PHASE DIAGRAMS AND CRITICAL PROPERTIES OF (CLASSICAL) COULOMB SYSTEMS

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I. INTRODUCTION

It seems good to start these lecture notes with the cautioning remark that, in a sense, their contents do not really fit into the topic "Rigorous Atomic and Molecular Physics", although the results we shall discuss are certainly rigorous and do concern atoms, ions and molecules to some extent. Traditionally, the underlying theoretical framework for atomic and molecular physics is thought to be quantum mechanics, and the number of degrees of freedom of the atomic and molecular systems which are considered is finite, at least if the radiation field is neglected which is what workers in that field do almost always.

In these lecture notes, quantum mechanics is not heard of, except that some models which we will mention briefly do provide idealized descriptions of certain quantum mechanical systems. In those instances, however, we will employ the imaginary-time, Feynman-Kac formulation of quantum mechanics which makes it look like classical, statistical mechanics.

The basic, theoretical framework underlying our lectures is the classical statistical mechanics of systems with infinitely many degrees of freedom. We think that we do describe methods and results which are relevant for the physics of systems composed of very many atoms and molecules, namely for condensed matter physics, but we leave it to the reader to judge.

Due to various circumstances it was not possible for us to produce somewhat detailed lecture notes which would include careful statements of results, proofs and discussion. Thus we can only hope the present notes will motivate the reader to consult the literature that is quoted in the text.

1.1. What are Coulomb systems ?

The underlying theory for the description of matter composed of nuclei and electrons forming ions, atoms and molecules is Quantum Electrodynamics (QED). Since in atomic, molecular and condensed matter physics the energies and velocities of the constituent particles are usually very moderate, non-relativistic QED [1] ought to provide a sufficient description. This point of view bears, however, some problems. Non-relativistic QED cannot correctly and consistently account for spin-dependent interactions. In particular, magnetic dipole interactions among constituent particles and the interactions of their magnetic moments with the quantized magnetic field have to be ignored or cutoff in some phenomenological way in order to prevent the theory from becoming mathematically meaningless. Some of these problems are briefly discussed e.g. in [2].

Thus, non-relativistic QED can only be expected to be an accurate description if

- nuclear charges are moderate;
- temperatures and densities are moderate;
- the interactions between charged particles and the radiation field do not excite high frequency modes of the electromagnetic field;
- spin-dependent forces are weak.

For the major part of these notes, spin and the radiation field will be neglected altogether, leaving us with charged particles interacting via two-body Coulomb forces. We shall call systems for which these approximations are valid Coulomb systems. Some of the models we shall discuss would normally not be called Coulomb systems. However, in those examples it turns out that one can isolate certain particle-like excitations with long-range Coulomb interactions. Among those models we shall mention ones which do feature an ultraviolet cutoff electromagnetic field : Simplified Landau-Ginsburg type models of superconductors.

Most of the models appearing in these lecture notes are quite naive caricatures of more realistic theories. Many of them are classical lattice models. However, in many instances, neither the assumption that the models be classical, nor the replacement of space by a lattice are really important, but are made because they are reasonable, or in order not to obscure the simplicity of an argument. The only serious requirement is that in order to render a classical, three-dimensional Coulomb system in thermal equilibrium mathematically meaningful, the Coulomb potential must be regularized at short distances. (The lattice is often a pedagogically and physically attractive regularization).

In spite of all these crude approximations we believe that the models, methods and results we discuss in these notes have quite a lot to do with physics and are chosen so as to exhibit certain interesting physical phenomena in a pure and simple form. (It is a good tradition in theoretical physics to replace a theory if it turns out to evade our comprehension by an approximate model which can be analysed in satisfactory depth).

Our most interesting methods, results and speculations appear in Sections III – V . Among them is a rigorous version of real-space renormalization group techniques powerful enough to establish the existence of Kosterlitz-Thouless (plasma → dipolar phase) transitions in a large variety of situations. Furthermore, we comment on "liquid crystal" phases in hard core Coulomb gases and the possible transitions in the three-dimensional Coulomb gas. See Sections II.2 and III.

The reader familiar with the basic definitions and notions concerning Coulomb systems may skip the remainder of the introduction and proceed to Section II.

I.2. Stability.

The ν-dimensional Coulomb potential is defined to be the Green's function, $V(x,y)$, of a ν-dimensional Laplacean, $-\Delta$. If particles in a Coulomb system are confined to a bounded region, Λ, some boundary conditions (b.c.) need to be specified at the boundary, $\partial\Lambda$, of the box. (Different b.c. in the Coulomb potential can yield different thermodynamic limits of the corresponding systems). We shall consider two types of b.c. :

(BC1) Insulating, or free b.c. :

$V(x,y) \equiv V(x-y)$ is the Green's function of the infinite volume Laplacean. The arguments, x and y, are constrained to be inside Λ. Physically, this corresponds to putting up walls at $\partial\Lambda$ which are perfect insulators.

(BC2) Conducting, or Dirichlet b.c. :

$V(x,y)$ is the Green's function of the Laplacean with 0-Dirichlet data at $\partial\Lambda$. The physical interpretation of these b.c. is that the walls of Λ are perfect conductors.

(In two dimensions, (BC1) and (BC2) can result in different thermodynamic limits of finite temperature, finite density Coulomb gases, [3]. We shall discuss further b.c. with yet more drastic effects in the thermodynamic limit, "roughening", in Section III).

For $\Lambda = \mathbb{R}^\nu$ we have

$$V(x) = \begin{cases} 1/2|x| \ , & \nu = 1 \\ 1/2\pi \ \ln(1/|x|) \ , & \nu = 2 \\ 1/\sigma_\nu |x|^{-(\nu-2)} \ , & \nu \geq 3 \ , \end{cases} \tag{I.1}$$

where σ_ν is the surface of the $(\nu-1)$ dimensional unit sphere. If configuration space is replaced by a lattice, $\mathbb{Z}^\nu$, we shall define the Coulomb potential to be the Green's function of the finite difference Laplacean (with free-, resp. Dirichlet b.c.). Instead, one could also define it to be the restriction of the continuum Coulomb potential to the lattice, with $V(x=0) \overset{e.g.}{=} 0$. The long range behaviour of the lattice Coulomb potential is still given by (I.1).

We also introduce a dipole potential : The potential between a dipole pointing in the $+\alpha$-direction at position x and one pointing in the $+\beta$-direction at y , both of unit strength, is given by

$$\mathring{W}_{\alpha\beta}(x,y) = -(\partial_\alpha \partial_\beta V)(x,y) \tag{I.2}$$

where $\partial_\alpha = \partial/\partial x^\alpha$, $\alpha = 1,\ldots,\nu$. (On the lattice ∂_α is a finite difference derivative). In the continuum, $\mathring{W}$ must be regularized at short distances : When $\Lambda = \mathbb{R}^\nu$

$$\mathring{W}_{\alpha\beta}(x,y) = (2\pi)^{-\nu/2} \int e^{ik(x-y)}(k_\alpha k_\beta/k^2) d^\nu k \ .$$

We replace $\mathring{W}$ by

$$W_{\alpha\beta}(x,y) = (2\pi)^{-\nu/2} \int e^{ik(x-y)}(k_\alpha k_\beta/k^2) f(k) d^\nu k \ , \tag{I.3}$$

where f is a non-negative function of rapid fall off, so that $\|W_{\alpha\beta}\|_\infty$ is finite. Of course, regularization is unnecessary on the lattice.

The Hamilton function (resp.- operator) of a system consisting of N point particles with masses $m_1,\ldots,m_N$, charges $q_1,\ldots,q_N$ and dipole moments $\mu_1,\ldots,\mu_N$,(where $\mu_j \propto S_j$, S_j is the spin operator of the j^{th} particle), is given by

$$\left.\begin{aligned} H^{(N)} &= T^{(N)} + U^{(N)} \ , \\ T^{(N)} &= \sum_{j=1}^{N} (1/2m_j) p_j^2 \ , \\ U^{(N)} &= \sum_{1\le i<j\le N} \{q_i q_j V(x_i - x_j) + \sum_{\alpha,\beta=1}^{\nu} \mu_i^\alpha \mu_j^\beta W_{\alpha\beta}(x_i - x_j)\} \end{aligned}\right\} \tag{I.4}$$

If interactions with the radiation field, described by a vector potential, A, in the Coulomb gauge (i.e. $\nabla \cdot A = 0$), are to be taken into account, $p_j^2/2m_j$ is replaced by $(p_j - q_j A(x_j))^2/2m_j$, and a term $\sum_{j=1}^{N} \mu_j \cdot B(x_j)$, $B(x) = (\nabla \wedge A)(x)$, is added.

The first basic problem to be studied is the <u>stability problem</u>. One assumes that m_j, q_j and $||\mu_j||$ are bounded uniformly in $j = 1,\ldots,N$ and $N = 2,3,\ldots$, and asks whether

$$H^{(N)} \geqq -\text{const. } N\,, \tag{I.5}$$

for some finite, N-independent constant. A system satisfying (I.5) is said to be <u>H-stable</u>. Classical, continuum Coulomb systems of point particles are never H-stable, unless $\nu = 1$, or $q_j \geqq 0$, for all j. If all charges are positive the system does however <u>not</u> behave thermodynamically because of the long range of the Coulomb potential. In fact, overall neutrality is important.

For three-dimensional, quantum mechanical systems with $\mu_j = 0$, for all j, H-stability has been established, provided all negatively charged particles are Fermions, and is known to fail if all particles are Bosons. These matters are discussed in W. Thirrings contribution and in [2].

We emphasize that H-stability depends <u>only</u> on the <u>short range singularity</u> of the the two-body potential, i.e. H-stability is the <u>ultraviolet</u> (not the infrared) problem of statistical mechanics; see e.g. [2,4]. If the Coulomb potential is cutoff at short distances - as we have done with the dipole potential - H-stability holds, and the proof is very simple, [5]. Thus, on the lattice, (I.5) is always true.

Three-dimensional, non-relativistic, quantum-mechanical matter, with negatively charged particles assumed to be Fermions, coupled to an ultraviolet cutoff, quantized electromagnetic field is <u>stable</u> if the spin of all particles is zero, but <u>unstable</u> if spin is included. It is unlikely that stability is restored when the ultraviolet cutoff is removed. (We thank Erhard Seiler for a discussion which helped to clarify this point).

Another notion of stability, equally basic for statistical mechanics, is Ξ-stability : Consider a system of m different species of particles, the total number of particles being arbitrary. The ℓ^{th} species is supposed to consist of particles with mass m_ℓ, charge $q_\ell, \ldots$ and activity (= fugacity) $z_\ell = e^{-\beta\mu_\ell}$, where $\beta = 1/kT$ is the inverse temperature and μ_ℓ the chemical potential. Let $\Xi_\Lambda(\beta, z_1, \ldots, z_m)$ denote the grand canonical partition function of this system, the Hamiltonian being given by (I.4). See [4,5,6] for the definition of Ξ_Λ. The system is said to be Ξ-stable if

$$(1 \leq) \Xi_\Lambda(\beta, z_1, \ldots, z_m) \leq e^{\text{const.}|\Lambda|} , \qquad \text{(I.6)}$$

for some finite constant; $|\Lambda|$ is the volume of the box Λ containing the system.

The notions of H-stability and Ξ-stability are not equivalent : The two-dimensional, classical neutral Coulomb gas with two species of particles of charge $\pm q$ is never H-stable, but is Ξ-stable if $\beta q^2 < 4\pi$. This is the result of [4] . (For some extensions see [6]).

However, if the Coulomb potential is regularized at short distances, a Coulomb system of finitely many species of Bosons, with positive and negative charges, is H-stable, but fails to be Ξ-stable when some of the activities are large enough; (in fact Ξ_Λ is infinite when some of the activities exceed critical values). See [7] .

Non-relativistic, quantum-mechanical matter in three dimensions, with negatively charged particles = Fermions, is Ξ-stable [8] ; (see also [7]). Classical, H-stable systems are always Ξ-stable, [5] ; in particular classical lattice Coulomb gases are Ξ-stable.

Quantum mechanically, the implication tends to go the other way around).

I.3. Thermodynamic Functions.

The basic results concerning the existence of the thermodynamic functions of Coulomb systems are due to Lieb and Lebowitz [8] . For various extensions of their methods see [2] and refs. given there. The problem of the thermodynamic limit is the "infrared problem" of statistical mechanics, and it is equally hard classically and quantum mechanically. Under certain restrictive conditions, the proof of existence of the thermodynamic limit for e.g. the pressure of Coulomb systems is simple, (much simpler than the proofs in [8], although the results are not quite as strong) :

A system composed of $2m$ species of particles is said to be charge conjugation invariant iff

$$m_{2j} = m_{2j+1} \;,\; q_{2j} = -q_{2j+1} \;,\; z_{2j} = z_{2j+1} \;,$$

and if the system is quantum mechanical the statistics of the particles in the $2j^{th}$ and $(2j+1)^{st}$ species are the same; for all $j = 1,\ldots,m$. (If, in addition, the particles have dipole moments, μ , it is required that μ_{2j} and $-\mu_{2j+1}$ have identical distributions, $j = 1,\ldots,m$) .

For charge conjugation invariant systems a simple proof of existence for the thermodynamic limit of the pressure has been given in [7] , extending an idea of Griffiths [9] . For such systems, the screening properties of the Coulomb potential emphasized in [8,2] are actually unimportant for the existence of the thermodynamic limit of the pressure, although sensitive dependence on shape and boundary conditions must be expected for potentials like the dipole potential which cannot be screened. As an example, we mention that in three dimensions the thermodynamic limit of the pressure of a charge conjugation invariant system with two-body potential

$V(x) \underset{|x|\to\infty}{\approx} |x|^{-\varepsilon}, \varepsilon > 0$, of positive type exists, although for $\varepsilon \neq 1$ there is no screening. See [7] . For additional methods involving correlation inequalities see [6] .

I.4. Equilibrium States.

A third basic problem concerning Coulomb systems is the question of existence and properties of the thermodynamic limit of equilibrium states, in particular of the correlation functions of classical systems, resp. the reduced density matrices or imaginary-time Green's functions of quantum mechanical systems.

For a rather large class of classical and quantum Coulomb systems locally normal equilibrium states in the thermodynamic limit can be constructed by means of a weak compactness argument, provided suitable boundary conditions (periodic b.c.) are imposed. (A proof of this can be based on constructive field theory methods of Glimm and Jaffe). In many physically interesting situations not even such a weak result is known to hold ! Moreover, the problem of constructing the time evolution for infinite Coulomb systems "near equilibrium" is essentially entirely open, except in very special, physically unrealistic cases.

After these rather depressing remarks we now recall some positive results among which the most impressive ones are due to Brydges [10] and Brydges and Federbush [11] : For a large class of classical, dilute Coulomb systems in two or more dimensions they have constructed the thermodynamic limit of the correlation functions (with Dirichlet, i.e. conducting b.c.), and they have established Debye screening in the form of exponential cluster properties. This remarkable development is reviewed in detail in the lectures given by D. Brydges.

Another construction of the thermodynamic limit of correlation functions, resp. reduced density matrices or imaginary-time Green's functions valid for all values of the thermodynamic parameters for which the system behaves thermodynamically is given in [6,7] . That method is based on correlation inequalities first used in a related context in [12] . The hypotheses under which those inequalities are known to hold are unfortunately rather restrictive :

- Exact charge conjugation invariance.
- The two-body potential is of positive type; (n-body potentials vanish for $n > 2$).
- The system is classical or quantum mechanical with Boltzmann - or Bose-Einstein statistics.

The first and the third hypothesis are physically awkward. However, the inequalities hold for arbitrary values of β and z and a large class of potentials including the Coulomb potential and ones with slower decrease than the Coulomb potential. Moreover, they are strong enough to provide some general information about the properties of the thermodynamic limit, [6,7]. They also permit to include the radiation field and supply some general information of interest in superconductivity and Bose-Einstein condensation, [7]. In spite of the many encouraging results alluded to above and discussed in more detail in the lectures by Aizenman, Brydges, Lebowitz, Lieb and Thirring it should be clear that the mathematical foundations of the theory of Coulomb systems and non-relativistic matter - starting from first principles - are still quite incomplete. Several topics, such as non-relativistic QED, may have been undeservedly neglected.

In the remainder of these notes we shall study highly idealized systems of excitations with Coulomb interactions about which detailed statements can be made. We shall concentrate on the discussion of the Kosterlitz-Thouless transition and other aspects of the phase diagram of two - (and higher) dimensional Coulomb systems.

II. Generalities about Classical Coulomb Gases.

Throughout the remaining sections we study classical lattice Coulomb gases, but many of our results extend to continuum gases, provided the Coulomb potential is regularized at short distances, some also to quantum mechanical gases. We concentrate our attention on monopole gases but at various places mention results on dipole gases.

We first recall the sine-Gordon (or Siegert) transformation [3,4,6,13]. The end of the section contains an outlook on what is discussed in subsequent sections, in particular a phase diagram of a hard core Coulomb lattice gas in two, resp. three dimensions which we shall establish in part.

II.1. The sine-Gordon transformation.

We consider Coulomb gases on the lattice $\mathbb{Z}^\nu$. The Coulomb potential, V , is the Green's function of the finite difference Laplacean, Δ . Unless stated otherwise, free, i.e. insulating, b.c. are imposed. (Other b.c. are treated in the references quoted in the text).

We start by considering systems in a finite region $\Lambda \subset \mathbb{Z}^\nu$. A configuration of such a system is a function

$$q_\Lambda : \Lambda \to \mathbb{Z} \ , \ \Lambda \ni j \to q(j) \in \mathbb{Z} \ ,$$

where $q(j)$ is interpreted as the total electric charge concentrated at site j . The a priori distribution of $q(j)$ is given by a measure $d\lambda$ on $\mathbb{Z}$. We shall be interested in the following choices of $d\lambda$:

A) Hard core gas :

$$d\lambda(q) = \{\delta(q)+z/2[\delta(q-1)+\delta(q+1)]\}dq \ , \qquad (2.1)$$

where δ is the Dirac function and z the (bare) activity.

B) Standard lattice gas without hard cores :

$$d\lambda(q) = \{ \sum_{n\in \mathbb{Z}} I_n(z)\delta(q-n)\} dq \quad , \tag{2.2}$$

where $I_n(z)$ is the n^{th} modified Bessel function, i.e. the n^{th} Fourier coefficient of $\exp(z\cos\phi)$, and z is a(bare) activity.

C) Villain gas :

$$d\lambda(q) = \{ \sum_{n\in \mathbb{Z}} \delta(q-n)\} dq \tag{2.3}$$

We note that this measure is the limit of $I_o(z)^{-1} d\lambda(q)$, as $z \to +\infty$, with $d\lambda$ given by (2.2).

Clearly there are other interesting choices for $d\lambda$, but here $d\lambda$ will usually be given by A).

$$\begin{aligned} E(q_\Lambda) &= \tfrac{1}{2} \sum_{i,j} q(i)\, q(j)\, V(i-j) \\ &\equiv \tfrac{1}{2} \left(q_\Lambda, (-\Delta)^{-1} q_\Lambda \right). \end{aligned} \tag{2.4}$$

The functional $E(q_\Lambda)$ is the electrostatic self-energy of the configuration q_Λ , self-energies of charges included.

The equilibrium distribution for the configuration q_Λ is given by

$$\left. \begin{aligned} & Z_\Lambda^{-1} \exp\left[-\beta E(q_\Lambda)\right] \prod_{j\in\Lambda} d\lambda(q(j)) \\ & Z_\Lambda = \int \exp\left[-\beta E(q_\Lambda)\right] \prod_{j\in\Lambda} d\lambda(q(j)). \end{aligned} \right\} \tag{2.5}$$

Note, by a finite redefintion of $d\lambda$, self-energies of charges can be excluded in the definition of $E(q_\Lambda)$.

Next we consider the Fourier transform of the equilibrium measure introduced in (2.5). Let $\phi : \mathbb{Z}^\nu \to \mathbb{R}$ be a Gaussian random field on $\mathbb{Z}^\nu$ with distribution

$$d\mu_{\beta V}(\phi) = N^{-1} \exp\left[\tfrac{1}{2\beta}(\phi, \Delta\phi)\right] \prod_j d\phi(j) , \qquad (2.6)$$

where

$$(\phi, \Delta\phi) = -\sum_{|i-j|=1} (\phi(i) - \phi(j))^2 ,$$

and

$$N = \det\left(- \Delta/2\pi\beta\right)^{-1/2}$$

is a normalization factor. Mathematically, $d\mu_{\beta V}$ is defined to be the Gaussian measure with mean 0 and covariance βV. In one and two dimensions, $d\mu_{\beta V}$ is only defined, a priori, when integrated against bounded functions of

$$\{\phi(f) : \operatorname{supp} f \text{ bounded}, \sum_j f(j) = 0\} , \qquad (2.7)$$

with $\phi(f) \equiv \sum_j \phi(j) f(j)$. This is because

$$\hat{V}(k) = \left[2\left(\nu - \sum_{\alpha=1}^{\nu} \cos k^{\alpha}\right)\right]^{-1} , \qquad (2.8)$$

(the Fourier transform of $V(j)$) is not integrable at $k = 0$ when $\nu = 1$ or 2. See e.g. [4] for details. Thus

$$\int d\mu_{\beta V}(\phi) e^{i\phi(f)} = \begin{cases} \exp[-\beta/2 (f, Vf)], & \sum_j f(j) = 0, \\ 0, & \sum_j f(j) \neq 0. \end{cases} \qquad (2.9)$$

In $\nu \geqq 3$ dimensions, V is positive definite and no constraints arise. By (2.4) and (2.9)

$$\exp[-\beta E(q_\Lambda)] = \int d\mu_{\beta V}(\phi) e^{i\phi(q_\Lambda)} , \qquad (2.10)$$

provided $Q(q_\Lambda) \equiv \Sigma q(j) = 0$ when $\nu = 1,2$. (If $Q(q_\Lambda) \neq 0$, $\nu = 1$ or 2, we set $E(q_\Lambda) \equiv +\infty$, and (2.10) remains true). Note that the variable $\phi(j)$ is <u>conjugate</u> to the charge variable $q(j)$. Thus

$$Z_\Lambda = \int \exp[-\beta E(q_\Lambda)] \prod_{j\in\Lambda} d\lambda(q(j))$$
$$= \int d\mu_{\beta V}(\phi) \prod_{j\in\Lambda} \hat{\lambda}(\phi(j)), \tag{2.11}$$

where

$$\hat{\lambda}(\phi) = \int d\lambda(q)\, e^{iq\phi}. \tag{2.12}$$

In the hard core gas (2.1) ,

$$\hat{\lambda}(\phi) = 1 + z\cos\phi \tag{2.1'}$$

In the standard gas (2.2),

$$\hat{\lambda}(\phi) = \exp[z\cos\phi], \tag{2.2'}$$

and in the Villain gas (2.3)

$$\hat{\lambda}(\phi) = \sum_{n\in\mathbb{Z}} \delta(\phi - 2\pi n). \tag{2.3'}$$

We denote by $\langle - \rangle_\Lambda(\beta,\lambda)$ both, expectations in the equilibrium measure (2.5), and expectations in the (generally non-positive) measure

$$Z_\Lambda^{-1} \prod_{j\in\Lambda} \hat{\lambda}(\phi(j))\, d\mu_{\beta V}(\phi). \tag{2.13}$$

The interpretation of correlations $\langle F(q_\Lambda)\rangle_\Lambda(\beta,\lambda)$, where F is a bounded function of q_Λ , is obvious. (It is an expectation of a sort we are familiar with from lattice spin systems).

In order to interpret expectations such as

$$\langle \exp i\phi(f)\rangle(\beta,\lambda),$$

note that, by (2.9) - (2.11),

$$\langle \exp i\phi(f)\rangle_\Lambda(\beta,\lambda) =$$
$$Z_\Lambda^{-1} \int \exp[-\beta E(q_\Lambda + f)] \prod_{j\in\Lambda} d\lambda(q(j)). \tag{2.14}$$

Thus $<\exp i\phi(f)>_\Lambda(\beta,\lambda)$ measures the correlations between external charges, $f(j)$, located at different sites $j \in \Lambda$, which are put into the system from the outside. More precisely, $-1/\beta \log<e^{i\phi(f)}>_\Lambda$ is the average amount of free energy needed to pump the charges $\{f(j)\}_{j\in\Lambda}$ into a system of charges in thermal equilibrium at inverse temperature β. Of particular interest in the following is the <u>fractional charge correlation</u>

$$G_\Lambda(x) = \langle \exp i\gamma\,(\phi(0) - \phi(x)) \rangle_\Lambda (\beta,\lambda) \tag{2.15}$$

which measures the correlation between two fractional charges, one charge γ located at $0 \in \Lambda$ and one, $-\gamma$, located at $x \in \Lambda$, $0 < \gamma < 1$, put into the system. The behaviour of the fractional charge correlation $G_\Lambda(x)$ reflects the screening properties of the Coulomb gas; see [3] and Section III.

Next, we briefly sketch how to extend the formalism developed here to the simplest example of a dipole gas. See [3,6] for more details. As our dipole potential we choose e.g.

$$W_{\alpha\beta}(x,y) = -\left(\partial_\alpha \partial_\beta V\right)(x-y), \tag{2.16}$$

where ∂_α is the finite difference derivative in direction α. Let $\Lambda_d \subseteq \Lambda$ be some sublattice of Λ, e.g. $\Lambda_d = \ell\mathbb{Z}^\nu \cap \Lambda$, $\ell = 1,2,3,\ldots$. A <u>configuration</u> of a lattice dipole gas is described by a function $\mu_\Lambda : \Lambda_d \to \mathbb{R}^\nu$, $\Lambda_d \ni j \to \mu(j) \in \mathbb{R}^\nu$, where $\mu(j)$ is the total dipole moment at site $j \in \Lambda_d$. The dipoles are non-overlapping if $\ell \geq 2$. The energy of μ_Λ is given by

$$E_d(\mu_\Lambda) = \tfrac{1}{2} \sum_{i,j} \sum_{\alpha,\beta} \mu^\alpha(i)\mu^\beta(j) W_{\alpha\beta}(i,j), \tag{2.17}$$

the equilibrium distribution by

$$Z_\Lambda^{-1} \exp[-\beta E_d(\mu_\Lambda)] \prod_{j\in\Lambda_d} d\lambda(\mu(j)),$$

where $d\lambda$ is a finite measure on $\mathbb{R}^\nu$, e.g.

$$d\lambda(\mu) = \{\delta(\mu) + z\,\delta(\mu^2 - 1)\}\, d^\nu\mu \tag{2.18}$$

Combining (2.9) with (2.16) and (2.17) one gets

$$\exp[-\beta E_d(\mu_\Lambda)] = \int d\mu_{\beta V}(\phi) \prod_{j \in \Lambda_d} \exp[i\mu(j)\cdot(\nabla^*\phi)(j)], \tag{2.19}$$

where $\nabla^* \equiv (\partial_1^*, \ldots, \partial_\nu^*)$, and ∂_α^* is the adjoint of ∂_α. For $j \in \Lambda_d$, let $\{\phi\}_j = \{\phi(i) : |i-j| \leq 1\}$. We set

$$\hat{\lambda}(\{\phi\}_j) = \int d\lambda(\mu) \exp[i\mu\cdot(\nabla^*\phi)(j)].$$

The dipole measure in the ϕ-variables is then given by

$$Z_\Lambda^{-1} \prod_{j \in \Lambda_d} \hat{\lambda}(\{\phi\}_j)\, d\mu_{\beta V}(\phi). \tag{2.20}$$

The definition and interpretation of the expectations $\langle - \rangle_\Lambda(\beta,\lambda)$ and correlations $\langle F(\mu_\Lambda)\rangle_\Lambda(\beta,\lambda)$, $\langle \exp i\phi(\nabla\cdot h)\rangle_\Lambda(\beta,\lambda)$, h an $\mathbb{R}^\nu$-valued function on Λ, is analogous as in the previous case of monopole gases. Note that, by (2.6) and (2.20), dipole gases have the continuous symmetry $\phi(\cdot) \mapsto \phi(\cdot) + \text{const.}$ which is always broken by the boundary conditions. As a consequence one can show [3] that there exist Goldstone excitations and that correlations in dipole gases do not decay exponentially.

We conclude this subsection by recalling the standard integration by parts formula, (e.g. [3] and refs. given there).

We do this for the monopole gases; for dipoles see e.g. [3]. First, we recall the well known identity

$$\int \phi(j)\, G(\phi)\, d\mu_{\beta V}(\phi) = \beta \sum_i V(j-i) \int \frac{\partial G}{\partial \phi(i)}\, d\mu_{\beta V}(\phi) \tag{2.21}$$

which a physicist calls <u>Wick's theorem</u>.

Clearly
$$\frac{\partial \hat{\lambda}}{\partial \phi}(\phi) = i\int d\lambda(q)\, q\, e^{iq\phi}.$$

Thus, by (2.13) and (2.21)

$$\langle \phi(j)\, G(\phi)\rangle_{\Lambda}(\beta,\lambda) = \tag{2.22}$$
$$\beta \sum_{i} V(j-i)\left\{\left\langle \frac{\partial G}{\partial \phi(i)}(\phi)\right\rangle_{\Lambda}(\beta,\lambda) + i\langle q(i)\, G(\phi)\rangle_{\Lambda}(\beta,\lambda)\right\}$$

This equation shows that $(1/i\beta)\phi(j)$ is the <u>effective potential</u> felt by an infinitesimal test charge at site j. By setting $G(\phi) = \phi(\ell)$ and repeating (2.22) one gets

$$\langle \phi(j)\phi(\ell)\rangle_{\Lambda}(\beta,\lambda) = \beta V(j-\ell) \tag{2.23}$$
$$- \beta^2 \sum_{i,m} V(j-i)\, V(\ell-m)\langle q(i)\, q(m)\rangle_{\Lambda}(\beta,\lambda)$$

If on the r.s. of (2.23) integrations over q and ϕ are interchanged one obtains

$$\langle \phi(j)\phi(\ell)\rangle_{\Lambda}(\beta,\lambda) = \beta V(j-\ell)$$
$$- \beta^2 \sum_{i} V(j-i)\, V(\ell-i)\langle \gamma(\phi(i))\rangle_{\Lambda}(\beta,\lambda) \tag{2.24}$$
$$+ \beta^2 \sum_{i,\ell} V(j-i)\, V(\ell-m)\langle \sigma(\phi(i))\sigma(\phi(m))\rangle_{\Lambda}(\beta,\lambda)$$

for some functions γ and σ on the real line which are determined by $\hat{\lambda}$ and are real-valued if $\hat{\lambda}$ is real, [3]. If $\hat{\lambda}$ is non-negative (resp. a "renormalized" version of $\hat{\lambda}$ is non-negative, see Section II.2) formulas (2.23) and (2.24) provide a surprisingly powerful tool.

Finally, we wish to add a remark on the existence of the thermodynamic limit : For a large variety of Coulomb monopole <u>and</u> dipole gases the correlation inequalities in [6] can be used to construct the thermodynamic limit of the states $\langle - \rangle_\Lambda(\beta,\lambda)$, as $\Lambda \nearrow \mathbb{Z}^\nu$.

In particular, assume that

$$\hat{\lambda}(\phi) = \exp G(\phi) \tag{2.25}$$

where $G(\phi)$ is real-valued and of positive type. Impose free (insulating) or Dirichlet (conducting) b.c. on the Coulomb potential V . Then

$$\lim_{\Lambda \nearrow \mathbb{Z}^\nu} \langle - \rangle_\Lambda (\beta,\lambda) \equiv \langle - \rangle (\beta,\lambda) \tag{2.26}$$

exists. The limiting state, $\langle - \rangle(\beta,\lambda)$, is translation invariant and, in $\nu \geq 3$ dimensions, clustering, [6] . (In two dimensions it clusters on observables which are functions of

$$\{ \phi(f) : \text{supp} f \text{ bounded} , \sum_j f(j) = 0 \} \text{).}$$

Necessary conditions for (2.25) to hold are exact charge conjugation invariance of the system, i.e. $d\lambda(q) = d\lambda(-q)$, and positivity of $\hat{\lambda}(\phi)$. It is easy to see that (2.25) holds for the standard and the Villain gas for which $d\lambda$ is given by (2.2), (2.3), respectively. It fails for the hard core gas, although that gas is charge conjugation invariant, and $\hat{\lambda}(\phi) > 0$, for $z < 1$.

For charge conjugation invariant, strictly neutral systems with periodic b.c. at $\partial\Lambda$, a translation-invariant, thermodynamic limit, $\langle - \rangle(\beta,\lambda)$, can be constructed by passing to subsequences. Every limiting state obtained in this way has strong regularity properties - as a state on bounded functions of the charge variables $\{q(j)\}_{j \in \mathbb{Z}^\nu}$ - reminiscent of superstability estimates. This can be proven by means of chessboard estimates [3,14] . For any translation-invariant limiting state, $\langle - \rangle(\beta,\lambda)$, we obtain from (2.23) by Fourier transformation

$$\langle |\hat{\varphi}(k)|^2 \rangle(\beta,\lambda) = \beta \hat{V}(k) - \beta^2 \hat{V}(k)^2 \langle |\hat{q}(k)|^2 \rangle(\beta,\lambda)$$

with

$$\hat{V}(k) = \left[2\left(\nu - \sum_{\alpha=1}^{\nu} \cos k^{\alpha}\right)\right]^{-1} \tag{2.27}$$

Existence of thermodynamic limits will not be discussed any further (see [6-11]), but there certainly are still many interesting open problems.

II.2. The phase diagram of lattice Coulomb gases.

We consider the hard core lattice Coulomb gas with $d\lambda$ given by (2.1). The equilibrium state of this system is henceforth denoted by $\langle - \rangle(\beta,z)$. Our purpose here is to describe what is known about the phase diagram of this interesting system in two and three dimensions. The relevant thermodynamic parameters are the inverse temperature β and the activity z. Here are portraits of the phase diagrams in $\nu = 2$ and 3 dimensions.

$\nu = 2$

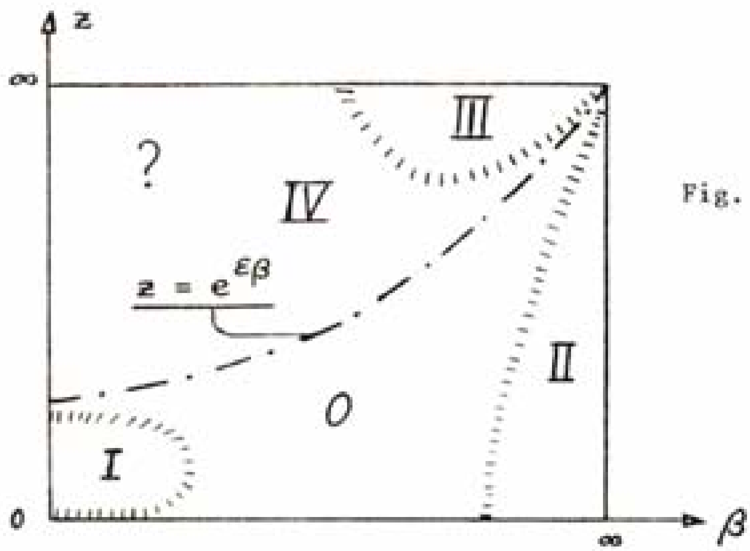

Fig. 1

$\underline{\nu = 3}$

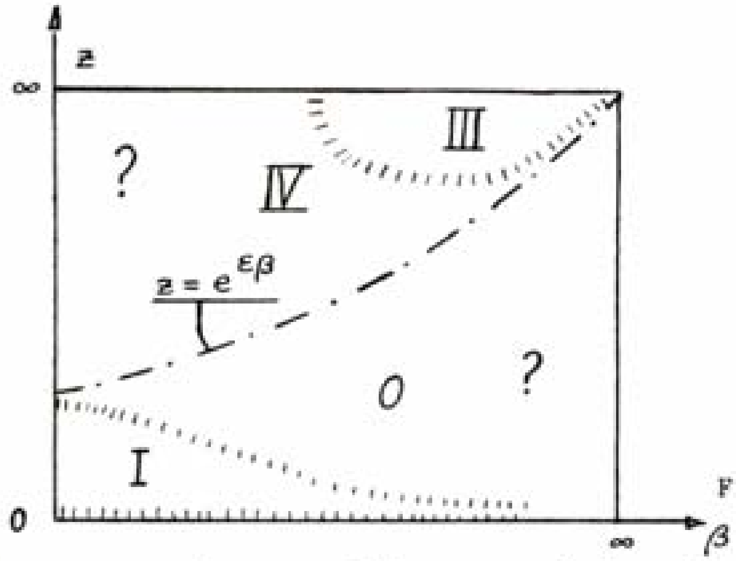

Fig. 2

We first discuss common features of the two- and three dimensional diagrams and then discuss striking differences known to arise within domain 0 .

Domain 0 is an open region bounded by the lines $\beta = 0$, $z = 0$ $\beta = \infty$ and $z = e^{\varepsilon\beta}$, with $\varepsilon \geqq 1/8\nu$ ($= 1/16$, for $\nu = 2$, $= 1/32$, for $\nu = 3$) . It is characterized by the existence of a translation invariant state which is a limit of states with periodic b.c. (For $z < 1$ that limit is clustering). The charge-charge correlation $\langle q(0)q(x)\rangle(\beta,z)$ tends to 0 , as $|x| \to \infty$, (absence of long range order). Furthermore there is no short range order, in the sense that for $x = ne$, e a unit lattice vector, $n = 1,2,3,\ldots$,

$$\langle q(0)q(ne)\rangle(\beta,z) \text{ is negative and concave.} \tag{2.28}$$

The local charge, $q_\Lambda = \sum_{j\in\Lambda} q(j)$, has abnormal fluctuation, i.e. $\langle q_\Lambda^2\rangle \leqq O(\partial\Lambda)$.

On each line $z = z_o = \text{const.}$, the exponential decay rate, $m(\beta)$, of $\langle q(0)q(x)\rangle(\beta,z)$, the inverse of the correlation length $\xi(\beta)$, is known to satisfy the inequality

$$m(\beta) \leqq \text{const.}\, e^{-\delta\beta}, \tag{2.29}$$

for some $\delta > 0$ which depends on z_o and ν . These results are proven in [3].

The basic tools used in the proofs are the existence of self-adjoint transfer matrices, i.e. reflection positivity in the ϕ- and q-representations, [3,14], and the fact that for $z < e^{c\beta}$

$$\langle |\hat{\phi}(k)|^2 \rangle(\beta,z) \geq 0 . \tag{2.30}$$

Thus by (2.27)

$$\langle |\hat{q}(k)|^2 \rangle(\beta,z) \leq [\beta \hat{V}(k)]^{-1} \leq \beta^{-1} k^2 . \tag{2.31}$$

Absence of long range order and abnormality of the fluctuations of q_Λ follow from (2.31) by Fourier transformation. For $z < 1$, the state $\langle - \rangle(\beta,z)$ is given by a positive measure in the ϕ-variables, so that (2.30) holds. When $z > 1$, $\langle - \rangle(\beta,z)$ does not correspond to a positive measure in the ϕ's, and (2.30) is not obvious. In order to deal with large bare activities z, $1 < z < e^{c\beta}$, we must apply a simple renormalization transformation : Between two nearest neighbor sites, i and j, we introduce an additional site, ij, and replace the factor $\exp -1/2\beta(\phi(i)-\phi(j))^2$ in the Gaussian measure $d\mu_{\beta V}(\phi)$ by

$$\text{const.} \exp - \tfrac{1}{\beta}(\phi(i) - \psi(ij))^2 \exp - \tfrac{1}{\beta}(\phi(j) - \psi(ij))^2 \, d\psi(ij)$$

We then interchange the integrations over ψ- and ϕ-variables, first integrating out all $\phi(j)$, $j \in \mathbb{Z}^\nu$. The ϕ-integrations can be done explicitly by using the identities

$$\begin{aligned} \int e^{iq\phi} \prod_{\alpha=1}^{2\nu} e^{-(1/\beta)(\phi-\psi_\alpha)^2} d\phi \\ = e^{-\beta q^2/8\nu} \; e^{iq\overline{\Psi}} , \end{aligned} \tag{2.32}$$

where $\overline{\Psi} = (1/2\nu) \sum_{\alpha=1}^{2\nu} \psi_\alpha$, and each ψ_α stands for a variable $\psi(ij)$

associated with the new site in the middle of the link ij. By differentiating in q and adding the resulting identities for $\pm q$ we obtain

$$\int \phi \cos(q\phi) \prod_{\alpha=1}^{2\nu} e^{-1/\beta(\phi-\psi_\alpha)^2} d\phi$$
$$= e^{-\beta q^2/8\nu}[\bar{\psi}\cos(q\bar{\psi}) - (\beta q/4\nu)\sin(q\bar{\psi})] \tag{2.33}$$

Thus, under this renormalization transformation, the activity, $\lambda(\{q\})$, of $e^{iq\phi}$ is multiplied by $\exp-(\beta q^2/8\nu)$, in particular, in the hard core gas,

$$1 + z\cos\phi(i) \longmapsto 1 + z e^{-\beta/8\nu} \cos\bar{\psi}(i)$$

$$\phi(i)\,(1 + z\cos\phi(i)) \longmapsto \tag{2.34}$$

$$\bar{\psi}(i)\,(1 + z e^{-\beta/8\nu}\cos\bar{\psi}(i)) - (\beta q/4\nu) e^{-\beta/8\nu}\sin\bar{\psi}(i)$$
$$\bar{\psi}(i) = \sum_{j:|j-i|=1/2} \psi(ij), \qquad i \in \mathbb{Z}^\nu.$$

In the ψ-variables, the state $\langle - \rangle(\beta,z)$ is given by a positive measure, provided $ze^{-\beta/8\nu} < 1$, i.e. $z < e^{\epsilon\beta}$, with $\epsilon = (8\nu)^{-1}$. Then (2.34) clearly implies (2.30).

A sequence of renormalization transformations of this type, driving down bare activities, is a crucial tool in [15]. See also Section IV and § 5 of [3].

Next we discuss domain I which is contained in domain 0. Its main characteristics is the existence of a state with exponential Debye screening: The infinite volume limit, $\langle - \rangle(\beta,z)$, of the family of states $\{\langle - \rangle_\Lambda(\beta,z)\}$ with Dirichlet b.c. exists, and correlations in $\langle - \rangle(\beta,z)$ cluster exponentially. For small β, $m(\beta,z) \approx \sqrt{\beta z}$, [10,11], to be compared with the large β behaviour (2.29). What we have described here corresponds to a plasma phase of the Coulomb gas. It is discussed in detail in the lectures of D. Brydges.

We now describe the differences in the behaviour of the two- and three-dimensional Coulomb gas, including domain II for the two-dimensional gas. The two-dimensional Coulomb potential between a positive and a negative charge separated by a distance ℓ grows like $(1/2\pi)\ln(\ell+1)$. This is a confining potential, and in the absence of other charges the two charges form a tightly bound, neutral dipole. At finite temperature and density, this dipole may break up, due to interactions with other charges in the system. The probability of this event can be estimated heuristically as follows : The Boltzmann factor of the two charges is

$$\exp\left[-\beta/2\pi \, \ln(\ell+1)\right] \tag{2.35}$$

The entropy S of the configuration is

$$S \propto \ell \cdot v \tag{2.36}$$

where ℓ estimates the order of magnitude of the number of possible positions of the negative charge, for a fixed position of the positive charge, and v is the mean area over which the position of the positive charge may vary. It is shown in [15] that $v \propto \ell^p$, for some $p > 0$. At densities low enough that the lattice structure is not felt on large scales, dimensional analysis gives $v \propto \ell^2$. Now observe that

$$\ell^{p+1} \exp\left[-(\beta/2\pi)\ln(\ell+1)\right] \tag{2.37}$$

is summable in ℓ, for $\beta > 2\pi(p+2)$, (i.e. for $\beta > 8\pi$ if $v \propto \ell^2$ which is exact in the continuum limit). This means that the probability of separating the negative charge from the positive one by a distance ℓ tends to 0, as $\ell \to \infty$, "integrably fast", provided β is large enough, i.e. stable, neutral "molecules" are expected to form among which neutral dipoles may be expected to be the dominant configurations of the gas if the density is low enough. We have shown in [3] that dipole gases have correlations with power law decay, i.e. Debye screening breaks down in this low temperature-low density dipolar phase. A refined version of this somewhat rough picture is justified rigorously in [15] by means of an inductive renormalization group scheme. Thus, in two dimensions, domain II corresponds to the Kosterlitz-Thouless dipolar phase characterized by power fall-off of correlations and scaling. It is clear that - and why - the mechanism described here fails in $\nu \geq 3$ dimensions : The Boltzmann factor for a neutral multipole of point charges, e.g. a dipole of lenght ℓ , $\approx \exp(+\beta/4\pi\ell)$, does not tend to 0 , as the diameter d $(=\ell)$ tends to ∞ . For this reason, all neutral multipoles are unstable, no matter how small the temperature and density are, and the gas is a mixture of free charges forming a plasma and "unstable molecules". Therefore one expects that exponential Debye screening persists throughout a domain in the (β,z) plane essentially as large as domain 0 . This is not quite what Brydges and Federbush [10,11] are able to show. Their methods only establish screening for small enough densities, depending on the value of β . Instead of the Berezinski-Kosterlitz-Thouless transition which the two-dimensional gas undergoes when (β,z) is moved from domain I to domain II one expects that the three-dimensional gas exhibits what one might call a roughening transition when β and z are increased. We shall briefly comment on this kind of transition in Section III.

We now continue our discussion of the phase diagrams with domain III, corresponding to low temperatures and high densities. It is characterized by the existence of at least two ordered states, $\langle - \rangle_{\pm}(\beta,z)$, with

$$\langle q(x)\rangle_{\pm}(\beta,z) = \pm(-1)^{|x|}, \quad |x| \equiv \sum_{\alpha=1}^{\nu} x^{\alpha}, \tag{2.38}$$

i.e. the charge density is staggered and the charges are arranged in a crystal of the $NaC\ell$ type. On the boundary of region III at least three states coexists, two ordered ones and a state describing a low density phase. This has been proven in [14] by means of a Peierls argument inspired by the one in [16] . An analogous (more difficult) result for hard core lattice dipole gases has been proven in [3] .

It is reasonable to conjecture that domain IV contains a region of parameters (β,z) corresponding to equilibrium states that describe a high density liquid phase with short range order, $(\langle q(0)q(x)\rangle(\beta,z)$ is staggered in $x)$, but without long range order in the charge-charge correlation $\langle q(0)q(x)\rangle(\beta,z)$. Presently, we know of no analytical method that would permit to investigate domain IV rigorously, except that some of the ideas in [15] may be useful. This is proposed to the reader as a challenging open problem.

Finally, if - instead of the hard core Coulomb gas - the standard or the Villain gas are considered, domain 0 extends over the whole quadrant $\{(\beta,z) : \beta > 0 , z > 0\}$, because in these models $\hat{\lambda}(\phi) \geq 0$, so that reflection positivity in the ϕ- and the q-representation and the inequality $\langle|\hat{\phi}(k)|^2\rangle(\beta,z) \geq 0$ hold. The boundary line $z = \infty$ of the standard model is the Villain model, (e.g. [3]) . Domains I and II extend up to that line. For general results on dipole gases, see [3,6].

III. Screening and Roughening

In this section we sketch some of the features of the equilibrium states when (β,z) is in the Debye-Hückel domain I, resp. in 0 . Our remarks are intended to be somewhat complementary to the lectures of D. Brydges.

First, we give several different characterizations of Debye screening and then we speculate on a "roughening transition" in the three-dimensional Coulomb gas. We consider the gases introduced in (2.1) - (2.3).

i) Strong screening, [10,11] .

Let $A(\phi),B(\phi)$ be bounded functions depending only on finitely many of the random variables $\{\phi(j)\}_{j\in\mathbb{Z}^\nu}$. Let $B(\phi)_x$ denote the translation of $B(\phi)$ by a vector $x\in\mathbb{Z}^\nu$. Strong screening is the statement that

$$\langle A(\phi)B(\phi)_x\rangle(\beta,\lambda) \xrightarrow[|x|\to\infty]{} \langle A(\phi)\rangle(\beta,\lambda)\langle B(\phi)\rangle(\beta,\lambda) \tag{3.1}$$

exponentially fast.

By interchanging the order of the q- and ϕ-integrations, one derives from this exponential clustering of q-correlations. In one-dimensional Coulomb gases this strong form of screening always fails for suitably chosen A and B . (See the lectures by Aizenman and refs. given there). For $\nu \geq 2$ dimensional gases, (3.1) is established in [10,11] in the (β,z)-domain which we have denoted by I , and for Dirichlet b.c.. We now interpret this result for the fractional charge correlation

$$G(x) = \langle \exp i\gamma(\phi(0)-\phi(x))\rangle(\beta,\lambda), \tag{3.2}$$

$0<\gamma<1$, introduced in (2.15) . Let $\langle A;B\rangle$ be a short hand for $\langle AB\rangle - \langle A\rangle\langle B\rangle$. By (3.1)

$$G^c(x) \equiv \langle e^{i\gamma\phi(0)}; e^{-i\gamma\phi(x)}\rangle(\beta,\lambda) \leq \text{const}\, e^{-m|x|}, \tag{3.3}$$

for some $m>0$, provided $(\beta,z)\in I$.

Next, we derive a lower bound on $G_\Lambda(x)$. Let

$$\delta\rho_{ox} \equiv \gamma(\delta_{jo}-\delta_{jx}) \quad \text{and suppose} \quad d\lambda(q) = d\lambda(-q) \tag{3.4}$$

By equations (2.4) and (2.14)

$$G_\Lambda(x) = Z_\Lambda^{-1} \int \prod_{j\in\Lambda} d\lambda(q(j)) \exp[-\beta E(q_\Lambda)] \cdot$$
$$\cdot \exp[-\beta(q_\Lambda, (-\Delta)^{-1}\delta\rho_{ox})] \cdot$$
$$\cdot \exp[-\beta/2 (\delta\rho_{ox}, (-\Delta)^{-1}\delta\rho_{ox})]$$

$$\geqq \exp[-\beta \langle (q_\Lambda, (-\Delta)^{-1}\delta\rho_{ox}) \rangle_\Lambda (\beta,\lambda)] \cdot \qquad (3.5)$$
$$\cdot \exp[-\beta/2 (\delta\rho_{ox}, (-\Delta)^{-1}\delta\rho_{ox})]$$

$$= \exp[-\beta\gamma^2 (V(0) - V(x))] \;, \qquad (3.6)$$

where (3.5) follows from Jensen's inequality, and (3.6) from the fact that $\langle (q_\Lambda, (-\Delta)^{-1}\delta\rho_{ox}) \rangle_\Lambda(\beta,\lambda) = 0$, by (3.4). Thus, for $\nu \geqq 2$, $G(x)$ does not approach 0 exponentially fast, as $|x| \to \infty$. This and (3.3) imply that in the thermodynamic limit

$$G(x) \xrightarrow[|x|\to\infty]{} |\langle e^{i\gamma\phi(0)} \rangle (\beta,\lambda)|^2 > 0 \qquad (3.7)$$

The interpretation of inequalities (3.3) and (3.7)is that the Coulomb potential of a pair of fractional charges brought into the Coulomb gas from the outside is screened exponentially fast by the particles in the gas, although their charge is integer. In particular, the mean free energy needed to bring the pair of fractional charges corresponding to $\delta\rho_{ox}$ into the system does not diverge, as $|x| \to \infty$, as one would at first expect in two dimensions, because of the logarithmic growth of the Coulomb potential and the fact that fractional charges are not screened easily by integer charges. Thus the pair of external charges can break up in two essentially free charges. Note that by (3.6),

$$G(x) \xrightarrow[|x|\to\infty]{} \text{const.} \geqq \exp[-\beta\gamma^2 V(0)] > 0 \tag{3.8}$$

in three dimensions, for arbitrary β and z, i.e. a fractional-charge dipole can always break up. In contrast, in one dimension

$$\exp[-\beta\gamma^2|x|] \leqq G(x) \leq \exp[-c|x|], \tag{3.9}$$

for some constant c which is positive for all β and z. Thus, in one dimension the electrostatic potential of fractional charges is never screened.

The behaviour of the two-dimensional gas interpolates between the one of the one- and the one of the three-dimensional gas, as (β,z) is varied. We have shown in [15] that in domain II, the low temperature, low density Kosterlitz-Thouless domain,

$$a(1+|x|)^{-\beta\gamma^2/2\pi} \leqq G(x) \leq b(1+|x|)^{-c}, \tag{3.10}$$

for some constants a,b and $c > 0$. Together with (3.3) and (3.7) this proves the existence of a Kosterlitz-Thouless transition which is further discussed in Section IV.

ii) Screening of integral charges.

This is (3.1) for observables, A and B, which are periodic in $\phi(j)$ with period 2π, for all j. I.e. the Fourier transforms of A and B only contain integral charges in their support. The one-dimensional Coulomb gas generally does screen integral charges, for arbitrary β and z. This is discussed in Aizenman's lectures and refs. given there.

iii) Weak screening of external charges.

This notion of screening involves studying the expectation of the charge density, $q(j)$, in the presence of external charges, described by a charge density, $\rho(j)$, of bounded support, not assumed to be integer-valued. Let

$$Z(\rho) \equiv \langle e^{i\phi(\rho)}\rangle(\beta,\lambda).$$

We consider

$$I(j) \equiv Z(\rho)^{-1}(-i)\langle\phi(j)e^{i\phi(\rho)}\rangle(\beta,\lambda) \tag{3.11}$$

We apply integration by parts, namely identity (2.22). This yields

$$I(j) = \beta\sum_{\ell} V(j-\ell)\{\rho(\ell) + Z(\rho)^{-1}\langle q(\ell)e^{i\phi(\rho)}\rangle(\beta,\lambda)\} \tag{3.12}$$

Fourier transformation yields

$$\hat{I}(k) = \beta\hat{V}(k)\{\overline{\hat{\rho}(k)} + Z(\rho)^{-1}\langle\overline{\hat{q}(k)}e^{i\phi(\rho)}\rangle(\beta,\lambda)\}$$

We now assume that

$$I(j) \xrightarrow[|x|\to\infty]{} -i\langle\phi(0)\rangle(\beta,\lambda),$$

by a power $\varepsilon > 0$ faster than $V(x)$ decays. This is clearly true in domain I, where strong screening (3.1) holds. Then

$\hat{I}(k)\,\hat{V}^{-1}(k) \longrightarrow 0$, as $|k| \to 0$, so that

$$\beta^{-1}[\hat{I}\hat{V}^{-1}](0) = 0 = Q(\rho) + Z(\rho)^{-1}\langle\hat{q}(0)e^{i\phi(\rho)}\rangle(\beta,\lambda)$$

i.e.

$$\sum_{j} Z(\rho)^{-1}\langle q(j)e^{i\phi(\rho)}\rangle(\beta,\lambda) = -Q(\rho), \tag{3.13}$$

where $Q(\rho) = \Sigma\rho(\ell) = \hat{\rho}(0)$ is the total charge of ρ. This says that the external charges, $\{\rho(\ell)\}_{\ell\in\mathbb{Z}^\nu}$, are screened completely, asymptotically, by particles in the system (even if ρ is not integer-valued). Stronger statements, e.g. on the decay of the effective electric field of ρ, are obtained if the decay assumptions for $I(j)+i\langle\phi(0)\rangle(\beta,\lambda)$, as $|j|\to\infty$, are refined. For a related, "axiomatic" discussion, see [17] .

By using the sine-Gordon (ϕ-) representation, it is easy to see that an evaluation of $I(j)$ in mean field approximation leads precisely to the Debye-Hückel equation. The methods of Brydges and Federbush [10,11] can be used, in principle, to estimate systematic corrections.

iv) Dipole layers.

An interesting variant of the discussion in iii) is the following : Let

$$D_\Lambda(\gamma\phi) \equiv \prod_{j\in\Lambda\subset\mathbb{Z}^{\nu-1}} e^{i\gamma(\phi(j,0)-\phi(j,-1))},$$

where Λ is a square array in the $j^\nu = 0$ plane. The state

$$\langle - \rangle_\gamma(\beta,\lambda) \equiv \lim_{\Lambda\nearrow\mathbb{Z}^{\nu-1}} \frac{\langle - D_\Lambda(\gamma\phi)\rangle(\beta,\lambda)}{\langle D_\Lambda(\gamma\phi)\rangle(\beta,\lambda)},$$

$\langle - \rangle(\beta,\lambda)$ an infinite volume state with Dirichlet b.c., describes a ν-dimensional Coulomb gas in the presence of an infinite, planar dipole layer located on the $j^\nu = 0$ plane. We wish to estimate the effective potential

$$\psi_\gamma(j) \equiv -i/\beta\langle\phi(j)\rangle_\gamma(\beta,\lambda). \tag{3.14}$$

In vacuo, the graph of ψ_γ as a function of $m \equiv j^\nu$ is as shown in Fig.3.

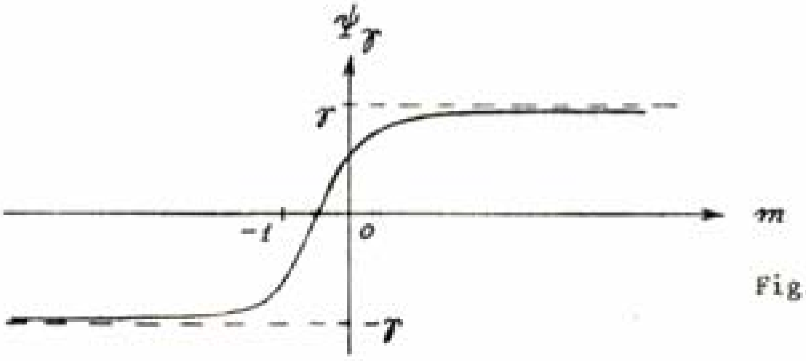

Fig. 3

If the particles in the Coulomb gas form a perfect plasma the graph of ψ_γ has the shape displayed in Fig. 4,

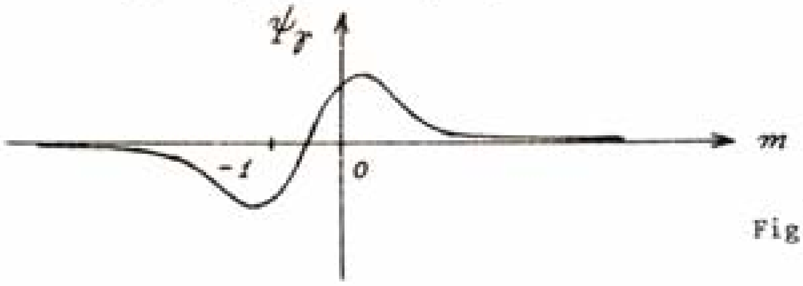

Fig.4

$$\psi_\gamma(m) \xrightarrow[|m|\to\infty]{} 0 \text{ , exponentially fast.} \tag{3.15}$$

One way of analyzing transitions in the three-dimensional Coulomb gas in domain O of Fig.2 is to investigate the behaviour of $\psi_\gamma(m)$ for different values of (β,z). It seems likely, that one can prove (3.15) in the domain of convergence of the expansion of [10,11]. (For the Villain gas (2.3) a proof is actually simple). Outside domain I, $\psi_\gamma(m)$ may approach 0 only like some inverse power of $|m|$, (or have the shape shown in Fig.3). We have no idea of how to prove this. It is more rewarding to replace γ by $-i\gamma$ and $-i\phi(j)$ by $\phi(j)$. One then considers the function

$$\psi_{-i\gamma}(m) = 1/\beta \langle \phi(\cdot,m)\rangle_{-i\gamma}(\beta,\lambda) \tag{3.16}$$

which is real-valued. For the Dirichlet b.c. state of the Villain gas in the thermodynamic limit, the graph of $\psi_{-i\gamma}$ is just as shown in Fig.4, with $\psi_{-i\gamma}(m) \xrightarrow[|m|\to\infty]{} 0$, exponentially fast, provided β is small enough.

However, there are indications that

$$\psi_{-i\gamma}(m) \longrightarrow \pm\gamma, \text{ as } m \longrightarrow \pm\infty, \tag{3.17}$$

if β is large. This phenomenon would be the analogue of the roughening transition in the three-dimensional Ising model [18]. In two dimensions, (3.17) is the expected behaviour for all β, but the surface tension, positive for small β, is expected to vanish for large β. The functions $\psi_{-i\gamma}(m)$ and

$$\langle e^{i\alpha(\phi(m)-\phi(-m))}\rangle_{-i\gamma}(\beta,\lambda)$$

have nice physical interpretations in the three-dimensional Villain gas in the ϕ-representation,(i.e. the "discrete Gaussian model"), and, for $\gamma = 1/\beta$, in the dual U(1) lattice gauge theory. (In the Coulomb gas only ψ_γ, resp. $\langle e^{i\alpha(\phi(m)-\phi(-m))}\rangle_\gamma(\beta,\lambda)$ have a natural, physical interpretation). These matters will have to be discussed in more detail, elsewhere.

IV. The Kosterlitz-Thouless Transition in the Two-Dimensional Coulomb Gas.

In this section we briefly sketch a rigorous argument [15] establishing the Berezinski-Kosterlitz-Thouless transition [19] in a class of models, including the two-dimensional Coulomb gas, the plane rotator and higher dimensional, abelian lattice gauge theories.

The main idea is to use the sine-Gordon (ϕ-) representation to rewrite correlations in the Coulomb gas as convex combinations of correlations in dilute gases of neutral multipoles of variable size, at random positions. Such gases are known not to exhibit Debye screening [3,15]. Here we study the behaviour of the fractional charge correlation $G(x)$ defined in (2.15) which we discussed already in Section III.1.

Our aim is to sketch the proof of (3.10), i.e.

$$a\,(1+|x|)^{-\beta\gamma^2/2\pi} \leq G(x) \leq b\,(1+|x|)^{-c}, \tag{4.1}$$

for some $c > 0$, provided $z < e^{\varepsilon'\beta}$ and $\beta > \beta_c$, for some $\varepsilon' > 0$ and $\beta_c < \infty$; (a and b are finite positive constants). We use the ϕ-representation (2.11), (2.12), (2.1') of the hard core gas : Thus $\langle - \rangle_\Lambda(\beta,z)$ denotes the expectation in the measure, (signed for $z > 1$),

$$Z_\Lambda^{-1} \prod_{j\in\Lambda} (1 + z\cos\phi(j))\, d\mu_{\beta V}(\phi), \tag{4.2}$$

and we impose free (i.e. insulating) b.c.. The more interesting case of Dirichlet b.c. is only slightly more difficult; see [15].

The lower bound in (4.1) has already been established in Section III, so we concentrate on the proof of the upper bound. We note that, by the $\phi \to -\phi$ symmetry of (4.2)

$$G_\Lambda(x) = Z_\Lambda^{-1} \int I_1(\phi_\Lambda)\, d\mu_{\beta V}(\phi), \tag{4.3}$$

where

$$\left.\begin{aligned} &I_\alpha(\phi_\Lambda) = \cos\phi(\alpha\,\delta\rho_{ox}) \prod_{j\in\Lambda} (1 + z\cos\phi(j)), \\ \text{with}\quad &\delta\rho_{ox}(j) = \gamma\,(\delta_{jo} - \delta_{jx}), \quad 0 < \gamma < 1. \end{aligned}\right\} \tag{4.4}$$

Our proof of (4.1) is based on applying the following elementary identities to $I_\alpha(\phi_\Lambda)$:

$$\begin{aligned}(1 + K_1\cos\alpha_1)(1 + K_2\cos\alpha_2) &= 1/3 \sum_{\kappa=1,2} (1 + 3K_\kappa\cos\alpha_\kappa) \\ &\quad + 1/6 \sum_{\varepsilon=\pm 1} (1 + 3K_1K_2\cos(\alpha_1 + \varepsilon\alpha_2))\end{aligned} \tag{4.5}$$

$$\cos\alpha_o\,(1 + K_1\cos\alpha_1) = 1/2 \sum_{\varepsilon=\pm 1} (\cos\alpha_o + K_1\cos(\alpha_o + \varepsilon\alpha_1)) \tag{4.6}$$

and

$$\begin{aligned}&(\cos\alpha_o + K_1\cos(\alpha_o + \alpha_1))(1 + K_2\cos\alpha_2) \\ &= 1/3\,(\cos\alpha_o + 3K_1\cos(\alpha_o + \alpha_1)) \\ &\quad + 1/6 \sum_{\varepsilon=\pm 1} \{(\cos\alpha_o + 3K_2\cos(\alpha_o + \varepsilon\alpha_2)) + \\ &\qquad (\cos\alpha_o + 3K_1K_2\cos(\alpha_o + \alpha_1 + \varepsilon\alpha_2))\}\end{aligned} \tag{4.7}$$

A function ρ of finite support contained in Λ with values in $\{+1,-1\}$ is called a charge density, and $Q(\rho) \equiv \sum_j \rho(j)$ is the total charge of ρ ; ρ restricted to a proper subset of supp ρ is called a constituent of ρ . A family of charge densities with disjoint supports is called an ensemble.

First (4.5) is applied to

$$I_\alpha(\phi_\Lambda)=\sum_m c_{\mathcal{E}_m^{(o)}} \cos\phi(\alpha\delta\rho_{ox}) \prod_{\rho\in\mathcal{E}_m^{(o)}} (1+K^{(o)}(\rho)\cos\phi(\rho)) \tag{4.8}$$

where $c_{\mathcal{E}_m^{(o)}} = \delta_{m1}$, each charge density $\rho \in \mathcal{E}_1^{(o)}$ has support on a single site, $j(\rho)$, where it takes the value 1,

$\bigcup_{\rho\in\mathcal{E}_1^{(o)}} \{j(\rho)\} = \Lambda$, and $K^{(o)}(\rho) = z$.

The rules for applying (4.5) are as follows : Group all ρ's in $\mathcal{E}_1^{(o)}$ in pairs (ρ_1,ρ_2) supported on nearest neighbor sites in an otherwise arbitrary way. For a given pair (ρ_1,ρ_2) apply (4.5) to

$$(1+K^{(o)}(\rho_1)\cos\phi(\rho_1))(1+K^{(o)}(\rho_2)\cos\phi(\rho_2))$$

The result is then inserted on the r.s. of (4.8), for all possible (ρ_1,ρ_2) , and the resulting expression is expanded out. This yields

$$I_\alpha(\phi_\Lambda)=\sum_m c_{\mathcal{E}_m^{(1)}} \cos\phi(\alpha\delta\rho_{cx}) \prod_{\rho\in\mathcal{E}_m^{(1)}} (1+K^{(1)}(\rho)\cos\phi(\rho)) \tag{4.9}$$

with $c_{\mathcal{E}_m^{(1)}} > 0$, for all m , and each ρ on the r.s. of (4.9) is supported on one site, with a nearest neighbor site empty, or on a pair of nearest neighbors and takes values ± 1 . Notice that each application of (4.5) to a pair (ρ_1,ρ_2) produces a term, $1+3K^{(o)}(\rho_1)K^{(o)}(\rho_2)\cos\phi(\rho_1-\rho_2)$, with the property that the total charge $Q(\rho_1-\rho_2)$ vanishes. The density $\rho' \equiv \rho_1-\rho_2$ is interpreted as a neutral dipole. Another term that is produced is $1+3K^{(o)}(\rho_1)\cos\phi(\rho_1)$ which has the property that the charge ρ_2

has been eliminated. Thus the resulting ensembles, $E_m^{(1)}$, tend to contain neutral dipole densities and tend to be sparser than $E_1^{(o)}$. This is a feature common to all subsequent steps in an inductive expansion of $I_\alpha(\phi_\Lambda)$: During those steps charge densities ρ,ρ' are combined to larger densities $\rho\pm\rho'$, with a chance of 1/2 that the total charge is lowered, or one of the densities ρ,ρ' is eliminated which makes the ensemble sparser, at the prise of increasing the unrenormalized activities, $K(\rho)$, by factors of 3.

In the next step, the ρ's in each $E_m^{(1)}$ are paired among each other or with $\alpha\delta\rho_{ox}$, and identities (4.5), resp. (4.6) are applied to all such pairs, the resulting expression is expanded and yields a class $\{E_{m'}^{(2)}\}$ of ensembles derived from $\{E_m^{(1)}\}$ by combining densities ρ,ρ' , with $\mathrm{dist}(\rho,\rho') \equiv \mathrm{dist}(\mathrm{supp}\rho, \mathrm{supp}\rho') = 1$, to a larger density $\rho\pm\rho'$, resp. cancelling either ρ or ρ' , for all m' . After a finite number of steps, depending on an integer $k = 1,2,3,\ldots,$ in this inductive scheme one obtains

$$I_\alpha(\phi_\Lambda) = \sum_m c_{E_m^k} \prod_{\rho\in E_m^k\sim\rho_m^*} (1+K^k(\rho)\cos\phi(\rho)) \cdot \\ \cdot (\cos(\alpha\delta\rho_{ox}) + K^k(\rho_m^*)\cos\phi(\rho_m^* + \alpha\delta\rho_{ox})) \tag{4.10}$$

where $c_{E_m^k} > 0$, for all m , and each ensemble E_m^k is the union two sub-ensembles N_m^k and J_m^k , defined as follows :

Define the diameter, $d(\rho)$, of a charge density ρ in some ensemble E to be the smallest integer of the form 2^ℓ, $\ell = 1,2,3,\ldots,$ such that $\mathrm{supp}\,\rho$ can be covered by a single square in $\mathbb{Z}^2$ with sides of length $d(\rho)$. Furthermore, $d(\rho_m^*)$ is defined as the diameter of $\rho_m^* + \alpha\delta\rho_{ox}$.

The sub-ensembles N_m^k are now defined by the properties :

a) $N_m^k \supseteq N_m^{k-1}$, $N_m^o = \emptyset$;

b) each $\rho \in N_m^k$ is <u>neutral</u>, i.e. $Q(\rho) = 0$;

c) For all ρ, ρ' in N_m^k, $\rho \neq \rho'$,
$\mathrm{dist}(\rho,\rho') \geq M \min(d(\rho),d(\rho'))^{\alpha}$, for some constants $M > 0$ and $\alpha \in (3/2,2)$, e.g. $\alpha = 5/3$, to be chosen appropriately, [15]. (Here $\mathrm{dist}(\rho,\rho') \equiv \mathrm{dist}(\mathrm{supp}\,\rho, \mathrm{supp}\,\rho')$).

d) $\mathrm{dist}(\rho,\rho'') \geq Md(\rho)^{\alpha}$, for all $\rho \in N_m^k$ and all $\rho'' \in J_m^k$. □

The expansion described between (4.9) and (4.10) terminates for all ρ's in N_m^k, for all m, because the ρ's in N_m^k are <u>neutral</u>, see b), and far separated from other charge densities, see c), d).

The sub-ensembles J_m^k are defined by

i) $J_m^k \cap N_m^k = \emptyset$, $J_m^k \cup N_m^k = E_m^k$;

ii) for arbitrary densities ρ and $\rho' \neq \rho$ in J_m^k,

$$\mathrm{dist}(\rho,\rho') \geq 2^k ,$$

for all m. Thus, k labels a <u>distance scale</u>. Identity (4.10) and properties a) - d), resp. i) - ii) are obvious for $k = 0$, and we have already outlined how they are checked for $k = 1$. In order to do the induction step, $k \to k+1$, we split J_m^k into two disjoint subsets, ${}^{>}J_m^k$ and ${}^{<}J_m^k$, where ${}^{<}J_m^k$ has the property that, given any $\rho \in {}^{<}J_m^k$, there exists some $\rho' \in J_m^k$, with $\mathrm{dist}(\rho,\rho') < 2^{k+1}$. We set ${}^{>}J_m^k = J_m^k \sim {}^{<}J_m^k$. Densities in ${}^{>}J_m^k$ are separated from other densities in J_m^k by a distance $\geq 2^{k+1}$ and will participate in the expansion only on scales $\geq 2^{k+1}$. Next, a charge density $\rho \in {}^{<}J_m^k$ is paired with some $\rho' \in {}^{<}J_m^k$ for which $\mathrm{dist}(\rho,\rho') < 2^{k+1}$. Then identity (4.5), resp. identity (4.7) is applied to the factors labelled by (ρ,ρ'), and the resulting expression is expanded. This operation increases the activity of the resulting density by a factor of 3. Subsequently a new pair of densities within distance $< 2^{k+1}$ from each other is formed, etc... After finitely many operations, (4.10) is recovered, with k increased to $k+1$. See [15] for details. By induction in k and a series of combinatorial arguments one obtains

Theorem 1, [15] .

(1)
$$I_\alpha(\phi_\Lambda) = \sum_m c_{N_m} \prod_{\rho\in N_m \sim \rho_m^*} (1+K(\rho)\cos\phi(\rho)) \cdot$$
$$\cdot \left(\cos\phi(\alpha\delta_{\rho_{0x}}) + K(\rho_m^*)\cos\phi(\rho_m^* + \alpha\delta_{\rho_{0x}})\right),$$

where $c_{N_m} > 0$, for all m ; all ρ's in N_m , except possibly one density $\rho = \rho_c$ which is charged, are neutral and satisfy conditions b) - d) formulated above.

(2) For all $\rho \in N_m$, $\rho \neq \rho_c$,
$$K(\rho) \leq z^{|\rho|} \exp\left[c \sum_{n=0}^{n(\rho)} A_n(\rho)\right], \tag{4.11}$$

where $|\rho| = \sum_j |\rho(j)|$, $A_n(\rho)$ is the minimal number of $2^n \times 2^n$ squares necessary to cover the support of ρ , $n(\rho) \leq c' \ell n d(\rho)^\alpha$, and c, c' are finite constants independent of N_m .

(3) If some $\rho \in N_m$ contains a constituent ρ_1 such that $\mathrm{dist}(\rho_1, \rho-\rho_1) \geq 2Md(\rho_1)^\alpha$ then $Q(\rho_1) \neq 0$. □

Remarks. Part (1) follows from (4.10) by induction in k , and (3) is a fairly simple consequence of conditions b) - d) satisfied by N_m . The hard part is (2) : Since, by condition a) above, each N_m is an inductive limit of a family $\{N^k_{n(m,k)}\}_{k=1,2,3,\ldots}$, each neutral $\rho \in N_m$ belongs to some N^k_m , for a finite k . Thus, the term $1+K(\rho)\cos\phi(\rho)$ is produced after a finite number, N , of applications of identiy (4.5). That identity then yields

$K(\rho) \leqq z^{|\rho|} 3^N$, and a somewhat complicated estimate on N yields (4.11). (Same comment on term labelled by ρ^*_m) . The interpretation of the quantity $c \cdot \sum_{n=o}^{n(\rho)} A_n(\rho)$ is that of an <u>entropy</u> of ρ . See also (2.36). For details we refer to [15] .

Next, we note that

$$\left.\begin{aligned} Z_\Lambda &= \int I_{\alpha=0}(\phi_\Lambda)\, d\mu_{\beta V}(\phi), \\ Z_\Lambda G_\Lambda(x) &= \int I_{\alpha=1}(\phi_\Lambda)\, d\mu_{\beta V}(\phi). \end{aligned}\right\} \tag{4.12}$$

Since the algebraic structure of the expansion of $I_\alpha(\phi_\Lambda)$ is independent of α , the expansions of Z_Λ and $Z_\Lambda G_\Lambda$ involve the same N_m , ρ^*_m and c_{N_m} . For free b.c., the contribution of all terms containing a factor $K(\rho_c)\cos\phi(\rho_c)$, $Q(\rho_c) \neq 0$, to the r.s. of (4.12) vanishes; see (2.9). Thus

$$\begin{aligned} \int I_\alpha(\phi_\Lambda)\, d\mu_{\beta V}(\phi) = \sum_m c_{N'_m} \int \prod_{\rho \in N'_m \sim \rho^*_m} (1 + K(\rho)\cos\phi(\rho)) \cdot \\ \cdot \left(\cos\phi(\alpha\delta_{\rho_{ox}}) + K(\rho^*_m)\cos\phi(\rho^*_m + \alpha\delta_{\rho_{ox}})\right) d\mu_{\beta V}(\phi) \end{aligned} \tag{4.13}$$

where all ρ's in N'_m and ρ^*_m are neutral, for all m .

Our goal is now to replace $\prod_{\rho \in N'_m \sim \rho^*_m} (1+K(\rho)\cos\phi(\rho))$ by a new product

$$\prod_{\rho \in N'_m \sim \rho^*_m} (1 + \zeta(\rho,\beta)\cos\phi(\bar{\rho})), \tag{4.14}$$

where $\bar{\rho}$ is a renormalized charge density, and $\zeta(\rho,\beta)$ a renormalized activity, <u>without changing the values of</u> Z_Λ <u>and</u> $Z_\Lambda G_\Lambda$. An essential ingredient in the proof of (4.1) is that (4.14) defines a <u>positive</u> function of ϕ . This is manifestly true if

$$\zeta(\rho,\beta) < 1 \text{ , for all } \rho \in N'_m \text{ and all } m . \tag{4.15}$$

One of the main technical results of [15] is that $\bar{\rho}$ can be chosen such that (4.15) holds for $z < e^{\varepsilon'\beta}$, for some $\varepsilon' > 0$, and β large enough.

The idea of the renormalization $\rho \to \bar{\rho}$ is as follows : First, we carry out the renormalization transformation described in Section II.2, (2.32) - (2.34). Next, let $\rho \in N'_m$ contain a charged constituent ρ_1 separated from $\rho-\rho_1$ by a distance $\geq 2M$. Then ρ_1 is replaced by a new charge density σ_1 concentrated near the surface of a domain Σ_1 containing $\operatorname{supp}\rho_1$ but not intersecting $\operatorname{supp}(\rho-\rho_1)$, in such a way that the electrostatic interactions of ρ_1 and σ_1 with $\rho-\rho_1$ and all other $\rho' \in N'_m$ are <u>unchanged.</u> This is achieved by a change of variables on the r.s. of (4.13), $\phi(j) \mapsto \phi(j)+ia_{\rho_1}(j)$, for some real function a_{ρ_1}. This is the method of <u>complex translations</u> introduced in [20] . As a result, $K(\rho)\cos\phi(\rho)$ is replaced by $\exp[-\beta(E(\rho_1)-E(\sigma_1))]K(\rho)\cos\phi(\sigma_1+\rho-\rho_1)$, where $E(\rho')$ is the electrostatic energy of ρ' ; see (2.4). In order to locate the charged constituents of some $\rho \in N'_m$, one uses part (3) of Theorem 1. Let $S_n(\rho)$ be a minimal collection of $2^n \times 2^n$ squares needed to cover ρ, and let $S'_n(\rho)$ be defined as

$$\{s \in \mathcal{S}_n(\rho) : dist(s,s') \geq 2M\,2^{\alpha n}, \text{ for all } s' \neq s \text{ in } \mathcal{S}_n(\rho)\}.$$

By Theorem 1, (3), each $s \in S'_n(\rho)$ covers a charged constituent, ρ_s, of ρ. The renormalization procedure described above is now applied to ρ_s, for each $s \in S'_n(\rho)$, in such a way that $\operatorname{supp}\sigma_s$ is contained in the interior of a $2^{n+1} \times 2^{n+1}$ square covering s. This permits one to apply the same renormalization transformation inductively on all length scales 2^n, $1 \leq n \leq n(\rho)$. One obtains

<u>Theorem 2,</u> [15] .

For $\alpha > 3/2$, M large enough and for all m,

$$\int I_\alpha(\phi_\Lambda)\,d\mu_{\beta V}(\phi) = \int F_m(\phi_\Lambda)\,(\cos\phi(\alpha\,\delta\rho_{\alpha x}) +$$

$$K(\rho_m^*)\cos\phi(\rho_m^* + \alpha\,\delta\rho_{\alpha x})\,d\mu_{\beta V}(\phi)$$

where $F_m(\phi_\Lambda) = \prod_{\rho\in N_m' \sim \rho_m^*} (1 + \xi(\rho,\beta)\cos\phi(\bar\rho))$, (4.16)

and

$$\xi(\rho,\beta) \leqq K(\rho)e^{-\varepsilon\beta|\rho|}\exp\Big[-c''\beta\sum_{n=1}^{n(\rho)} \mathrm{card}(\mathcal{O}_n'(\rho))\Big], \quad (4.17)$$

with c'' independent of N_m'. □

We now sketch, how Theorems 1 and 2 are used to prove (4.15) and the upper bound on $G(x)$ in (4.1). First, one proves a combinatorial lemma saying that, for $\alpha < 2$, there exists some $c''' > 0$ such that

$$|\rho| + \sum_{n=1}^{n(\rho)} \mathrm{card}(\mathcal{O}_n'(\rho)) \geqq c''' \sum_{n=o}^{n(\rho)} A_n(\rho), \quad (4.18)$$

with c''' independent of N_m' ; see [15] .

Moreover, $\sum_{n=o}^{n(\rho)} A_n(\rho) \geqq \mathrm{const.}\,\ell n\, d(\rho)$. (4.19)

It now follows from (4.17), (4.11), (4.18) and (4.19) that for $z < e^{\varepsilon'\beta}$, for some $\varepsilon' > 0$, there exist constants $c > 0$ and $d < \infty$ such that

$$\xi(\rho,\beta) \leqq \exp[-(c\beta - d)\,\ell n\, d(\rho)], \quad (4.20)$$

which proves (4.15) for $\beta > d/c$. The inductive renormalization transformation described above can of course be applied to $\cos(\delta\rho_{ox})$ and $\cos(\rho_m^* + \delta\rho_{ox})$, too, for all m. Together with (4.15), i.e. $F_m(\phi) \geqq 0$, this yields

$$Z_\Lambda G_\Lambda(x) = \sum_{m} c_{n'_m} \int F_m(\phi_\Lambda)\,(\xi' \cos\phi(\overline{\delta\rho_{ox}}) + \xi'' \cos\phi(\overline{\rho^*_m + \delta\rho_{ox}}))\, d\mu_{\beta V}(\phi)$$

$$\leqq (\xi' + \xi'') \int F_m(\phi_\Lambda)\, d\mu_{\beta V}(\phi), \quad (4.21)$$

where $0 < \xi', \xi'' \leqq \exp[-(c_\gamma\beta - d)\ln \ell]$,

and $$\ell \equiv \min\left(d(\delta\rho_{ox}), d(\rho^*_m + \delta\rho_{ox})\right) \geqq |x|, \quad (4.22)$$

with $0 < c_\gamma < c$, for $0 < \gamma < 1$, (see (4.4)). It follows immediately from (4.16) and (2.9) that

$$\int F_m(\phi_\Lambda)\, d\mu_{\beta V}(\phi) \leqq \int F_m(\phi_\Lambda)(1 + K(\rho^*_m)\cos\phi(\rho^*_m))\, d\mu_{\beta V}(\phi) = Z_\Lambda$$

so that, with (4.21) and (4.22),

$$Z_\Lambda G_\Lambda(x) \leqq 2 \exp[-(c_\gamma\beta - d)\ln|x|]\, Z_\Lambda$$

which, for β large enough, yields (4.1), by taking $\Lambda \nearrow \mathbb{Z}^2$. Together with the material in Section III, i), in particular (3.3) and (3.7), this completes the proof of existence of a Kosterlitz-Thouless transition in the hard core Coulomb gas, as (β, z) is varied.

V. Other Models with Kosterlitz-Thouless Transitions.

Here is a list of models for which transitions of the Kosterlitz-Thouless type (as some thermodynamic parameters are varied) are established in [15] .

1) Hard core-, standard- and Villain Coulomb gas in two dimensions.

2) The two-dimensional plane rotator model. It is shown in [15] that, for large enough β , $\langle \vec{s}_0 \cdot \vec{s}_x \rangle(\beta) \geq a(1+|x|)^{-c}$, in zero external field, for some finite c . An upper bound with power fall-off was previously proven in [20] . Some further results may be found in [21] .

3) The two-dimensional $\mathbb{Z}_n$ models, for n large enough : Existence of a massless phase for $T_- < T < T_+$, for some finite, positive T_-, T_+ .

4) The two-dimensional solid-on-solid model, for which it is shown e.g. that $\langle(\phi(0)-\phi(x))^2\rangle \geq a\ln|x|$, at sufficiently high temperatures.

5) A three-dimensional, non-compact lattice Higgs model (a Landau-Ginsburg lattice theory) for which the existence of a transition from a superconducting, massive to a massless QED phase is verified; (see also [22] and refs.).

6) The four-dimensional, pure $U(1)$-lattice gauge theory : Breakdown of confinement for large β . This was first shown in [23] , by a more complicated argument.

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