UNBOUNDED, SYMMETRIC SEMIGROUPS ON A SEPARABLE HILBERT SPACE ARE ESSENTIALLY SELFADJOINT

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Introduction and summary.

The main result of this paper is the following

Theorem I.

Let $\mathcal{H}$ be a separable Hilbert space, and $\mathcal{D}$ a linear subspace dense in $\mathcal{H}$.

Consider a semigroup, $(\overset{\circ}{\Gamma}_t)_{t \geq 0}$, of (possibly unbounded) linear operators, $\overset{\circ}{\Gamma}_t$, on $\mathcal{H}$ with the following properties :

(i) *For each vector $\Phi \in \mathcal{D}$, there exists $\varepsilon(\Phi) > 0$ such that Φ belongs to the domain of $\overset{\circ}{\Gamma}_t$, for all $t \in [0, \varepsilon(\Phi))$, and*

$$\operatorname*{s-lim}_{t \searrow 0} \overset{\circ}{\Gamma}_t \Phi = \Phi .$$

(ii) *If $\Phi \in \mathcal{D}$, and s,t, s+t belong to $[0, \varepsilon(\Phi))$ then $\overset{\circ}{\Gamma}_t$ is defined on $\overset{\circ}{\Gamma}_s \Phi$, and*

$$\overset{\circ}{\Gamma}_t (\overset{\circ}{\Gamma}_s \Phi) = \overset{\circ}{\Gamma}_{t+s} \Phi .$$

(iii) *The operators $(\overset{\circ}{\Gamma}_t)_{t \geq 0}$ are hermitean.*

Under these hypotheses, the operators $\overset{\circ}{\Gamma}_t$ have unique selfadjoint extensions, Γ_t, and $(\Gamma_t)_{t \geq 0}$ is a semigroup of (possibly unbounded) selfadjoint operators on $\mathcal{H}$. □

A precise formulation and the proof of this result are given in Section I. *)

It should be noted right away that "locally densely defined", unitary (semi-) groups, $(\overset{\circ}{U}_t)$, do in general not have unique, globally defined

*) Theorem I has been quoted in [1,2] . It is included, together with our proof, in a forthcoming book by Davies [3] .

unitary extensions; (see Section I.1).

As a corollary of Theorem I we find a very simple proof of Nelson's analytic vector theorem; (see Theorem I.3, Section I.2).

Remarks on the semi-analytic vector theorem and a criterion for essential selfadjointness of the generator of the semigroup $(\Gamma_t)_{t \geqq 0}$ constructed in Theorem I on the domain $\mathcal{D}$ conclude Section I.

In Section II we apply Theorem I to the theory of Laplace transforms and to the proof of the essential selfadjointness of some class of Schrödinger operators, including Stark Hamiltonians.

Some comments on the kind of problems that motivated us to conjecture and prove Theorem I may be helpful : These problems concern the reconstruction of equilibrium (KMS) states and a unitary representation of time translations from imaginary-time Green's functions in quantum equilibrium statistical mechanics. Such a reconstruction was achieved by Ruelle [4] under somewhat restrictive hypotheses. With Theorem I as one basic tool the author found a general version of Ruelle's theorem, [5] . The results of [5] can be viewed as an extension of the Osterwalder-Schrader reconstruction theorem in relativistic quantum field theory [6,7] to equilibrium statistical mechanics. They have been summarized in a somewhat special context in [8] . Recently, a result closely related to Theorem I and its applicability to the reconstruction of equlibrium states from imaginary time Green's functions were rediscovered by Klein and Landau [9,10]. (The proof of Theorem I presented below is very different from their's and may have the advantage of being rather short and entirely selfcontained).

We conclude this introduction by describing a general, mathematical problem met in the reconstruction of equilibrium states and the time evolution from Green's functions and the way Theorem I can be used to solve it.

Let V be some topological vector space, and * an anti-linear conjugation on V. Let $(\alpha_t)_{t \geqq 0}$ be a locally defined semigroup of endomorphisms of V with the property that for each $v \in V$, there exists some $\varepsilon(v) > 0$ such that $\alpha_t(v) \in V$, for all $t \in [0, \varepsilon(v))$.

Let F be a bilinear form on $V \times V$ with the properties :

(a) F is continuous in each argument, and

$$F(v^*, v) \geqq 0$$

It follows that

$$\langle v, w \rangle \equiv F(v^*, w) \tag{1}$$

is a positive semi-definite inner product on V.

It is assumed that V is separable in the topology determined by the semi-norm

$$\| v \| = \sqrt{\langle v, v \rangle} \tag{2}$$

(b) For v and w in V, $F(w, \alpha_t(v))$ is a measurable function of t on $[0, \varepsilon(v)]$ with the property that

$$\lim_{t \searrow 0} F(w, \alpha_t(v)) = F(w, v) \text{ , and}$$

$$F(\alpha_t(v)^*, \alpha_t(v)) < \infty \text{ , for all } t \in [0, \varepsilon(v)).$$

(c) For $0 \leqq u < \varepsilon(W)$, $0 \leqq s < \varepsilon(V)$ and $0 \leqq t < \min(\varepsilon(W)-u, \varepsilon(V)-s)$

$$F(\alpha_u(W)^*, \alpha_t \circ \alpha_s(V)) = F(\alpha_t \circ \alpha_u(W)^*, \alpha_s(V)).$$

The problem is to show that $F(W, \alpha_t(V))$ is the restriction of an analytic function of one complex variable to the real interval $(0, \varepsilon(V))$.

Theorem II.

Let (V, α_t, F) be as above, with F satisfying (a) - (c). Then the function $F(W, \alpha_t(V))$ is the restriction of a function $F_{W,V}(z)$ analytic in z on the strip $\{\zeta : 0 < \mathrm{Re}\,\zeta < \varepsilon(V)\}$ to the real interval $(0, \varepsilon(V))$. □

This is a slight generalization of Theorem II.1, and the proof is an adaptation of the arguments in Section II.1. Here are the main-ideas :

Let $N \subset V$ be the kernel of the inner product $\langle \cdot , \cdot \rangle$, defined in (1). By (a), $\mathcal{H} = \overline{V/N}^{\|\cdot\|}$, with $\|\cdot\|$ the norm defined in (2), is a separable Hilbert space. Let Φ be the obvious injection map of V into $\mathcal{H}$ with kernel N. By construction, $\mathcal{D} \equiv \Phi(V)$ is a dense, linear subspace of $\mathcal{H}$. It follows from (b) and (c) that N is invariant under $(\alpha_t)_{t \geq 0}$. Therefore the equation

$$\mathring{\Gamma}_t\, \Phi(V) = \Phi(\alpha_t(V)), \quad 0 \leq t < \varepsilon(V),$$

determines a locally defined semigroup $(\mathring{\Gamma}_t)_{t \geq 0}$, on $\mathcal{H}$. Using (b) and (c) one easily checks that it satisfies the hypotheses of Theorem I. Thus, it has a unique selfadjoint extension, $(\Gamma_t)_{t \geq 0}$. Theorem II now follows from the spectral theorem for $(\Gamma_t)_{t \geqq 0}$.

An alternate proof of Theorem II can be deduced from some general results of Glaser [11]. For applications see [8, 5, 10], and [12] and refs. given there, (where Theorems I and II are applied to relativistic quantum field theory).

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I am deeply indebted to E. Nelson for discussions relating to Theorem I.1 and for having contributed a crucial idea used in the proof of that result. I also thank E.B. Davies and E. Seiler for discussions and the former for a copy of a chapter in [3] where our main result and proof are quoted.

I. *Unbounded semigroups, analytic and semi-analytic vectors.*

I.1. *An unbounded symmetric semigroup on a separable Hilbert space has a unique selfadjoint extension.*

The mathematical structure studied in this section consists of the following objects :

(1) A separable Hilbert space, $\mathcal{H}$; (vectors in $\mathcal{H}$ are denoted by capital Greek letters, Φ , Ψ , ... The scalar product of Φ with Ψ is denoted by $\langle\Phi, \Psi\rangle$, and $\|\Phi\| = \sqrt{\langle\Phi, \Phi\rangle}$ is the norm of Φ . The domain of a linear operator, A , on $\mathcal{H}$ is denoted by D(A) , its range by R(A)) .

(2) A linear subspace, $\mathcal{D}$, dense in $\mathcal{H}$.

(3) A local semigroup $(\overset{\circ}{\Gamma}_t)_{t \geq 0}$ on $\mathcal{H}$ with the following properties :

(i) (Domain property). For each $\Phi \in \mathcal{D}$, there exists some $\varepsilon(\Phi) > 0$ such that $\Phi \in D(\overset{\circ}{\Gamma}_t)$, for all $t \in [0, \varepsilon(\Phi))$; $\overset{\circ}{\Gamma}_t \Phi$ is weakly measurable in t on $[0, \varepsilon(\Phi))$; and

$$\operatorname*{s-lim}_{t \searrow 0} \overset{\circ}{\Gamma}_t \Phi = \Phi . \tag{I.1}$$

(ii) (Semigroup law). If s, t and $s+t$ all belong to $[0, \varepsilon(\Phi))$ then $\overset{\circ}{\Gamma}_s \Phi \in D(\overset{\circ}{\Gamma}_t)$, and

$$\overset{\circ}{\Gamma}_t (\overset{\circ}{\Gamma}_s \Phi) = \overset{\circ}{\Gamma}_{t+s} \Phi . \tag{I.2}$$

(iii) (Symmetry). For Φ and Ψ in $\mathcal{D}$, $0 \leqq u < \varepsilon(\Phi)$, $0 \leqq s < \varepsilon(\Psi)$ and $0 \leqq t < \min(\varepsilon(\Phi) - u, \varepsilon(\Psi) - s)$

$$\langle \overset{\circ}{\Gamma}_u \Phi, \overset{\circ}{\Gamma}_t (\overset{\circ}{\Gamma}_s \Psi)\rangle = \langle \overset{\circ}{\Gamma}_t (\overset{\circ}{\Gamma}_u \Phi), \overset{\circ}{\Gamma}_s \Psi \rangle \tag{I.3}$$

Theorem I.1.

Let $(\mathcal{H}, \mathcal{D}, (\mathring{\Gamma}_t)_{t \geq 0})$ be as specified in (1)-(3), above. Then $\mathring{\Gamma}_t$ has a unique selfadjoint extension, Γ_t , and $(\Gamma_t)_{t \geqq 0}$ is a densely defined, selfadjoint semigroup on $\mathcal{H}$.

Proof :

1° Bounds on $\|\mathring{\Gamma}_t \Phi\|$:

Let $\Phi \in \mathcal{D}$, and define a function g on $[0, \varepsilon(\Phi))$ by

$$g(t) \equiv \log \|\mathring{\Gamma}_t \Phi\|.$$

We claim that, for s and u in $[0, \varepsilon(\Phi))$

$$g(\tfrac{1}{2} s + \tfrac{1}{2} u) \leqq \tfrac{1}{2} (g(s) + g(u)). \tag{I.4}$$

We assume that $s \leqq u$. Then by (I.2) and (I.3)

$$\begin{aligned} \|\mathring{\Gamma}_{s/2 + u/2} \Phi\|^2 &= \langle \mathring{\Gamma}_{s/2+u/2} \Phi, \mathring{\Gamma}_{s/2+u/2} \Phi \rangle \\ &= \langle \mathring{\Gamma}_{u/2 - s/2} (\mathring{\Gamma}_s \Phi), \mathring{\Gamma}_{s/2+u/2} \Phi \rangle \\ &= \langle \mathring{\Gamma}_s \Phi, \mathring{\Gamma}_{u/2 - s/2} (\mathring{\Gamma}_{s/2+u/2} \Phi) \rangle \\ &= \langle \mathring{\Gamma}_s \Phi, \mathring{\Gamma}_u \Phi \rangle \\ &\leqq \|\mathring{\Gamma}_s \Phi\| \cdot \|\mathring{\Gamma}_u \Phi\|, \end{aligned}$$

from which (I.4) follows by taking logarithms. An immediate consequence of (I.4) is that the maximum of g(t), and hence of $\|\mathring{\Gamma}_t \Phi\|$, restricted to the interval $[0, \varepsilon']$, with $\varepsilon' < \varepsilon(\Phi)$ is taken at 0 or ε' . Thus

$$\| \mathring{\Gamma}_t \Phi \| \leqq K_{\varepsilon'}, \text{with } K_{\varepsilon'} = \max \left(\| \Phi \|, \| \mathring{\Gamma}_{\varepsilon'} \Phi \| \right). \tag{I.5}$$

2° <u>Construction of a densely defined generator</u>, $\mathring{H}$, <u>for</u> $(\mathring{\Gamma}_t)_{t \geq 0}$:

Let $\{ \delta_n(t) \}$ be a family of functions with the properties :

$$\delta_n \geqq 0 , \qquad \delta_n \quad \text{is } C^\infty, \int \delta_n(t)\, dt = 1, \operatorname{supp} \delta_n \subseteq [0, \tfrac{1}{n}]. \tag{I.6}$$

(Note that $\delta_n \xrightarrow[n \to \infty]{} \delta$, the Dirac function).

Given $\Phi \in \mathcal{D}$, let $n_0 = n_0(\Phi)$ be so large that $\frac{1}{n_0} < \varepsilon(\Phi)$, and define

$$\Phi_n = \int \delta_n(t)\, \mathring{\Gamma}_t \Phi \, dt ,$$

where the r.s. makes sense as a weak integral. For, $\mathring{\Gamma}_t \Phi$ is weakly measurable in t (see (3) (i)), and $\| \mathring{\Gamma}_t \Phi \| \leqq K_{1/n} \leqq K_{1/n_0} < \infty$ for $t \in \operatorname{supp} \delta_n$, by (I.5) and (I.6). Thus, by (I.6),

$$\| \Phi_n \| \leqq K_{1/n} . \tag{I.7}$$

Using (3) (i) and (I.6) we see that

$$\operatorname*{s-lim}_{n \to \infty} \Phi_n = \Phi . \tag{I.8}$$

Since $\mathcal{D}$ is dense in $\mathcal{H}$, the domain

$$\mathcal{D}_0 = \bigcup_{\Phi \in \mathcal{D}} \left\{ \Phi_{n_0(\Phi)+k} \right\}_{k=0}^{\infty} \tag{I.9}$$

is dense in $\mathcal{H}$, by (I.8) . We note, moreover, that, for each $\psi \in \mathcal{D}_0$, there exists some $\varepsilon'(\psi)$ such that $\psi \in D(\mathring{\Gamma}_t)$, for all $t \in [0, \varepsilon'(\psi))$.

Next, we claim that for each $\psi \in \mathcal{D}_0$ and $0 \leq t < \varepsilon'(\psi)$,

$$\frac{d}{dt} \mathring{\Gamma}_t \psi = \operatorname*{s-lim}_{h \searrow 0} \frac{1}{h} \left(\mathring{\Gamma}_{t+h} - \mathring{\Gamma}_t \right) \psi \equiv - \mathring{H} \mathring{\Gamma}_t \psi \tag{I.10}$$

exists and determines a densely defined, linear operator, $\mathring{H}$.

Since $\psi \in \mathcal{D}_0$, there exists some $\Phi \in \mathcal{D}$ and a positive integer $n \geqq n_0(\Phi)$ such that $\psi = \Phi_n$. Therefore $\psi \in D(\mathring{\Gamma}_t)$, for $0 \leqq t < \varepsilon(\Phi) - \frac{1}{n} \equiv \varepsilon'(\psi)$. Thus for $0 < h \leqq \frac{1}{2}(\varepsilon'(\psi) - t)$,

$$\begin{aligned} \frac{1}{h}(\mathring{\Gamma}_{t+h} - \mathring{\Gamma}_t)\psi &= \frac{1}{h}\int \delta_n(s)(\mathring{\Gamma}_{s+t+h} - \mathring{\Gamma}_{s+t})\Phi\, ds \\ &= \int \frac{1}{h}(\delta_n(s-h) - \delta_n(s))\,\mathring{\Gamma}_{s+t}\,\Phi\, ds, \end{aligned} \tag{I.11}$$

in the sense of weak integrals. Here we have used (I.2). By (I.5)

$$\left\| \frac{1}{h}(\mathring{\Gamma}_{t+h} - \mathring{\Gamma}_t)\psi + \int \delta_n'(s)\,\mathring{\Gamma}_{s+t}\,\Phi\, ds \right\|$$

$$\leqq K_{\varepsilon''} \int \left| \frac{1}{h}(\delta_n(s-h) - \delta_n(s)) + \delta_n'(s) \right| ds$$

with $\varepsilon'' = \frac{1}{2}(\varepsilon'(\psi) + t)$, uniformly in $0 < h \leqq \frac{1}{2}(\varepsilon'(\psi) - t)$.

As h tends to 0 , the r.s. tends to 0 , for all $n < \infty$; see (I.6). Hence

$$\begin{aligned} \underset{h \searrow 0}{s\text{-}\lim}\ \frac{1}{h}(\mathring{\Gamma}_{t+h} - \mathring{\Gamma}_t)\psi &= -\int \delta_n'(s)\,\mathring{\Gamma}_{s+t}\,\Phi\, ds \\ &= -\mathring{\Gamma}_t \int \delta_n'(s)\,\mathring{\Gamma}_s\,\Phi\, ds \equiv -\mathring{H}\,\mathring{\Gamma}_t\,\psi, \end{aligned}$$

in the sense of weak integrals. This proves (I.10). Let

$$\mathcal{D}_1 \equiv \{ \mathring{\Gamma}_t \psi : 0 \leqq t < \varepsilon'(\psi),\ \psi \in \mathcal{D}_0 \} \supseteq \mathcal{D}_0 .$$

Note that for each $\theta \in \mathcal{D}_1$ there is an $\varepsilon''(\theta) > 0$ such that $\theta \in D(\mathring{\Gamma}_t)$, $0 \leqq t < \varepsilon''(\theta)$. Let now θ and Ξ be arbitrary vectors in $\mathcal{D}_1$, and $0 < h < \min(\varepsilon''(\theta), \varepsilon''(\Xi))$. Then

$$\left\langle \theta, \frac{1}{h}(\mathring{\Gamma}_h - \mathbb{1})\Xi \right\rangle = \left\langle \frac{1}{h}(\mathring{\Gamma}_h - \mathbb{1})\theta, \Xi \right\rangle.$$

follows from (I.3) (Symmetry) and the construction of $\mathcal{D}_1$; (represent θ and Ξ as weak integrals!) Thus $\mathring{H} = -\underset{h \searrow 0}{s\text{-}\lim}\frac{1}{h}(\mathring{\Gamma}_h - \mathbb{1})$ is symmetric on $\mathcal{D}_1$.

3° Construction of a selfadjoint extension of $\mathring{H}$.

We now come to the main step of the proof. In 2° we have constructed the generator $\mathring{H}$ and shown that it is symmetric on $\mathcal{D}_1$. However, we do not know, a priori, whether its deficiency indices are equal, i.e. it is not clear, whether $\mathring{H}$ has a selfadjoint extension. This difficulty is circumvented as follows, [13] : We identify $\mathcal{H}$ with the subspace $\mathcal{H} \oplus \{0\}$ of the direct sum $\overline{\mathcal{H}} \equiv \mathcal{H} \oplus \mathcal{H}$ of two copies of $\mathcal{H}$. Clearly $\overline{\mathcal{D}_1} = \mathcal{D}_1 \oplus \mathcal{D}_1$ is dense in $\overline{\mathcal{H}}$. Consider the operator $\begin{bmatrix} -\mathring{H} & 0 \\ 0 & \mathring{H} \end{bmatrix}$ on $\overline{\mathcal{H}}$. By (I.10) its domain contains $\overline{\mathcal{D}_1}$, and it has clearly equal deficiency indices. Thus it has a selfadjoint extension, $-\overline{H}$, with

$$-\overline{H}\,\Big|_{\overline{\mathcal{D}_1}} = \begin{bmatrix} -\mathring{H} & 0 \\ 0 & \mathring{H} \end{bmatrix}\Big|_{\overline{\mathcal{D}_1}} \tag{I.12}$$

Let E_m denote the spectral projection of $\overline{H}$ associated with the interval $[-m, m]$, $m = 1, 2, 3, \cdots$, and set $\overline{H}_m = E_m \overline{H} = E_m \overline{H} E_m$. Note that if f is a continuous function on $\mathbb{R}$

$$E_m f(\overline{H}) = f(\overline{H}) E_m = f(\overline{H}_m) , \tag{I.13}$$

by the spectral theorem.

Next, choose $\psi \in \mathcal{D}_0$ and define

$$\overline{\psi}_m(t) = E_m \begin{pmatrix} \mathring{\Gamma}_t \psi \\ 0 \end{pmatrix}, \quad 0 \leq t < \varepsilon'(\psi). \tag{I.14}$$

Then by (I.10)

$$\begin{aligned} \frac{d}{dt} \overline{\psi}_m(t) &= E_m \begin{pmatrix} -\mathring{H} \mathring{\Gamma}_t \psi \\ 0 \end{pmatrix} \\ &= E_m \begin{bmatrix} -\mathring{H} & 0 \\ 0 & \mathring{H} \end{bmatrix} \begin{pmatrix} \mathring{\Gamma}_t \psi \\ 0 \end{pmatrix} \\ &= -E_m \overline{H} \begin{pmatrix} \mathring{\Gamma}_t \psi \\ 0 \end{pmatrix}, \quad \text{by (I.12)} \\ &= -\overline{H}_m \overline{\psi}_m(t) \quad , \quad \text{by (I.13)} \end{aligned}$$

Since $\overline{H}_m$ is a bounded operator on $\overline{\mathcal{H}}$, we conclude

$$\overline{\Psi}_m(t) = e^{-t\overline{H}_m}\,\overline{\Psi}_m(0)$$

$$= E_m e^{-t\overline{H}}\,\overline{\Psi}_m(0), \qquad \text{by (I.13)}$$

$$= e^{-t\overline{H}} E_m \overline{\Psi}_m(0)$$

$$= e^{-t\overline{H}}\,\overline{\Psi}_m(0), \qquad \text{by (I.14)}.$$

Because $\{E_m\}$ tends strongly to $\mathbb{1}$, as $m \to \infty$, we have

$$\operatorname*{s-lim}_{m\to\infty} \overline{\Psi}_m(t) = \begin{pmatrix} \overset{\circ}{\Gamma}_t \psi \\ 0 \end{pmatrix} \equiv \overline{\Psi}(t),$$

for all $0 \leqq t < \varepsilon'(\psi)$. Since $e^{-t\overline{H}}$ is selfadjoint, it is closed, so that by taking m to ∞ we obtain

$$\overline{\Psi}(t) = e^{-t\overline{H}}\,\overline{\Psi}(0). \tag{I.15}$$

This equation says, in particular, that for $\psi \in \mathcal{D}_0$, $e^{-t\overline{H}}\begin{pmatrix}\psi\\0\end{pmatrix} \in \mathcal{H} \oplus \{0\}$, for $0 \leqq t < \varepsilon'(\psi)$.

Since $\overline{H}$ is selfadjoint, the spectral theorem permits us to analytically continue $e^{-t\overline{H}}\begin{pmatrix}\psi\\0\end{pmatrix}$ to $\{t : 0 < \mathrm{Re}\,t < \varepsilon'(\psi)\}$. By analyticity in t we thus conclude that

$$e^{is\overline{H}}\begin{pmatrix}\psi\\0\end{pmatrix} \in \mathcal{H} \oplus \{0\}, \qquad \text{for all real } s.$$

Since $\mathcal{D}_0$ is dense in $\mathcal{H}$, we have

$$e^{is\overline{H}}(\mathcal{H} \oplus \{0\}) \subseteq \mathcal{H} \oplus \{0\}, \tag{I.16}$$

i.e. $\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$ commutes with $e^{is\overline{H}}$. We may thus define

$$H \equiv \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\overline{H}. \tag{I.17}$$

By (I.12) and (I.16), H is a selfadjoint extension of $\mathring{H}$, $H|_{\mathcal{D}_1} = \mathring{H}|_{\mathcal{D}_1}$, and by (I.15) and the definition (I.17) of H,

$$\mathring{\Gamma}_t \psi = e^{-tH} \psi, \tag{I.18}$$

for all $\psi \in \mathcal{D}_o$, $0 \leqq t < \varepsilon'(\psi)$.

4° Uniqueness of H.

Let H_1 and H_2 be two selfadjoint extensions of $\mathring{H}$ satisfying (I.18). Choose an arbitrary $\theta \in \mathcal{H}$. Then

$$F(t) \equiv \langle \theta, \mathring{\Gamma}_t \psi \rangle = \langle \theta, e^{-tH_i} \psi \rangle, \quad i = 1, 2,$$

is the restriction of functions $F_i(z) \equiv \langle \theta, e^{-zH_i} \psi \rangle$, holomorphic on $\{z : 0 < \mathrm{Re}\, z < \varepsilon'(\psi)\}$, to the real interval $(0, \varepsilon'(\psi))$. Since on that interval they agree,

$$F_1(z) = F_2(z)$$

for all z with $0 < \mathrm{Re}\, z < \varepsilon'(\psi)$, by the identity principle for holomorphic functions. Thus

$$\langle \theta, e^{isH_1} \psi \rangle = \langle \theta, e^{isH_2} \psi \rangle,$$

for almost all real s, (hence for all real s, by continuity), for all $\theta \in \mathcal{H}$ and all $\psi \in \mathcal{D}_o$. Since (e^{isH_1}) and (e^{isH_2}) are unitary groups, and $\mathcal{D}_o$ is dense, $H_1 = H_2$.

5° Completion of the proof

We define $\Gamma_t = e^{-tH}$, where H is the unique extension of $\mathring{H}$ constructed in 3° and 4°. For $\psi \in \mathcal{D}_o$ we have

$$\Gamma_t \psi = \mathring{\Gamma}_t \psi, \quad 0 \leqq t < \varepsilon'(\psi)$$

see (I.18).

Clearly, each $\Phi \in \mathcal{D}$ can be approximated by a sequence $\{\Phi_n\} \subset \mathcal{D}_0$ such that

$$s\text{-}\lim_{n\to\infty} \Phi_n = \Phi \,, \quad s\text{-}\lim_{n\to\infty} \mathring{\Gamma}_t \Phi_n = \mathring{\Gamma}_t \Phi \,,$$

for $0 \leqq t < \varepsilon(\Phi)$. Since Γ_t is closed,

$$\Gamma_t \Phi = \mathring{\Gamma}_t \Phi \,, \quad \text{for} \quad 0 \leqq t < \varepsilon(\Phi),$$

for all $\Phi \in \mathcal{D}$.

■

We note that it follows from the above proof that the selfadjoint extension H of $\mathring{H}$ is uniquely determined by the vectors

$$\{ \mathring{\Gamma}_t \psi : 0 \leq t < \varepsilon'(\psi), \psi \in \mathcal{D}_0 \} = \mathcal{D}_1 \,.$$

Thus $\mathcal{D}_1$ is a core for H . Since

$$H/_{\mathcal{D}_1} = \mathring{H}/_{\mathcal{D}_1} \quad , \text{ see (I.12) , (I.17) ,}$$

$\mathring{H}$ is essentially selfadjoint on $\mathcal{D}_1$.

Theorem I.1 has the following straightforward.

Corollary I.2.

Let $\mathring{H}$ be a symmetric operator defined on a domain $\mathcal{D}_1$ dense in $\mathcal{H}$. Suppose that for all $\psi \in \mathcal{D}_1$ there exists $\varepsilon'(\psi) > 0$ such that the equation

$$\frac{d}{dt} \psi(t) = -\mathring{H} \psi(t) \tag{I.19}$$

has a solution (not, a priori, unique) satisfying $s\text{-}\lim_{t \searrow 0} \psi(t) = \psi$, $\psi(t) \in \mathcal{D}_1$, for all $0 \leqq t < \varepsilon'(\psi)$. Then $\Gamma_t : \psi \mapsto \psi(t)$ defines a unique selfadjoint semigroup , and $\mathring{H}$ is essentially selfadjoint on $\mathcal{D}_1$.

Proof :

We may repeat the construction in step 3° of the proof of Theorem I.1 to construct a selfadjoint operator $\overline{H}$ (not necessarily unique) on $\mathcal{H} \oplus \mathcal{H}$ extending $\begin{bmatrix} -\mathring{H} & 0 \\ 0 & \mathring{H} \end{bmatrix}$. Let $\overline{\mathcal{D}}_1 = \mathcal{D}_1 \oplus \mathcal{D}_1$. Since $\psi(t) \in \mathcal{D}_1$, for

$0 \leq t < \varepsilon'(\psi)$ and $\bar{H}\big|_{\bar{\mathcal{D}}_1} = \begin{bmatrix} -\mathring{H} & 0 \\ 0 & \mathring{H} \end{bmatrix}\bigg|_{\bar{\mathcal{D}}_1}$, $\bar{H}$ has the property

$$\frac{d}{dt}\begin{pmatrix} \psi(t) \\ 0 \end{pmatrix} = -\bar{H}\begin{pmatrix} \psi(t) \\ 0 \end{pmatrix}$$

which, as in step 3° above, implies

$$\begin{pmatrix} \psi(t) \\ 0 \end{pmatrix} = e^{-t\bar{H}}\begin{pmatrix} \psi \\ 0 \end{pmatrix}.$$

As in step 3° it is seen that $\mathcal{H} \oplus \{0\}$ reduces $\bar{H}$. Thus

$$\psi(t) = e^{-tH}\,\psi \quad \text{, with} \quad H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\bar{H}. \tag{I.20}$$

Since, for a given extension H of $\mathring{H}$, (I.20) holds for every solution $\psi(t)$ of (I.19) with $\psi(t) \xrightarrow{s} \psi$, as $t \searrow 0$, we conclude that the solution of (I.19) must be unique. The remainder of the proof is contained in the one of Theorem I.1.

■

<u>Remark</u> :

It is quite clear that the analogue of Theorem I.1. for <u>unitary</u> groups is <u>false</u>. To see this we give a counterexample.*)

Let $\mathcal{H} = L^2(\mathbb{R} \times [-1,1], dx\,dy)$. Let $\mathcal{D}$ be the dense domain consisting of those square-integrable functions, $f(x,y)$, for which there exists some $\varepsilon(f) > 0$ such that $f(x,y) = 0$ if $|x| \leq \varepsilon(f)$.
For each $f \in \mathcal{D}$ we define

$$(\mathring{U}_t f)(x,y) = f(x-t,y) \ , \quad 0 \leq |t| < \varepsilon(f),$$

i.e. $\mathring{U}_t$ translates f by an amount t in the x-direction. The <u>locally defined</u> group $(\mathring{U}_t)$ has, however, infinitely many different globally defined, unitary extensions from $\mathcal{D}$ to $\mathcal{H}$. One extension is the obvious one : The group of translations in the x-direction, (U_t^1) . Another one is defined as follows : $U_t^{-1} : f \mapsto f_t$ with

*) An example of this type was suggested to me by M. Aizenman.

$$f_t(x,y) = \begin{cases} f(x-t,y), \text{ if } x<0, x-t<0, \text{ or } x>0, x-t>0 \\ f(x-t,-y), \text{ if } x>0, x-t\leqq 0, \text{ or } x<0, x-t\geqq 0 \end{cases} \tag{I.21}$$

More generally, let φ be a measure preserving map from $[-1,1]$ onto $[-1,1]$. Replace in (I.21) "$-y$" by "$\varphi(y)$". This yields a globally defined extension (U_t^φ) of $(\mathring{U}_t)$.

I.2. *Nelson's analytic vector theorem* [14,15a].

In this section we show that the methods used in the proof of Theorem I.1. can be used to give a simple proof of Nelson's analytic vector theorem.

Theorem I.3.

Let $\mathring{H}$ be a symmetric operator on a dense domain $\mathcal{D}\subset\mathcal{H}$, and $\mathring{H}\mathcal{D}\subseteq\mathcal{D}$. Suppose that, for all $\Phi\in\mathcal{D}$, there exists some $\varepsilon(\Phi)>0$ such that

$$\sum_{n=0}^{\infty} \lambda^n \|\mathring{H}^n\Phi\|/n! < \infty ,$$

for all $0\leqq\lambda<\varepsilon(\Phi)$.

Then $\mathring{H}$ is essentially selfadjoint on $\mathcal{D}$.

Proof : Let $\Phi\in\mathcal{D}$. Then

$$\Phi(t) \equiv s\text{-}\lim_{N\to\infty} \sum_{n=0}^{N} \frac{(-t\mathring{H})^n}{n!}\Phi$$

exists for all $|t|<\varepsilon(\Phi)$. Let $\overline{\mathcal{H}}=\mathcal{H}\oplus\mathcal{H}$, $\overline{\mathcal{D}}=\mathcal{D}\oplus\mathcal{D}$, as in the proof of Theorem I.1. Then $\begin{bmatrix}-\mathring{H} & 0\\ 0 & \mathring{H}\end{bmatrix}$ leaves $\overline{\mathcal{D}}$ invariant, and for each $\Psi=\begin{pmatrix}\Psi_1\\ \Psi_2\end{pmatrix}\in\overline{\mathcal{D}}$ there exists $\overline{\varepsilon}(\Psi)=\min(\varepsilon(\Psi_1),\varepsilon(\Psi_2))$ such that

$$\sum_{n=0}^{\infty} \frac{|t|^n}{n!} \left\| \begin{bmatrix}-\mathring{H} & 0\\ 0 & \mathring{H}\end{bmatrix}^n \Psi \right\| < \infty, \quad \text{for } |t|<\overline{\varepsilon}(\Psi).$$

Since $\begin{bmatrix}-\mathring{H} & 0\\ 0 & \mathring{H}\end{bmatrix}$ has equal deficiency indices, it has a selfadjoint extension,

$-\bar{H}$. Moreover, for $\psi \in \bar{\mathcal{D}}$

$$(-\bar{H})^m \psi = \begin{bmatrix} -\mathring{H} & 0 \\ 0 & \mathring{H} \end{bmatrix}^m \psi, \text{ for all } m = 1, 2, 3, \cdots .$$

Thus, for $|t| < \varepsilon(\psi)$,

$$e^{-t\bar{H}} \psi = \sum_{n=0}^{\infty} \frac{(-t\bar{H})^n}{n!} \psi = \sum_{n=0}^{\infty} \frac{t^n}{n!} \begin{bmatrix} -\mathring{H} & 0 \\ 0 & \mathring{H} \end{bmatrix}^n \psi . \tag{I.22}$$

If $\psi = \begin{pmatrix} \Phi \\ 0 \end{pmatrix}$, $\Phi \in \mathcal{D}$, (I.22) yields

$$e^{-t\bar{H}} \begin{pmatrix} \Phi \\ 0 \end{pmatrix} = \begin{pmatrix} \Phi(t) \\ 0 \end{pmatrix} .$$

It is shown as in step 3° that this implies that $\mathcal{H} \oplus \{0\}$ reduces $\bar{H}$. Thus, with $H = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \bar{H}$, $\Phi(t) = e^{-tH} \Phi$, for $|t| < \varepsilon(\Phi)$.

Since $\Phi(t)$ is uniquely determined by $\{ \mathring{H}^m \Phi : m = 0, 1, 2, \cdots \}$, one shows as in step 4° above that H is the unique selfadjoint extension of $\mathring{H}$, i.e. $\mathring{H}$ is essentially selfadjoint on $\mathcal{D}$. ■

We do not consider other, well-known reformulations of Theorem I.2, but see [15] .

1.3. *A remark on the semi analytic vector theorem* [15] .

Although the material in this section has little to do with the main theme of this paper it is clearly connected with the one in the previous section. Moreover the following arguments are very short.

In [16] A. Sokal has rediscovered and extended the following

Theorem BS (Nevanlinna [17])

Let f be a function analytic in the circle

$$C_R = \{ z : \operatorname{Re} z^{-1} > R^{-1} \}$$

and satisfy

$$f(z) = \sum_{n=0}^{N-1} a_k z^k + R_N(z), \text{ with } |R_N(z)| \le A b^N N! \, |z|^N$$

uniformly in N and in $z \in C_R$. Then $B(t) \equiv \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}$ converges for $|t| < b^{-1}$ and has an analytic continuation to $S_b \equiv \{t : dist(t, \mathbb{R}_+) < b^{-1}\}$, and f can be represented by the absolutely convergent integral

$$f(z) = z^{-1} \int_0^{\infty} e^{-tz^{-1}} B(t)\, dt,$$

for any $z \in C_R$. In particular, $f(z)$ is uniquely determined by $\{a_k\}_{k=0}^{\infty}$, for all $z \in C_R$. □

The name "BS" of this theorem is intended to recall that it is fundamental in the theory of Borel summability. Here it is applied to prove

Theorem I.4.

Let $\mathring{H}$ be a positive operator on a domain $\mathcal{D}$ dense in $\mathcal{H}$, with $\mathring{H}\mathcal{D} \subseteq \mathcal{D}$. Suppose that, for each $\Phi \in \mathcal{D}$, there exists some $\varepsilon(\Phi) > 0$ such that

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^2} \|\mathring{H}^n \Phi\| < \infty, \text{ for all } 0 \leq \lambda < \varepsilon(\Phi). \tag{I.23}$$

Then $\mathring{H}$ is essentially selfadjoint on $\mathcal{D}$.

Proof : Since $\mathring{H}$ is positive, it has a positive selfadjoint extension, H ; see [15] . Let $\Phi \in \mathcal{D}$, $\psi \in \mathcal{H}$, and consider the function $f(z) = \langle \psi, e^{-zH} \Phi \rangle$ analytic in z for $\mathrm{Re}\, z > 0$. Its boundary value on the imaginary axis is a continuous function. Furthermore

$$\begin{aligned} \frac{d^m f}{dz^m}(z) &= \langle \psi, (-H)^m e^{-zH} \Phi \rangle \\ &= \langle \psi, e^{-zH} (-H)^m \Phi \rangle \\ &= \langle \psi, e^{-zH} (-\mathring{H})^m \Phi \rangle , \end{aligned}$$

because H is an extension of $\mathring{H}$, and $\mathring{H}^m \mathcal{D} \subseteq \mathcal{D}$. Using the fact that $\|e^{-zH}\| \leq 1$, for $\mathrm{Re}\, z > 0$, and (I.23), we see that the hypotheses of Theorem BS are satisfied. Thus $f(z)$ is uniquely determined by (and computable in terms of)

$$a_k = (1/k!) \langle \psi, (-\mathring{H})^k \Phi \rangle. \tag{I.24}$$

By taking the boundary value we see that $f(it) = \langle \psi, e^{-itH} \Phi \rangle$ is uniquely determined by $\{a_k\}_{k=0}^{\infty}$, given by (I.24). Since $\mathcal{D}$ is dense in $\mathcal{H}$, it follows that e^{-itH} is uniquely determined by

$$\{ \langle \psi, (-\mathring{H})^k \Phi \rangle : \psi \in \mathcal{H}, \Phi \in \mathcal{D}, k = 0, 1, 2, \cdots \}.$$

Therefore H is the unique selfadjoint extenion of H , i.e. H is essentially selfadjoint on $\mathcal{D}$.

By combining Theorem I.1. with Theorem BS one obtains

Theorem I.5.

Let $(\mathcal{H}, \mathcal{D}, (\mathring{\Gamma}_t)_{t \geq 0})$ satisfy hypotheses (1) - (3) of Theorem I.1. Let H be the unique selfadjoint generator of $(\mathring{\Gamma}_t)_{t \geq 0}$ constructed in Theorem I.1. Assume, moreover, that for each $\Phi \in \mathcal{D}$

$$\underset{t \searrow 0}{s\text{-}\lim} \frac{d^n}{dt^n} \mathring{\Gamma}_t \Phi \text{ exists, for all } n = 1, 2, 3, \cdots,$$

and for $\lambda > 0$ small enough, depending on Φ ,

$$\sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^2} \left\| \left[\frac{d^n}{dt^n} \mathring{\Gamma}_t \Phi \right]_{t=0} \right\| < \infty . \tag{I.25}$$

Then $H/_{\mathcal{D}}$ is essentially selfadjoint, i.e. $\mathcal{D}$ is a core for H .

Proof : Let Φ and ψ be in $\mathcal{D}$, $0 \leq t < \min(\varepsilon(\Phi), \varepsilon(\psi))$. Consider the function

$$f_{\psi,\Phi}(t) = \langle \psi, \mathring{\Gamma}_t \Phi \rangle = \langle \psi, \Gamma_t \Phi \rangle .$$

By Theorem I.1. it is the restriction of a function, $f_{\psi,\Phi}(z)$, analytic in z for $0 < \mathrm{Re}\, z < \min(\varepsilon(\Phi), \varepsilon(\psi))$ to real values of z . Assumption (I.25) and $|\langle \psi, \Gamma_z H^m \Phi \rangle| = |\langle \Gamma_{\bar{z}} \psi, H^m \Phi \rangle| \leq \| \Gamma_{\mathrm{Re}\, z} \psi \| \cdot \| H^m \Phi \|$, together with (I.5), show that $f_{\psi,\Phi}(z)$ satisfies the hypotheses of Theorem BS, with $R < \min(\varepsilon(\Phi), \varepsilon(\psi))$. Thus $f_{\psi,\Phi}(z)$ and its boundary value $f_{\psi,\Phi}(is)$, s real, are uniquely determined by $\{a_n\}_{n=0}^{\infty}$ with

$$a_n = \frac{1}{n!} \langle \psi, H^n \Phi \rangle = \frac{1}{n!} \langle \psi, \left[\frac{d^n}{dt^n} \mathring{\Gamma}_t \Phi \right]_{t=0} \rangle .$$

Since $\left(e^{isH}\right)_{s\in\mathbb{R}}$ is a unitary group, and $\mathcal{D}$ is dense in $\mathcal{H}$, $\left(e^{isH}\right)$ is uniquely determined by $f_{\Psi,\Phi}(is)$, Ψ and Φ in $\mathcal{D}$, and thus by

$$\{\langle\Psi, H^n\Phi\rangle : \Psi, \Phi \text{ in } \mathcal{D},\ n = 0,1,2,3,\cdots\} \qquad \blacksquare$$

Remark : The point of our use of Theorem BS is not so much that it gives the above short proofs of Theorems I.4 and I.5, but that it really provides us with a constructive tool to calculate $e^{-tH}\Phi$, for $\Phi\in\mathcal{D}$, namely

$$e^{-tH}\Phi = t^{-1}\int_0^\infty e^{-st^{-1}} B_\Phi(s)\,ds\,, \tag{I.26}$$

where $B_\Phi(s)=\sum_{n=0}^{\infty}\frac{s^n}{(n!)^2}(-H)^n\Phi$, and the integral on the r.s. of (I.26) converges in norm under the hypotheses of Theorem I.4. and weakly on $\mathcal{D}$ under the ones of Theorem I.5.

The techniques of this section can also be used in the context of derivations on operator algebras [5] .

II. Miscellaneous applications of Theorem I.1.

II.1 *An application of Theorem I.1. to the theory of Laplace transforms.* *)

Let $F(t)$ be a distribution on the interval $[\beta_0, \beta]$. Without loss of generality we set $\beta_0 = 0, \beta = 2\pi$. Let $\mathcal{L}_+$ be the space of C^∞ test functions, f, with $\operatorname{supp} f \subset (0,\pi)$. We propose to study that class of functions, F, which have the positivity property

$$\int \overline{f(t)}\, F(t+s)\, f(s)\, dt\, ds \geqq 0, \tag{II.1}$$

for all $f\in\mathcal{L}_+$, and which are such that $\int\overline{f(t)}\,F(t+s)\,f(s)\,dt\,ds$ is continuous in f in a topology of $\mathcal{L}_+$ in which $\mathcal{L}_+$ is separable. Thanks to (II.1),

$$(g,f)\longmapsto\langle g,f\rangle\equiv\int\overline{g(t)}\,F(t+s)\,f(s)\,dt\,ds,$$

g, f in $\mathcal{L}_+$, equips $\mathcal{L}_+$ with a continuous inner product. Let $\mathcal{N}$ be its kernel. Then

$$\mathcal{H} = \overline{\mathcal{L}_+/\mathcal{N}}^{\|\cdot\|}$$

---

*) The following discussion was motivated by a general theorem, due to V. Glaser [11] brought to our attention by H. Epstein who emphasized connections between Glaser's result and Theorem I.1.

is a separable Hilbert space. Let

$$\Phi : f \in \mathcal{L}_+ \longmapsto \Phi(f) \in \mathcal{H}$$

be the natural map from $\mathcal{L}_+$ into $\mathcal{H}$. Clearly $\mathcal{D} \equiv \Phi(\mathcal{L}_+)$ is dense in $\mathcal{H}$, by construction. Given a test function, f, let f_t be defined by

$$f_t(s) = f(s-t).$$

Note that, for each $f \in \mathcal{L}_+$, there exist positive numbers $\varepsilon(f)$ and $\varepsilon'(f)$ such that

$$f_t \in \mathcal{L}_+, \quad for \quad -\varepsilon'(f) < t < \varepsilon(f), \tag{II.2}$$

and, for g and f in $\mathcal{L}_+$ and
$-\min(\varepsilon'(f), \varepsilon'(g)) < t < \min(\varepsilon(f), \varepsilon(g))$,

$$\begin{aligned} \langle g, f_t \rangle &= \int \overline{g(u)}\, F(u+s)\, f(s-t)\, du\, ds \\ &= \int \overline{g(u'-t)}\, F(u'+s')\, f(s')\, du'\, ds' \\ &= \langle g_t, f \rangle. \end{aligned} \tag{II.3}$$

Thus the mapping : $f \longmapsto f_t$, $f \in \mathcal{L}_+$, leaves $\mathcal{N}$ invariant. This permits us to define a semigroup $(\mathring{\Gamma}_t)_{t \geqq 0}$ on $\mathcal{D}$ by

$$\mathring{\Gamma}_t \Phi(f) = \Phi(f_t), \quad for\ all\ f \in \mathcal{L}_+,\ 0 \leqq t < \varepsilon(f).$$

Clearly, $$\mathring{\Gamma}_t(\mathring{\Gamma}_s \Phi(f)) = \Phi(f_{t+s}) = \mathring{\Gamma}_{t+s} \Phi(f). \tag{II.4}$$

if t, s and $t+s$ are in $[0, \varepsilon(f))$. Moreover, since for all $f \in \mathcal{L}_+$, $f_{-t} \in \mathcal{L}_+$, for $0 \leqq t < \varepsilon'(f)$,

$$\Phi(f) = \mathring{\Gamma}_t \Phi(f_{-t}). \tag{II.5}$$

By (II.3)-(II.5), $(\mathring{\Gamma}_t)_{t\geq 0}$ satisfies hypotheses (1)-(3) of Theorem I.1. Therefore it has a unique selfadjoint extension $\Gamma_t = e^{-tH}$. We now have

$$\int g(u)\, F(u+s+t)\, f(s)\, du\, ds$$
$$= \langle e^{-t'H}\Phi(g),\, e^{-t''H}\Phi(f)\rangle,$$

for $t = t'+t''$, $0 \leq t' < \varepsilon(g)$, $0 \leq t'' < \varepsilon(f)$. By the spectral theorem we thus have analyticity in the strip $\{t : 0 < \mathrm{Re}\, t < \varepsilon(f)+\varepsilon(g)\}$. By choosing $g = f \overset{e.g.}{=} \delta_n$, $n = 1,2,3,\cdots$, where the functions δ_n were introduced in step 2° of the proof of Theorem I.1, and appealing to standard limiting arguments [6,7] we recover the following result, originally due to Widder [18], which is a special case of Glaser's theorem.

<u>Theorem II.1</u>

<u>The distribution</u> F <u>is the restriction of a function, denoted</u> $F(z)$, <u>which is analytic in the strip</u> $0 < \mathrm{Re}\, z < 2\pi$, <u>to the interval</u> $z = t$ <u>real</u>, $0 < t < 2\pi$, <u>and</u> $F(t) = \int e^{-t\lambda} d\mu(\lambda)$, <u>for some positive measure</u> $d\mu$ <u>on</u> $\mathbb{R}$.

<u>Remarks</u> :

1) We are informed that Klein and Landau [9] have recently used Theorem II.1 to give an alternate proof of Theorem I.1. (Our study of the relations between Theorems I.1 and II.1 is however independent of theirs).

2) In statistical mechanics, functions F of the type characterized in Theorem I.5 arise in connection with KMS states. They have the following additional property :

$$F(t) = F(2\pi - t), \tag{II.6}$$

(in particular, F is periodic). By Theorem II.1 and (II.6).

$$F(t) = \int e^{-\lambda t} d\mu(\lambda) = F(2\pi - t) = \int e^{-\lambda(2\pi - t)} d\mu(\lambda),$$

i.e.

$$F(t) = \frac{1}{2} \int [e^{-\lambda t} + e^{-\lambda(2\pi - t)}] d\mu(\lambda)$$

If we set

$$d\rho(\lambda) \equiv \lambda (1 - e^{-2\pi\lambda}) d\mu(\lambda)$$

we obtain

$$F(t) = \int d\rho(\lambda) [1 - e^{-2\pi\lambda}]^{-1} (2\lambda)^{-1} [e^{-\lambda t} + e^{-\lambda(2\pi - t)}].$$

This is an integral representation of $F(t)$ in terms of the Green's functions of $\left(-\frac{d^2}{dt^2} + \lambda^2\right)$, $\lambda \neq 0$, with periodic boundary conditions at $t = 0, 2\pi$. (Physically speaking, these Green's functions correspond to harmonic oscillator two-point correlations at inverse temperature $\beta = 2\pi$).

II.2 *Some applications to quantum mechanics.*

(1) First, we sketch an application of Theorem I.1. to non-relativistic quantum mechanics. See also [9] for some general, independent but related results. Theorem I.1. can be used to construct natural self-adjoint extensions of Schrödinger operators which are unbounded below, e.g. the Stark Hamiltonian. Consider, for example

$$\mathcal{H} = L^2(\mathbb{R}^{3N}), \quad \mathcal{D} = \mathcal{D}(\mathbb{R}^{3N}). \tag{II.7}$$

Points, x_N , in $\mathbb{R}^{3N}$ represent the coordinates of N particles, i.e. $x_N = (\vec{x}_1, \cdots, \vec{x}_N)$, $\vec{x}_j = (x_j^1, x_j^2, x_j^3) \in \mathbb{R}^3$. We define

$$H_0 = -\sum_{j=1}^{N} (1/2m_j) \Delta_j \, , \quad \Delta_j = \frac{\partial^2}{(\partial x_j^1)^2} + \frac{\partial^2}{(\partial x_j^2)^2} + \frac{\partial^2}{(\partial x_j^3)^2} \, . \tag{II.8}$$

Let $\Omega_N = \Omega^{\times N}$ be the path space for the Wiener process associated with H_0 ; see e.g. [19] . Let $P^t_{x_N, y_N}(d\omega_N)$ be the Wiener measure determined by H_0, conditioned on paths $\omega(\tau)_N \in \Omega_N$, $0 \leqq \tau \leqq t$, with the properties $\omega(0)_N = x_N$, $\omega(\tau)_N = y_N$.

Let $V(x_N)$ be e.g. a continuous function on $\mathbb{R}^{3N}$, with

$$|V(x_N)| < a \sum_{j=1}^{N} \vec{x}_j^{\,2} + b. \tag{I.35}$$

Then, for $f \in \mathcal{D}(\mathbb{R}^{3N})$ and $t > 0$ sufficiently small, f_t, defined by

$$f_t(x_N) \equiv \int \Big\{ \int_{\Omega_N} P^t_{x_N, y_N}(d\omega_N)\, e^{-\int_0^t V(\omega(\tau)_N)\, d\tau} \Big\} f(y_N)\, dy_N , \tag{I.36}$$

with $dy_N \equiv \prod_{j=1}^{N} d^3 y_j$, is in $\mathcal{H}$.

To show this, one first regularizes V at ∞, ($V \longmapsto V_R$, $|V_R(x_N)| \leqq R$), but we suppresses such regularization and the discussion of $R \nearrow \infty$ which is straightforward. Note that

$$|f_t(x_N)| \leqq \int \Big\{ \int_{\Omega_N} P^t_{x_N, y_N}(d\omega_N)\, e^{-\int_0^t V(\omega(\tau)_N)\, d\tau} \Big\} |f(y_N)|\, dy_N$$

$$\leqq t^{-1} \int_0^t d\tau \iint p^{\tau}(x_N, z_N)\, e^{-tV(z_N)}\, p^{t-\tau}(z_N, y_N)\, |f(y_N)|\, dy_N\, dz_N ,$$

where $p^{\tau}(x_N, z_N)$ is the integral kernel of $e^{-\tau H_0}$, and we have used Jensen's inequality with respect to $t^{-1}\int_0^t d\tau$ —. From this and the well known decay properties of $p^{\tau}(x_N, z_N)$ our assertion follows easily. We leave it as an exercise to the reader to verify that the locally defined, symmetric semi-group $(\mathring{\Gamma}_t)_{t \geqq 0}$ given by $\mathring{\Gamma}_t : f \in \mathcal{D}(\mathbb{R}^{3N}) \longmapsto f_t \in \mathcal{H}$ satisfies all hypotheses of Theorem I.1. (This is conveniently done by introducing the regularization $V \longmapsto V_R$ and then taking $R \nearrow \infty$). Theorems I.2 and I.5 can be used to construct domains of essential self-adjointness for $H = "H_0 + V"$ defined as the unique, selfadjoint generator of $(\mathring{\Gamma}_t)_{t \geqq 0}$.

Among applications we mention that the above observations yield natural selfadjoint extensions of general Stark Hamiltonians [20] or of the Hamiltonian with $V(x_N) = -\sum_{j=1}^{N} a_j |\vec{x}_j|^2$, $a_j > 0$ [21,15b], etc.. Using Ito stochastic integrals, as e.g. in [19], electromagnetic vector potentials can be included. The techniques outlined above can also be applied when

$$H_0 = \sum_{j=1}^{N} (-\alpha_j \Delta_j + \beta_j)^{\gamma_j}, \quad \alpha_j > 0, \beta_j > 0,$$

and $0 < \gamma_j < 1$, $V(x_N) \leq a \sum_{j=1}^{N} |\vec{x}_j| + b$.

This is seen by noticing that the integral kernel of $e^{-\tau H_0}$ in x-space is positive and has exponential decay (a consequence of the Payley-Wiener theorem).

(2) One can ask whether the two - (resp. several) Hilbert space technique used in the proofs of Theorems I.1 (see step 3°) and I.3 can be applied to other problems. In a general context, this method has been discussed in some detail in Achieser-Glasmann [13]. Davies and Simon [22] have recently used those techniques to prove absence of singular continuous spectrum for some class of Schrödinger operators, (the "twisting trick" in [22]). One can think of other applications in the same spirit ; (e.g. the decoupling of local singularities from long range potentials). Another example is a two Hilbert space proof, due to Nelson, of von Neumann's theorem saying that if C is a densely defined, closed operator on $\mathcal{H}$ then C^*C and CC^* are selfadjoint. The method has been frequently applied in operator algebra theory.

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