{19}$$

for some constants $\varepsilon \in (0,1)$ (arbitrarily small) and $k < \infty$ independent of a. In the limit $a = 0$, inequality (19) follows from the Källen-Lehmann representation with $\varepsilon \geqq d-2$, if $\lim\limits_{a\to 0} < \varphi_x\varphi_y >_{\beta,a}$ is Euclidean invariant. By (8), (14) and (19)

$$0 < < \varphi_x\varphi_y >_{\beta,a} \leqq \zeta(a)^{-p}\ k^{1-p}\ |x-y|^{-2p-\varepsilon(1-p)} \quad , \tag{20}$$

with $p \overset{\text{e.g.}}{=} (4-3\varepsilon)(4-2\varepsilon)^{-1} < 1$, $d = 4$.

Thus, by (19) and (20)

$$\begin{aligned} &\zeta(a)^2 \sum_{z;z',z''}^{>} a^d < \varphi_{x_{P(1)}}\varphi_z >_{\beta,a} \cdots < \varphi_{z''}\varphi_{x_{P(4)}} >_{\beta,a} \\ &\leqq C^V\ \zeta(a)^{\varepsilon/(2-\varepsilon)} \quad . \end{aligned} \tag{21}$$

Thus, at non-coinciding arguments,

$$\lim_{a\to 0} u^{(4)}_{\beta,a}(x_1,\dots,x_4) = 0 \quad , \tag{22}$$

provided

$$\lim_{a\to 0} \zeta(a) = 0 \quad . \tag{23}$$

We now recall that if renormalized perturbation theory were asymptotic for the coefficients in the Callan-Symanzik equation then

$$(\lim_{a\to 0} \langle\varphi_x\varphi_y\rangle_{\beta,a})\cdot|x-y|^2 \to +\infty, \text{ as } y \to x \quad , \tag{24}$$

for small values of the renormalized coupling constant, g , defined e.g. by

$$g = \bar{u}^{(4)}\ \chi^{-2}\ \xi^{-4} \quad ,$$

where

$$\bar{u}^{(4)} = \sum_{x,y,z} \lim_{a\to 0} u^{(4)}_{\beta,a}(0,x,y,z) \quad ,$$

$$\chi = \sum_{x} \lim_{a\to 0} \langle \varphi_0\varphi_x \rangle_{\beta,a} \quad ,$$

and ξ is the correlation length of the limiting theory [9]. By (8), (24) holds only if

$$\lim_{a\to 0} \zeta(a) = 0 \quad ,$$

which implies triviality, i.e. (22). This, however, contradicts the asymptoticity of perturbation theory. Thus, in <u>four dimensions</u>, there does not exist any family of $\lambda\varphi^4$ theories with the property that renormalized perturbation theory is asymptotic to the coefficients in the Callan-Symanzik equation, and any $\lambda\varphi_4^4$ theory with infinite field strength renormalization, i.e. non-canonical ultraviolet dimension, is trivial.

If $\lim_{a\to 0} \zeta(a) = \zeta > 0$,

i.e. dim $[\varphi] = 1$, the limiting theory cannot be Euclidean - and scale - invariant, unless it is a free field. This is a general theorem, due to Pohlmeyer [16] (extending the Federbush-Johnson theorem to the massless case). Thus, a non-trivial $\lambda\varphi_4^4$ theory would have to be <u>asymptotically free</u>. This possibility is presumably ruled out by an inequality which sharpens (15), (see sect. 6, inequalities (46) and (47), and [15]), but there is, to date, no complete proof.

4. Next, we summarize some results on critical exponents for the Ising - and the $\lambda\varphi^4$ models on the lattice $\mathbb{Z}^d = \mathbb{Z}^d_{a=1}$, $d \geqq 5$. Similar results have been found independently by Aizenman [1]. Previously, Sokal already proved that the specific heat of these models in five or more dimensions is finite [17].

We fix λ and μ and study the behaviour of the correlations when $\beta \nearrow \beta_c$, where β_c is the inverse of the critical temperature. (It is shown in [7] that $\beta_c < \infty$, for $d \geqq 3$). We define

$$\chi(\beta) = \sum_x < \varphi_o \varphi_x >_\beta \quad , \quad \text{(susceptibility)} \quad . \tag{25}$$

It is shown in [2,18] that

$$\chi(\beta) \nearrow \infty \quad , \quad \text{as} \quad \beta \nearrow \beta_c \quad .$$

One new result is the following : In five or more dimensions , for the Ising model or the $\lambda\varphi^4$ lattice theory,

$$\chi(\beta) \sim (\beta_c - \beta)^{-1} \quad , \quad \text{as} \quad \beta \nearrow \beta_c \quad , \tag{26}$$

i.e. the critical exponent, γ , of $\chi(\beta)$ takes its <u>mean field value</u> $\gamma = 1$. (See also [1]).

The proof involves deriving a lower and an upper bound, $\propto \chi(\beta)^2$, for $\frac{\partial}{\partial\beta}\chi(\beta)$, by using the Lebowitz inequality and the new inequality (15), or related inequalities in [1,14] . Combining these inequalities with an argument, due to Glimm and Jaffe [2b], one obtains (under slightly more restrictive hypotheses on the coupling constants)

$$\frac{\partial}{\partial\beta}\xi(\beta)^{-2} \sim Z(\beta) \quad , \tag{27}$$

for $\beta < \beta_c$ and $\beta_c - \beta$ small. Here $\xi(\beta)$ is the correlation length (inverse physical mass), and $Z(\beta)$ is the strength of the one particle pole in $<\varphi_x\varphi_y>_\beta$, at zero spatial momentum.

We conjecture that

$$Z(\beta) \geqq Z_o > 0 \quad , \tag{28}$$

for $\beta < \beta_c$, in $d \geqq 5$ dimensions. This would imply

$$\xi(\beta) \sim (\beta_c-\beta)^{-1/2} \quad ,$$

which is the mean field result.

Conjecture (28) is a stronger form of the conjecture that

$$\eta = 0 \quad , \quad \text{for} \quad d \geqq 5 \quad , \tag{29}$$

where η is the critical exponent defined by

$$< \varphi_x\varphi_y >_\beta \underset{|x|\to\infty}{\simeq} \quad \text{const.} \quad |x-y|^{2-d-\eta} \quad .$$

In [15] we propose a strategy for proving (29) , but there is no complete proof. The proofs of (26) and (27) are straightforward consequences of the lower and upper bounds on $u_\beta^{(4)}$ and are given in [15] .

5. We now describe some of the ideas which go into the proof of the basic lower bound (12) on $u_\beta^{(4)}$. We start by recalling the random walk representation of the correlation functions of a lattice $\lambda\varphi^4$ theory or an Ising model developed in [5,6] which is inspired by Symanzik's work [4] . Without loss of generality we may set $a = 1$.

We define

$$d\rho_n(s) = \begin{cases} \delta(s)ds & \text{if } \ n = 0 \\ \\ \dfrac{s^{n-1}}{(n-1)!} \cdot \chi_{[0,\infty)}(s)ds & \text{if } \ n = 1,2,3,\dots , \end{cases}$$

where $\chi_{[0,\infty)}$ is the characteristic function of the positive half axis. Let ω be a random walk starting at some site x in $\mathbb{Z}^d$ and ending at another site y. Let $n_j(\omega)$ be the number of visits of ω at some site j. We define

$$d\rho_\omega(t) \equiv \prod_j d\rho_{n_j(\omega)}(t_j) \quad . \tag{30}$$

Let $|\omega|$ be the total number of nearest-neighbor steps made by ω. We note that

$$dP(\omega;t) = \beta^{|\omega|} \prod_j e^{-(2d\beta+m^2)t_j} d\rho_{n_j(\omega)}(t_j)$$

is the random walk analogue of the Wiener measure conditioned on random walks with killing rate m starting at x and ending at y. The variable t_j is the total amount of time spent by ω at site j.

We now define a t-dependent partition function

$$Z(t) \equiv \int e^{-\beta H(\varphi)} \prod_j g(\varphi_j^2 + 2t_j)d\varphi_j \quad , \tag{31}$$

where $g(\varphi^2) = \frac{d\lambda(\varphi)}{d\varphi}$, see (1). Furthermore

$$z(t) \equiv \frac{Z(t)}{Z} > 0 \quad . \tag{32}$$

Then

$$< \varphi_x \varphi_y >_\beta = \sum_{\omega: x\to y} \beta^{|\omega|} \int d\rho_\omega(t)\, z(t) \ . \tag{33}$$

If $\lambda = 0$, $\mu = -(2d\beta+m^2)$ we find

$$z(t) = \prod_j e^{-(2d\beta+m^2)t_j} ,$$

so that

$$< \varphi_x\varphi_y >_{\beta,\lambda=0} = \sum_{\omega:x\to y} \int dP(\omega;t) = (-\beta\Delta+m^2)^{-1}_{xy} ,$$

where Δ is the finite difference Laplacean. As expected, this is the two-point function of the free (Gaussian) lattice field with mass $\frac{m}{\sqrt{\beta}}$.

For the four-point function one finds

$$u^{(4)}_\beta(x_1,\ldots,x_4) = \sum_P \sum^P_{\omega_1,\omega_2} \beta^{|\omega_1|+|\omega_2|} \int d\rho_{\omega_1}(t^1)d\rho_{\omega_2}(t^2)\cdot \tag{34}$$

$$\cdot [z(t^1+t^2) - z(t^1)z(t^2)]$$

where $\sum^P_{\omega_1,\omega_2}$ ranges over all walks ω_1 and ω_2 , with $\omega_1 : x_{P(1)} \to x_{P(2)}$, $\omega_2 : x_{P(3)} \to x_{P(4)}$. Formulas (31)-(34), along with many applications, can be found in [5] .

We now define

$$\mathring{z}(t) = \prod_j e^{\lambda t_j^2} z(t) . \tag{35}$$

If we write $\ell n\, \mathring{z}(t^1+t^2)$ as an integral over derivatives in t^1 and t^2 and use Ginibre's inequality [8] to estimate the integrand we obtain [15]

$$\mathring{z}(t^1+t^2) \geq \mathring{z}(t^1)\, \mathring{z}(t^2) . \tag{36}$$

By combining (34)-(36) we obtain the lower bound

$$u^{(4)}_\beta(x_1,\ldots,x_4) \geq \sum_P \sum^P_{\omega_1,\omega_2} \beta^{|\omega_1|+|\omega_2|} \int d\rho_{\omega_1}(t^1)z(t^1)\cdot \tag{37}$$

$$\cdot\, d\rho_{\omega_2}(t^2)z(t^2)[\exp(-2\lambda \sum_j t_j^1 t_j^2)-1] .$$

Note that if $j \notin \omega_i$, $t^i_j = 0$, i.e. the r.h.s. of (37) vanishes, unless

$$\omega_1 \cap \omega_2 \neq \emptyset \quad . \tag{38}$$

If $z(t) = \prod_j \exp[-2d\beta t_j]$ a term on the r.h.s. of (37) corresponding to a given P is proportional to the probability that a standard random walk $\omega_1 : x_{P(1)} \longrightarrow x_{P(2)}$ and a walk $\omega_2 : x_{P(2)} \longrightarrow x_{P(3)}$ intersect. It is well known that, in the scaling limit $(a \to 0)$, that probability vanishes in four or more dimensions. [19] This fact together with representation (37) represent the basic intuition behind the proof of the vanishing of $u^{(4)}_{\beta,a}$ in the limit $a \to 0$. In order to make it precise and derive the lower bound in (12) we must resum the r.h.s. of (37). From (37) and (38) we obtain

$$u^{(4)}_\beta(x_1,\ldots,x_4) \geq - \sum_P \sum^P_{\omega_1,\omega_2} \beta^{|\omega_1|+|\omega_2|} \chi(\{\omega_1,\omega_2 : \omega_1 \cap \omega_2 \neq \emptyset\}) .$$

$$\cdot \int d\rho_{\omega_1}(t^1) d\rho_{\omega_2}(t^2)\; z(t^1)\; z(t^2) \quad .$$

By the exclusion-inclusion principle,

$$\begin{aligned} &\chi(\{\omega_1,\omega_2 : \omega_1 \cap \omega_2 \neq \emptyset\}) \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \sum_{\substack{z_1,\ldots,z_n \\ z_i \neq z_j}} \chi(\{\omega_1,\omega_2 : \omega_1 \cap \omega_2 \ni \{z_1,\ldots,z_n\}) \\ &\leq \sum_{n=1}^{2m+1} \frac{(-1)^{n-1}}{n!} \sum_{\substack{z_1,\ldots,z_n \\ z_i \neq z_j}} \chi(\{\omega_1,\omega_2 : \omega_1 \cap \omega_2 \ni \{z_1,\ldots,z_n\}) \quad , \end{aligned} \tag{39}$$

for $m = 0,1,2,\ldots$.

If we set $m = 0$ we obtain

$$u_\beta^{(4)}(x_1,\dots,x_4) \geqq - \sum_P \sum_{z\in\mathbb{Z}^d} \sum_{\substack{\omega_1,\omega_2 \\ \omega_1\cap\omega_2\ni z}}^{P} \beta^{|\omega_1|+|\omega_2|} \cdot \tag{40}$$

$$\cdot \int d\rho_{\omega_1}(t^1)z(t^1)\cdot \int d\rho_{\omega_2}(t^2)z(t^2) \ .$$

Next, if $z \notin \{x_1,x_2,x_3,x_4\}$ we decompose ω_i , $i = 1,2,$ into two paths

$$\begin{aligned} &\omega_1' : x_{P(1)} \longrightarrow z \ , \ \omega_1'' : z' \longrightarrow x_{P(2)} \ , \text{ with } (zz') \in \omega_1 \\ &\omega_2' : x_{P(3)} \longrightarrow z \ , \ \omega_2'' : z'' \longrightarrow x_{P(4)} \ , \text{ with } (zz'') \in \omega_2 \end{aligned} \tag{41}$$

and then sum independently over ω_1' , ω_2', z' , z'' , ω_1'' , ω_2'' . In this summation we overcount the terms on the r.h.s. of (40) and therefore obtain a further lower bound on $u_\beta^{(4)}$. (A separate, but straightforward analysis is required when $z \in \{x_1,\dots,x_4\}$, but the corresponding contribution vanishes when $a \to 0$). We now use the following identity : Let $\omega' : x \to z$, $\omega'' : z' \to y$, with (zz') nearest neighbors. Then

$$\begin{aligned} &\sum_{\omega',\omega''} \beta^{|\omega'|+|\omega''|+1} \int d\rho_{\omega'\circ(zz')\circ\omega''}(t)\ z(t) \\ &= \beta \sum_{\omega',\omega''} \beta^{|\omega'|+|\omega''|} \int d\rho_{\omega'}(t')\ d\rho_{\omega''}(t'')\ z(t'+t'') \\ &= \beta < \varphi_x\varphi_z >_\beta < \varphi_{z'}\varphi_y >_\beta + \\ &\beta \sum_{\omega',\omega''} \beta^{|\omega'|+|\omega''|} \int d\rho_{\omega'}(t')d\rho_{\omega''}(t'')\ [z(t'+t'')-z(t')z(t'')], \end{aligned} \tag{42}$$

where $\omega'\circ(zz')\circ\omega''$ is the path obtained by composing ω' , (zz') and ω'' . We define a t-dependent two-point function [5]

$$< \varphi_x\varphi_y >_\beta (t) \equiv Z(t)^{-1} \int e^{-\beta H(\varphi)} \varphi_x\varphi_y \prod_j g(\varphi_j^2+2t_j)d\varphi_j$$

$$= z(t)^{-1} \sum_{\omega:x\to y} \beta^{|\omega|} \int d\rho_\omega(s)\; z(s+t) \tag{43}$$

Using (43), the second term on the r.h.s. of (42) can be resummed to yield

$$\beta \sum_{\omega',\omega''} \beta^{|\omega'|+|\omega''|} \int d\rho_{\omega'}(t')d\rho_{\omega''}(t'')[z(t'+t'')-z(t')z(t'')]$$

$$= \beta \sum_{\omega'} \beta^{|\omega'|} \int d\rho_{\omega'}(t')z(t')[< \varphi_{z'}\varphi_x >_\beta(t')-< \varphi_{z'}\varphi_x >_\beta] \;.$$

By Ginibre's inequality [8]

$$< \varphi_{z'}\varphi_x >_\beta (t') \leqq < \varphi_{z'}\varphi_x >_\beta \quad , \tag{44}$$

for $t' \geqq 0$. Thus the second term on the r.h.s. of (42) is negative. (For more details, see [5,14,15]) .

If we insert (42) on the r.h.s. of (40), using (41) and (44), we finally obtain

$$u_\beta^{(4)}(x_1,..,x_4) \geqq -\beta^2 \sum_P \sum_{z;z',z''} < \varphi_{x_{P(1)}}\varphi_z >_\beta < \varphi_{z'}\varphi_{x_{P(2)}} >_\beta \cdot$$

$$\cdot <\varphi_{x_{P(3)}}\varphi_z > < \varphi_{z''}\varphi_{x_{P(4)}} >_\beta + E(\beta)$$

where $E(\beta)$ takes into account the correction required when $z \in \{x_1,x_2,x_3,x_4\}$. This completes our sketch of the proof of (12).

6. There are various refinements of inequality (12) which we believe might be important for the analysis of mean-field behaviour in $d \geq 5$ dimensions and of the four-dimensional $\lambda\varphi^4$ theory :

I) By using the upper bound (39) for $\chi(\{\omega_1,\omega_2 : \omega_1 \cap \omega_2 \neq \emptyset\})$ we obtain a lower bound on $u^{(4)}$ for each choice of $m = 0,1,2,\ldots$. Given m , that lower bound can actually be expressed as a sum over all "skeleton diagrams" ($\equiv$ Feynman diagrams without any self-energy subdiagrams) of order $2m+1$, computed according to the following "Feynman rules" :

i) Each propagator is given by the full two-point function, $< \varphi_x\varphi_y >_\beta$.

ii) Each vertex corresponds to

$$-\beta^2 \sum_{x_1,\ldots,x_4} \delta_{x_1 z}\, \delta_{x_2 z}\, \delta_{x_3 z'}\, \delta_{x_4 z''} \quad ,$$

where $x_1,\ldots,x_4$ are arguments of propagators attached to the given vertex which is localized at z , and z' , z'' are nearest neighbors of z .

These lower bounds appear to be useful in a refined analysis of the lattice $\lambda\varphi^4_d$ theory or the Ising model in $d \geq 5$ dimensions, (with $a = 1$) for which all skeleton diagrams are infrared convergent, because the renormalized propagator is square summable ; see (8) .

Odd order lower bounds and even order upper bounds on $u_\beta^{(4)}$ in terms of skeleton diagrams with vertices proportional to the coupling constant λ have previously been proven in [14] . For applications, see [14,15] .

II) We now outline another refinement of the basic lower bound (15) on $u_\beta^{(4)}$ which is potentially useful to complete the analysis of the four-dimensional $\lambda\varphi^4$ theory and to show that it is trivial even if $\lim_{a\to 0} \zeta(a) > 0$. We reinstall the lattice spacing, a . First, we recall the lower bound (40) on $u^{(4)}_{\beta,a}$, i.e.

$$u^{(4)}_{\beta,a}(x_1,..,x_4) \geqq - \sum_P \sum_{z\in\mathbb{Z}^d_a} \sum^{P}_{\substack{\omega_1,\omega_2 \\ \omega_1\cap\omega_2\ni z}} \beta^{|\omega_1|+|\omega_2|} \cdot$$

$$\cdot \int d\rho_{\omega_1}(t^1)z(t^1) \int d\rho_{\omega_2}(t^2)z(t^2) \quad .$$

We orient ω_1 to be directed from $x_{P(1)}$ to $x_{P(2)}$. Given some fixed path ω_2 , choose z to be the last point at which ω_1 intersects ω_2 . Adhering to the notations introduced in (41) we then obtain

$$u^{(4)}_{\beta,a}(x_1,..,x_4) \geqq -\beta \sum_P \sum_{z;z'} \sum_{\omega_1',\omega_1''} \sum_{\omega_2:x_{P(3)}\to x_{P(4)}} \beta^{|\omega_1'|+|\omega_1''|+|\omega_2|} \cdot$$

$$\cdot \chi(\{\omega_1'',\omega_2:(zz')\circ\omega_1''\cap\omega_2=\{z\}\}) \int d\rho_{\omega_1'}(t')d\rho_{\omega_1''}(t'')z(t'+t'') \cdot$$

$$\cdot \int d\rho_{\omega_2}(t^2)z(t^2) \quad . \qquad (45)$$

We now appeal to an argument used in [5] (new proof of Lieb's inequality [20]) to show that the r.h.s. of (45) can be resummed over ω_1' and ω_1'' to yield

$$u^{(4)}_{\beta,a}(x_1,\dots,x_4) \geq -\beta \sum_P \sum_{\substack{z\in\mathbb{Z}^d_a \\ z':|z'-z|=a}} \sum_{\omega_2} \beta^{|\omega_2|} \cdot \qquad (46)$$

$$\cdot < \varphi_{x_{P(1)}} \varphi_z >_{\beta,a} < \varphi_{z'}, \varphi_{x_{P(2)}} >^{(\omega_2)}_{\beta,a} \int d\rho_{\omega_2}(t^2) z(t^2),$$

where $< \varphi_x \varphi_y >^{(\omega_2)}_{\beta,a}$ is the two-point function of the theory with Dirichlet boundary conditions along the walk ω_2 , i.e.

$$< \varphi_x \varphi_y >^{(\omega_2)}_{\beta,a} = \lim_{t\to\infty} < \varphi_x \varphi_y >_{\beta,a} (t^{(\omega_2)}) ,$$

with

$$t^{(\omega_2)}_j = \begin{cases} t & \text{if} \quad j \in \omega_2 \\ 0 & \text{otherwise} \end{cases} .$$

We expect that, for $|x_{P(2)}-z'| \geq \varepsilon > 0$ (measured in cm) ,

$$< \varphi_{z'}, \varphi_{x_{P(2)}} >^{(\omega_2)}_{\beta,a} \leq K_\varepsilon (\log a^{-1})^{-\kappa(\omega_2)} , \qquad (47)$$

for some constant K_ε which is independent of a and finite for all $\varepsilon > 0$, and for some $\kappa(\omega_2)$ which is positive for almost all ω_2 , (with respect to the weight $\beta^{|\omega_2|} \int d\rho_{\omega_2}(t^2)\, z(t^2)$) . Although we have no proof of (47) the following heuristic considerations suggest that it ought to be true.

a) If $\lim_{a\to 0} \zeta(a) > 0$ we expect that $<\varphi_x \varphi_y>^{(\omega_2)}_{\beta,a}$ behaves qualitatively like $(-\Delta^{(\omega_2)}_a)^{-1}_{x,y}$, where $\Delta^{(\omega_2)}_a$ is the finite difference Laplacean on $\mathbb{Z}^d_a$ with zero Dirichlet data imposed on the set of sites visited by ω_2 .

b) In the scaling limit $(a \to 0)$ the Hausdorff dimension of a typical path is 2 . We therefore expect that

$$< \varphi_{z'} \varphi_{x_{P(2)}} >^{(\omega_2)}_{\beta,a} \sim (-\Delta_a^{(\omega_2)})^{-1}_{z',x_{P(2)}}$$

$$\sim (\log a^{-1})^{-\kappa(\omega_2)} \quad ,$$

with $\kappa(\omega_2) > 0$, almost surely. This bahaviour is expected, since $|z'-x_{P(2)}| \geqq \varepsilon > 0$ and dist $(z',\omega_2) = a \to 0$. (Our considerations are motivated by the analogy with Brownian motion [19]).

The renormalization condition (19) and inequalities (15), (46) and (47) would imply (under suitable assumptions on the behaviour of $\kappa(\omega_2)$) that, in four dimensions,

$$\lim_{a\to 0} u^{(4)}_{\beta,a}(x_1,\ldots,x_4) = 0 \quad ,$$

at non-coinciding arguments, even if $\lim_{a\to 0} \zeta(a) > 0$.

7. We conclude this note by describing some related problems and results, (in particular concerning the self-avoiding random walk).

a) An alternative way of constructing the continuum limit of a lattice field theory consists of keeping the lattice spacing, a , fixed and analyzing the <u>scaling limit</u>. One then works with bare fields, φ^o , introduces a scale parameter $\theta = 1,2,3,\ldots$ and chooses functions $\beta(\theta)$ and $\alpha(\theta)$ with the following properties :

i) $\beta(\theta) \nearrow \beta_c$, as $\theta \nearrow \infty$, (e.g. in such a way that the scaled correlation length, $\theta^{-1}\xi(\beta(\theta))$, remains bounded); and

ii) $\lim_{\theta\to\infty} \alpha(\theta)^2 < \varphi^o_{\theta x} \varphi^o_{\theta y} >_{\beta(\theta)} \equiv G(x-y)$ exists.

From the infrared bound, i.e.

$$< \varphi_x \varphi_y >_\beta \leqq \text{const. } \beta^{-1} |x-y|^{2-d} \quad ,$$

for $\beta \leqq \beta_c$, $d \geqq 3$, we conclude that

$$\alpha(\theta) \geqq \theta^{(d-2)/2} \quad . \tag{48}$$

By setting $a = \theta^{-1}$, $\varphi_x = \alpha(\theta)\, \varphi^0_{\theta x}$, this approach is seen to be completely equivalent to the one used in this note, where the lattice spacing a tends to 0 , distances are kept fixed in physical units, and the renormalized field, φ , is used. See e.g.[9]. (Our results then yield triviality of the scaling limit, $\theta \to \infty$, in $d \underset{(=)}{\geq} 4$ dimensions).

b) If in formulas (33) and (34) one sets

$$z(t) \equiv z_{\lambda,\xi}(t) = \prod_j e^{-\xi t_j - \lambda t_j^2}$$

one obtains the Edwards model of the self-suppressing random walk [21] on the lattice. The function

$$G_\beta(x,y) \equiv \sum_{\omega: x \to y} \beta^{|\omega|} \int d\rho_\omega(t) z_{\lambda,\xi}(t)$$

is a weighted sum over all random walks with self-suppression starting at x and ending at y . It corresponds to the two-point function $< \varphi_x \varphi_y >_\beta$ of the $\lambda\varphi^4$ theory. We also define

$$G^{(4)}_\beta(x_1,y_1;x_2,y_2) \equiv \sum_{\substack{\omega_1: x_1 \to y_1 \\ \omega_2: x_2 \to y_2}} \beta^{|\omega_1|+|\omega_1|} \int d\rho_{\omega_1}(t^1) z_{\lambda,\xi}(t^1) \;.$$
$$\cdot \; d\rho_{\omega_2}(t^2) z_{\lambda,\xi}(t^2) [e^{-2\lambda \sum_j t^1_j t^2_j} - 1] \;. \tag{49}$$

We note that the integrand on the r.h.s. of (49) vanishes, unless $\omega_1 \cap \omega_2 \neq \emptyset$. Obviously the function G^4_β, more precisely

$$\sum_P G^{(4)}_\beta (x_{P(1)}, x_{P(2)}; x_{P(3)}, x_{P(4)})$$

is the analogue of $u^{(4)}_\beta$. Since

$$z_{\lambda,\xi}(t^1) z_{\lambda,\xi}(t^2) [e^{-2\lambda \Sigma_j t^1_j t^2_j} - 1]$$

$$= z_{\lambda,\xi}(t^1 + t^2) - z_{\lambda,\xi}(t^1) z_{\lambda,\xi}(t^2) ,$$

inequality (37) is saturated in the present model. Therefore all inequalities previously derived for the connected four-point function, $u^{(4)}_\beta$, of the lattice $\lambda\varphi^4$ theory extend to $G^{(4)}_\beta$. (The proof follows directly from (37), (39) and the repulsive character of the interaction between different random walks). These inequalities are useful in trying to show that in five or more dimensions, in the scaling limit, two self-suppressing random walks with non-coinciding starting - and ending points do never intersect with probability 1. This would follow by showing that $G^{(4)}_\beta$ vanishes in the scaling limit. As remarked already, $G^{(4)}_\beta$ satisfies the upper and lower bound in (12). The obstruction against completing the proof that $G^{(4)}_\beta$ vanishes in the scaling limit is that one does <u>not</u> know for this model that $G_\beta(x,y)$ satisfies a spin wave upper bound of the form of (8).

For fixed λ and ξ, let β_c be the supremum over all values of β for which $G_\beta(x,y)$ has exponential decrease. We conjecture that in three or more dimensions

$$G_{\beta_c}(x,y) \leq \text{const. } |x-y|^{2-d-\eta} , \tag{50}$$

for some $\eta \geqq 0$.

If (50) turned out to be true, as expected, then the function $\alpha(\theta)$ introduced above would satisfy inequality (48), and it would then follow from (12) that $G_\beta^{(4)}$ (at non-coinciding arguments) vanishes in the scaling limit, (i.e. two self-suppressing random walks do not intersect), in five or more dimensions.

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