HIGGS PHENOMENON WITHOUT SYMMETRY BREAKING ORDER PARAMETER

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Abstract.

We discuss symmetry breaking order parameters, e.g. $\langle\varphi\rangle$, in gauge theories with Higgs scalars, φ, in suitable gauges. We show that, typically, $\langle\varphi\rangle = 0$. A complete set of gauge-invariant, observable composite fields for such theories, local ones and ones localized near strings (paths) is constructed. We then examine the validity of standard perturbation theory, based on assuming that $\langle\varphi\rangle \neq 0$, and reformulate it in terms of our gauge-invariant fields and without assuming that $\langle\varphi\rangle \neq 0$. Finally, we classify classical field configurations with non-trivial topology ("defects") in such theories and propose a defect-gas approach to predict their effects.

Introduction

The continuum formulation based on perturbation methods and the lattice (Wilson) formulation[1] of gauge quantum field theories seemingly lead to contradictory results, in particular when applied to Higgs models, since in the Wilson formulation all the gauge-dependent Green's functions vanish and there cannot be spontaneous symmetry breaking[2]. These conflicting pictures are reexamined in Sect. 1; the differences between the two formulations disappear if a gauge fixing is introduced on the lattice (Sect.2). We then investigate the existence of a "symmetry breaking order parameter" in Higgs models. The Higgs expectation value $< \varphi >$ is shown to vanish in the temporal gauge, and, more generally, the gauge-invariant two-point function of the Higgs field is shown to have an exponential decay (Sect. 3). In Sect. 4, the vanishing of $< \varphi >$ is shown to be a disorder effect induced by "defects", i.e. by field configurations with non-trivial topology (instantons, etc.).

The above results necessitate a change in the usual treatment of the Higgs phenomenon, in the continuum formulation. It is shown that a complete description of the Higgs phenomenon in terms of gauge-invariant fields is <u>always</u> possible; (in particular, the physical states and the mass spectrum can be obtained by using only gauge-invariant fields). This description does not rely on the existence of a symmetry breaking order parameter, but only on the existence of a non-trivial orbit minimizing the Higgs potential. The mass matrix of the standard (perturbative) formulation is reproduced when one neglects the quantum fluctuations of the Higgs fields. The existence and construction of the complete set of gauge-invariant fields is obtained as a general group-theoretical result. No assumption is therefore required about the Higgs representation or about the non-existence of phase transitions between the Higgs-and the confinement regimes (Sects. 5, 6).

In Sect. 7, we discuss the validity of a perturbative expansion in the presence of defects. It is argued that the defect density is asymptotically zero when $g \to 0$, and a perturbation expansion of the functional integral for gauge-invariant Green's functions is asymptotic and coincides with the standard perturbation theory based on a non-zero order parameter $\bar{\varphi} = \langle \varphi \rangle$. Furthermore, the Green's functions of gauge-invariant fields reduce to the Green's functions of the corresponding gauge-dependent fields of the standard formulation when the Higgs quantum fluctuations are much smaller than the radius of the minimizing orbit. Finally some typical features of defect gases and some physical effects of defects are sketched.

1. Gauge-invariant description of gauge quantum field theories.

By definition, a gauge theory is a quantum field theory with a gauge group acting as a local symmetry group. Conventionally, it is formulated in terms of fields which transform non-trivially under the action of local gauge transformations. However, there is an important physical constraint: All observable quantities are required to be gauge-invariant. Therefore, it must be possible to describe the entire physical contents of a gauge quantum field theory in terms of gauge-invariant (possibly non-local) fields.

A possible approach towards implementing this proq-program consists of formulating gauge theories on the lattice in terms of a manifestly gauge-invariant action, as proposed by Wilson[(1)]. In Wilson's approach the only non-vanishing Green's functions are those which are invariant under all local gauge transformations. In particular, in a lattice theory with Higgs fields, the expectation value, $< \varphi >$, of the scalar field φ vanishes, and it is impossible to have a spontaneous breaking of the gauge group[(2)]. At first sight, this obscures the interpretation of the lattice theories in terms of conventional wisdom[(3)] concerning theories like QED, the Higgs model or grand-unified theories, and connections with standard perturbation theory are no longer evident.

One of the main purposes of this paper is to compare the gauge-invariant lattice formulation with the conventional one and to clarify those connections.

The conventional approach to gauge quantum field theory does not automatically realize a manifestly gauge-invariant formulation of gauge theories, since it involves gauge fixing and the use of gauge-dependent fields and their Green's functions in an essential way. In fact, gauge fixing and gauge-dependent Green's functions are the basic building blocks of the perturbative expansion and the calculation of S-matrix elements in terms of Feynman diagrams. Implicit in the standard approach is the classification of the physical states in terms of their transformation properties under constant gauge transformations. In the light of the remarks made at the beginning it is difficult to understand the physical meaning of a gauge group which acts trivially on all the observables, so that states related by a gauge transformation are not physically distinguishable.

The rôle of gauge-dependent Green's functions appears particularly puzzling in the conventional treatment of the Higgs mechanism in terms of the gauge-dependent order parameter $< \varphi >$.

The resolution of this puzzle is somewhat important if one wants to understand whether a gauge-invariant order parameter exists and whether the Higgs mechanism is accompanied by some sort of phase transition and "symmetry breaking", without relying on semi-classical and perturbative arguments. (In perturbation theory, the order parameter $< \varphi >$ is put in "by hand", as suggested by the classical Higgs potential, but there is really no guarantee that "symmetry breaking" occurs in a non-perturbative treatment).

A non-perturbative approach to the Higgs phenomenon is important in order to achieve a better understanding of physical problems like gauge hierachies, generation of fermion masses (which, in the conventional approach, depend on the value of the gauge-dependent order parameter $< \varphi >$), the occurrence of very different mass scales, the existence of elementary particles associated with the Higgs field, etc.

2. Can the Higgs phenomenon be characterized by a symmetry breaking local order parameter?

In order to discuss the problems mentioned in Section 1 we use the formulation of (gauge) quantum field theory in terms of Euclidean functional integrals. We shall usually think of a regularization of these functional integrals by means of a lattice cutoff, i.e. we adopt Wilson's strategy[(1)], and we think of the continuum theory as a scaling limit of the infinite volume lattice theory (the existence of that limit being assumed or taken for granted[a)]). Since we do not want to rely on perturbation theory, the lattice regularization appears particularly attractive.

In order to understand the formal relation between lattice gauge theories and the usual (Faddeev-Popov) continuum formulation[(4)] we start with a few well known comments on gauge fixing in lattice theories.

The functional measure of a lattice theory is given by

$$d\mu(\chi) = Z^{-1}\ e^{-\beta A(\chi)}\ d\chi$$

where χ is a family of lattice fields, including a lattice gauge field, $d\chi$ is their apriori distribution, and A is a gauge-invariant action <u>without</u> gauge fixing terms. In this case the measure $d\mu$ is well known to be <u>invariant under local gauge transformations</u> (irrespectively of what boundary conditions are used to construct the infinite volume limit). This entails the <u>vanishing</u> of gauge-dependent Green's functions[(2)]:

For any gauge transformation, $g = g(x)$, localized inside the lattice region V,

$$d\mu^{g}_{V,\sigma}(\chi) \equiv d\mu_{V,\sigma}(\chi^{g}) = d\mu_{V,\sigma}(\chi)$$

(where σ indicates a choice of boundary conditions) so that for any function A of the fields χ localized inside V

$$\langle A\rangle_{V,\sigma} \equiv \int d\mu_{V,\sigma}(\chi)\, A(\chi) = \int d\mu^{g^{-1}}_{V,\sigma}(\chi)\, A(\chi)$$
$$= \int d\mu_{V,\sigma}(\chi)\, A^{g}(\chi) \equiv \langle A^{g}\rangle_{V,\sigma}$$

Thus, in this approach, gauge-dependent Green's functions which play a basic rôle in the standard formulation of continuum gauge quantum field theories[b)] are actually zero.

As emphasized by Faddeev and Popov[(4)] a gauge fixing appears necessary in the continuum formulation. In order to compare it with the lattice formulation it is desirable to study gauge fixing for lattice theories, as well. The basic property of a gauge fixing F is to modify the functional integral in such a way that expectation values of gauge-invariant functions of the field variables χ remain unchanged, so that the physical contents of the theory is unaffected. Thus, if $e^{-\beta A}$ is replaced by $e^{-\beta A}F$ in the expression for $d\mu$ the condition stated above is equivalent to

$$\int \prod_{x} dg(x)\, F^{g} = 1$$

This is essentially the Faddeev-Popov condition, with F standing for the product of a gauge fixing times the Faddeev-Popov determinant. The latter can be represented in terms of an integral over Faddeev-Popov ghosts, but that representation is not particularly useful outside perturbation theory. (See[5] for a discussion of gauge fixing in lattice theories.)

In most of the following we use only gauge fixings with the following two additional properties[c)]: a) F is a local function, and b) F is invariant under global (constant) gauge transformations.

As examples of gauge fixings on the lattice satisfying a) and b) we mention:

1) $F = \prod_x \delta(g_{x,x+e_0})$ with g_{xy} the gauge field on the lattice (formally $g_{xy} = P\exp[ig\int_x^y A_\mu(x')dx'^\mu]$) and e_0 a unit vector in the time direction, which reproduces the temporal gauge on the lattice[5], and

2) $F = \exp\frac{1}{\xi}\sum_x \mathrm{Tr}\{\prod_\mu (g^{-1}_{x-e_\mu,x}\, g_{x,x-e_\mu})\}$, (formally $F \to \exp\{-\frac{1}{\xi}\int dx \sum_a (\partial^\mu A^a_\mu(x))^2\}$ as the lattice spacing $a \to 0$) corresponds to the ξ-gauges.

The lattice formulation <u>with gauge fixing</u> makes it possible to discuss non-perturbative effects in the continuum formulation. In particular, one may now answer the question of the existence of a symmetry breaking local order parameter ($\langle\varphi\rangle \neq 0$) as a possible caracterization of the Higgs phenomenon, once the gauge has been fixed. Since now the value of $\langle\varphi\rangle$ depends in general on the boundary conditions, the question is whether the gauge fixing gives rise to a sufficiently strong coupling with the boundary. We remark that the relevance of such an analysis for the continuum theory relies on the assumption that the Euclidean continuum theory (in a pure phase) can be obtained as a scaling limit of the infinite volume lattice theory. We also note that the above general strategy makes use of the analytic continuation of the correlation functions from Euclidean to Minkowski space, a property which is not under control for some gauges,(because of lack of positivity), except for gauge-invariant Green's functions.

In the next Section we will prove the vanishing of $< \varphi >$ in the temporal gauge. We shall actually prove a more general result, namely the exponential decay of the <u>gauge invariant</u> two point function of the Higgs field.

In Section 4 the vanishing of $< \varphi >$ is shown to be a disorder effect induced by non-perturbative,defect-like field configurations;(instantons, vortices etc.).

3. Vanishing of $< \varphi >$ in the temporal gauge and exponential decay of the gauge-invariant two-point function of the Higgs field.

Before dealing with the temporal gauge we will first introduce a change of variables which works in the general case and which simplifies the analysis of the gauge-invariant two-point function of the Higgs field. As a matter of fact, we will deduce the vanishing of $< \varphi >$ in the temporal gauge from the exponential decay of that gauge-invariant two-point function.

i) *Exponential decay of the gauge-invariant two-point function.*

The analysis will be done on a lattice with fixed spacing a. The Higgs field $\varphi(x)$ can be written as

$$\varphi(x) = R(h_x)\, r_x \quad , \tag{3.1}$$

where R denotes the Higgs representation, h_x is an element of the gauge group G and r_x is a point of the section of the orbits of the gauge group in the Higgs field representation. R may be reducible and it may contain orbits with non-trivial residual group. Clearly, given $\varphi(x)$, eq. (3.1) does not uniquely determine h_x; nevertheless one can show that the functional measure can be expressed in terms of the variables h_x, r_x and the gauge field variables $g_\mu(x)$. Now $g_\mu(x)$ can be written in terms of h_x and the gauge-invariant variables $\ell_{\mu,x}$:

$$g_\mu(x) \equiv g_{x,x+e_\mu} \equiv h_x\, \ell_{\mu,x}\, h^{-1}_{x+e_\mu} \tag{3.2}$$

and one can replace the old variables h_x, r_x, $g_\mu(x)$ by the new variables $\ell_{\mu,x}$, r_x and $g_0(x)$. Independence and completeness of the new variables follow from the equations

$$\prod_n g_0(x+ne_0) = h_x \prod_n \ell_{0,x+ne_0}\, h_B^{-1} \quad , \tag{3.3}$$

$$h_x = \left(\prod_n g_0(x+ne_0)\right) h_B \left(\prod_n \ell_{0,x+ne_0}\right)^{-1} , \tag{3.4}$$

where h_B denotes the value of h_x at the boundary of the lattice volume $(\varphi_B = R(h_B)r_B)$. In terms of the new variables the Faddeev-Popov condition for $F(\varphi,g)$

$$\int_G \prod_x d\chi_x \; F(\chi_x(g), \chi_x(\varphi)) = 1 ,$$

(χ_x denoting a gauge transformation), becomes

$$\int_G \prod_x d\chi_x \; \tilde{F}(\ell_\mu, \chi_x(g), r_x) = 1 . \tag{3.5}$$

Putting $\chi_x' \equiv \chi_x \, g_0(x) \, \chi_{x+e_0}^{-1}$ and using the invariance of the measure

$$\prod_x d\chi_x' = \prod_x d\chi_x ,$$

we get

$$\int \prod_x dg_0(x) \; \tilde{F}(\ell, g_0(x), r) = 1 . \tag{3.6}$$

We can now write the expectation value of a gauge-invariant variable A as

$$\langle A \rangle = Z^{-1} \int \prod_x d\mu(r_x) \prod_{\mu,x} d\ell_{\mu,x} \prod_x dg_0(x) \; A \; e^{\beta \sum_p \chi_p(\ell)} \cdot$$
$$\cdot \; e^{\beta' \sum_{\mu,x} (r_x, R(\ell_{\mu,x}) r_{x+e_\mu})} , \tag{3.7}$$

where $\chi_p(\ell) = \chi_p(g)$ is the plaquette character. The gauge-invariance of A implies that A is independent of g_0 and therefore by eq. (3.7) the integration over g_0 is trivial

$$\langle A \rangle = Z^{-1} \int \prod_x d\mu(r_x) \prod_{\mu,x} d\ell_{\mu,x} \; A \; e^{\beta \Sigma_p \chi_p(\ell)} \cdot$$
$$\cdot \; e^{\beta' \sum_{\mu,x} (r_x, R(\ell_{\mu,x}) r_{x+e_\mu})} . \tag{3.8}$$

Then, for fixed μ, the correlation functions of the gauge invariant variable $\ell_{\mu,x}$ are given by

$$\langle \ell_{\mu,x_1} \ell_{\mu,x_2} \dots \ell_{\mu,x_m} \rangle = Z^{-1} \int \prod_x d\mu(r_x) \prod_{\nu\neq\mu,x} d\ell_{\nu,x} \prod_x d\ell_{\mu,x}$$
$$\ell_{\mu,x_1} \dots \ell_{\mu,x_m} \; e^{\beta \Sigma_p \chi_p(\ell)} \; e^{\beta' \sum_{\lambda,x} (r_x, R(\ell_{\lambda,x}) r_{x+e_\lambda})}$$
$$= Z^{-1} \int \prod_x d\mu(r_x) \prod_{\nu\neq\mu,x} d\ell_{\nu,x} \; e^{\beta \sum_{P_\perp} \chi_{P_\perp}(\ell)}$$
$$e^{\beta' \sum_{\nu\neq\mu,x} (r_x, R(\ell_{\nu,x}) r_{x+e_\nu})} \int \prod_x d\ell_{\mu,x} \; e^{\beta \sum_{\nu\neq\mu,x} \chi(\ell_{\nu,x} \ell_{\mu,x+e_\nu} \ell^{-1}_{\nu,x+e_\mu} \ell^{-1}_{\mu,x})} \cdot$$
$$e^{\beta' \sum_x (r_x, R(\ell_{\mu,x}) r_{x+e_\mu})} \; \ell_{\mu,x_1} \dots \ell_{\mu,x_m} , \tag{3.9}$$

where $\sum_{P_\perp}$ is the sum over the plaquettes not containing bonds in the μ direction. For convenience, we call <u>horizontal planes</u> the subsets of the lattice points with fixed μ components $\{x; x_\mu = \text{const}\}$. In the last integral, for fixed r_x, $\ell_{\nu,x}$ $(\nu \neq \mu)$, there is no coupling between different horizontal planes. Furthermore, we notice that our expectation values are of the form

$$\int d\mu(x) \int d\nu_x(y)\, f(y) \,,$$

with $$\int d\mu(x) \int d\nu_x(y) = 1 \,. \tag{3.10}$$

Then, defining $Z_x \equiv \int d\nu_x(y)$, we can rewrite the above integral (3.10) as

$$\int d\mu(x)\, Z_x \int Z_x^{-1}\, d\nu_x(y)\, f(y) \equiv \int d\tilde{\mu}(x) \int d\tilde{\nu}_x(y)\, f(y) \,,$$

with

$$\int d\tilde{\nu}_x(y) = 1 \,, \qquad \forall x \,,$$
$$1 = \int d\tilde{\mu}(x) \int d\tilde{\nu}_x(y) = \int d\tilde{\mu}(x) \,. \tag{3.11}$$

Coming back to eq. (3.9), with this normalization we obtain an expression of the form

$$\int d\mu(\ell_{\nu\neq\mu}, \tau)\, Z_{\ell_{\nu\neq\mu},\tau} \int Z^{-1}_{\ell_{\nu\neq\mu},\tau}\, d\nu_{\ell_{\nu\neq\mu},\tau}(\ell_\mu)\, f(\ell_\mu) =$$
$$= \int d\tilde{\mu}(\ell_{\nu\neq\mu}, \tau) \int d\tilde{\nu}_{\ell_{\nu\neq\mu},\tau}(\ell_\mu)\, f(\ell_\mu) \,,$$

with $d\tilde{\mu}$, $d\tilde{\nu}$ satisfying eqs. (3.11).

The last integral is therefore the expectation value of a product of bond variables on a lattice, with coupling only along horizontal planes. Variables belonging to different planes are therefore statistically independent. In conclusion, if the points $x_1 \ldots x_n$ in eq. (3.9) belong to different planes

$$\langle \ell_{\mu,x_1} \ldots \ell_{\mu,x_n} \rangle = \int \langle \ell_{\mu,x_1} \rangle_{d\tilde{\nu}} \cdots \langle \ell_{\mu,x_n} \rangle_{d\tilde{\nu}}\, d\tilde{\mu}$$

Now $< \ell_{\mu,x} >_{d\tilde{\nu}}$ is analogous to a magnetization at finite temperature (if β, β' are finite), and one can prove that

$$\left| < \ell_{\mu,x} >_{d\tilde{\nu}_{\ell_{\nu\neq\mu},r}} \right| \leq \theta < 1 , \tag{3.12}$$

uniformly in the variables $\ell_{\nu\neq\mu}$, r.

Therefore

$$\left| < \ell_{\mu,x_1} \ldots \ell_{\mu,x_n} > \right| \leq \theta^n \int d\tilde{\mu} = \theta^n, \tag{3.13}$$

(x_1, ... x_n belonging to different horizontal planes $(x_i)_\mu \neq (x_j)_\mu$, $\forall$ i,j).

Equ. (3.13) also holds when ℓ is replaced by $R(\ell)$, where R is <u>any</u> representation different from the trivial one. To simplify notations, we shall omit the symbol R in the following.

By using the definitions (3.1), (3.2), we have

$$h_x^{-1}\, g_{x,x_1}\, g_{x_1,x_2} \ldots g_{x_n,y}\, \ell_y \equiv h_x^{-1} \prod_j g_{x_j,\, x_j+e_{\mu_j}}\, \ell_y = \prod_j \ell_{\mu_j,x_j} . \tag{3.14}$$

Hence, the gauge invariant two-point function of the Higgs field becomes

$$< h_x^{-1}\, (\text{string})_{xy}\, h_y > = < \prod_j \ell_{\mu_j,x_j} > \tag{3.15}$$

To compute the above expectation value we fix a direction μ such that the number of $\mu_j = \mu$ on the r.h.s. of eq. (3.15) is greater than $|y - x|/4$. We consider only strings with a fraction of "regular links" (in the direction μ) greater than some fixed number α; a link in the direction μ is regular if no other link in the direction μ lies in the same horizontal plane (with respect to the direction μ).

Under these conditions we have

$$< M > \equiv < \prod_j \ell_{\mu_j,x_j} > = \int d\tilde{\mu}\,(\ell_{\nu\neq\mu}, r) \int d\tilde{\nu}_{\ell_{\nu\neq\mu},r}\,(\bar{\ell}_\mu)$$

$$\int d\tilde{\nu}_{\ell_{\nu\neq\mu},r}\,(\ell_\mu^{reg})\;\; M\,(\ell_{\nu\neq\mu}, \bar{\ell}, \ell^{reg}) =$$

$$= \int d\tilde{\mu} \int d\tilde{\nu}(\bar{\ell}, r)\ M\left(\ell_{\nu+\mu}, \bar{\ell}, \langle \ell^{reg} \rangle_{d\tilde{\nu}(\ell^{reg}, r)} \right)$$

where we have separated the integration over regular (ℓ^{reg}) and "non-regular" ($\bar{\ell}$) links.

Hence

$$|\langle \prod_j \ell_{\mu_j, x_j} \rangle| \le \int d\tilde{\mu} \int d\tilde{\nu}(\bar{\ell}, s) \prod |\langle \ell^{reg}_{\mu, x} \rangle_{d\tilde{\nu}(\ell^{reg}, r)}| \le \\ \le \theta^{\alpha|x-y|/4a} = e^{-\alpha|\log\theta|\,|x-y|/4a} \tag{3.16}$$

Thus the gauge invariant two-point function has an exponential decay; in particular, one cannot build from it a gauge-invariant order parameter.

We remark that this statement is at least formally stable under taking the continuum limit. In fact, in this case one can show that the string (3.15) becomes the string of a massive vector field with mass $M^2 = \beta^{-1}\beta'\langle r_x \rangle^2$.

ii) *Vanishing of* $\langle \varphi \rangle$ *in the temporal gauge.*

The above formalism becomes very simple in the temporal gauge because the gauge fixing restricts $g_0(x) = 1$, and the expectation values of gauge independent variables are easy to discuss since one is left only with integration over gauge independent variables (the integration over $g_0(x)$ being trivial). The gauge dependent order parameter $\langle h_x \rangle$ vanishes in this gauge as can be easily seen by using eq. (3.4)

$$h_x = \prod_n g_0(x + ne_0)\, h_B \left(\prod_n \ell_{0, x+ne_0} \right)^{-1}$$

and the previous (gauge-invariant) result on the expectation value of $\prod_n \ell_{0, x+ne_0}$. In fact, applying eq. (3.16)

$$|\langle \prod_j \ell_{\mu_j, x_j} \rangle| \leq e^{-\alpha |\log \theta| |x-y| / 4a}$$

to a string connecting h_x with h_B in the time direction one gets the exponential decoupling of h_x from the boundary value h_B.

We remark that the gauge freedom left by the gauge fixing $g_0(x) = 1$ (i.e. the freedom of making gauge transformations independent of time) disappears as soon as one specifies the (euclidean) boundary condition $\varphi = \varphi_B$. Therefore, one cannot simply appeal to the above gauge freedom to conclude that $\langle \varphi \rangle = 0$.[d)]

More general results on the vanishing of expectations of gauge-dependent operators, based on a technique in statistical mechanics, due to Dobrushin and Shlosman, will be discussed elsewhere.

4. Restauration of symmetry as a disorder effect induced by configurations with non-trivial topology.

In this section we show that under general assumptions the vanishing of $\langle \varphi \rangle$ can be understood as a disorder effect ("restauration of symmetry") induced by non-perturbative field configurations with non-trivial topology. Such configurations will be called "defects" in analogy to defects in ordered systems. By "defects" we mean classical field configurations which in some sense dominate the functional integral. (On the lattice the same rôle is played by configurations approximating defect configurations of the continuum theory; see Sect.7). A general analysis of defects and their uses in an approximate evaluation of functional integrals in Higgs models is contained in Section 7.

We now give an argument showing that topological defects may cause $\langle \varphi \rangle$ to vanish. Our argument only involves "point defects", i.e. instantons, but the contributions of other types of defects (vortices, magnetic flux lines) increase the disorder. Since disorder effects caused by instantons increase with the instanton density ρ, it is enough to consider a low density approximation. Actually, in Higgs models, the instanton size is essentially bounded by the scale appearing whenever the Higgs potential has a non-trivial minimum; in turn this is related to the existence of dimensional terms in the effective Higgs potential (e.g. the "mass" term). As a consequence one expects the instanton density ρ to be finite and of the order of $m_H^4 \, e^{-c/g^2}$, where m_H is a typical mass parameter occurring in the Higgs potential. The smallness of the instanton size and their low density make it plausible that one may neglect the interactions between (anti)instantons. This is actually all what is needed for the following argument (a more general and more abstract argument will be presented at the end of this Section). Under the hypothesis discussed above one can prove that necessarily $\langle \varphi \rangle = 0$. In fact

$$\langle \varphi(x) \rangle = Z^{-1} \int d\varphi \, e^{-S(\varphi)} \varphi \simeq$$

$$Z^{-1} \sum_{n_+ = n_- = 0}^{\infty} (n_+!)^{-1} (n_-!)^{-1} \int dX \, dY \, e^{-S_{cl}(X,Y)} \langle \varphi(x) \rangle_{X,Y} , \tag{4.1}$$

where $Z \equiv \int d\varphi \, e^{-S(\varphi)} \simeq e^{const. V} \to \infty$ as the (lattice) volume $V \to \infty$, S_{cl} is the classical value of the action for configurations corresponding to n_+ instantons localized in $(x_1, \dots x_{n_+}) \equiv X$ and to n_- antinstantons localized in $(y_1, \dots y_{n_-}) \equiv Y$, so that

$$S_{cl}(X,Y) \simeq (n_+ + n_-) S_0 . \tag{4.2}$$

One can evaluate $\langle \varphi \rangle_{X,Y}$ approximately as follows:

First, note that in a Higgs theory the (anti-)instanton has a finite scale size $r \simeq O(m_H^{-1})$. Thus it occupies a space-time volume $v \simeq r^4$ (See Sect. 7.4).

We now consider an instanton configuration with instantons at positions X and anti-instantons at positions Y. Suppose x is a space-time point separated from $\{X,Y\}$ by a distance $d \gg r$. Then the gauge field has the form of a pure gauge (apart from a contribution which decays rapidly in d), i.e.

$$A(x) \approx \prod_{i=1}^{n_+} g_{x_i}(x) \prod_{j=1}^{n_-} \bar{g}_{y_i}(x) \, \partial_\mu \left(\prod_{j=1}^{n_-} \bar{g}_{y_j}^{-1}(x) \prod_{i=1}^{n_+} g_{x_1}^{-1}(x) \right)$$

Thus

$$\langle \varphi(x) \rangle_{X,Y} \approx \prod_{i=1}^{n_+} R_\varphi(g_{x_i}(x)) \prod_j^{n_-} R_\varphi(\bar{g}_{y_j}(x))\, \varphi_0 \,, \tag{4.4}$$

where R_φ is the representation of the gauge group G under which the Higgs scalar transforms, and φ_0 = const. is a minimum of the Higgs potential.

Now

$$V^{-1} \Big\| \int_{u \in V,\, |u-x| > r} d^4u\, R_\varphi(\overset{(-)}{g}_u(x)) \Big\| \le 1 - \delta \equiv \xi \,, \tag{4.5}$$

for some $\delta > 0$, uniformly in V, for V large enough. (Since $\overset{(-)}{g}_u(x)$ is a function of $x-u$ the average is actually over $x-u$ and ξ vanishes except for small boundary effects).

If we now insert (4.4) and (4.5) in (4.1) we obtain

$$|\langle \varphi(x) \rangle| \le Z_V^{-1} \sum_{n_+ = n_- = 0}^{\infty} (n_+!)^{-1} (n_-!)^{-1} e^{-(n_+ + n_-)S_0} V^{n_+ + n_-} \cdot$$

$$\cdot \left[\left(1 - \frac{v}{V}\right)\xi + \eta \frac{v}{V} \right]^{n_+ + n_-} \|\varphi_0\| \,, \tag{4.6}$$

where

$$Z_V = \sum_{n_+ = n_- = 0}^{\infty} (n_+!)^{-1} (n_-!)^{-1} e^{-(n_+ + n_-)S_0} V^{n_+ + n_-} .$$

As $V \to \infty$, the r.s. of (4.6) clearly tends to 0, for arbitrary η and any $\xi < 1$. This completes our argument.

Remarks. a) In the above argument the lattice cutoff avoids ultraviolet divergences, but the conclusions are expected to remain true in the continuum limit for asymptotically free theories. β) The above result can be easily generalized to show that if B is any field variable which has no component invariant under global gauge transformations, its expectation value vanishes, i.e. the global gauge symmetry group[e] G is "not broken"; γ) It is important to stress that the above result is different from the E-DDG result[2], since the action integrals on which the two results are based are very different, and in fact globally gauge-invariant Green's functions can here be non-zero even if they are not locally gauge-invariant, whereas they vanish in the E-DDG case; δ) when confronted with the standard (perturbative) approach to gauge symmetry breaking, the above result shows that there are non-perturbative effects, which "restore the global symmetry" (a somewhat misleading expression) and therefore, strictly speaking, the standard perturbation expansion based on a non-zero order parameter is not really justified.

As a consequence, even when confronted with very definite and practical questions, like that of laying down a sound framework for calculating the physical properties of a Higgs theory[f], one is led back to the general problems raised in Sect. 1, namely the ones of finding a gauge invariant characterization of the physical phenomena exhibited by a gauge q.f.t.

5. Gauge-invariant description.

According to the general philosophy discussed at the beginning, namely that all the physical states of the theory should be described in a gauge-invariant way, it is natural to look for a description in terms of gauge-invariant field operators in the sense that:

i) neutral states (i.e. states without superselected charges) are obtained by applying gauge-invariant field operators to the vacuum;

ii) charged states labelled by charges obeying a local Gauss law are obtained from neutral states by removing one charge to infinity.

In the simple SU(2) gauge model with Higgs fields φ and fermions ψ in the fundamental representation this program is implemented by using the fields

$$
\begin{aligned}
&\varphi^*_\alpha \, \vec{\sigma}_{\alpha\beta} \, \varphi_\beta \cdot \vec{F}_{\mu\nu} && (\text{"}W^{\circ}\text{"}) \\
&\varphi_\alpha \, \epsilon_{\alpha\gamma} \, \vec{\sigma}_{\gamma\beta} \, \varphi_\beta \cdot \vec{F}_{\mu\nu} && (\text{"}W^{+}\text{"}) \\
&\varphi^*_\beta \, \vec{\sigma}_{\beta\gamma} \, \epsilon_{\gamma\alpha} \, \varphi^*_\alpha \cdot \vec{F}_{\mu\nu} && (\text{"}W^{-}\text{"}) \qquad (5.1) \\
&\varphi^*_\alpha \, \psi_\alpha && (\text{"}e\text{"}) \\
&\varphi_\alpha \, \epsilon_{\alpha\beta} \, \psi_\beta && (\text{"}\nu\text{"}) \\
&\varphi^*_\alpha \, \varphi_\alpha && (\text{"Higgs particle"})
\end{aligned}
$$

where $\varepsilon_{\alpha\beta}$ are the matrix elements of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Remark. Independently, Banks and Rabinovic[7] and 't Hooft[8] have recently also proposed to use such invariant fields.

Notice that the fields introduced in (3.1) are invariant under local gauge transformations. They are therefore local quantum fields (i.e. they are expected to obey local commutativity). They reduce to the standard, gauge-dependent fields if one expands them around $\varphi_\alpha(x) = \bar{\varphi}_\alpha = const.$ The fields defined in (5.1) are sufficient to obtain a complete description of the physics of the theory, since they separate field configurations which are not gauge-equivalent (i.e. related by a local gauge transformation).

We emphasize that the equivalence of the gauge-invariant description we are about to develop and the standard approach is a dynamical problem involving a careful analysis of the existence of a locally effective order parameter (playing the rôle of $\langle \varphi \rangle$ in the old approach). While the relation between the two approaches appears to be relatively simple in models where the Higgs scalar has only one orbit under the action of G (up to equivalence), their equivalence may fail, partially, in more complicated theories. See Section 7.

5.1. *Existence and completeness of the gauge-invariant fields.*

The existence of a gauge-invariant description based on gauge-invariant fields in the general case is the result of the discussion which follows.

Let the Higgs field transform under a representation R_φ of a (gauge) group G; R_ψ be an irreducible representation of G under which a given set of fields ψ (e.g.

fermions, vector bosons etc.) transform. To simplify the notation, here and in the following we use the same symbol R to denote a representation of the gauge group and the corresponding representation space. The aim of the Theorem below is to show that a gauge-invariant description is always possible in the sense that there is a correspondence between the standard gauge dependent fields ψ and the "gauge-invariant" ones. This is obtained by making reference to a specific orbit $\{\bar{\varphi}\}$ in the representation R_φ, such that the "gauge-invariant" fields, when restricted to the point $\varphi = \bar{\varphi}$, coincide with the gauge dependent fields ψ of the standard approach, with $< \varphi > = \bar{\varphi}$. In the applications that we will discuss such an orbit will be a minimum of the Higgs potential. We denote by $G_{\{\bar{\varphi}\}}$ the (abstract) "residual" group of $\{\bar{\varphi}\}$ [g]. As it will be clear in the following, the rôle of the orbit $\{\bar{\varphi}\}$ is only that of choosing between different parametrizations of the same (gauge-invariant) kinematics, (see however the remarks after Part II).

THEOREM. Part I. [h] If $G_{\{\bar{\varphi}\}}$ is trivial there is a linear correspondence between the fields of R_ψ and the linear space of G-invariant (composite) fields $\mathcal{P}(\varphi)\cdot\psi$ which are polynomials in the Higgs scalars and linear in the fields ψ which transform under R_ψ. The correspondence is one to one modulo fields of the same form which vanish on $\{\bar{\varphi}\}$.

If $G_{\{\bar{\varphi}\}} \neq$ identity, in the standard description the physical content of the representation R_ψ can only be described modulo transformations of $G_{\bar\varphi}$. In the string language this amounts to describe the physical content of R_ψ in terms of $G_{\bar\varphi}$-invariant strings. The result of the following Parts II, III is that <u>there is a one-to-one correspondence between the $G_{\bar\varphi}$-invariant strings of the standard formulation and the G-invariant strings which are linear in ψ and polynomial in the Higgs field φ</u>. The above correspondence can be obtained by first noting that in the standard picture the physical content of R_ψ is described by projecting onto irreducible $G_{\bar\varphi}$-representations contained in R_ψ. Concretely, such projections are constructed by fixing a point $\bar\varphi$ in the orbit $\{\bar\varphi\}$ and by correspondingly decomposing R_ψ into subspaces carrying an irreducible representation of the stability group $G_{\bar\varphi}$ of $\bar\varphi$. The corresponding projections will be denoted by $P^i_{\bar\varphi}$. The analogue of the gauge-covariant fields $\mathcal{P}(\varphi(x))$ of Part I is now given by covariant projections $P(\varphi)$:

$$P(R_\varphi(g)\varphi) = R_\psi(g)\, P(\varphi)\, R_\psi^{-1}(g)\,. \tag{5.2}$$

In order to provide an acceptable kinematics, at least at the classical level, such covariant projections should be defined, continuous and non-vanishing almost everywhere. For the purpose of quantization it is actually required that they be polynomials in φ, so that composite fields built out of them make sense after quantization.

<u>THEOREM. Part II.</u> <u>To any $G_{\bar\varphi}$-invariant irreducible pro-</u>

jection $P^i_{\bar{\varphi}}$ of R_ψ there corresponds a G-covariant polynomial projection of R_ψ, i.e. a projection-valued polynomial, $P(\varphi)$, which has the property (5.2), coincides with $P^i_{\bar{\varphi}}$ when restricted to $\varphi = \bar{\varphi}$ and is irreducible in the sense that on the orbit $\varphi \in \{\bar{\varphi}\}$ it cannot be written as the sum of two G-covariant projections.

The converse is obvious, i.e. any G-covariant projection, which is non-vanishing and irreducible for $\varphi \in \{\bar{\varphi}\}$, for each fixed $\varphi \in \{\bar{\varphi}\}$, defines a projection P_φ onto an irreducible representation of G_φ contained in R_ψ.

Remark. Since a critical orbit has in general a larger residual group than those of the neighbour orbits, the corresponding set of projections is smaller than that obtained by using a neighbour orbit and therefore the kinematical descriptions based on a critical orbit involve a smaller number of fields. This is because on critical orbits some linear combinations of covariant polynomials vanish. Whether a description based on critical orbits is convenient is a problem strictly related to the dynamics of (non-abelian) gauge theories.

THEOREM. Part III. By using the G-covariant projections of Part II and the G-invariant fields of Part I one can replace the original (gauge dependent) fields $\Psi = (\psi_1, \ldots \psi_n)$ of R_ψ by a set of fields, one for each representation of the residual group $G_{(\bar{\varphi})}$ contained in R_ψ. Each of the

<u>new fields is a linear function of the old ones with coefficients which are polynomials in</u> φ <u>and either it is invariant under local gauge transformations or it transforms under a representation of</u> G <u>equivalent to</u> R_ψ. <u>In the string language this means that one can replace each</u> $G_{\bar\varphi}$<u>-invariant string of the standard picture by a G-invariant string</u>.

Trivial representations of $G_{\{\bar\varphi\}}$ are replaced by fields invariant under local gauge transformations, whereas (fields belonging to) each non-trivial representation of $G_{\{\bar\varphi\}}$ is replaced by (fields belonging to) a representation of G equivalent to R_ψ. The explicit construction of these fields is the following: for each fixed $\bar\varphi$, i) the trivial irreducible $G_{\bar\varphi}$-representations contained in R_ψ give rise, according to Part II, to G-covariant one-dimensional projections, which are actually projections onto G-covariant vectors $V(\varphi)$, (see also the proof of Part I), and $(V(\varphi), \psi)$ is the required invariant field; ii) each G-covariant projection $P(\varphi)$ corresponding to a non-trivial, irreducible representation of G_φ (contained in R_ψ), when applied to the original fields $\Psi = (\psi_1, \dots \psi_n)$ of R_ψ, yields the field $P(\varphi)\Psi \equiv \Psi(\varphi)$ which transforms under R_ψ.

It is worthwhile to note the following advantages of the description. The identification of the "physical" fields in the old picture is based on the choice of a point $\bar\varphi$ in the orbit $\{\bar\varphi\}$, i.e. on a local order parameter. On the contrary, the new fields depend only on the orbit, no local parameter being needed for their identification. In particular, in the case in which the residual group is the identity ("total breaking" case) the new fields are gauge-invariant, and they are in one-to-one correspondence with the physical one-particle spectrum (associated with the representation R). This new description also provides a solution of the problem raised in Section 2: Although $<\varphi> = 0$ (more generally, although only Green's functions invariant under global gauge transformations can be non-vanishing, even after gauge fixing, i.e. no "symmetry breaking", in the standard language), there are in general no G-multiplets. The absence of G-multiplets can be explained by the circumstance that physical particles are coupled to the vacuum by fields not related to each other by the action of global gauge transformations.

In the case of a non-trivial residual group, G_φ, the fields neutral with respect to G_φ of the standard formalism can be replaced by gauge-invariant fields as in the previous case. The problems of a gauge-invariant description of the multiplets transforming non trivially under G_φ in the standard picture is now (via Part III of the above Theorem) reduced to the construction of gauge-invariant fields out of the covariant fields $\Psi(\varphi)$. This problem has very little to do with the features of the Higgs phenomenon but is the familiar problem of giving a manifestly gauge-invariant description of theories with "unbroken groups" like QED, QCD etc. This is usually done by using fields localized on "strings". Here, we may construct such fields out of each covariant field $\Psi(\varphi)$ constructed in Part III of the above Theorem. Indeed, let $\Psi(\varphi)$ and $\Psi'(\varphi)$ be two covariant fields transforming under the same representation R_ψ of G. Then we form the field

$$\overline{\Psi_\alpha(\varphi)(x)}\ P\left(\exp \int_{\gamma_{xy}} A_\mu(\xi)\, d\xi^\mu\right)_{\alpha\beta}\ \Psi'_\beta(\varphi)(y)$$

which is localized on the path ("string") γ_{xy} connecting two points x and y. Such fields are believed to be suitable for the construction of states carrying a superselected charge by taking the limit $y \to \infty$. When implemented rigorously, this strategy would completely solve the problem mentioned at the beginning, namely the possibility of obtaining all the physical states associated to a g.q.f.t. in terms of gauge-invariant fields which are local or localized on strings.

The picture advocated in the above Theorem also sheds light on the problem mentioned in Sect. 2, namely the relation between the gauge-invariant formulation and the standard gauge-dependent approach. In fact, by fixing a point $\varphi = \bar{\varphi}$ on the orbit, the new fields introduced above reduce exactly to the fields of the standard picture usually constructed in terms of a vacuum expactation value $< \varphi > = \bar{\varphi}$. Thus, the rôle of the local order parameter $\bar{\varphi}$ in the standard picture appears merely as a way of fixing a system of local coordinates, with the result that the physical degrees of freedom are described by multiplets of fields which, since they depend on such a coordinate system in field space, are gauge-dependent. That rôle of the parameter $\bar{\varphi}$ (of the standard picture) is also in agreement with the result[9] that there is no phase transition between the confinement and the Higgs regime.

5.2. *An example:* SU(2) × U(1)

To illustrate the gauge-invariant formulation advocated above we consider the following example

<u>EXAMPLE: The SU(2) × U(1) model</u>

The standard approach is based on the use of the following gauge dependent fields:

(complex) Higgs field	$\varphi = \begin{pmatrix} \varphi_+ \\ \varphi_0 \end{pmatrix}$	transforming as	$(\frac{1}{2}, Y = 1)$
vector meson field	$\vec{W}_\mu$	" "	$(1, Y = 0)$
"	B_μ	" "	$(0, Y = 0)$
lepton fields	$\psi_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$	" "	$(\frac{1}{2}, Y = -1)$
"	$\psi_R = e_R$	" "	$(0, Y = -2)$

The theory is based on $\langle\varphi\rangle = \bar{\varphi} = \begin{pmatrix} 0 \\ \bar{\varphi}_0 \end{pmatrix}$, with a U(1) residual group gnerated by the "electric charge" $Q = T_3 + \frac{1}{2} Y$.

Our formulation is based on the following fields

a) gauge-invariant fields (corresponding to the electrically neutral fields):

i) $N \lambda_1 \varphi^* \vec{\sigma} \varphi \vec{F}^{W}_{\mu\nu} + \lambda_2 F^{B}_{\mu\nu}$, with N a normalization constant and λ_1, λ_2 arbitrary constants which can be chosen in such a way that the mass matrix is diagonal. When $\varphi = \bar{\varphi}$ such fields reduce to $\lambda_1 F^3_{\mu\nu} + \lambda_2 F^B_{\mu\nu}$.

ii) $i N \varphi \sigma_2 \psi = \det(\varphi \psi)$ which reduces to ν_L (neutrino)

iii) $N \varphi^* \varphi$ " φ_0 (Higgs neutral field)

b) gauge-covariant fields in terms of which one may construct the SU(2) × U(1) invariant bilocal fields or strings:

$N \varphi_\alpha \varphi^* \psi$, which reduces to e_L,

$\psi_R = e_R$, " e_R,

$N \varphi^* \sigma_i \sigma_2 \varphi^* (\varphi \sigma_2 \sigma_j \varphi F^j_{\mu\nu})$, " $F^1_{\mu\nu} + i F^2_{\mu\nu}$,

$N \varphi \sigma_2 \sigma_i \varphi (\varphi^* \sigma_j \sigma_2 \varphi^* F^j_{\mu\nu})$, " $F^1_{\mu\nu} - i F^2_{\mu\nu}$.

5.3. *A gauge-invariant expression: the mass matrix*

The gauge-invariant formulation and the absence of a symmetry breaking local order parameter indicate that the occurrence of different masses should be explained in terms of different gauge-invariant fields (or strings). It may be worthwhile to verify this fact explicitly. For concreteness we consider the fermion mass matrix and in order to avoid unnecessary complications due to strings we restrict to the case of trivial residual group (total symmetry breaking). In the standard gauge dependent approach, fermion masses arise through the Yukawa coupling with the Higgs field, of the form

$$\bar{\psi}_\alpha \psi_\beta \Gamma^\ell_{\alpha\beta} \varphi_\ell ,$$

with Γ denoting suitable gauge-invariant coupling matrices, by substituting for φ a non-vanishing expectation value $\bar{\varphi} = \langle\varphi\rangle$

$$M_{\alpha\beta} = \Gamma^\ell_{\alpha\beta} \langle \varphi_\ell \rangle .$$

Here the existence of a non-vanishing order parameter $\langle \varphi \rangle$ seems to play a crucial rôle. Actually, it is not difficult to write the Higgs-fermion Yukawa coupling in terms of the fields of the gauge invariant formulation. In the case of trivial residual group, the G-invariant projections $P^i(\varphi)$ are of the form

$$P^i_{\alpha\beta}(\varphi) = \mathcal{P}^i_\alpha(\varphi)\, \mathcal{P}^i_\beta(\varphi)^*$$

with $\mathcal{P}^i(\varphi)$ a G-covariant vector, and the gauge-invariant fields $\Psi^i(\varphi)$ are given by $\mathcal{P}^i_\beta(\varphi)^* \psi_\beta$. By using the completeness, with a suitable normalization, $\sum_i \mathcal{P}^i_{\alpha\beta} = \delta_{\alpha\beta}$, and the orthogonality of the projections P^i, we easily obtain

$$\begin{aligned}
\bar{\psi}_\alpha \psi_\beta \Gamma^\ell_{\alpha\beta} \varphi_\ell &= \sum_i \sum_j \overline{(P^i_{\alpha\gamma} \psi_\gamma)} (P^j_{\beta\delta} \psi_\delta) \Gamma^\ell_{\alpha\beta} \varphi_\ell \\
&= \sum_{i,j} \overline{(P^i_{\rho\gamma} \psi_\gamma)} (P^j_{\sigma\delta} \psi_\delta)\, \overline{P^i_{\alpha\rho}}\, P^j_{\beta\sigma} \Gamma^\ell_{\alpha\beta} \varphi_\ell \\
&= \sum_{i,j} \overline{\Psi^i}(\varphi)\, \Psi^j(\varphi)\, \overline{\mathcal{P}^i_\alpha}(\varphi)\, \mathcal{P}^j_\beta(\varphi)\, \Gamma^\ell_{\alpha\beta} \varphi_\ell .
\end{aligned}$$

Now, the operator

$$M^{ij} \equiv \mathcal{P}^i_\alpha(\varphi)\, \mathcal{P}^j_\beta(\varphi)\, \Gamma^\ell_{\alpha\beta} \varphi_\ell$$

is invariant under local gauge transformations and its vacuum expectation (which may therefore be non-vanishing) defines a mass matrix for the gauge-invariant fields $\Psi^i(\varphi)$. Clearly, when $\varphi = \bar{\varphi}$ the mass matrix $\langle M^{ij}(\varphi)\rangle$ reduces exactly to the mass matrix of the corresponding gauge dependent fields of the standard formulation.

For example, in the $SU(2) \times U(1)$ model with breaking leaving a $U(1)$ residual group the G-covariant projections may be written in the form

$$P^i_{\alpha\beta}(\varphi) = \mathcal{P}^i_\alpha(\varphi)\, \mathcal{P}^i_\beta(\varphi)^*,$$

with $\mathcal{P}^i(\varphi)$ a vector which is G-covariant modulo $U(1)$ transformations (this feature is actually common to all models where the residual group is $U(1)$). The Higgs-fermion Yukawa coupling involves two different representations (a left doublet and a right singlet) and clearly only the projectors corresponding to the left doublet are non trivial

$$\mathcal{P}^1_\alpha(\varphi) = N\varphi_\alpha \quad , \quad \mathcal{P}^2_\alpha(\varphi) = N\epsilon_{\alpha\beta}\,\varphi^*_\beta \quad , \quad N^2 \equiv \langle \varphi^*\varphi\rangle^{-1}.$$

The mass matrix then takes the following form

$$\langle M^e(\varphi)\rangle = Nf\, \langle \varphi^*_\alpha\, \delta^\ell_\alpha\, \varphi_e\rangle = f\, \langle \varphi^*\varphi\rangle^{1/2} ,$$

$$\langle M^\nu(\varphi)\rangle = Nf\, \langle \epsilon_{\alpha\beta}\,\varphi_\beta\, \delta^\ell_\alpha\, \varphi_e\rangle = 0 .$$

A similar analysis can be done for the mass matrix of the vector bosons.

The above discussion should make clear that the crucial feature of the Higgs phenomenon is not the existence of a symmetry breaking local order parameter, but rather the existence of a minimizing orbit $\{\bar{\varphi}\}$ with a residual group smaller than G.

5.4. *Complementarity principle and phase diagrams.*

The theorems discussed in this section show that the description of the Higgs phenomenon in terms of gauge-invariant fields (symmetric picture), advocated[11) 6) 7)10)] as an alternative formulation when the Higgs fields are in the fundamental representation, is actually possible in the general case and indeed required by the occurrence of non-perturbative effects which prevent the existence of a symmetry breaking local order parameter. On the other hand, the so called Higgs picture even if useful from a pragmatic point of view is not defendable on general grounds.

We point out that the existence and structure of a complete set of gauge-invariant fields have been proved in this section as a general group-theoretical result, and our construction does not require any conjecture about the dynamics. In this sense, the results discussed above may be regarded as a proof of the so called complementarity principle. They also put the symmetric picture in a sharper perspective. The condition of the Higgs field belonging to the fundamental representation does not seem to play a crucial rôle; the group-theoretical proof presented below shows that the important feature permitting to construct local, gauge-invariant fields is rather the triviality of the residual group defined by the minimizing orbit. (Except for the simple SU(2) case this property is not guaranteed by the condition that the Higgs field is in the fundamental representation, unless a somewhat artificial duplication is used.) Actually also in the discussion of phase diagrams in gauge theories the relevant feature is the structure of the residual group.

In conclusion, in both the Higgs and the confinement regime, a complete description is provided by the "gauge-invariant" fields introduced above on the basis of group theoretical results, independently of whether the dynamical behaviour of the theory does or does not exhibit a phase transition between the Higgs and the confinement region, depending on the structure of the residual group. A phase transition is likely to occur when the Higgs representation has more than one orbit, and additional parameters are needed to describe the phase diagram.

6. Proof of the Theorem

LEMMA. *Let $\{\bar\varphi\}$ be an orbit of a representation R_H of a compact Lie group G, $\{\bar\varphi\} \equiv \{\varphi = R_H(g)\bar\varphi,\ g \in G\}$; then any function $F(\varphi)$ defined on the orbit $\{\bar\varphi\}$ and transforming under a representation R of G (briefly R-covariant):*

$$F(R_H(g)\varphi) = R(g)\,F(\varphi),$$

is the restriction to that orbit of an R-covariant polynomial.

Proof. Let V_R be the vector space of the functions on the orbit $\{\bar\varphi\}$, which transform under R. Since G is a Lie group, V_R is a space of continuous functions on $\{\bar\varphi\}$. This follows from

$$|F(\varphi_1) - F(\varphi_2)| = |(1 - R(g_{12}))\,F(\varphi_1)|,$$

where g_{12} is chosen in such a way that $\varphi_2 = g_{12}\varphi_1$ and such that $g_{12} \to$ identity when $\varphi_2 \to \varphi_1$. V_R has at most dimension equal to the dimension of the representation R, since an R-covariant function is completely determined by its value on a fixed point $\bar\varphi \in \{\bar\varphi\}$.

Since the orbit is a compact space, by the Stone-Weierstrass theorem each element of V_R can be uniformly approximated by polynomials $P_n(\varphi)$ restricted to $\{\bar\varphi\}$. Actually, the R-covariant polynomials are sufficient for the approximation. In fact, if $P_n(\varphi) \to F(\varphi)$ on the orbit $\{\bar\varphi\}$, the R-covariant polynomials

$$P_n^R(\varphi) \equiv (Vol(G))^{-1} \int_G dg\, R^{-1}(g)\, P_n(R_H(g)\varphi) \quad , \quad Vol(G) \equiv \int_G dg \, ,$$

also converge to $F(\varphi)$ on $\{\bar{\varphi}\}$ since

$$|P_n^R(\varphi) - F(\varphi)| = |(Vol(G))^{-1}| \int_G dg (R^{-1}(g)\, P_n(R_H(g)\varphi) - F(\varphi)| \leq$$

$$\leq |(Vol(G))^{-1} \int_G dg\; R^{-1}(g) [P_n(R_H(g)\varphi) - F(R_H(g)\varphi)| \leq$$

$$\leq \sup_{\varphi \in \{\bar{\varphi}\}} |P_n(\varphi) - F(\varphi)| \to 0 \, .$$

Since V_R is finite dimensional and obviously contains the vector space $\mathcal{V}_R$ of R-covariant polynomials restricted to $\{\bar{\varphi}\}$, (the elements of $\mathcal{V}_R$ are actually the equivalence classes with respect to the property of being equal on $\{\bar{\varphi}\}$), the two vector spaces must coincide.

<u>Proof of THEOREM. Part I.</u> $(G_{\{\bar{\varphi}\}}$ = identity)

Any G-invariant local field, constructed in terms of $\varphi \in R_H$ and linear in $\psi \in R$ is of the form of a scalar product $(F(\varphi), \psi)$ with $F(\varphi)$ transforming under R. Therefore, by fixing a point $\bar{\varphi}$ of the orbit, the above invariant yields a definite component of R. Conversely, let us fix a (normalized) vector $v_i \in R$. For any point $\varphi \in \{\bar{\varphi}\}$, since $G_{\{\bar{\varphi}\}}$ is the identity, there is exactly one element $g_\varphi \in G$ such that

$$R_H(g_\varphi)\,\bar{\varphi} = \varphi \, .$$

Clearly if $\varphi' = R_H(h)\varphi$, then $g_{\varphi'} = h\, g_\varphi$. We then define

$$F(\varphi) \equiv R(g_\varphi)\, v_i \,.$$

Thus

$$F(R_H(h)\varphi) = R(g_{R_H(h)\varphi})\, v_i = R(h\, g_\varphi)\, v_i =$$
$$= R(h)\, R(g_\varphi)\, v_i = R(h)\, F(\varphi),$$

i.e. $F(\varphi)$ transforms under the representation R. By the above Lemma $F(\varphi)$ is the restriction of a polynomial $\mathcal{P}(\varphi)$ to the orbit $\{\bar\varphi\}$ and $(\mathcal{P}(\varphi), \psi)$ is a G-invariant local field which reduces to $\psi_i \equiv (v_i, \psi)$ when $\varphi = \bar\varphi$.

<u>Proof of Part II</u>. ($G_{\{\bar\varphi\}} \neq$ identity)

Given a G-covariant irreducible projection $P(\varphi)$, for each fixed $\bar\varphi$, $P(\bar\varphi)$ is a projection which is invariant under the subgroup $G_{\bar\varphi} \subset G$ which leaves $\bar\varphi$ stable.

To see the irreducibility of $P(\bar\varphi)$ we note that if $P(\bar\varphi)$ can be written as a sum of two $G_{\bar\varphi}$-invariant projections: $P(\bar\varphi) = P_1 + P_2$, then if we prove (see below) that any $G_{\bar\varphi}$-invariant projection can be obtained by putting $\varphi = \bar\varphi$ in a (suitable) G-covariant projection $P(\varphi)$, we would have $P(\bar\varphi) = P_1(\bar\varphi) + P_2(\bar\varphi)$ with $P_1(\varphi)$ and $P_2(\varphi)$ G-covariant projections. G-covariance implies $P(\varphi) = P_1(\varphi) + P_2(\varphi)$, for any $\varphi \in \{\bar\varphi\}$, contrary to the irreducibility of $P(\varphi)$.

We now prove that to any $G_{\bar\varphi}$-invariant projection P one can associate a G-covariant projection $P(\varphi)$ such that $P = P(\bar\varphi)$. In fact, each point $\varphi \in \{\bar\varphi\}$ defines an equivalence class $[g_\varphi]$ of elements of G through the equation

$$R_H(g)\bar\varphi = \varphi \qquad \text{iff} \quad g \in [g_\varphi].$$

Then, if $g_1, g_2 \in [g_\varphi]$ one has

$$R_H(g_1^{-1}g_2)\bar\varphi = R_H(g_1^{-1})R_H(g_2)\bar\varphi = R_H(g_1^{-1})\varphi = \bar\varphi,$$

i.e.

$$g_1^{-1}g_2 \in G_{\bar\varphi} \quad \text{and} \quad [g_\varphi] = g_\varphi G_{\bar\varphi},$$

(this means that $[g_\varphi]$ is a (right) coset of $G_{\bar\varphi}$).

Moreover, if $\varphi' = R_H(h)\varphi$, the class $[g_{\varphi'}]$ consists of the set of solutions g of the equation

$$R_H(g)\bar\varphi = \varphi' = R_H(h)\varphi = R_H(h)\,R_H(g_\varphi)\bar\varphi =$$

$$= R_H(hg_\varphi)\bar\varphi, \qquad \forall\, g_\varphi \in [g_\varphi],$$

so that

$$[g_{\varphi'}] = h\,g_\varphi G_{\bar\varphi} = h\,[g_\varphi]. \tag{6.1}$$

Thus, given a $G_{\bar\varphi}$-invariant projection P we now define for any $\varphi \in \{\bar\varphi\}$ the projection

$$P(\varphi) \equiv R(g)\,P\,R^*(g), \qquad g \in [g_\varphi].$$

Note that $P(\varphi)$ is well defined, since if $g_1, g_2 \in [g_\varphi]$, then $g_1^{-1}g_2 \in G_{\bar\varphi}$ and

$$R(g_1)P\,R^*(g_1) = R(g_1)\,R(g_1^{-1}g_2)P\,R^*(g_1^{-1}g_2)\,R^*(g_1) =$$

$$= R(g_2)\,P\,R^*(g_2).$$

By eq. (6.1), for any $h \in G$,

$$\mathcal{P}(R_H(h)\varphi) = R(hg_\varphi)\mathcal{P}R^*(hg_\varphi) = R(h)P(\varphi)R^*(h),$$

i.e. $P(\varphi)$ transforms covariantly under G. By the above Lemma $P(\varphi)$ is the restriction of a projection-valued polynomial $\mathcal{P}(\varphi)$ to the orbit $\{\bar{\varphi}\}$.

Proof of Part III.

The construction follows immediately from the proofs of Parts I, II.

7. Defects and perturbation expansion

The purpose of this Section is to discuss effects of "defects" on the validity of perturbation theory for gauge-invariant correlation functions as an asymptotic expansion. The drawback of the standard perturbation expansion for gauge dependent functions is indicated. The results of the standard approach are recovered in perturbation theory through the use of the gauge-invariant fields introduced in Sect. 5.

7.1. *Perturbation expansion for gauge-invariant correlation functions*

We recall that in order to develop a perturbation expansion in terms of Feynman diagrams one has to introduce a gauge fixing F. As usual we suppose that the Higgs potential defines a (unique) orbit of absolute minima. Each point $\bar{\varphi}$ of such an orbit gives rise to a field configuration which minimizes the action and defines a perturbation theory as an expansion around that configuration. Thus for each point $\bar{\varphi}$ one has the standard perturbation theory based on $< \varphi > = \bar{\varphi}$ (which is, order by order, finite as a consequence of the renormalizability of the chosen gauge). However, since in general the boundary conditions do not single out a point $\bar{\varphi}$ of the orbit (except for special gauge fixings) all the points $\varphi \in \{\bar{\varphi}\}$ are on equal footing and they must all be considered in the perturbative expansion of the functional integral, in accordance with the fact that there is no order parameter (see Sects. 3, 4). Such a "degeneracy" does not play any rôle for the expectation value of a gauge-invariant operator A, since in any order n of the perturbation expansion

$$\langle A \rangle^{(n)}_{F,\bar{\varphi}} = \langle g(A) \rangle^{(n)}_{gF, g\bar{\varphi}} = \langle A \rangle^{n}_{gF, g\bar{\varphi}} \tag{7.1}$$

where g denotes a global gauge transformation.
This equation follows from the fact that the Feynman propagators depend on $\bar{\varphi}$ in a covariant way.

It is worthwhile to stress that the non-existence of an order parameter ($\langle \varphi \rangle = 0$, see Sects. 3, 4) does not affect the form of the perturbation expansion for gauge-invariant correlation functions which is exactly the same as the one built on a "symmetry breaking parameter" $\langle \varphi \rangle = \bar{\varphi}$. The asymptotic validity of the formal expansion is discussed in the next two subsections.

7.2. *Standard perturbation expansion for gauge dependent correlation functions*

According to the discussion given in 7.1, since by Sects. 3, 4 the boundary conditions do not destroy the invariance under global gauge transformations, the perturbative expansion of the functional integral for gauge dependent correlation functions leads to a group average of the various perturbation expansions labelled by the various points $\bar{\varphi}$ of the orbit. This leads to the vanishing of all correlation functions without a component invariant under global gauge transformations (Sects. 3, 4) . Clearly, this conclusion differs from the one based on the standard perturbation expansion. We recall that the latter is based on a single fixed point $\bar{\varphi}$ of a minimizing orbit, with the implicit (unjustified) assumption that there is "symmetry breaking", with $\langle \varphi \rangle = \bar{\varphi}$. Actually, breaking only occurs when there is no coupling to the gauge fields, ($g = 0$). However, since for any $g \neq 0$ $\langle \varphi \rangle_g = 0$, one has $\lim_{g \to 0} \langle \varphi \rangle_g = 0$. In other words the limit $g \to 0$ leads to a mixed phase of the scalar theory (without gauge fields). One should therefore expand around the mixed, symmetric phase, rather than around the pure ones. One is thus led to a phase diagram of the following type

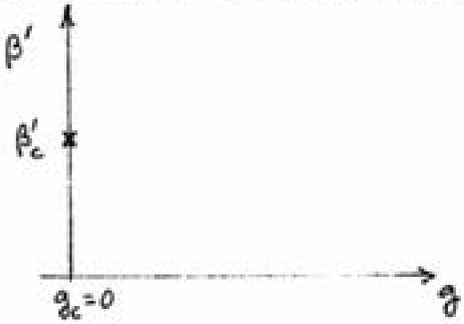

Fig. 1

with a critical point on the $g=0$ axis but no line of critical points emerging from β'_c (Fig. 1) above which there is a non-zero $\langle \varphi \rangle$.

In conclusion, the standard perturbation expansion cannot be asymptotic to gauge-dependent correlation functions.

7.3. *Relations between the perturbation expansion of gauge-invariant correlation functions and the standard perturbation expansion of gauge-dependent correlations.*

As advocated in Sect. 5 the "gauge-invariant" fields introduced there should provide a complete gauge-invariant description of the theory. We show here that there is a simple relation between the correlation functions of the gauge-invariant fields constructed in Sect. 5 and the corresponding gauge-dependent ones calculated in the standard perturbation expansion. This relation is based on the results of Sect. 5 and the discussion in the present section.

For simplicity, we consider the case of neutral fields ψ_i in the standard, gauge-dependent formulation. They correspond to gauge-invariant fields of the form

$$N \mathcal{P}^i(\varphi)\,\psi\,, \qquad N^{-1} = \|\mathcal{P}^i(\varphi)\|\,,$$

in the sense that

$$N\,\mathcal{P}^i(\bar\varphi)\psi = \psi_i\,. \tag{7.2}$$

We then consider the correlation function for the gauge-invariant fields $A_i = N\,\mathcal{P}^i(\varphi)\cdot\psi$. By the above discussion we expect that its perturbation expansion (with gauge fixing F) based on one fixed point $\bar\varphi$ is asymptotic and does not depend on F and on $\bar\varphi$.

Putting $\mathcal{P}^i(\varphi(x)) = \mathcal{P}^i(\bar\varphi) + \Delta\mathcal{P}^i(\varphi(x))$ we have

$$\langle A_1 \ldots A_N \rangle^{(n)}_{F,\bar\varphi} = \Big(\prod_{i=1}^{N} \|\mathcal{P}^i(\bar\varphi)\|\Big)^{-1} \Big\{ \prod_i \mathcal{P}^i(\bar\varphi) \langle \psi_1 \cdots \psi_N \rangle^{(n)}_{F,\bar\varphi} + \tag{7.3}$$
$$+ \sum_{k=0}^{N-1} \underbrace{\mathcal{P}(\bar\varphi) \ldots \mathcal{P}(\bar\varphi)}_{k} \langle \ldots \underbrace{\Delta\mathcal{P} \ldots \Delta\mathcal{P} \ldots \Delta\mathcal{P}}_{N-k} \ldots \rangle_{F,\bar\varphi}\,.$$

Thus, by eq. (7.2) the first term is exactly the correlation function of the standard gauge-dependent field ψ_i calculated in the gauge F in the perturbation expansion based on a fixed point $\bar\varphi$. The other terms are dominated by terms of the form

$$\frac{|\langle \ldots \psi\, \Delta\mathcal{P}^1 \ldots \psi \ldots \Delta\mathcal{P}^k \psi \rangle_{F,\bar\varphi}|}{\|\mathcal{P}^1(\bar\varphi)\| \ldots \|\mathcal{P}^k(\bar\varphi)\|}$$

and therefore they are small if the Higgs radius is much bigger than the mean (perturbative) fluctuations; the standard perturbation expansion is recovered approximately for those "gauges" in which that is guaranteed (the quantum fluctuations along the orbit strongly depend on the gauge fixing). Clearly, the renormalization procedure plays an important rôle in establishing the validity of this property. The renormalizability of the perturbation expansion (7.3) for the Green's functions of the "composite" fields A_i is expected to hold for asymptotically free theories.

7.4. *Field configurations with non-trivial topology (defects) in Higgs models*

In this Section we will discuss a classification of field configurations with non-trivial topology in Higgs models, in terms of homotopy groups. The point of such an analysis is that it allows to discuss the existence of defects, without a detailed knowledge of the dynamics of the solutions of classical (euclidean) equations.(12) Reviews of topological methods suitable for the classification of defects in ordered material media can be found in Refs.[13][14].

The idea is that a description of the configurations on which the functional measure is concentrated can be done by looking at the space of local parameters where the field variables take values with highest probability (determined by the functional measure). This means that a typical field configuration will take values in that parameter space in (some region R which covers) a large fraction of the space-time volume, and, due to the presence of the kinetic term in the action integral, it will be a continuous function there. This is true in one-dimensional models and can be justified in higher dimensions at least when suitable smearing and ultraviolet cutoffs are introduced. When restricted to R such field configurations are the exact analogue of the local order parameter used in the classification of defects in ordered media . The continuous deformability of one configuration into another gives rise to a natural topology in this configuration space, and topological defects are then characterized by homotopy classes. Clearly such structures may arise only when the region R has a non trivial topology.

We first classify defects in pure Yang-Mills models. Since the action integral vanishes only for pure gauge configurations

$$A_\mu(x) = g^{-1}(x)\,\partial_\mu\, g(x)\,, \tag{7.4}$$

the probability distribution, induced by the functional measure on $A_\mu(x)$ is peaked on pure gauge configurations and on small oscillations around them. The local parameter space can then be taken as the space of functions $g(x)$, taking values in G.

The topologically simplest regions R, on which a field configuration A_μ may behave as a pure gauge or a small oscillation around it, gives rise to the following defects.

1) *Point defects*

When R is topologically equivalent to $\mathbb{R}^4 - \{0\}$, the non-trivial homotopy classes of $g(x)$ are given by $\pi_3(G)$ and they identify point defects. For G compact, simple and simply connected $\pi_3(G) = Z$, Z the group of integers. Field configurations having the topology of a point defect and with finite action are called instantons.

2) *Line defects*

They arise when R is topologically equivalent to $\mathbb{R}^4$ - {line}. The homotopy classes of the g's are now elements of $\pi_2(G)$. For any Lie group $\pi_2(G) = 0$, so that there are no line defects in pure Yang-Mills Theories without matter fields

3) *Defects of dimension two (vortices)*

They are classified by $\pi_1(\overline{G}/K)$, where $\overline{G}$ is the universal covering group of G and K a discrete subgroup of $\overline{G}$ (typically its center). The exact sequence (13,14)

$$0 = \pi_1(\overline{G}) \to \pi_1(\overline{G}/K) \to \pi_o(K) \to \pi_o(\overline{G}) = 0$$

implies

$$\pi_1(\overline{G}/K) = \pi_o(K) = K \quad .$$

An example is given by the 't Hooft vortices in pure Yang-Mills theory. (8,12)

4) *Defects of dimension three (wall-defects)*

They are classified by $\pi_0(G)$, which is zero for any connected group G; in general π_0 counts the number of connected components of G.

We now turn to the more general case when also Higgs fields are present. The probability distribution for $\varphi(x)$ is peaked on the orbit $\{\bar{\varphi}\}$, (or the orbits), which minimizes the Higgs potential. The kinetic term $(D_\mu\varphi)^2$ gives rise to a non-trivial relation between the Higgs parameter space and the A_μ parameter space, so that, in general, one must consider configurations for A_μ other than those of pure gauge. In the following, for simplicity we will consider the case when G acts transitively on the orbit $\{\bar{\varphi}\}$. The parameter space can then be taken to be characterized by $\{\varphi(x) = R_H(g(x))\bar{\varphi}$, $g(x)$ continuous modulo H, $H \equiv$ residual group of $\bar{\varphi}$; $A_\mu(x)$ such that $D_\mu\varphi(x) = 0\}$. Clearly the condition $D_\mu\varphi = 0$ on A_μ is equivalent to the vanishing of $G_{\mu\nu}$ in all group directions other than those of H. A particular case of such configurations are those of the form

$$A_\mu = g^{-1}(x)\partial_\mu g(x) , \qquad \varphi = R(g(x))\bar{\varphi} ,$$

with g continuous, and clearly the corresponding defects are those classified before.

When $A_\mu \neq g^{-1}\partial_\mu g$ the defects may be classified in terms of the local order parameter $\varphi(x)$, taking values in G/H. (If there is more than one orbit, $\varphi(x)$ takes values in $(G/H_1)\cup(G/H_2)\cup \ldots$.)

1') *Point defects*

They are classified by $\pi_2(G/H)$. As a consequence of the exact sequence

$$\mathbb{Z} = \pi_3(G) \to \pi_3(G/H) \to \pi_2(H) = 0,$$

one concludes that $\pi_3(G/H)$ is a subgroup of $\pi_3(G)$. Therefore such defects arise from configurations with $A_\mu = g^{-1}\partial_\mu g$, with g continuous; they have already been classified.

2') *Line defects*

They are classified by $\pi_2(G/H)$. For all simply connected Lie groups G the exact sequence

$$0 = \pi_2(G) \to \pi_2(G/H) \to \pi_1(H) \to \pi_1(G) = 0$$

implies

$$\pi_2(G/H) = \pi_1(H).$$

If G is not simply connected one may use its universal covering group $\bar{G}$.

A well known example of such defects is given by the 't Hooft-Polyakov monopole;(in $\mathbb{R}^4$ it is a line defect!)

3') *Defects of dimension two*

They are classified by $\pi_1(G/H)$. If G is simply connected and connected the exact sequence

$$0 = \pi_1(G) \to \pi_1(G/H) \to \pi_0(H) \to \pi_0(G) = 0$$

yields

$$\pi_1(G/H) = \pi_0(H).$$

4') *Defects of dimension three*

They are classified by $\pi_0(G/H)$. If there is more than one orbit, say n, $\pi_0((G/H_1)\cup(G/H_2)\cup\ldots)$ contains at least n elements corresponding to the n different orbits.

We remark that in the presence of Higgs fields the action of point defects in four dimensions depends on a scale parameter and attains its lower bound for "zero size"[15]. Due to quantum fluctuations the relevant field configurations will then have a finite small size of the order of m_H^{-1}, the typical mass parameter occurring in the Higgs potential. This fact has been used in Sect. 4.

We note that, in a g.q.f.t. with an intrinsic mass scale, like a Higgs theory, field configurations dominating the Euclidean functional integral can be expected to be decomposable into a classical field configuration plus a small, essentially Gaussian fluctuation field, (in which one will try to expand perturbatively). The classical field configuration describes a typical configuration of a gas of defects classified by homotopy groups, (up to additional, internal structures), as explained above. Because we consider a g.q.f.t. with intrinsic mass scale (e.g. depending on a fairly large mass parameter, m_H) , an individual defect, δ_k , labelled by an element of some k^{th} homotopy group, π_k , in d space-time dimensions has an approximate geometrical locus which is some (d-k-1)-dimensional, closed, compact surface, S_k . Its mean action, $\overline{A}$, is typically proportional to the area (= const. when $k = d-1$, = length when $k = d-2$, = surface area when $k = d-3$, ...), $|S_k|$ of S_k , i.e.

$$\overline{A}(\delta_k) \underset{\approx}{>} a^{(k)} m_H^k |S_k| \tag{7.5}$$

where $a^{(k)}$ depends on the internal structure of the defect, and m_H is a typical mass scale. Its entropy is

$$S(\delta_k) \approx \sigma m_H^k |S_k| \tag{7.6}$$

where σ is some (essentially geometrical) constant. Its "chemical potential", μ $(= -\ln(\text{activity}))$, which determines the density $(\propto m_H^d \cdot e^{-\mu})$ of this gas of defects is

$$\mu(\delta_k) \approx ((1/g^2)a^{(k)}-\sigma)M_H^k|S_k| \tag{7.7}$$

This type of defect is dilute i.e. they form a low density gas) and has small size if

$$(1/g^2)a^{(k)}-\sigma \gg 0 \quad . \tag{7.8}$$

When g passes through a critical value,

$$g_c^{(k)} \approx \sqrt{a^{(k)}/\sigma}$$

a transition, characterized by the condensation of this type of defects, to a phase of extreme high density is expected to occur.

This can be shown in a class of lattice theories of defects, where one takes into account interactions between different defects of variable dimension. Those transitions appear to be most important for the understanding of quark confinement[8,12], the existence, or absence, of massive, stable magnetic monopoles in four-dimensional Higgs theories, the breaking of chiral invariance (related to a phase transition in Higgs theories with θ-vacua, at $\theta = \pi$ [16]) and possibly of gauge hierarchies.

The last three applications (effective actions for defect gases and their phase diagrams) will be discussed in more detail elsewhere. Here we just want to point out some theoretical problems of the approach described here :

- classical configurations contributing to the functional integrals contain an infinite number of defects (the total action and entropy are infinite) of finite density. They have therefore no well-defined, prescribed behaviour at spatial infinity. They have to be analyzed in space-time regions small with respect to the inverse defect density.

- It is then clear that dominant classical configurations do typically not correspond to local minima of the global action, but rather to approximate, local minima of the action density integrated over bounded space-time regions. For this reason, the fluctuation field has in general zero and negative frequency modes, so that non-Gaussian corrections have to be taken into account. This makes quantitative, semiclassical calculations extremely tedious.

- The approach based on interpreting the classical field configuration as a configuration of a gas of individual defects with definite chemical potentials and short range ($m_H >> 0$!) interactions is only reliable when (7.8) holds, i.e. at very low densities. This makes a reliable, quantitative investigation of e.g. the condensation of monopoles in this approach impossible. Nevertheless, it appears promising to study the qualitative features of defect condensation transitions and their physical effects in phenomenological (lattice) models of interacting defect gases which are well defined for all values of g.

7.5. *Validity of perturbation expansion and defects*

The previous estimate (see Sect. 4) of the instanton density implies that in space time regions of size d the probability of finding a point defect inside is smaller than $C\,(d/L)^4 = C\,(d\,m_H)^4\, e^{-4A/g^2}$ (C and $A>0$ constants). Thus we have

i) regions of size $d \ll L$ are essentially "defect-free". Since $L \simeq m_H^{-1}\, e^{A/g^2}$ this applies to regions of size comparable to inverse Higgs masses;

ii) as a function of g, $\rho \sim L^{-4}$ vanishes at the origin ($g = 0$) together with all its derivatives, so that $\rho = 0$, in each order of the perturbation expansion.

The above two properties and (7.5) - (7.8) suggest that the non-perturbative effects originated by defects vanish at $g = 0$ together with all their derivatives and are in any case small in regions of size $d \ll L$. We expect therefore that a) the perturbation expansion is asymptotic; b) non-perturbative effects are important only for correlation distances of order $\gtrsim L \approx m_H^{-1} e^{\mu}$.

The features discussed above seem to have a general character. In situations in which the action has more than one minimum and the boundary conditions are not able to fix one of them the standard perturbation expansion of the functional integral around one fixed minimum neglects those field configurations which cannot be regarded as small oscillations around that minimum. This is not correct if the boundary conditions do not fix the asymptotic behaviour of the field configurations with non-zero functional measure. However, for models with exponentially small defect density, if one considers any fixed space time region D, the functional measure defined on the field configurations with support in D by "integrating over all the configurations outside D", is essentially concentrated on the "small oscillations" around each of the minima with corrections which vanish asymptotically together with all its derivatives as $g \to 0$. Therefore one expects that a perturbation expansion taking into account all the minima is asymptotic for correlation functions inside any fixed region D. Even if the corrections due to non-zero defect density are exponentially small, they may lead to very interesting effects like the generation of small masses for the gauge bosons of the residual group. They are expected to play a rôle in understanding the occurrence of gauge hierarchies.

It is important to point out that the above features of the functional integral are exactly realized in the well known double-well oscillator model described by the Hamiltonian

$$H(g) = -\frac{d^2}{dx^2} + x^2 + 2gx^3 + gx^4$$

The expansion around one of the minima of the potential is the expansion in the parameter g. It is known that a) the perturbation expansion for the energy levels is asymptotic and b) energy levels come in pairs with separation exponentially small in g $\left(\sim E_0 e^{-A/g^2} \cong L^{-1}\right)$[17], so that such non-perturbative effects are relevant only for correlation times of order $\gtrsim L$. We also note that the non-symmetric correlation functions vanish (reflection symmetry is not broken) whereas the standard perturbation expansion around one minimum gives a non zero value for them, so that for non-symmetric correlation functions the expansion around <u>one</u> minimum cannot be asymptotic.

FOOTNOTES

a) It is believed that asymptotic freedom is the crucial property that entails the existence of the continuum limit. It has by now become a quite generally accepted requirement and it is valid (perturbatively) in grand unified theories where because of the occurrence of very different scales a non-perturbative understanding of the Higgs phenomenon becomes a crucial issue.

b) In particular, conventional perturbation theory in the continuum limit can only be formulated by introducing gauge-dependent Green's functions and fixing a gauge, even if one wants to calculate a gauge-invariant Green's function.

c) Most of the gauge fixings used in the literature have these properties. The 't Hooft non-linear gauge and the unitary gauge do not satisfy b).

d) The temporal gauge has been discussed in detail by G.C. Rossi and M. Testa[(6)] by extensively exploiting the validity of a Gauss' law. It is not clear to us whether their argument gives $\langle \varphi \rangle = 0$ when the boundary conditions are specified.

e) As emphasized before this group acts non trivially only on gauge dependent (unobservable) field variables and its non trivial action is made possible just by the introduction of a gauge fixing.

f) In particular one is facing the puzzle of understanding the experimental success of the standard picture, like in the Glashow-Weinberg-Salam model, for which the breaking of the global gauge group is crucial.

g) Technically, $G_{\{\bar{\varphi}\}}$ is the abstract group isomorphic to the stability group of any point of the orbit $\{\bar{\varphi}\}$.

h) All the proofs are postponed to Sect. 6.

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