

AXIONS, QUANTUM MECHANICAL PUMPING, AND PRIMEVAL MAGNETIC FIELDS

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Axions, Quantum Mechanical Pumping, and Primeval Magnetic Fields

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Abstract

We discuss the ordinary quantum Hall effect and a higher-dimensional cousin. We consider the dimensional reduction of these effects to $1+1$ and $3+1$ space-time dimensions, respectively. After dimensional reduction, an axion field appears, which plays the rôle of a space-time dependent difference of chemical potentials of chiral modes. As applications, we sketch a theory of quantum pumps and a mechanism for the generation of primeval magnetic fields in the early universe.

1 Introduction

In these notes, we clarify the rôle played by certain pseudo-scalar fields related to “axions” in some transport- or pumping processes in semiconductor devices and in the early universe. These processes are similar to ones observed in quantum Hall systems. We therefore start by recalling some key features of the theory of the quantum Hall effect. We then consider transport processes in very long, narrow rectangular Hall samples with constrictions, as shown in Figure 1.

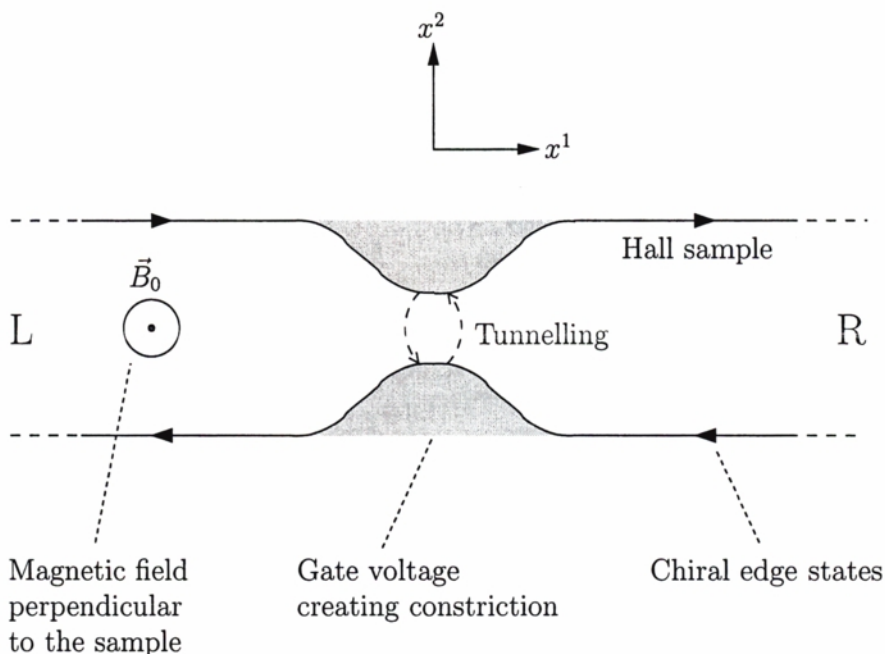


Figure 1: A constriction in a quantum Hall sample.

For samples of this kind, filled with an incompressible Hall fluid, the component, A_2 , of the electromagnetic vector potential, A , parallel to the short axis, 2, of the sample can be interpreted as a pseudo-scalar field analogous to the *axion* known from elementary particle physics [1]. In the region where the sample has a constriction, *tunnelling processes* between the chiral edge modes on the upper and lower edge of the sample may occur. It is interesting to consider the effect of turning on a time-dependent voltage drop in the 2-direction. Not surprisingly,

we find that when such a voltage drop, $V(t)$, with

$$\int_{-\infty}^{+\infty} V(t)dt =: \delta\varphi \quad (1)$$

is turned on, an electric charge δQ proportional to $\delta\varphi$ is transported through the constriction from the left, L , to the right, R . This system thus realizes a simple “*quantum pump*”. Due to the tunnelling processes between edge states of opposite chirality, the state of the pump exhibits a periodicity in $\delta\varphi$ proportional to the inverse electric charge of the charge carriers in the sample. Thus, such a pump can be used, in principle, to explore properties of the quasi particles in incompressible quantum Hall fluids, such as their electric charges, [2].

Our model can also be used to describe quantum wires carrying a Luttinger liquid. The rôle of the constriction is then played by impurities mixing left and right movers.

We then proceed to studying a five-dimensional analogue of the quantum Hall effect. If four-dimensional physics is described by dimensional reduction from a five-dimensional slab to two parallel boundary “3-branes”, the axion can be interpreted as the component of the five-dimensional electromagnetic vector potential transversal to the branes. Tunnelling of chiral fermions from one to the other brane, due e.g. to a mass term, generates a periodic axion potential. It is then argued that the dynamics of the axion may trigger the growth of large-scale primeval magnetic fields in the early universe. In other words, axion dynamics - which is coupled to the dynamics of the curvature tensor of space-time - can be viewed as a realization of a quantum-field theoretical “pump” driving the growth of large-scale primeval magnetic fields, [3][4][5]; see also [6]. Whether this mechanism plays a rôle in explaining the observed large-scale magnetic fields in the universe is, however, still uncertain; see [7].

2 Brief Recap of the Quantum Hall Effect

We consider a uniform 2-dimensional electron gas of density n forming at the interface between a semiconductor and an insulator when a gate voltage is applied in the direction perpendicular to the interface. We imagine that a homogeneous magnetic field, $\vec{B}_0$, perpendicular to the interface is turned on. Let $\nu := (nhc)/(e|\vec{B}_0|)$ denote the “filling factor”. From the experiments of von Klitzing et al. [8] and Tsui et al. [9] we have learnt that, for certain values of ν , the 2-dimensional electron gas forms an incompressible fluid, in the sense that the longitudinal resistance, R_L , of the system vanishes. We consider the response of such a system to turning on a small external electromagnetic field $(\vec{E}, B)$, where $\vec{E}$ denotes the in-plane component of the electric field, and $B_{\text{tot}} = B_0 + B$ is the component of the total magnetic field perpendicular to the plane of the fluid. By $\vec{j}(x)$ we denote the current density in the plane of the 2-dimensional electron gas, and by $j^0(x) = \rho(x) + en$ the deviation of the electric charge density from the uniform background charge density, en ; (here $x = (\underline{x}, t)$, where $\underline{x}$ is a point in the sample and t is time).

By combining Hall's law (for $R_L = 0$), i.e.,

$$j^k(x) = G_H \varepsilon^{kl} E_l(x) \quad , \quad (2)$$

where G_H is the bulk Hall conductance, with the continuity equation for j_0 and $\underline{j}$ and Faraday's induction law, one easily finds that

$$j^0(x) = G_H B(x) \quad , \quad (3)$$

see [10]. Denoting by $F = (F_{\mu\nu})$ the electromagnetic field tensor over the $(2+1)$ -dimensional space-time, Λ , of the sample and by $J \equiv (J_{\mu\nu} = \varepsilon_{\mu\nu\lambda} j^\lambda)$ the 2-form dual to the charge-current density $(j^0, \underline{j})$, eqs. (2) and (3) can be summarized in

$$J = G_H F \quad , \quad (4)$$

the field equation of "Chern-Simons electrodynamics" [11]. Defining the dimensionless Hall conductivity, σ_H , by

$$\sigma_H = \frac{h}{e^2} G_H \chi_\Lambda(x) \quad (5)$$

and using units such that $e^2/h = 1$, the field equations of Chern-Simons electrodynamics are

$$J(x) = \sigma_H(x) F(x) \quad . \quad (6)$$

Taking the exterior derivative of eq. (6), we find that

$$dJ = d\sigma_H \wedge F \quad , \quad (7)$$

because $dF = d(dA) = 0$. The gradient $d\sigma_H$ is transversal to the boundary, $\partial\Lambda$, of the sample's space-time. Eq. (7) tries to tell us that electric charge is *not* conserved in an incompressible Hall fluid, because dJ , the dual of $\partial_\mu j^\mu = \partial_t j^0 + \text{div} \underline{j}$, does *not* vanish. The origin of this false impression is that, so far, we have neglected the *diamagnetic edge current*, J_{edge} , in our equations. This current is localized on $\partial\Lambda$ and is dual to a vector field $i = (i_\mu)$ with support on $\partial\Lambda$ and parallel to $\partial\Lambda$. The edge current J_{edge} saves electric charge conservation:

$$d(J + J_{\text{edge}}) = 0 \quad . \quad (8)$$

Eqs. (8) and (7) then yield

$$\partial_\mu i^\mu = \sigma_H^{\text{edge}} E^\parallel \quad , \quad (9)$$

where $E^\parallel$ is the component of the electric field parallel to the boundary of the sample, and the "edge" conductivity, σ_H^{edge} , is equal to $-\sigma_H$, the "bulk" conductivity, as follows from (7). Eq. (9) describes the $(1+1)$ -dimensional chiral anomaly [12]. Apparently, the edge current, i , is an anomalous (chiral) electric current localized on the boundary of the sample; (the chirality of i depends on the direction of $\vec{B}_0$ and the sign of the electric charge of the fundamental charge carriers).

Equations (9) and (6) can be derived from an action principle. If $S_{\Lambda}^{\text{eff}}(A)$ denotes the *effective action*, i.e., the generating functional of the current Green functions, of an incompressible Hall fluid confined to a three-dimensional space-time region Λ , in the presence of an external electromagnetic field $(\underline{E}, B)$ with vector potential A , then

$$S_{\Lambda}^{\text{eff}}(A) \approx \frac{\sigma_H}{2} \int_{\Lambda} A \wedge dA + \Gamma_{\partial\Lambda}(a) \quad , \quad (10)$$

where $a = A^{\parallel}|_{\partial\Lambda}$ is the restriction of A to the boundary, $\partial\Lambda$, of Λ , and “ $\approx$ ” means that only the *leading contributions* (in the sense of dimensional analysis) to the effective action are displayed on the R.S. The first (bulk) term on the R.S. of (10) is the *Chern-Simons action*, the second (edge) term turns out to be the *anomalous chiral action* [12] in two space-time dimensions. Its gauge variation fixes the value of σ_H^{edge} by

$$\left. \frac{d}{d\varepsilon} \right|_{\varepsilon=0} \Gamma_{\partial\Lambda}(a + \varepsilon d\chi) = \chi \sigma_H^{\text{edge}} E^{\parallel} \quad . \quad (11)$$

Electromagnetic gauge invariance is a fundamental property of non-relativistic many-body theory. Thus, $S_{\Lambda}^{\text{eff}}(A)$ must be gauge invariant, i.e.,

$$S_{\Lambda}^{\text{eff}}(A) = S_{\Lambda}^{\text{eff}}(A + d\chi) \quad , \quad (12)$$

for an arbitrary function χ on Λ . Individually, the Chern-Simons action, $\frac{\sigma_H}{2} \int_{\Lambda} A \wedge dA$, and the boundary action $\Gamma_{\partial\Lambda}(a)$ are *not* invariant under a gauge transformation, χ , *not* vanishing on the boundary $\partial\Lambda$; but the R.S. of (10) is gauge invariant precisely if $\sigma_H = -\sigma_H^{\text{edge}}$.

Since $S_{\Lambda}^{\text{eff}}(A)$ is the generating functional of the current Green functions, we have that

$$j^{\mu}(x) = \frac{\delta S_{\Lambda}^{\text{eff}}(A)}{\delta A_{\mu}(x)} \quad , \quad i^{\mu}(x) = \frac{\delta \Gamma_{\partial\Lambda}(a)}{\delta a_{\mu}(x)} \quad . \quad (13)$$

These expressions, together with eq. (10) for $S_{\Lambda}^{\text{eff}}(A)$, reproduce the basic equations (6) and (9).

The boundary action $\Gamma_{\partial\Lambda}(a)$ is known to be the generating functional of the chiral Kac-Moody current operators of current algebra with gauge group $U(1)$. It is then a natural idea [13][14] that the boundary degrees of freedom of an incompressible Hall fluid are described by a *chiral conformal field theory*. Under the natural assumptions that

- (i) sectors of physical states of this theory are labelled by their electric charge and, possibly, finitely many further quantum numbers (e.g. spin) with finitely many possible values; and
- (ii) excitations of this theory with even/odd electric charge (in units where $e = 1$) obey Bose/Fermi statistics,

one shows that σ_H is necessarily a *rational number*, and one obtains a table of values of σ_H that compares well with those of the dimensionless Hall conductivity of experimentally established incompressible Hall fluids, [13][14]. Moreover, one can systematically work out the spectrum of fractionally charged quasi-particles propagating along the edge of the sample. The smallest fractional electric charge turns out to be given by $q = er/d_H$, where r is a positive integer - and, for many fluids, $r = 1$ - and d_H is the integer denominator of σ_H , (writing $\sigma_H = n_H/d_H$, with n_H and d_H relatively prime integers); see [13][14].

3 Hall Samples with Constriction and Quantum Wires

In this section we consider a very long, narrow rectangular Hall sample, as shown in Figure 1. The axis parallel to the long side of the sample is taken to be the 1-axis, the one parallel to the short side is the 2-axis, and we set $\underline{x} = (x^1, x^2) \equiv (x, y)$. We define a field φ by

$$\varphi(x, t) = \int_{\gamma_u} A_2(x, y, t) dy \quad , \quad (14)$$

where $\mathbf{l} = (x, y_1)$ is a point on the lower edge of the sample, $\mathbf{u} = (x, y_u)$ is a point on the upper edge, and γ_u is the straight line from $\mathbf{l}$ to $\mathbf{u}$. We assume that the 1-component,

$$E \equiv E_1 = \partial_0 A_1 - \partial_1 A_0 \quad , \quad (15)$$

of the in-plane electric field, $\underline{E}$, is independent of y . It is convenient to choose a gauge such that $A \equiv (A_0, A_1)$ is independent of y . Then the effective action in equation (10) becomes

$$S^{\text{eff}}(\varphi, A) \approx \sigma_H \int dt \int_I dx \varphi E \quad , \quad (16)$$

where I is the interval on the x -axis which the Hall sample is confined to. The terms corresponding to the upper and the lower edge in the boundary action $\Gamma_{\partial\Lambda}$ on the R.S. of (10) cancel each other, because A_0, A_1 are independent of y , up to a manifestly gauge-invariant term proportional to $\int dt \int dx (A^T)^2$. The action $S^{\text{eff}}(\varphi, A)$ describes the coupling of an “axion field” $\varphi(x, t)$ to the electric field $E(x, t)$ of (1+1)-dimensional QED. For the current, I , through the sample and the charge density, P , in an external axion field configuration φ , we find the expressions

$$I(x, t) = \frac{\delta S^{\text{eff}}(\varphi, A)}{\delta A_1(x, t)} = -\sigma_H \dot{\varphi}(x, t) \quad , \quad (17)$$

$$P(x, t) = \frac{\delta S^{\text{eff}}(\varphi, A)}{\delta A_0(x, t)} = \sigma_H \varphi'(x, t) \quad , \quad (18)$$

provided $E = 0$, so that there are no contributions from the boundary action. (Here $I(x, t) = \int j^1(x, y, t)dy$, $P(x, t) = \int j^0(x, y, t)dy$. We observe that (17) and (18) imply the continuity equation $\dot{P} + I' = 0$.)

The action $S^{\text{eff}}(\varphi, A)$ in eq. (16) yields an accurate description of charge transport in a long, narrow sample filled with an incompressible Hall fluid with Hall conductivity σ_H if the electric field in the 1-direction vanishes (so that the term proportional to $\int dt \int dx (A^T)^2$ does not contribute), as long as tunnelling processes between the upper and the lower edge can be neglected. However, for a sample with a constriction, as shown in Figure 1, such tunnelling processes *do* occur. In a description of the Hall fluid in terms of an action that displays the edge degrees of freedom explicitly, tunnelling processes between the two edges are described by terms of the form

$$\int \left[t(x) \sum_{\alpha} \bar{\psi}_{\text{left}, \alpha}(x, y_u, t) e^{2\pi i q_{\alpha} \int_{y_1}^{y_u} A_2(x, y, t) dy} \psi_{\text{right}, \alpha}(x, y_1, t) \right] dx dt + \text{h.c. (left} \leftrightarrow \text{right)} \quad , \quad (19)$$

where α labels the different species of charged quasi-particles described by left chiral fields, $\psi_{\text{left}, \alpha}$, $\bar{\psi}_{\text{left}, \alpha}$, on the upper edge and by right chiral fields $\psi_{\text{right}, \alpha}$, $\bar{\psi}_{\text{right}, \alpha}$, on the lower edge, and $q_{\alpha}e$ is the electric charge of a quasi-particle of species α . Setting

$$\overset{(-)}{\psi}_{L, \alpha}(x, t) = \overset{(-)}{\psi}_{\text{left}, \alpha}(x, y_u, t) \quad , \quad \overset{(-)}{\psi}_{R, \alpha}(x, t) = \overset{(-)}{\psi}_{\text{right}, \alpha}(x, y_1, t) \quad , \quad (20)$$

and recalling eq. (14), the term (19) can be written as

$$\int \left[t(x) \sum_{\alpha} \bar{\psi}_{L, \alpha}(x, t) e^{2\pi i q_{\alpha} \varphi(x, t)} \psi_{R, \alpha}(x, t) + \text{h.c. (L} \leftrightarrow \text{R)} \right] dx dt \quad . \quad (21)$$

The function $t(x)$ is a measure for the strength of the amplitude of tunnelling between the two edges; $|t(x)|$ is “large” for x close to the constriction, and tends to 0 rapidly, as the distance of x to the constriction increases.

Besides (21), the action for the edge degrees of freedom contains terms not mixing the left- and right-moving degrees of freedom. These terms do *not* depend on φ . Integrating (or “tracing”) out all edge degrees of freedom, we obtain an effective “boundary action”, $\tilde{\Gamma}(\varphi, A \equiv (A_0, A_1))$, which now depends on φ ! It is *periodic* in φ : if φ_0 is the smallest real number such that

$$q_{\alpha} \varphi_0 = n_{\alpha} \quad , \quad n_{\alpha} \in \mathbf{Z} \quad , \quad (22)$$

for all species α , then

$$\tilde{\Gamma}(\varphi(\cdot, \cdot) + \varphi_0, A) = \tilde{\Gamma}(\varphi(\cdot, \cdot), A) \quad . \quad (23)$$

This follows immediately from the form of (21) of the tunnelling terms in the boundary action.

The remarks on the relation between fractional charges and the value of the Hall conductivity σ_H at the very end of Section 2 lead to the equation

$$\sigma_H \varphi_0 = \frac{n_H}{r} \quad , \quad (24)$$

where n_H is the Hall numerator and r is an integer, (see [13]; actually $r = 1$, for the Laughlin- and the simple Jain fluids with $\sigma_H = n/(2pn+1)$, $p, n = 1, 2, \dots$).

The total effective action is given by

$$S^{\text{eff}}(\varphi, A) \approx \sigma_H \int dt \int_I dx \varphi E + \tilde{\Gamma}(\varphi, A) \quad . \quad (25)$$

The periodicity property (23) of $\tilde{\Gamma}(\varphi, A)$ implies that, if the 1-component of the electric field vanishes $E = 0$, the macroscopic state of this system depends periodically on the external “axion field” φ , with period φ_0 , and that eqs. (17) and (18) for the electric current $I(x, t)$ and the charge density $P(x, t)$ continue to hold *in average* when the system is driven through several cycles. Indeed, because of the invariance of $\tilde{\Gamma}(\varphi, A)$ under a gauge transformation $A' = A + d\chi$, one has

$$\partial_t \frac{\delta}{\delta A_0(x, t)} \tilde{\Gamma}(\varphi, A) + \partial_x \frac{\delta}{\delta A_1(x, t)} \tilde{\Gamma}(\varphi, A) = 0 \quad , \quad (26)$$

and one can write

$$\frac{\delta}{\delta A_1(x, t)} \tilde{\Gamma}(\varphi, 0) = \partial_t U(\varphi, x, t) \quad , \quad (27)$$

where the function $U(\varphi, x, t)$, which depends on the axion field configuration φ and the spacetime point (x, t) , is given by

$$U(\varphi, x, t) = - \int_{-\infty}^x dy \frac{\delta}{\delta A_0(y, t)} \tilde{\Gamma}(\varphi, 0) \quad . \quad (28)$$

The function U is periodic in φ , with period φ_0 ,

$$U(\varphi(\cdot, \cdot) + \varphi_0, x, t) = U(\varphi(\cdot, \cdot), x, t) \quad , \quad (29)$$

and does not depend on time explicitly,

$$U(\varphi(\cdot, \cdot + \Delta t), x, t + \Delta t) = U(\varphi(\cdot, \cdot), x, t) \quad . \quad (30)$$

Consider a pump which works with a period T , i.e., a pump driven by an axion field $\varphi(\cdot, \cdot)$ which fulfills

$$\varphi(\cdot, \cdot + T) = \varphi(\cdot, \cdot) + n\varphi_0 \quad (31)$$

for some integer n . One then finds that the charge transport due to the second term on the R.H.S. of (25) vanishes, since

$$\int_t^{t+T} dt \frac{\delta}{\delta A_1(x, t)} \tilde{\Gamma}(\varphi, 0) = U(\varphi(\cdot, \cdot + T), x, t + T) - U(\varphi(\cdot, \cdot), x, t) = 0 \quad . \quad (32)$$

We now recall the physical meaning of the axion field φ . By eq. (14),

$$\dot{\varphi}(x, t) = \int_{\gamma_u} \dot{A}_2(x, y, t) dy = \int_{\gamma_u} E_2(x, y, t) dy = V(x, t) \quad , \quad (33)$$

where $V(x, t)$ is the *voltage drop* at x between the lower and the upper edge of the sample; (we are using that $E_2 = \partial_0 A_2 - \partial_2 A_0 = \partial_0 A_2$, because A_0 is independent of y). Let $\varphi(t)$ be an x -independent configuration of the “axion field”, with

$$\delta\varphi := \int_{-\infty}^{+\infty} \dot{\varphi}(t) dt = \int_{-\infty}^{+\infty} V(t) dt \quad . \quad (34)$$

Then eq. (17) tells us that the total amount, δQ , of electric charge transported from the left (L) to the right (R) of the sample is given by

$$\delta Q = \int_{-\infty}^{+\infty} I(x, t) dt = -\sigma_H \delta\varphi \quad , \quad (35)$$

(and $P(x, t) \equiv 0$, by eq. (18)). Thus, a sample with a time-dependent voltage drop between the upper and lower edge can be viewed as a “quantum pump” transporting electric charge from the left to the right. The macroscopic state of this pump is periodic in $\delta\varphi$ with period φ_0 . Thus, when the pump is operated over $n = 1, 2, \dots$ cycles, a total amount, δQ_n , of electric charge

$$\delta Q_n = -\sigma_H n \varphi_0 = -\frac{nn_H}{r} \quad (36)$$

is transported from the left to the right; (here we have used eq. (24)). Since φ_0^{-1} is the smallest fractional electric charge of a quasiparticle tunnelling through the constriction, a measurement of this charge can be obtained from independent measurement of the charge δQ_n transported from the left to the right in n cycles and of σ_H . Whether a given voltage pulse $\delta\varphi = \int_{-\infty}^{+\infty} V(t) dt$ corresponds to an integer number of cycles of the pump can be inferred from the *fluctuations* of the charge, δQ , transported from the left to the right during that pulse around its mean value $-\sigma_H \delta\varphi$: if, on the left and right ends the sample is connected to free-electron leads then (independently of $\delta\varphi$) δQ must be an integer (multiple of e). If $\sigma_H \delta\varphi$ is *not* an integer then δQ will exhibit fluctuations around its mean value $-\sigma_H \delta\varphi$. But if $\delta\varphi$ corresponds to exactly nr cycles, $n = 1, 2, \dots$, then $-\sigma_H \delta\varphi = nn_H$ is an integer, and hence the fluctuations of δQ in this process will essentially *vanish*.

Typical features of the effective action $\tilde{\Gamma}(\varphi)$, with $E = 0$, can be determined by measuring the tunnelling current I_T through the constriction: when $E = 0$

$$I_T(t) = \int dx \lambda(x) \frac{\delta \tilde{\Gamma}(\varphi)}{\delta \varphi(x, t)} \quad , \quad (37)$$

where $\lambda(x)$ is the width of the sample at x . A tunnelling current I_T can be generated e.g. by a modulation of the magnetic field perpendicular to the plane of

the sample. The expression for the tunnelling current in terms of $\tilde{\Gamma}$ given above shows that, from measurements of I_T and of the voltage drop V as functions of time, one can infer the period φ_0 of $\tilde{\Gamma}$ and, hence, the smallest fractional electric charge of the quasi-particles. Furthermore, one can argue that the fluctuations of I_T are proportional to the fractional charge of the quasi-particles tunnelling through the constriction - an effect used in the experiments described in [2] to measure the fractional charges of quasi-particles.

If the magnetic field is set to 0 our considerations can also be used to describe *quantum wires*. Then $e\dot{\varphi}(x, t)$ has the interpretation of a (space-time dependent) *difference of chemical potentials* between left- and right-moving modes in the wire. Eq. (17) then says that if there isn't any chirality-reversing scattering in the wire (i.e. $\tilde{\Gamma}(\varphi, A = 0) = 0$), and for $E = 0$,

$$I(x, t) = -G\dot{\varphi}(x, t) = \frac{G}{e}[\mu_L - \mu_R](x, t) \quad , \quad (38)$$

where G now has the interpretation of a *longitudinal conductance*. If all quasi-particles in the wire have integer electric charge then G is an integer multiple of e^2/h ; (see [15][16]).

If there are tunnelling processes mixing left- and right-movers, due e.g. to *impurities* in the wire, then the term $\tilde{\Gamma}(\varphi, A)$ on the R.S. of (25) does not vanish, even if $E = 0$. The general expression for the current I in the wire is given by the equation

$$I(x, t) = \frac{G}{e}[\mu_L - \mu_R](x, t) + \frac{\delta\tilde{\Gamma}(\varphi, A)}{\delta A_1(x, t)} \quad , \quad (39)$$

with $e\dot{\varphi} = \mu_L - \mu_R$ and $\dot{A}_1 = E$. If scattering at the impurities converts left-into right-movers, and conversely, the second term on the R.S. of eq. (39) does not vanish even if $E = 0$, and hence conductance is not quantized, anymore, in accordance with experiment. However, charge transport over long periods of time still exhibits "quantization", provided $E = 0$, due to the periodicity of $\tilde{\Gamma}$ in φ .

A more detailed account of our results and an analysis of the Luttinger liquids in quantum wires in the presence of impurities will be given elsewhere.

4 A 5-dimensional analogue of the Quantum Hall Effect, and Primeval Magnetic Fields in the Early Universe

Imagine, for a moment, that our world corresponds to a stack of 3-branes in a 5-dimensional space-time. We suppose that all electrically charged modes propagating through the 5-dimensional bulk have a large mass (comparable, e.g., to the Planck mass) and have parity-violating dynamics. We may then ask whether there is an analogue of the quantum Hall effect in the $(4+1)$ -dimensional bulk. To be specific, we assume that there are two parallel, flat 3-branes separated by a $(4+1)$ -dimensional slab Λ of width λ representing the bulk

of the system. Let $\hat{A}$ denote the 5-dimensional electromagnetic vector potential and A the restriction to the boundary, $\partial\Lambda$, of the slab of the components of $\hat{A}$ parallel to $\partial\Lambda$. Assuming that only the graviton and the photon are massless modes, and dropping the gravitational contribution, the effective action of such a system is given by

$$\begin{aligned} S_{\Lambda}^{\text{eff}}(\hat{A}) &= \frac{\varepsilon}{4\lambda} \int_{\Lambda} \hat{F} \wedge * \hat{F} \\ &+ \frac{\sigma_T}{3} \int_{\Lambda} \hat{A} \wedge \hat{F} \wedge \hat{F} \\ &+ \Gamma_{\partial\Lambda}(A) + \text{irrelevant terms} \quad , \end{aligned} \quad (40)$$

with $\xi = (\xi^0, \xi^1, \dots, \xi^4) \equiv (t, \underline{x}, \xi^4)$, where ξ^4 is the coordinate perpendicular to the boundary 3-branes, which are located at $\xi^4 = 0, \lambda$, respectively, and ε is a dimensionless constant. The first term on the R.S. of (40) is a Maxwell term, which is the dominant term, the second term is the 5-dimensional Chern-Simons action, the last term is the 4-dimensional anomalous chiral (boundary) action, which ensures that $S_{\Lambda}^{\text{eff}}(\hat{A})$ is gauge-invariant. From the theory of the chiral anomaly we infer that

$$\sigma_T = \frac{e^3}{8\pi^2} \sum_{\alpha} q_{\alpha}^3 \quad , \quad (41)$$

where the q_{α} 's are the charges of the chiral fermions propagating along $\partial\Lambda$. The action $\Gamma_{\partial\Lambda}$ is the 4-dimensional version of the boundary action $\Gamma_{\partial\Lambda}$ in eq. (10). It is the generating functional of the Green functions of chiral currents $j_{L/R}^{\mu}$ satisfying

$$\partial_{\mu} j_{L/R}^{\mu} = \pm \frac{\sigma_T}{3} \underline{E} \cdot \underline{B} \quad , \quad (42)$$

where $(\underline{E}, \underline{B})$ is the electromagnetic field on the boundary 3-branes. Modes of opposite chirality are localized on the two opposite 3-branes, (at $\xi^4 = 0$ and $\xi^4 = \lambda$, respectively).

Imagine that the fields $\hat{F}_{\mu\nu}$, $\mu, \nu = 0, 1, 2, 3$ are *independent* of ξ^4 . We define the *axion field*, φ , by

$$\varphi(\underline{x}, t) = \int_0^{\lambda} A_4(t, \underline{x}, \xi^4) d\xi^4 \quad . \quad (43)$$

After dimensional reduction to the boundary 3-branes, $\partial\Lambda$, the effective action in (40) becomes

$$S_{\Lambda}^{\text{eff}}(\varphi, A) \approx \frac{\varepsilon}{4} \int_{\partial\Lambda} F_{\mu\nu} F^{\mu\nu} d^4x + \frac{\varepsilon}{2\lambda^2} \int_{\partial\Lambda} \partial_{\mu} \varphi \partial^{\mu} \varphi d^4x \quad (44)$$

$$+ \sigma_T \int_{\partial\Lambda} \varphi F \wedge F \quad (45)$$

$$+ \tilde{\Gamma}_{\partial\Lambda}(\varphi, A) \quad . \quad (46)$$

If tunnelling between the two boundary 3-branes is suppressed completely the boundary action $\tilde{\Gamma}_{\partial\Lambda}(\varphi, A)$ is independent of φ and can be combined with the

Maxwell term to renormalize its coefficient. But if tunnelling processes mixing fermions of opposite chirality are present then $\tilde{\Gamma}_{\partial\Lambda}(\varphi, A)$ depends on φ and is $\neq 0$ even if $F = 0$. Tunnelling processes generate a small mass, proportional to $Me^{-\lambda/l_P}$, of boundary fermions; (here M is a typical bulk mass scale, and l_P is the Planck length). By repeating the arguments explained in Section 3, one finds that $\tilde{\Gamma}_{\partial\Lambda}(\varphi, A)$ is *periodic* in φ with period φ_0 proportional to q_*^{-1} , where q_* is the smallest electric charge of modes propagating along the 3-branes.

We recall that $S^{\text{eff}}(\varphi, A)$ is the generating functional of the Green functions of the pseudo-scalar density and the electric current density; in particular, $j^\mu = \delta S^{\text{eff}}(\varphi, A)/\delta A_\mu$. Plugging the expressions for $j^\mu = \langle \mathcal{J}^\mu \rangle$ and for $\langle \bar{\psi}\gamma^5\psi \rangle$ obtained from (46) into Maxwell's equations and the equations of motion for the axion field, we find the following equations of motion:

$$\begin{aligned} F_{[\mu\nu;\sigma]} &= 0 \quad , \\ F_{;\sigma}^{\mu\nu} &= 2\beta\sigma_T(\varphi\tilde{F}^{\mu\nu})_{;\nu} + \beta\frac{\delta\tilde{\Gamma}(\varphi, A)}{\delta A_\mu} \quad , \\ \square\varphi \equiv \varphi_{;\mu}^{\mu} &= -\beta'\lambda^2 \left[\sigma_T F\tilde{F} + \frac{\delta\tilde{\Gamma}(\varphi, A)}{\delta\varphi} + k\text{tr}(R\tilde{R}) \right] \quad , \end{aligned} \quad (47)$$

where β and β' are dimensionless constants, and the term $k\text{tr}(R\tilde{R})$, where R is the Riemann tensor, comes from a term $k \int \varphi \text{tr}(R\tilde{R})$ in the effective action describing the coupling of the axion to space-time curvature (which has not been displayed in eq. (46)). If there exist magnetic monopoles the first equation in (47) must be replaced by $F_{[\mu\nu;\sigma]} = j_{\mu\nu\sigma}^M$, where j^M is the magnetic current 3-form. In conventional vector analysis notation, eqs. (47) take the form

$$\begin{aligned} \underline{\nabla} \cdot \underline{B} &= 0 \\ \underline{\nabla} \wedge \underline{E} &= \underline{\dot{B}} \\ \underline{\nabla} \cdot \underline{E} &= 2\beta\sigma_T \underline{\nabla}\varphi \cdot \underline{B} + \beta\frac{\delta\tilde{\Gamma}(\varphi, A)}{\delta A_0} \\ \underline{\nabla} \wedge \underline{B} &= -\underline{\dot{E}} + \sigma_L \underline{E} + 2\beta\sigma_T [\dot{\varphi}\underline{B} + \underline{\nabla}\varphi \wedge \underline{E}] + \beta\frac{\delta\tilde{\Gamma}(\varphi, A)}{\delta \underline{A}} \\ \square\varphi &= -\beta'\lambda^2 \left[2\sigma_T \underline{E} \cdot \underline{B} + \frac{\delta\tilde{\Gamma}(\varphi, A)}{\delta\varphi} + k\text{tr}(R\tilde{R}) \right] \quad , \end{aligned} \quad (48)$$

where, in the fourth equation of (48), the term $\sigma_L \underline{E}$ has been added to describe a dissipative current parallel to $\underline{E}$, with σ_L the longitudinal conductivity; (Ohm's law).

It is clear from eq. (43) that the time derivative, $e\dot{\varphi}$, of the axion field has the interpretation of a (space-time dependent) *difference of chemical potentials* of right-handed and left-handed charged modes propagating on the “upper” and the “lower” brane, respectively.

Absorbing the leading A -dependent contribution to $\tilde{\Gamma}(\varphi, A)$ into a renormalization of the constant β , the leading term in $\tilde{\Gamma}(\varphi, A)$ is independent of A and

has the form

$$\tilde{\Gamma}(\varphi, A) = \int U(\varphi(x)) d^4x \quad , \quad (49)$$

where U is a (temperature-dependent) *periodic* function with period φ_0 . Plugging eq. (49) into (48), we find that a special solution of (48) is given by $\underline{E} = \underline{B} = 0$ and φ solving the equation

$$\square\varphi = -\beta'\lambda^2 \left[U'(\varphi) + k\text{tr}(R\tilde{R}) \right] \quad . \quad (50)$$

As mentioned above, U actually depends on the temperature of the universe: $U \approx 0$ at temperatures well above the electro-weak phase transition; while, at temperatures below the electro-weak scale, U is a non-constant, periodic function of φ with minima at $\varphi = n\varphi_0$, $n \in \mathbf{Z}$. Thus, at the time t_* of the electro-weak phase transition, the configuration $\varphi(t_*, \underline{x})$ corresponds, approximatively, to a solution of

$$\square\varphi = -\beta'\lambda^2 \text{tr}(R\tilde{R}) \quad , \quad (51)$$

and there is no reason why $\varphi(t_*, \underline{x})$ should be close to a minimum of the function $U(\varphi)$, or why $\dot{\varphi}(t_*, \underline{x})$ should be small. The source term $-\beta'\lambda^2 \text{tr}(R\tilde{R})$ on the R.S. of (51) does not vanish, provided there are gravitational waves propagating through the universe. For a Friedman universe, it is proportional to the amplitude of gravitational waves: thus, such waves can, in principle, feed the growth of the axion field.

At times $t > t_*$, the equation of motion of the axion is given by

$$\square\varphi = -\beta'\lambda^2 \left[U'(\varphi) + k\text{tr}(R\tilde{R}) \right] \quad , \quad (52)$$

with $U \neq \text{constant}$. Assuming that gravitational waves eventually disperse away, the term proportional to $\text{tr}(R\tilde{R})$ will approach 0, for a Friedman universe. Let us suppose that, after inflation, $\varphi(t, \underline{x}) \approx \varphi(t)$ varies slowly over space. Then eq. (52) reduces to an ordinary differential equation

$$\ddot{\varphi} = -\beta'\lambda^2 U'(\varphi(t)) \quad , \quad (53)$$

to be solved for essentially random initial conditions, $(\varphi(t_*), \dot{\varphi}(t_*)) \neq (n\varphi_0, 0)$, $n \in \mathbf{Z}$. Eq. (53) is the equation of motion for a pendulum in a potential force field, $-U'(\varphi(t))$. Solutions of (53) are given by

$$\varphi(t) = \frac{\mu_L - \mu_R}{e} t + \alpha(t) \quad , \quad (54)$$

where α is a periodic function of t .

Next, we linearize eqs (48) around the special solution $\underline{E} = \underline{B} = 0$, $\varphi(t)$ as in (54). This is not a difficult task. One finds that for sufficiently small wave vectors, $\underline{k}$, ($|\underline{k}| < 2\beta\sigma_T(\mu_L - \mu_R)/e$, for $\alpha = 0$), there are *exponentially growing transverse modes*, $\underline{\hat{B}}(\underline{k}, t)$, of the magnetic field with non-vanishing magnetic helicity. One expects that axion field configurations which are slowly varying

in space lead to qualitatively similar instabilities. When combined with the galactic dynamo mechanism they might provide an explanation of the large-scale magnetic fields observed in the universe; (but see [7] for discussion of some of the difficulties with this and other scenarios). We hope to present a more detailed account of our results, in particular of the possible rôle of gravitational waves, elsewhere.

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