SOME COMMENTS ON THE CROSSOVER BETWEEN STRONG AND WEAK COUPLING IN SU(2) PURE YANG-MILLS THEORY*

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Abstract.

Some of the main approaches towards an understanding of quark confinement are described. One circle of results concerning the confinement of (static) quarks at moderately strong coupling and the crossover between the strong and weak coupling regime in four dimensional pure Yang-Mills theory with SU(2) gauge group are summarized in more detail. In particular, the crossover interval is located.

Some connections between our approach and a theory of non-relativistic strings are briefly described.

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1. It is widely believed that QCD with an SU(3) colour gauge group is a good theory of the strong interactions at energies which are not so enormous that grand unification of the fundamental interactions or gravitation (e.g. a complicated micro-structure of space-time) would become important.

QCD is an asymptotically free theory, so that at high energies (renormalization group improved) perturbation theory becomes reliable. However, at small energies - i.e. large distance scales - QCD is far from a conventional free field theory, perturbation theory is not applicable, and to date there are no really satisfactory algorithms permitting to calculate the low energy behaviour (hadron spectrum, etc.) of QCD.

Among the important qualitative features of strong interaction physics at large distance scales which QCD ought to explain are

(1) Quark confinement. (The quark fields which transform non-trivially under the center $Z = \mathbb{Z}_3$ of the gauge group $G = SU(3)$ do not couple the vacuum to physical one particle states. More generally, there are no physical, asymptotic states which transform non-trivially under global transformations in Z).

(2) The known physical hadrons are "bound states" of two or three (but not more) quarks, i.e. of the minimal number of quarks that suffices to construct a state transforming trivially under Z .

Clearly states transforming trivially under the center Z of G may still transform non-trivially under "global gauge transformations" in G/Z , i.e. they may have colour. However, their colour can be shown to be <u>screened</u> by the colour of gluons, [1,2,3,4]. (This observation contains already half an explanation of (2) !).

In these notes we briefly sketch some approaches towards understanding (1).

2. We simplify this task by studying a (by now standard) caricature of the full theory, namely pure Yang-Mills theory with quarks put in as external, static colour sources. This caricature is reasonable when one studies the behaviour of a theory with very heavy quarks and retains some qualitative, predictive power even in the case where light quarks are coupled to the colour gauge field. Moreover, in the estimates presented below, colour SU(3) is usually replaced by colour SU(2) . This is clearly an unjustified simplification, but it has the virtue that only the confinement of q-$\bar{q}$ pairs (i.e. two-quark bound states) needs to be studied and that certain technical estimates simplify drastically.

3. We shall adopt the Euclidean description of relativistic quantum field theory. Euclidean space-time is $\mathbb{E}^D$, with D the (space-time) dimension, the gauge group is some compact, simple Lie group G (later on $=SU(2)$) , and Z denotes its center.

Let $\{\chi_i\}_{i\in I}$ be a list of all unitary characters of G , and suppose χ_0 is the character of a faithful, unitary representation of G (the fundamental representation when $G=(S)U(N)$, $N=1,2,3,\ldots$).

Let Ω be the space of at least twice differentiable, oriented loops free of self intersections, (i.e. "circles").

Given a classical gauge field, A , (a connection), let

$$g:\ \omega\in\Omega\longmapsto g_\omega=\mathcal{P}\left\{\exp\int_\omega A_\mu(x)\,dx^\mu\right\}\in G \qquad (1)$$

denote the map from loops into parallel transporters; (g is a gauge field configuration). It has been shown in [5] that, in classical, Euclidean pure Yang-Mills theory, the gauge-invariant functions

$$\{ \chi_0(g_\omega) : \omega \in \Omega \}$$

generate a complete algebra of functions which specify A up to gauge equivalence; i.e. they specify the orbit [A] of A under all possible gauge transformations.

Thus, in quantum theory it is natural to try to associate a (gauge-invariant!) "field operator" with each $\chi_0(g_\omega)$ (resp. all $\chi_i(g_\omega)$, $i \in I$), where ω is an arbitrary space-like loop.

Euclidean quantization consists of converting the functions $\chi_i(g_\omega)$ into random fields,

$$y_i(\omega) = N(\chi_i(g_\omega)), \tag{2}$$

where N stands for some "normal ordering" (see e.g. [4,6]), on the loop space Ω. The distribution of these random fields is supposed to be given by a probability measure, $d\mu([A])$, on some space, $\mathcal{O}$, of gauge orbits of all possible connections.

The properties of the quantum theory are coded into the sequence of Euclidean Green's functions

$$S_n(y_{j_1}(\omega_1), \cdots, y_{j_n}(\omega_n)) = \text{"}\int_{\mathcal{O}} \prod_{i=1}^{n} y_{j_i}(\omega_i)\, d\mu([A])\text{"}, \tag{3}$$

(provided $\{S_n\}_{n=0}^{\infty}$, $S_0 \equiv 1$, can be shown to satisfy certain postulates or axioms discussed in [7]).

Of particular importance for the understanding of the qualitative features of the resulting quantum theory are the one- and two-loop functions

$$S_1(\mathcal{Y}_j(\omega)) \quad , \quad S_2(\mathcal{Y}_{j_1}(\omega_1), \mathcal{Y}_{j_2}(\omega_2)) :$$

$S_1(\mathcal{Y}_j(\omega))$, χ_j non-trivial on Z , yields information on the $q-\bar{q}$ potential and consequently on the two-quark bound state spectrum (in the limit of very heavy quarks); $S_2(\mathcal{Y}_{j_1}(\omega_1), \mathcal{Y}_{j_2}(\omega_2))$ contains information about the low-lying mass spectrum of pure Yang-Mills theory, [6].

4. Let $\omega = \omega_{L\times T}$ be the loop consisting of two "vertical" pieces of length T and two "horizontal"pieces, γ_i (connecting 0 to (L,0,...,0)) and γ_f (connecting (0,...,0,T) to (L,0,...,0,T)) . See Fig. 1.

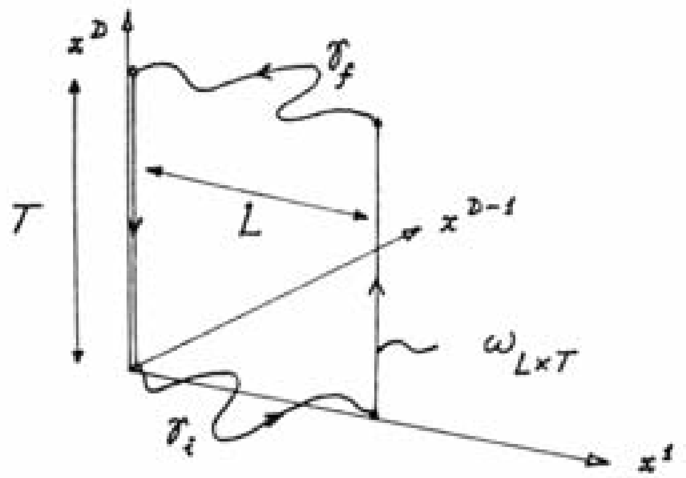


Fig. 1

The static $q-\bar{q}$ potential is defined by

$$\mathring{V}(L) \equiv \mathring{V}_{\chi_j}(L) = \inf_{\{\gamma_i,\gamma_f\}} \lim_{T\to\infty} -\frac{1}{T} \ln\left| S_1(\mathcal{Y}_j(\omega_{L\times T}))\right| \quad (4)$$

(It may be regarded as an expression of some sort of "nuclear democracy" that in a strongly coupled lattice gauge theory $S_1(\mathcal{Y}_j(\omega_{L\times T}))$ resembles the transition function,

$$p_T(\gamma_i, \gamma_f) \tag{5}$$

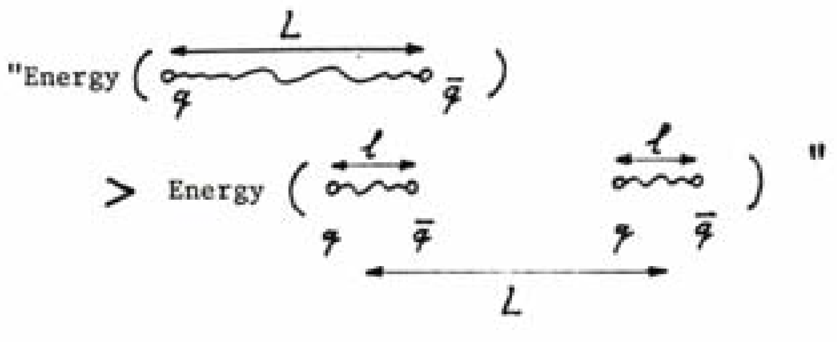
for the diffusion of a non-relativistic string [4,6], so that the (low-lying) bound state spectrum of two <u>very heavy</u> quarks of mass M_q should resemble the excitation spectrum of a non-relativistic string, shifted up by $2M_q$) .

Seiler [8] has shown that

$$\mathring{V}(L) \leq \text{const. } L\,, \text{ as } L \to \infty. \tag{6}$$

Suppose now that one couples quarks of mass $\geq M_q = M_{\bar{q}}$ to the pure Yang-Mills theory described above, with M_q rather large compared to a typical (low energy) mass scale of the pure Yang-Mills theory. Let V(L) still be given by (4), but now calculated in the full theory.*)

One is entitled to expect that quarks are confined if



(7)

with $L \gg M_q^{-1}$, $\ell \approx O(M_q^{-1})$

Now, the contribution of the interactions between the two disconnected q-$\bar{q}$ pairs to the r.s. of (7) can be expected to tend to 0 , as $L \to \infty$. Thus (7) holds if

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*) The correct definition of the q-$\bar{q}$ potential is more involved.

$$2M_q + V(L) > 4M_q + V(\text{const.}\, M_q^{-1}), \text{ or}$$
$$V(L) > \text{const.}\, M_q + \text{const.}', \tag{8}$$

as $L \to \infty$.

In a non-abelian, pure Yang-Mills theory in $D \leqq 4$ dimensions one hopes to prove

$$\mathring{V}(L) \nearrow \infty, \quad \text{as } L \to \infty, \tag{9}$$

or, more ambitiously,

$$\mathring{V}(L) \geqq \text{const.}\, L, \text{ as } L \to \infty. \tag{10}$$

But if dynamical quarks are coupled to the theory inequality (8) is the best that can possibly be true, since breaking the string by the creation of a q-$\bar{q}$ pair is possible dynamically !

However, if e.g. (10) holds for the pure Yang-Mills theory one may be confident that (8) holds for M_q large.

5. In the remainder of this note we review arguments in favour of (10) for a pure SU(2) Yang-Mills theory in ($D \leqq$) 4 dimensions and discuss the crossover between strong and weak coupling in the behaviour of $W(\omega) = S_1(y_j(\omega))$, (for χ_j non-trivial on Z).

There are various approaches to this problem of which we mention three; (the 1/N-expansion, approaches based on consideration of gauge-dependent Green's functions, e.g. the gluon propagator - see Mandelstam's contribution - high temperature expansions and numerical results are not included in our review).

(A) The "topological" approach, [9,10] (and [11]).

We shall (for simplicity) suppose that the gauge group G is simply connected and has a discrete center.

The topological approach is based on analyzing the statistical mechanics of excitations in the gauge field configurations g , analogous to the defects in ordered systems of bulk matter, which can be characterized by topological properties.

In a four dimensional, pure Yang-Mills theory such excitations fall into two classes :

(A1) Instantons, labelled by the elements of $\pi_3(G)$, the third homotopy group of the gauge group G . Instantons are analogous to point defects in a four dimensional ordered system. By themselves, they cannot cause confinement of static quarks in the sense of inequalities (9) or (10). (They may however yield significant corrections to perturbation theory. See Callan's contribution).

The main significance of instantons is that the structure of the physical vacuum depends on their statistical properties in an important way, (θ-vacua, [12]). Moreover, if the "activity" (the statistical weight) of instantons is sufficiently large they can cause a two-fold vacuum degeneracy at $\theta = \pi$ which can be understood in terms of "wall defects". The order parameter is given by the "instanton density"($\propto \operatorname{tr} F(x)\tilde{F}(x)$). This has first been noticed in [13,14].

(A2) Vortices, labelled by the elements of $\pi_1(G/Z)$. If G is simply connected and Z discrete then

$$\pi_1(G/Z) = \pi_0(Z) = Z . \tag{11}$$

Vortices correspond to codimension 2 defects, i.e. sheets for $D = 4$, line defects for $D = 3$, and for $D = 2$ point defects (= two dim. "instantons". When $D = 2$ the vortices also give rise to a vacuum angle θ, with $e^{i\theta} \in Z$, and a two-fold vacuum degeneracy at $\theta = \pi$, [13,14,4]). Vortices are expected (and can often be shown) to play a very significant rôle in the mechanism causing confinement of static quarks, in the sense of inequs. (9), (10); [9,10,15]:

(i) In two-dimensional theories the statistics of vortices causes permanent confinement by a linear potential; [16,14,17].

(ii) In three-dimensional $U(N)$ theories, $N = 1,2,3,\ldots$, the statistics of vortices yields permanent confinement by a potential growing at least logarithmically, ($\overset{o}{V}(L) \geqq$ const. $\log L$, as $L \to \infty$) ; [10,17]. These results are rigorous for lattice gauge theories.

(iii) Mack and Petkova have shown [10] that <u>condensation of vortices</u>, (i.e. the probability that gauge field configurations, g, contain vortices is large and <u>essentially independent</u> of their length $(D = 3)$, resp. surface $(D = 4)$) implies linear confinement. In order to make this precise we consider a lattice gauge theory :

Let Λ be a D-dimensional, connected region in Euclidean space-time with the property that the boundary $\partial\Lambda$ is homeomorphic to $S^{D-2} \times S^1$, i.e.

$$\partial\Lambda = \Phi \times \Sigma$$

where Φ can be interpreted as the locus of a closed vortex sheet homeomorphic to S^{D-2} and Σ is homeomorphic to S^1. (In [10] such regions Λ are called "vortex containers"). Let $g_{\partial\Lambda}$ be some fixed gauge field configuration on $\partial\Lambda$, let $Z(g_{\partial\Lambda})$ be the partition function of the theory confined to Λ with given boundary condition $g_{\partial\Lambda}$, and let $Z_z(g_{\partial\Lambda})$ be the partition function of the same theory but with a <u>vortex</u> labelled by $z \in Z$ added in the interior of Λ.

Let

$$P_z(g_{\partial\Lambda}) = Z_z(g_{\partial\Lambda})\Big[\sum_{z' \in Z} Z_{z'}(g_{\partial\Lambda})\Big]^{-1}. \tag{12}$$

Let χ be the character of a representation of G which is non-trivial on Z. Define

$$\hat{P}_\chi(g_{\partial\Lambda}) = \sum_{z \in Z} \chi(z)\, P_z(g_{\partial\Lambda}). \tag{13}$$

Suppose now that, for $|\Sigma|$ (= length of Σ) large enough,

$$\sup_{\{g_{\partial\Lambda}\}} |\hat{P}_\chi(g_{\partial\Lambda})| \leq \xi < 1, \tag{14}$$

uniformly in Λ, provided $|\Phi|$ (= surface of Φ) is sufficiently large. Then

$$\mathring{V}_\chi(L) \geq \text{const.}\, L, \text{ as } L \to \infty. \tag{15}$$

(If $$\sup_{\{g_{\partial\Lambda}\}} |\hat{P}_\chi(g_{\partial\Lambda})| \leq \exp\{-\text{const.}\, \text{diam.}(\Phi)^{-1}\}, \tag{16}$$

for diam (Φ) large enough, then

$$\mathring{V}_\chi(L) \geq \text{const.} \log L, \text{ as } L \to \infty.) \tag{15'}$$

These results (whose proofs are quite simple) are contained in [10]; see also [9] .

For lattice gauge theories with gauge group G replaced by Z (i.e. by a finite abelian group) it has been shown that if the probability that a vortex (of "thickness" 1) of length n (D = 3) , resp. surface n (D = 4) appears is bounded above by

$$\exp\{-Kn\} , \tag{17}$$

with K sufficiently large (depending on D and Z) then

$$\mathring{V}(L) \leqq const., \text{ for all } L. \tag{18}$$

For finite, abelian gauge groups and $D \geqq 3$, (17) and consequently (18) are known to be true when $\beta \propto 1/g_0^2$ is large, i.e. at weak coupling, see [11] , whereas (14) (and hence confinement) is valid for small β .

A general formulation of the statistical mechanics of defects [A)] is presently being developed. (See [6,11] for a preliminary account).

In three dimensional theories the behaviour (17) is usually associated with the appearence of super-selection sectors labelled by a topological charge, $Q \in \hat{Z}$, (the character group of Z).

More generally, the existence of non-trivial super-selection sectors labelled by topological charges in a gauge theory is associated, in the Euclidean formulation, with topological line defects labelled by the elements of a homotopy group π_{D-2} and a non-zero statistical weight $\propto \exp\{-\text{const.} \times \text{length}\}$.

In four-dimensional theories these defects correspond to the 't Hooft-Polyakov magnetic monopoles which appear only in theories with matter fields, because π_2 of a Lie group vanishes. The mass of a particle in a sector with non-zero topological charge can be bounded below by a quantity which measures, roughly speaking, a free energy per unit length of an infinitely long line defect and which generalizes what one knows as surface tension in two-dimensional scalar theories with solitons, [18]. If that mass tends to 0, those particles condense and the corresponding topological charge gets confined. (A general analysis of such sectors in terms of "dual algebras" may be found in [19]).

(B) The "string" approach [20,21,22]

The idea of this approach is that at large distance scales (low energies) QCD may approach a kind of "non-interacting asymptote". Of course this asymptote cannot be a local free field theory. However, it is feasible that it resembles the dual resonance model, i.e. a theory of a free, relativistic string.

In order to substantiate such an idea one must first attempt to develop a Euclidean formulation of string theory. Here one meets serious difficulties, [23], presumably related to the existence of tachyons in the standard formulation.[B)] Nevertheless one can write down formal Schwinger-Dyson equations for n-string Euclidean Green's functions [24] and try to compare them with Schwinger-Dyson equations for n-loop Euclidean Green's functions of Yang-Mills theory (scaled to large distances).[C)]

In pure U(N) lattice gauge theory, the Schwinger-Dyson equation for $W(\omega) = S_1(y_0(\omega))$ has the following form :
Let ℓ be a link (nearest neighbor pair) through which an oriented loop, ω, passes. Let p be some plaquette, and ∂p denote the oriented

boundary of p and $(\partial p)^{-1}$ the same four-link loop as ∂p, but with orientation reversed. Suppose ∂p contains ℓ. We then define $\omega \circ \partial p$ and $\omega \circ (\partial p)^{-1}$ by the following diagrams

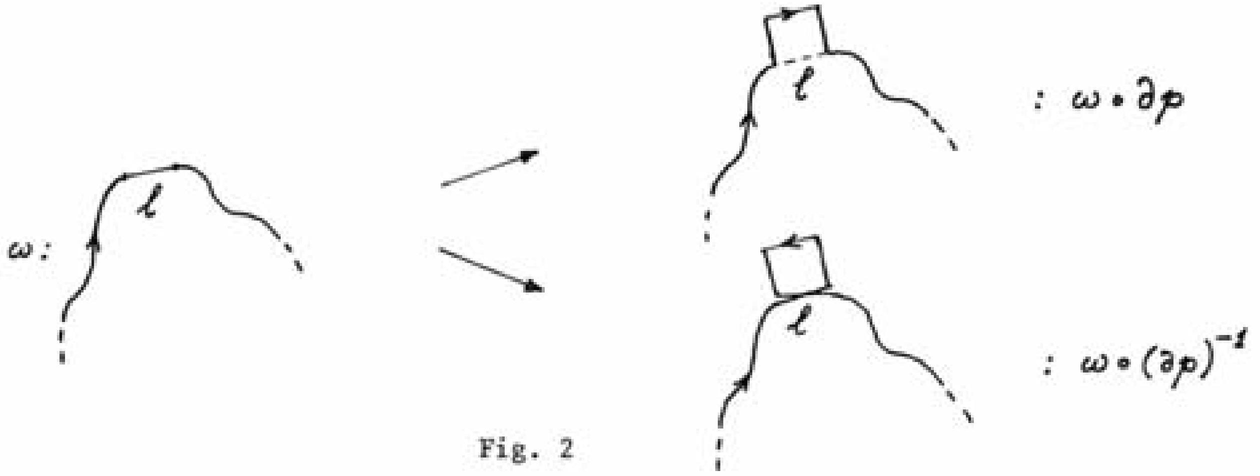

Fig. 2

Then the Schwinger-Dyson equation for $W(\omega)$ is [21] *)

$$W(\omega) = \frac{\beta}{N} \sum_{p \supset \ell} \{ W(\omega \circ \partial p) - W(\omega \circ (\partial p)^{-1}) \} \qquad (19)$$

$$+ \frac{1}{N} \sum_{\omega', \omega''} \delta_\ell(\omega / \omega', \omega'') S_2(y_0(\omega'), y_0(\omega'')).$$

The last term on the r.s. of (19) is a "contact term" :

$\delta_\ell(\omega/\omega', \omega'')$ vanishes if ω traverses ℓ only once, $\delta_\ell(\omega/\omega', \omega'') = 1$ if ω' and ω'' are two (disconnected)loops with the property that if ℓ is joined twice, with <u>different</u> orientation, to ω' and ω'' one obtains the loop ω.

(We note that ω' or ω'' may be empty, in which case S_2 is replaced by S_1 in the last term on the r.s. of (19). The "contact term" is <u>absent</u> if ω is free of self-intersections).

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*) Here we must allow ω to have self-intersection.

Equation (19) is a simple consequence of integration by parts in the functional integral expression for $W(\omega)$. Similar, coupled equations can be derived for $S_n(y_0(\omega_1), \cdots, y_0(\omega_n)),\ n \geq 2,$ [21]. Eguchi [21] has argued (using the non-planar character of $\omega \circ (\partial p)^{-1}$ and the factorization conjecture) that, in the $N \to \infty$ limit, (19) approaches the string equation, (after rescaling β). This is only correct modulo contact terms, unless at the same time an $N \to \infty$ limit of string theory is taken.

We note that, for small β, $W(\omega)$ fulfills a string equation, up to errors of $O(\beta^2)$. This follows easily from $\exp\{\beta \operatorname{Re}\chi(g)\} \approx 1 + \beta \operatorname{Re}\chi(g)$, β small, by taking conditional expectations.

If in the functional measure of the lattice theory [25] $\exp\{\beta \operatorname{Re}\chi(g_{\partial p})\}$ is replaced by $1 + \beta \operatorname{Re}\chi(g_{\partial p})$, for all plaquettes p, with $\beta < \chi(1)^{-1}$, (which is compatible with the existence of a positive-definite transfer matrix) then the class of random surfaces labelling terms in the high temperature expansion can be described explicitly.

These observations were used by the author as an argument in favour of the claim (Polyakov [21]) that Yang-Mills theory and the string theory have the same asymptote, as $\beta \searrow 0$, (i.e. in the strong coupling limit or, hopefully, at large distance scales), resp. $\alpha' \searrow 0$; see [4].

Some time ago (Cargèse '79), we proposed to look for a kind of "spectral representations" for $S_1(y(\omega))$ and $S_2(y_1(\omega_1), y_2(\omega_2))$ that would provide general information about the possible behaviour of Yang-Mills theory at large distance scales. An ansatz for $S_1(y(\omega))$ compatible with the string equation has been proposed by Polyakov [21], but there are no general results, yet. In [4,6] $S_1(y_0(\omega))$, with ω as in Fig. 1 and $G = SU(2)$, was related (for small β) to the transition function of a non-relativistic string, whose behaviour can be studied explicitly.

Subsequently, Lüscher [22] argued that the string equation for $S_1(y(\omega))$ would imply the area law. (He neglects however contact terms in that equation).

A more detailed and ambitious program for understanding the relations between Yang-Mills and strings is being pursued by Gervais and Neveu [20] to whose contribution we refer the reader for more details.

6. Next, we discuss a third approach to the study of confinement based on an "expansion in random surfaces". These surfaces can be thought of as describing the history (or trajectory) of a non-relativistic, open ended string, with end points tied down at 0 and $\ell = (L, 0, \cdots, 0)$, in a time - (x^D-) interval $[0, T]$. I.e. the surfaces are made out of intermediate states of that string.

(C) The "string history" approach, [4].

The starting point is a rewriting of the pure lattice Yang-Mills theory, with a gauge group G , on $\mathbb{Z}^D$ in terms of a product (extending over all values, u , of x^D) of non-linear $G \times G$ σ-models on the lattice $\mathbb{Z}^{D-1}$ in random external gauge fields.

A gauge field configuration g is given by a collection $\{g_{xy}\}$ of group elements $g_{xy} \in G$ attached to links xy (ordered pairs of nearest neighbors) in $\mathbb{Z}^D$ whose a priori distribution is given by the Haar measure, dg_{xy}, on G. Wilson's action [25] is

$$A_D^{YM} = - \sum_p \mathrm{Re}\,\chi(g_{\partial p}), \tag{20}$$

and the Euclidean functional measure, $d\mu$, is given by

$$d\mu(g) = Z^{-1} e^{-\beta_0 A_D^{YM}(g)} \prod_{xy} dg_{xy} \tag{21}$$ *)

where Z is the partition function and $\beta_0 \propto g_0^{-2}$. Let $\langle - \rangle_D^{YM} = \int - \, d\mu(g)$ denote the expectation with respect to $d\mu$.

We now introduce horizontal and vertical gauge fields, namely

$$\left.\begin{aligned} g_{xy} &=: h_{ij}(u), \text{ for } x=(i,u), \; y=(j,u), \\ g_{xy} &=: v_i(u), \text{ for } x=(i,u), \; y=(i,u+1), \end{aligned}\right\} \tag{22}$$

with i,j in $\mathbb{Z}^{D-1}$.

We define an (auxiliary) action

$$A^{\sigma}_{(b,t)} = - \sum_{ij \subset \mathbb{Z}^{D-1}} \mathrm{Re}\,\chi(v_i^{-1} b_{ij} v_j t_{ji}), \tag{23}$$

and a probability measure

$$d\rho^{\sigma}_{(b,t)}(v) = Z^{-1}_{(b,t)} e^{-\beta_0 A^{\sigma}_{(b,t)}(v)} \prod_i dv_i \tag{24}$$

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*) We take it forgranted that the r.s. of (21) is constructed as a limit of measures with space-time (volume)cutoff. Some limit always exists [4].

with $v_i \in G$, dv_i the Haar measure on G , and $(b,t) \in G \times G$.
We set

$$< - >^{\sigma}(b,t) = \int - \, d\rho^{\sigma}_{(b,t)}(v). \tag{25}$$

Clearly, (23) is the action of a $G \times G$ non-linear σ-model in an external gauge field (b,t) on the lattice $\mathbb{Z}^{D-1}$, and (24) is the corresponding Euclidean functional measure.

It follows directly from (20), (22) and (23) that

$$A^{YM}_{D}(g) = \sum_{u=-\infty}^{\infty} \Big\{ A^{YM}_{D-1}(h(u)) + A^{\sigma}_{(h(u),h(u+1))}(v(u)) \Big\} \tag{26}$$

This identity, and (21) and (24) yield

$$d\mu(g) = Z^{-1} \prod_{u=-\infty}^{\infty} \Big\{ \exp\left[-\beta_0 A^{YM}_{D-1}(h(u))\right] \prod_{ij} dh_{ij}(u) \cdot Z_{(h(u),h(u+1))} \, d\rho^{\sigma}_{(h(u),h(u+1))}(v(u)) \Big\} \tag{27}$$

Given an oriented path $\gamma \subset \mathbb{Z}^{D-1}$, let

$$h_\gamma = \prod_{ij \subset \gamma}^{\circlearrowleft} h_{ij}$$

denote the path-ordered product of h_{ij}'s in G along γ , i.e. the parallel transporter associated with γ . Let U be a unitary representation of G of dimension d_U , with character χ_0 . Let $\omega = \omega_{L \times T}$ be the oriented loop depicted in Fig. 1, with bottom part γ_i and top part γ_f .

We define the matrix elements

$$B_{n_0 m_0} = U(h_{\gamma_i}(u=0))_{n_0 m_0} ,$$

and
$$T_{m_T n_T} = U(h^{-1}_{\gamma_f}(u=T))_{m_T n_T}$$

Let $\underset{\sim}{n}, \underset{\sim}{m} = \{(n_u, m_u)\}_{u=0}^{T}$ be a sequence of labels of matrix elements; $n_u, m_u = 1, \ldots, d_U$. Clearly

$$\chi_0(g_{\omega_{L\times T}}) = \sum_{\underset{\sim}{n}, \underset{\sim}{m}} B_{n_0 m_0} T_{m_T n_T} \cdot \prod_{u=0}^{T-1} \{U(v_0(u)^{-1})_{n_{u+1} n_u} U(v_\ell(u))_{m_u m_{u+1}}\}, \tag{28}$$

where $0 = (0, \ldots, 0)$, $\ell = (L, 0, \ldots, 0)$; see Fig. 1.

It follows from (25), (27) and (28) that

$$\begin{aligned} S_1(y_0(\omega_{L\times T})) &= \langle \chi_0(g_{\omega_{L\times T}})\rangle_D^{YM} \\ &= Z^{-1} \sum_{\underset{\sim}{n}, \underset{\sim}{m}} \int \prod_{u=-\infty}^{\infty} \{e^{-\beta A^{YM}_{D-1}(h(u))} Z_{(h(u), h(u+1))} \prod_{ij} dh_{ij}(u)\} \cdot \\ &\quad \cdot B_{n_0 m_0} T_{m_T n_T} \prod_{s=0}^{T-1} \langle U(v_0^{-1})_{n_{s+1} n_s} U(v_\ell)_{m_s m_{s+1}} \rangle^{\sigma}_{(h(s), h(s+1))} \\ &= \sum_{\underset{\sim}{n}, \underset{\sim}{m}} \langle B_{n_0 m_0} T_{m_T n_T} \prod_{s=0}^{T-1} \langle U(v_0^{-1})_{n_{s+1} n_s} U(v_\ell)_{m_s m_{s+1}} \rangle^{\sigma}_{(h(s), h(s+1))} \rangle_D^{YM} \end{aligned} \tag{29}$$

Identity (29) and various general consequences concerning confinement have been found in [4] . In particular, if for some $\alpha > 0$

$$|\langle U(v_0^{-1})_{nn'} U(v_\ell)_{mm'} \rangle^{\sigma}(b,t)| \leq \text{const.}\, e^{-\alpha \cdot L} , \tag{30}$$

uniformly in (b,t) , then

$$|S_1(y_0(\omega_{L\times T}))| \leqq \text{const.}^T e^{-\alpha L\cdot T}, \text{ i.e.}$$
$$\mathring{V}(L) \geqq \alpha\cdot L, \text{ as } L\to\infty. \tag{31}$$

Inequality (30) implies that U , and thus χ_0, are non-trivial on the center Z of G ; [4] . In [4], an expansion of $\langle U(v_0^{-1})_* \; U(v_\ell)_* \rangle^\sigma(b,t)$ in terms of random paths, γ , joining 0 to ℓ has been developed for G = U(N) , O(N) , N = 1,2,3,..., and G = SU(2) .

For G = SU(2) and $\chi = \chi_0$ the isospin 1/2 character, our expansion has the following particularly simple form :

$$\langle U(v_0^{-1})_{nn'} \, U(v_\ell)_{mm'} \rangle^\sigma(b,t)$$

$$= \beta_0^{-1} \sum_{\gamma:0\to\ell} (2(D-1))^{-|\gamma|-1} U(b_{\gamma^{-1}})_{mn'} U(t_\gamma)_{nm'} F_\gamma(b,t) \tag{32}$$

Here γ is an arbitrary, connected path of links in $\mathbb{Z}^{D-1}$ starting at 0 and ending at ℓ , γ^{-1} is the same path, but with reversed orientation (i.e. $\gamma^{-1}: \ell \to 0$) , $|\gamma|$ is the number of links in γ (counted with multiplicity), and

$$F_\gamma(b,t) = Z_{(b,t)}^{-1} \, Z_{(b,t)}(\gamma),$$

where $Z_{(b,t)}(\gamma)$ is a certain path-dependent partition function; see [4].

Let γ_u be a path in $H_u = \{x^D = u\}$ starting at (0,...,0,u) and ending at (L,0,...,0,u) , and let γ_u^+ be the same path as γ_u but pushed up by one step to the hyperplane H_{u+1} . We set

$$\gamma_i \equiv \gamma_{-1}^+ \;, \; \gamma_f \equiv \gamma_T^{-1}.$$

Finally, let $\gamma^{+}_{u+1} \circ \gamma_u^{-1}$ be the loop obtained by composing γ^{+}_{u+1} with γ_u^{-1}.

From (29) and (32) we now obtain the following remarkable, <u>exact identity</u> :

$$\langle \chi_0 (g_{\omega_{L\times T}}) \rangle^{YM}_D = \sum_{\gamma_0,\cdots,\gamma_{T-1}} \prod_{u=0}^{T-1} \beta_0^{-1} (2(D-1))^{-|\gamma_u|-1} .$$

$$\cdot \left\langle \prod_{u=0}^{T-1} \left\{ F_{\gamma_u}(h(u),h(u+1)) \chi_0 (g_{\gamma^{+}_{u-1} \circ \gamma_u^{-1}}) \right\} \chi_0 (g_{\gamma^{+}_{T-1} \circ \gamma_T^{-1}}) \right\rangle^{YM}_D \tag{33}$$

The r.s. can be interpreted as a sum over intermediate states, $\gamma_0, \cdots, \gamma_{T-1}$ (i.e. a history) of a non-relativistic string with initial state γ_i, at $x^D = 0$, and final state γ_f, at $x^D = T$. (This is justified even semi-quantitatively when β_0 is small; [4,6]).

In [4] the following estimate on F_γ has been established :

"In measure"

$$\left. \begin{aligned} & 0 \leqq F_{\gamma_u}(h(u), h(u+1)) \leq \exp[-\alpha |\gamma_u|], \\ & \text{with} \quad \alpha > -\ln[(2D-3)(D-1)(4D-4)^{-1}\beta_0]. \end{aligned} \right\} \tag{34}$$

$$\left. \begin{aligned} & \text{Thus, when} \quad D = 3, \quad \alpha > 0 \quad \text{if} \quad \beta_0 < 4/3, \\ & \text{and when} \quad D = 4, \quad \alpha > 0 \quad \text{if} \quad \beta_0 < 4/5. \end{aligned} \right\} \tag{35}$$

From (34) and straightforward combinatorics we obtain

$$\sum_{\gamma : 0 \to \ell} (2(D-1))^{-|\gamma_u|-1} F_{\gamma_u}(h(u), h(u+1)) \leq \text{const.}\, e^{-\alpha \cdot L}, \quad (36)$$

i.e. the σ-model two-point function satisfies inequality (30); see (32). Thus, if $\alpha > 0$ the trivial estimate

$$|\chi_o(g_{\gamma^+_{u-1}} \circ \gamma_u^{-1})| \leq 2 \quad (37)$$

is sufficient to prove (31) , i.e. confinement. However, from [26] we infer that for $D = 4$, $(b,t) = (1,1)$ and $\beta_o > 1.01\cdot$,

$$\sum_{\gamma : 0 \to \ell} (2(D-1))^{-|\gamma|-1} F_\gamma(b,t) \geq \text{const.} > 0, \quad (38)$$

i.e. the two-point function of the three-dimensional, non-linear σ-model has long range order. In [4] arguments have been discussed which suggest that (38) ramains true for a large class of $(b,t) \neq (1,1)$, when $\beta_o > 1.01$. (In the case G = U(1) , (38) follows from a result of Guth [27] for large β_o). Therefore inequality (37) is too rough to prove confinement at values of β_o for which the three-dimensional σ-model has long range order. (It would give the perimeter law). In this regime another mechanism of quark confinement takes over : The mean values of "the random phase factors" $\chi_o(g_{\gamma^+_{u-1}} \circ \gamma_u^{-1})$, $u = 0, \cdots, T$ may be so small that confinement follows again.

Thus (34), (35) and (38) lead to the conclusion that the crossover between strong coupling (where inequalities (30) and (34) hold) and weak coupling (where confinement must be due to cancellation of random phase factors) for $D = 4$ must take place at $0.8 < \beta_0 \lessapprox 1$ (i.e. $1.6 < \beta \equiv 2\beta_0 \lessapprox 2$), in excellent agreement with numerical results.

Elaborating on [4] we have recently derived a series of estimates based on (33) for the behaviour of the average of the random phase factors : In the average (with respect to $d\mu$) one has, for example ,

$$\prod_{u=0}^{T} \chi_0 \left(g_{\gamma^+_{u-1} \circ \gamma_u^{-1}} \right) \lessapprox \prod_{u=0}^{T} \left(\tanh 2\beta_0 \right)^{\| \gamma^+_{u-1} \circ \gamma_u^{-1} \| / D-1} \tag{39}$$

which is valid for all β_0. (Here $\|\gamma \circ (\gamma')^{-1}\|$ is the length of the symmetric difference of γ and γ').

If $\beta_0 < 4/5$, is sufficiently small, and on the r.s. of (33) one replaces the F_{γ_u}'s by the r.s. of (34) and the $\chi_0 (g_{\gamma^+_{u-1} \circ \gamma_u^{-1}})$'s by the r.s. of (39) one obtains an expression closely related to the path space representation of the transition function $p_T(\gamma_i, \gamma_f)$ for the diffusion of a non-relativistic string , [4,6] .

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Notes :

A) By this we mean the equilibrium statistical mechanics (and diffusion theory) of a gas of interacting defects, labelled by elements of homotopy groups. It looks promising to try to describe such gases by effective, generalized lattice gauge theories; (see e.g. B. Julia and G. Toulouse, J. Physique-Lettres 16, 395, (1979), [6,11].)

B) D. Weingarten considers "Euclidean" lattice string theories, parametrized by matrices, U_{xy} , attached to links $xy \subset \mathbb{Z}^D$. The a priori distribution of U_{xy} is <u>Gaussian</u>, the action consists of a sum over terms coupling four U_{xy}'s , $xy \subset \partial p$. An alternative proposal studied by the author (unpubl.) consists of assigning to each link xy a matrix with <u>anti-commuting</u> (Grassmannian) matrix elements. This eliminates the pathologies found by Weingarten, but it is not clear, yet, whether the resulting models are good approximations to lattice gauge theory.

C) There is an alternative approach to relate Yang-Mills-to string theory, inspired by a suggestion of Nielsen and Olesen : In a four-dimensional gauge theory one could try to relate the vortex sheets bounded by a 't Hooft loop [9] to histories of relativistic strings. This might be promising in a phase of some gauge theory (with matter fields), where Z-monopoles are confined.

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