# Lectures on Yang-Mills Theory

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Contents :

This is a collection of three sets of lecture notes :

1 "Random Geometry and Yang-Mills theory", to appear in the proceedings of the "Colloquium on Random Fields", Esztergom (Hungary), June 24-30, 1979.

2 "Some Results and Comments on Quantized Gauge Fields", to appear in the proceedings of the Cargèse summer school on gauge theory, Cargèse (Corsica), August 27- September 7, 1979.

3 "On the Construction of Quantized Gauge Fields", to appear in the proceedings of the Kaiserslautern summer school, Kaiserslautern (Germany), August 13-24, 1979.

The purpose of these notes is to give an elementary and leisurely introduction to some mathematical terminology and techniques used in the study of gauge field quantization, to review some rigorous results on lattice gauge theory - in particular on the quark confinement problem, phase transitions and connections to string theories - and to briefly describe some results and methods in the construction of abelian gauge theories in the continuum limit.

What looks new in these notes is almost exclusively the result of collaboration with D. Brydges and E. Seiler (see 2 refs. 10 and 25) and with B. Durhuus (see 2 ref. 7). Numerous discussions with H. Epstein, E. Seiler and T. Spencer had a considerable influence.

The construction of the abelian Higgs model in two space-time dimensions, (2) ref. 25, which is briefly sketched in (3), Chap. III, has been described in two excellent reviews by E. Seiler : see (3) refs. 8 and 9.

(1) is a purely descriptive, introductory pahmphlet. (2) and (3) contain some details, but no proofs for which we must refer the reader to the references quoted in the text.

In (1) and (2) the use of random (or stochastic) geometry in the study of gauge field quantization is advertized and exemplified. It is indicated there why random geometry is a natural mathematical concept and useful tool in other branches of theoretical physics, as well, notably statistical mechanics and some class of dynamical systems. Examples in statistical mechanics were sketched in various seminars and - with Yang-Mills theory as the main subject matter - in a lecture at the "28ième Rencontre entre Physiciens Théoriciens et Mathématiciens" in Strasbourg, May 17-19, 1979, of which no notes exist.

For uses of stochastic geometrical methods in statistical mechanics, see e.g.:

- M. Aizenman, "Translation Invariance and Instability of Phase Coexistence in the Two-Dimensional Ising Systems", to appear in Commun. math. Phys.; Y. Higuchi, "On the Absence of Non-Translationally Invariant Gibbs States for the Two-Dimensional Ising System", Preprint 1979.

- M. Aizenman, F. Delyon and B. Souillard, "Lower Bounds on the Cluster Size Distribution ", Preprint 1979.

- (1), ref. 22; §§ 3 and 6 of (2), and other references quoted in the above papers.

We hope to present a more detailed account of the material described in §§ 3 and 6 of (2) and of some further applications to statistical mechanics problems elsewhere.

Another idea which is advertized in (1) is the use of renormalization group (Block spin) transformations with rigorous error estimates in the proof of stability of quantized Yang-Mills theory in two and three space-time dimensions, (starting from a theory on a lattice of arbitrarily small mesh). This method is not elaborated, in these notes, since the author has nothing concrete or definite to say about it. We wish to recommend, however, that the reader consult refs. 29 and 30 quoted in 2.

Finally, we wish to draw attention to the possibility of describing non-linear $\sigma$-models and Yang-Mills theory in terms of fields with values in a Grassmannian. This observation has found important applications in the construction of instanton solutions to the self-dual Yang-Mills equations. See 1 ref. 6 and references given there. We have investigated the use of that formalism for the quantization of gauge fields in "A New Look at Generalized, Non-Linear $\sigma$-Models and Yang-Mills Theory", to appear in the proceedings of the Bielefeld Symposium, December 1978, (L. Streit, ed.). Our conclusions were mostly negative, and that approach is not discussed in the present notes. We feel it still deserves to be kept in mind, however. It may e.g. have further applications on the classical level.

1.

# RANDOM GEOMETRY AND YANG-MILLS THEORY†

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This is a very brief report on a one-hour lecture I presented at the Colloquium on Random Fields of the Jańos Bolyai Mathematical Society.

For its larger part my lecture was rather experimental : I stated various problems and discussed a very few preliminary rigorous results in a branch of mathematics and mathematical physics which one might call random (or stochastic) geometry. Further more, I pointed out why random geometry is important in the quantization of Yang-Mills theory.

The main reason why my lecture was "experimental" is that I do not know any literature about random geometry, yet. (I recently learnt that I should study [1]) This branch of mathematics may already be alive and well, there possibly exist many interesting results, and most problems which I advertized or proposed to study may either have been solved before or may be ill-posed. Finally, the few rigorous results I sketched may be well-known and/or trivial for the experts.

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† Outline of lecture given at the "Colloquium on Random Fields", Esztergom (Hungary), June 1979.

The absence of possibly important references at the end of this report must be excused by my ignorance and by the circumstance that I spoke about recent developments in which I have personally been involved.

I am somewhat more confident that that part of my lecture concerning the study of (lattice) gauge theories and the uses of random geometry in the study of Yang-Mills fields - e.g. expansions in random surfaces, connections to dual strings (processes whose state space is the space of closed loops in a lattice), etc.- was reasonably serious scientific talking. At least, I concluded from the reactions of some parts of the audience that this was the case.

The ideas expressed in my talk have grown out of numerous, recent discussions and collaboration with E. Seiler, joint work with T. Spencer, experimenting with explaining the main concepts of quantized Yang-Mills theory, provided there is such a thing, to different audiences and my reading of "Physics Letters" which, towards the end of 1978 and at the beginning of 1979, published a number of stimulating papers describing connections between dual strings and Yang-Mills theory and some vague probabilistic concepts that might be useful in the study of those theories ; notably, [2,3,4,5] and others.

Topics discussed or mentioned in my lecture included :

1. Introduction to the main mathematical concepts involved in the study of quantized Yang-Mills fields :

- <u>Random (or stochastic) geometry</u>

- <u>Phase-space localization</u> (or micro-local analysis) in functional integrals. Problems with the compatibility of phase-space localization and local gauge invariance.

- <u>Renormalization group arguments</u> ; (approximate "block-spin transformations" with rigorous error estimates).

The latter two techniques are strongly interrelated. The main emphasis of the lecture was placed on random geometry.

2. Combinatorial geometry and combinatorial random geometry with sketches of applications.

3. Probabilistic formulation of Yang-Mills theory in the Euclidean region and Osterwalder-Schrader reconstruction.

4. Connections between $\nu$-dimensional Yang-Mills theory and $(\nu-1)$-dimensional, non-linear $\sigma$-models in an external gauge field.

5. Applications of 4 to the problem of confinement and phase transitions in Yang-Mills theory.

6. Lattice Yang-Mills theory and combinatorial random geometry : Expansion in random surfaces and connections with dual string models.

Short verbal summaries of parts 1-6 and remarks now follow.

Part 1:

Recently, numerous mathematicians (Atiyah,Drinfeld,Hitchin, Manin, Schwarz,Singer and others) have initiated a serious study of classical, Euclidean (time purely imaginary) Yang-Mills theory;and they have had much success:Among other results they have found a linear algebra construction of all solutions to the self-dual Yang-Mills equations (a system of first order elliptic equations). Since their work received much publicity,detailled references are unnecessary.See however [6] and references given there. In their work the mathematicians have used and advertized algebraic topology,differential and algebraic geometry,inverse scattering methods,etc., all very highbrow for a mathematical physicist.

Much of the motivation behind this work comes from semi-classical quantization (formal steepest descent in yet more formal functional integrals). This approach to the problem of quantizing Yang-Mills theory can hardly be considered very satisfactory, in spite of its great heuristic value and its many partial successes ; see e.g. [7] and refs. given there.

It is a rather wide spread opinion that one only understands those quantum theories which are quantizations of some underlying classical theories. (For example, many theoreticians have studied the "quantization of solitary waves" in two space-time dimensional, non-linear field theories. It is, however, a fact that the soliton sectors of the quantized versions of those theories could be constructed without knowledge of the solitary wave solutions of the classical field equations. See e.g. [8]).

There are many reasons - and beautiful mathematical theorems - to expect that a lot of detailed and explicit knowledge of some classical theory is very useful to quantize the theory and derive properties of the quantized theory, [9,10].

There are however quantum theories without an underlying classical theory. A prominent example is the theory of non-relativistic matter at finite density. In the realm of relativistic quantum field theory Euclidean field theory, as developped by Schwinger, Symanzik, Nelson and others (see e.g. [11]), is a direct approach towards constructing relativistic quantum theories. Much of Euclidean field theory is a branch of probability theory, in particular the theory of random fields and of functional integrals. In quantum field models not involving nonabelian gauge fields the Euclidean field theory approach makes use of only trivial information about the solutions to the classical, Euclidean field equations (=critical points of the Euclidean action). Yet, it serves to prove existence of relativistic quantum fields and supplies a lot of detailed information about their properties. Thus it is a natural and useful attempt to try to apply Euclidean field theory and Osterwalder-Schrader reconstruction also to the problem of constructing and analyzing abelian and non-abelian Yang-Mills (gauge) theories. A review of some recent results on the construction of super-renormalizable, abelian gauge theories is contained in E. Seiler's contribution to these proceedings.

It is one of Wilson's achievements to have proposed a direct and non-perturbative approach to constructing quantized gauge fields : lattice gauge theories [12]. Lattice gauge theories have the advantage of existing, preserving the whole structure of a relativistic quantum field theory, except Lorentz invariance, and of trivializing the problems associated with gauge groups of the second kind. Moreover they provide an ideal laboratory for testing the properties of gauge theories at long distances. It is thus natural that they have been studied intensely over the past five years. Mathematically speaking, the study of pure (lattice) gauge theories is the study of a particular class of random fields over a space of closed loops in the Euclidean space-time (lattice), namely traces of "normal (or Ito) ordered" holonomy operators on a (random) principal bundle with random connection. (A connection on a fibre bundle is what the physicists call a gauge field). A simple theorem says that a principal bundle over a connected base space, $\Gamma$, and a connection on it are uniquely determined (i.e. up to gauge equivalence) by the traces of all holonomy operators on the group of all closed loops containing an arbitrary, but fixed point of $\Gamma$ (i.e. by the unitary characters of the holonomy group at some point of $\Gamma$). This is presumably well known. (For a proof see e.g. [13].) Thus the study of quantized gauge fields at imaginary time is the study of random fields over a space of geometric objects, the closed loops in $\Gamma$, more precisely the study of random connections on (random) principal bundles over $\Gamma$. (These statements are only accurate when $\Gamma$ is a lattice. When $\Gamma$ is a continuum the situation is somewhat more complicated,but the above remarks remain a first order approximation to the truth).Therefore quantized gauge fields represent a particular example in the subject of random geometry.

The study of random fields is the study of functional integrals. Making sense of formal functional integrals which determine a Euclidean field theory,e.g. a Yang-Mills theory,is called non-perturbative renormalization theory. The most impressive contributions to that theory (in the framework of super-renormalizable quantum field models not involving gauge fields) are due to Glimm and Jaffe;see e.g. [14,11]. One key to their success was that they used phase-space localization to construct functional integrals,in the form of localizing random fields on classical phase space. They then could estimate partial functional integrals over components of the random field properly localized on phase space:This can be viewed as a "renormalization group transformation". One of the high lights of their approach was the non-perturbative renormalization of the $\lambda\varphi^4$ quantum field model in three space-time dimensions by means of an inductive construction, [14].

An important elaboration of these ideas which makes their intimate relation to microlocal analysis and the renormalization group much more transparent is due to Gallavotti et al. [15].

Other forms of phase-space localization in functional integrals consist of partial Fourier-Laplace transforms of measures on distribution spaces and, not unrelated to that, introducing random fields canonically conjugate to a given random field. A recent investigation of these techniques in the framework of Euclidean field theory is [16]. In the case of lattice theories, Fourier-Laplace transformation of functional measures (the distributions of lattice random fields) is called "duality transformation". In this context, phase-space localization is related to partial duality transformations. They have recently been used in a rather important way in an analysis of Coulomb lattice systems, [17].(Partial) duality transformations play a prominent rôle in the study of lattice gauge fields. This is one way of using phase-space localization in the study of quantized gauge fields. Unfortunately, phase-space localization in the sense of localizing (functions of) a gauge field in classical phase space (the cotangent bundle over Euclidean space-time Γ ) which has proven so powerful a tool is - in its conventional form - incompatible with gauge invariance. This is one major reason why the construction of quantized gauge fields in the continuum limit is so difficult. See E. Seiler's contribution and [18] for further discussion of these matters.

In passing I should like to emphasize that some sort of phase-space localization has been the key to numerous, other recent successes in mathematical physics among which I mention the work of V. Enss on quantum mechanical scattering theory [19] .

The way renormalization group ideas are used in the work of Glimm and Jaffe and of Gallavotti and coworkers requires the possibility of localizing the Euclidean random field on classical phase space. Since this appears impossible in the case of gauge fields, one must find other ways of applying renormalization group ideas which do not require more than the possibility of doing partial duality transformations. Presently there are no convincing proposals to that effect, except a general feeling that approximate "block-spin transformations" (see the contributions of Griffiths and Israel to these proceedings) applied in conjunction with partial duality transformations to lattice gauge theories on lattices of arbitrarily small mesh ought to be an important element.

These remarks serve to motivate my conviction that random geometry, phase-space localization in functional integrals and a rigorous version of the renormalization group will play a crucial rôle in the construction and analysis of quantum theories of Yang-Mills fields.

Part 2 :

Combinatorial geometry is the study of geometric objects (paths and loops, surfaces, hypersurfaces, clusters) consisting of the sites, links, plaquettes, elementary hypercubes of some $\nu$-dimensional lattice, $\Gamma$ , and of their topological and geometric properties. Moreover, it is the study of fibre - and principal bundles with base space $= \Gamma$ , or = some space $C_n(\Gamma)$ of n-dimensional geometric objects in $\Gamma; (n \leq \nu \equiv \dim(\Gamma))$.

Combinatorial random (or stochastic) geometry is the study of stochastic processes whose state space is a space, $C_n(\Gamma)$ , of geometric objects in $\Gamma$ , of random fields over $C_n(\Gamma)$ , of probability measures over $C_n(\Gamma)$ (or over $\overset{\infty}{\underset{m=0}{\oplus}} C_n(\Gamma)^{\times m}$), e.g. squares of quantum mechanical wave functions over $C_n(\Gamma)$ ; $n = 2,3,\ldots,\nu$. Moreover it is the study of random connections (or holonomy operators) over (random) bundles with base space $= \Gamma \equiv C_0(\Gamma)$ , or $C_1(\Gamma)$, or ... $C_\nu(\Gamma)$ , etc... [More ambitiously, one can envisage to convert the lattice (base space) $\Gamma$ and its intrinsic topological and geometric properties into random variables, too].

I have already explained why and how combinatorial random geometry is naturally used in the study of lattice gauge fields ; but see [13,20] for an extensive discussion and applications. Apart from lattice gauge theories, combinatorial random geometry is used in

-equilibrium statistical mechanics : see the contributions of Aizenmann and Eberlein to these proceedings, the work of Minlos, Pirogov and Sinai on phase transitions ; [21] and refs. given there. A further amusing example is the "balanced model" of Ising spins discussed in [22] , etc...

-(differentiable) dynamical systems [23],

-diffusion of clusters of "A-particles" in a medium of "B-particles" : Geometrical properties (size, shape, number of edges and vertices,...) of typical A-clusters ; diffusion of extended defects in crystals, etc.

-quantum mechanics of large, extended molecules (polymers), etc.

-dual resonance models (strings),"bag models", etc.

This is a rather modest selection of fields in theoretical and mathematical physics to which concepts and techniques of combinatorial random geometry can probably be applied successfully.

In [24,20] I have tried to initiate a reasonably systematic study of combinatorial random geometry as it arises naturally in the study of classical lattice systems, lattice Yang-Mills fields, lattice string - or bag models, etc. So far the results are rather modest, but I believe that these ideas have a future.

Part 3 :

Quantization of gauge fields (i.e. of connections, resp. holonomy operators on bundles with base space = physical space-time) which is presumably a physical necessity is, more mathematically speaking, an attempt at reconciling geometry, with probability theory and quantum mechanics. "Random continuum geometry" is the name of a mathematical science that is really needed when one tries to construct quantized gauge fields (except in the case of gauge fields with an abelian gauge group which superficially, or in $\nu < 4$ dimensional space-time,is easier).

Unfortunately, continuum random geometry does - it seems - not exist as a well-defined mathematical science, yet, in contrast to combinatorial random geometry. One of the reasons why random geometry of geometrical objects in manifolds must be very difficult is that, in the continuum, the description of geometrical objects like hypersurfaces (or geodesics, minimal surfaces...) in a manifold requires the use of parameters and local coordinates. (In contrast, on a lattice they are given by countable sets of sites, links, plaquettes, etc., and there is no need for parametrization and local coordinates).

In random geometry one may wish, for example, to convert n-dimensional closed hypersurfaces of a $\nu > n$ dimensional manifold into parametrization - independent and coordinate transformation covariant random currents (or operator-valued currents). Parametrization independence of physical observables and states will presumably require detailed knowledge of skew-adjoint representations of infinite dimensional Lie algebras and their integrability to unitary representations of infinite dimensional Lie groups, e.g. the group of gauge transformations in a gauge theory (bundle automorphisms) or the diffeomorphism group of the circle in dual string theory. (The representation theory of (a central extension of) the Lie algebra of this group, called Virasoro algebra, was first studied by theoretical physicists, Virasoro and others, interested in quantized fields over a space of loops in space-time = dual strings).

These algebraic and group theoretical problems are very hard ; see [25]. Moreover, they are just preliminaries for the development of the subject of random geometry proper. They are avoided completely when one studies combinatorial random geometry. This is why lattice gauge theories are such an attractive starting point for the study of quantized Yang-Mills theories. A reasonable program for the construction of a quantized gauge theory in the continuum limit, at purely imaginary time, might therefore consist of first constructing the expectation values of arbitrary products of traces of arbitrary "normal-ordered" random holonomy operators (Wilson loops) for a lattice gauge theory and then try to prove the existence of the limit of those expectation values as the lattice spacing tends to $0$ , (using e.g. phase-space localization and renormalization group transformations). Apart from proving the existence of the limit one major problem will be to show that the limiting expectations, denoted

$S_n(C_1,\ldots,C_n)$ , where $C_1,\ldots,C_n$ are closed, oriented loops in $\mathbb{E}^\nu$ , with $\mathrm{dist}(C_i,C_j) > 0$ , for $i \neq j$ , are continuous under small, smooth deformations of the loops and Euclidean invariant, for all $n = 1,2,3,..$; ($S_o = 1$).

Once this is shown, one can reconstruct from the "n-loop Euclidean Green's functions" $\{S_n(C_1,\ldots,C_n)\}_{n=0}^{\infty}$ a unique, Poincaré-covariant quantum gauge theory. This is called Osterwalder-Schrader reconstruction [26]. In the present context, O-S reconstruction involves proving some results concerning the analytic continuation of representations of Lie groups, resp. Lie semi-groups. A useful tool is a theorem that guarantees the existence of unique selfadjoint extensions of a large class of unbounded, symmetric one-parameter semi-groups on separable Hilbert spaces [27].[The main open problem concerning the reconstruction of quantum gauge theories from $\{S_n(C_1,\ldots,C_n)\}_{n=0}^{\infty}$ is a sharp formulation and proof of locality. All other problems can be solved].

One can argue that $S_1(C)$ tells one something about "confinement of static quarks" [12] and $S_1(C)$, $S_2(C,C')$ about the low-lying mass - and spin spectrum of Yang-Mills theory [20] ,i.e. a certain amount of physical information can be extracted directly from the Euclidean (imaginary time) Green's functions. See [12,28,24,20] for more details.

Part 4 :

This part was a brief report on the recent paper [24]. The main results are

A) a representation of $\nu$-dimensional, pure lattice gauge theories as integrals of products of $(\nu-1)$ - dimensional, non-linear $\sigma$-models in external gauge fields, with applications ;

B) an expansion of the n-loop Euclidean Green's functions, $S_n(C_1,\ldots,C_n)$, of lattice gauge theories in terms of random surfaces bounded by the loops, when the gauge group is $U(n)$ or $O(n)$ , n= 1,2,3,..., or $SU(2)$.

When applied to $S_1(C)$ this expansion exhibits two complementary mechanisms for confinement of static quarks and suggest an intimate connection between Yang-Mills theory and the theory of dual strings, including an educated guess about the low lying mass - and spin spectrum of Yang-Mills theory : ("approximate Regge trajectories"). See also [20].

Part 5 :

This part contained further applications of result A) of part 4. It was a brief report on some of the results of refs. [29,30,31,24]. In these references the following lattice models are studied :

(1) Classical, two-component, neutral lattice Coulomb gases and abelian lattice $\sigma$-models (Ising-, Potts- and classical rotator models).

(2) Abelian lattice gauge (Higgs) theories, in particular Landau-Ginsburg type theories.

(3) Non-linear $\sigma$-models on the lattice (e.g. a classical, ferromagnetic spin system with 4-component spins of length 1).

(4) Pure, non-abelian lattice gauge theories.

The main findings contained in the above references have the following flavour :

(i) A rigorous connection between (2) in $\nu$ dimensions and (1) in $(\nu-1)$ dimensions, and between (4) in $\nu$ dimensions and (3) in $(\nu-1)$ dimensions. (For example, the one-loop Green's function, $S_1(C)$, of a $\nu$-dimensional gauge theory can generally be bounded above by a product of two-point correlation functions of a $\sigma$-model in $(\nu-1)$ dimensions).

(ii) As one consequence of (i) one obtains a technique whereby the construction of a pure lattice gauge theory with gauge group $\mathbb{Z}_2$ on the three - and four - dimensional lattice is reduced to the one of a two- dimensional Ising model with random couplings in one direction, [24].

(iii) Rigorous results and conjectures about the phase diagram - phase transitions and critical properties - of (1) in two dimensions and (3) in three dimensions.

(iv) Consequences of (i) and (iii) for the theory of quark and monopole confinement, the Higgs mechanism, etc. in (2) and (4). The following theorem is a typical example of results that follow from combining (i) with (iii) : If the two-dimensional Coulomb gas undergoes a transition from a high temperature phase with Debye screening [32] to a low temperature, dipolar phase without Debye screening (for partial results see [17]) then the three dimensional, abelian lattice Higgs (Landau-Ginsburg) model undergoes a transition from a superconducting phase without confinement of fractional charges and heavy vortices, at small values of the electric charge, to a QED phase in which fractional charges are confined by a logarithmic potential and the photon is massless, at large electric charge. This is shown in [30].

(v) A comparison theorem relating a lattice Higgs theory with gauge group $G$ to a lattice Yang-Mills theory with gauge group = center of $G$ , [29,31]. The theorem implies that if the latter confines static quarks then so does the former. As one corollary one concludes permanent confinement of static quarks (with non-zero "electric charge") in all two-dimensional lattice gauge theories and in three-dimensional theories with gauge group $\mathcal{U}(n)$, n=1,2,3,...

Part 6 :

This part was an elaboration and application of result B) of Part 4. In particular, expansions of two-point correlation functions of non-linear $\sigma$-models in $(\nu-1)$ dimensions with fields taking values in a group $G$, $G = \mathcal{U}(n)$, $O(n)$, n=1,2,3,..., $SU(2)$, in terms of random walks were used to generate an expansion for the one-loop Green's function, $S_1(C)$, of a pure lattice gauge theory in $\nu$ dimensions with gauge group $G$ . These expansions are used to prove exponential clustering of correlations in those $\sigma$-models, resp. confinement of static quarks in Yang-Mills theory. Two basic mechanisms for confinement emerge from that expansion, and one of them might potentially yield confinement in continuum theories. These results can be found in [24]. Some elaborations of them and connections with the theory of dual strings are discussed in [20].

Final remarks.

This is my first set of notes to a lecture that does not contain a single formula or estimate or state (and prove) a theorem. My only purpose is to verbally discuss, explain and advertize some mathematical, in fact probabilistic concepts which I believe are going to play a somewhat crucial rôle in various branches of mathematical physics, in particular in quantized Yang-Mills theory which one hopes may be the theory of the fundamental interactions (except gravitation) of particle physics. These concepts may be labelled by the words : Random (stochastic) geometry, phase-space localization in functional integrals, renormalization group.

The papers quoted in the bibliography (not these lecture notes) permit the reader to develop his own ideas about what these concepts mean and why they might be useful.

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# SOME RESULTS AND COMMENTS ON QUANTIZED GAUGE FIELDS

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## ABSTRACT

A few basic facts concerning the geometry of classical gauge fields are summarized; in particular, it is asserted that a principal bundle with connection can be characterized uniquely by its "Wilson loops". The quantization of gauge fields is then shown to consist of converting the Wilson loops into "random fields" on a manifold of oriented loops, a problem in "random geometry". Other examples in random geometry are briefly sketched. A general theorem permitting to reconstruct quantized Wilson loops from a sequence of Schwinger functionals is stated, the quark-antiquark potential is introduced, and "disorder fields" are discussed in general terms. The status of the construction of quantized gauge fields in the continuum limit is indicated, and some random-geometrical arguments are applied to lattice gauge theories and used to derive estimates on the expectation of the Wilson loop, resp. the disorder field.

## 1. Introduction

These lecture notes are organized as follows:

§2. Some elementary facts about the geometry of gauge fields.

§3. Random (or stochastic) geometry.

§4. Schwinger functionals and relativistic quantum fields.

§5. Existence of quantized gauge fields.

§6. Random geometrical methods in lattice gauge theories.

§7. Conclusions and acknowledgments.

The purpose of these notes is to introduce the reader to some basic, conceptual aspects of the problem of constructing quantized gauge fields and to summarize some rigorous results concerning the general (axiomatic) theory of quantized gauge fields, the existence of models in the continuum limit and some physical properties of lattice gauge theories, in particular the confinement of static quarks, the proof of which is based on random geometrical methods. This is the content of §6 which, due to page limitations, has come out to be too short. (That material will probably be treated in more detail elsewhere. Much of it is contained in the references quoted in the text). Unfortunately, we were forced to omit all proofs and to even state some of the main results in a somewhat cavalier way, but the necessary precision can be achieved by consulting the references given in the text. Our choice of references does not represent a value judgment. It reflects the author's taste and ignorance and a certain emphasis on developments in which he has been involved.

The *main problem* of quantizing gauge fields and thereby constructing a mathematically consistent and physically realistic model of the fundamental interactions is among the central problems of theoretical physics. We are still far from having complete and satisfactory solutions to that problem, and the technical barrier separating super-renormalizable from renormalizable theories is still not overcome, at all, in spite of the advent of asymptotically free theories, the Yang-Mills theories. (Some progress may be in sight, though). In view of the main problem these notes and many of the references quoted may seem naive; (they certainly are). They may however help to see some conceptual and mathematical problems through a perspective which we hope is not completely useless.

## §2. Some Elementary Facts about the Geometry of Gauge Fields

In this section we briefly review some mathematical building blocks of the theory of classical matter and *gauge fields*, emphasizing some geometrical aspects. For mathematical details see e.g. |1| and the notes by Singer and Mitter. In |2| we have attempted to give a rather detailed, pedagogical "introduction for physicists" to this subject which might also be useful.

Let M be a manifold, the *space-(imaginary) time manifold*.

For particle physics one would choose $M = E^4$, but for well known, technical reasons one also chooses $M = E^\nu$ or $S^\nu$ , $\nu = 2,3,(4)$.

Let G be some compact Lie group, the *gauge group*. Let V be a topological space (typically a vector space carrying a representation of G or a homogeneous space) on which G acts as a homeomorphism group. Physically, V is the space of internal degrees of freedom of some matter field.

Let $\mathcal{F} = (B,M,V,G,\pi)$ be a fibre bundle with bundle space B, base space M, fibre V, group G and projection $\pi$. If V = G, the gauge group itself, we shall denote $\mathcal{F}$ by $\mathcal{P} = (P,M,G,\pi)$, and such a bundle is usually called a principal bundle. For definitions and results concerning fibre-and principal bundles see e.g. |1|. We propose to view classical matter fields as sections of a fibre bundle $\mathcal{F}$ and classical gauge fields as connections on a fibre-or principal bundle.

Let $\{\Omega_i\}_{i\varepsilon I}$ be a covering of M by open, simply connected coordinate neighborhoods such that the bundle space B restricted to $\Omega_i$ is homeomorphic to $\Omega_i \times V$, for all $i\varepsilon I$.

Let $\xi_{\Omega_i}, \xi'_{\Omega_i} : \Omega_i \times V \to \pi^{-1}(\Omega_i)$ be two coordinate functions.

Let $\xi^{(')}_{\Omega_i,x} := \xi^{(')}_{\Omega_i}(x,\cdot)$. By definition of a fibre bundle,

$\xi^{-1}_{\Omega_i,x}\xi'_{\Omega_i,x} \equiv h(x)$ is an element of the gauge group G, depending continuously on $x\varepsilon\Omega_i$, for all $i\varepsilon I$. The G-valued function h thus determines a change of coordinates and is called in physics a gauge transformation. Let $x\varepsilon\Omega_i\cap\Omega_j$. The gauge transformations $h_{ij}(x) := \xi^{-1}_{\Omega_i,x}\xi_{\Omega_j,x}$ are called transition functions. They determine the bundle uniquely and also serve to associate to each fibre bundle a principal bundle: the one with the $h_{ij}$'s as its transition functions.

A connection, A, on a fibre bundle is a family of 1-forms $\{A^{(i)}\}_{i\varepsilon I}$ with values in the Lie algebra $\mathcal{G}$ of G such that $A^{(i)}$ is defined on $\Omega_i$, $i\varepsilon I$, and for $x\varepsilon\Omega_i\cap\Omega_j \neq \emptyset$

$$A^{(i)}(x) = h^{-1}_{ji}(x)A^{(j)}(x)h_{ji}(x)-h^{-1}_{ji}(x)(dh_{ji})(x) \tag{2.1}$$

Moreover, if h is a gauge transformation defined on $\Omega_i$, $A^{(i)}$ transforms according to

$$A^{(i)} \to A^{(i)h} = h^{-1}A^{(i)}h-hdh \tag{2.2}$$

In physics, A is called gauge field potential. It serves to define the notion of parallel transport. To explain this we choose some $\Omega = \Omega_i$, $i \in I$. Let $\gamma_{yx} \subset \Omega$ be an oriented curve connecting $x \in \Omega$ to some point $y \in \Omega$. We propose to construct a homeomorphism $g^{(i)}_{\gamma_{yx}} \in G$ from V into V in terms of A which describes the parallel transport of some $\Phi(x) \in V$ from x to y along $\gamma_{yx}$. If $y = x+dx$ is infinitely proximate to x we set

$$g^{(i)}_{x+dx,x} \ \Phi(x) := (1_V + A^{(i)}(x))\,\Phi(x), \tag{2.3}$$

with $$A^{(i)}(x) = \sum_{j=1}^{\nu} A^{(i)}_j(x)\,dx^j,$$

(in local coordinates; $\nu = \dim M$).

If $\gamma_{yx}$ is bounded, oriented, continuous and piecewise smooth, (2.3) can be integrated along $\gamma_{yx}$, yielding

$$g^{(i)}_{\gamma_{yx}} = P\{\exp \int_{\gamma_{yx}} A^{(i)}_j(z)\,dz^j\}, \tag{2.4}$$

where the r.s. is an infinite product obtained as a limit of finite products of factors $(1+A^{(i)}_j(x_{k,m})\Delta^j_{k,m})$, with $|\Delta^j_{k,m}| \searrow 0$, as $m \to \infty$, for all j and k.

Under gauge transformations, h, $g^{(i)}_{\gamma_{yx}}$ transforms according to

$$g^{(i)}_{\gamma_{yx}} \to g^{(i)h}_{\gamma_{yx}} = h^{-1}(y)\,g^{(i)}_{\gamma_{yx}}\,h(x), \tag{2.5}$$

and if $\gamma_{yx} \subset \Omega_i \cap \Omega_j$

$$g^{(i)}_{\gamma_{yx}} = h^{-1}_{ji}(y)\,g^{(j)}_{\gamma_{yx}}\,h_{ji}(x). \tag{2.6}$$

Thanks to equations (2.5) and (2.6) one can now define parallel transport on the bundle space B of $\mathcal{F}$ as follows: If $\gamma_{yx} \subset \Omega_i$ parallel transport on B is given by a homeomorphism $\Gamma_{\gamma_{yx}}: V_x \to V_y$, with $V_x = \pi^{-1}(x)$ the fibre over x, which is defined by $\Gamma_{\gamma_{yx}} = \xi_{\Omega_i,y}\, g^{(i)}_{\gamma_{yx}}\, \xi^{-1}_{\Omega_i,x}$ . (2.7)

If $\gamma_{yx}$ is not contained in a single coordinate neighborhood $\Omega_i$, one cuts up $\gamma_{yx}$ into pieces $\gamma_{x_{m+1}x_m} \subset \Omega_{i_m}$, with $\Omega_{i_{m+1}} \cap \Omega_{i_m} \neq \emptyset$, and sets

$$\Gamma_{\gamma_{yx}} = \Gamma_{\gamma_{x_N x_{N-1}}} \Gamma_{\gamma_{x_{N-1}x_{N-2}}} \cdots \Gamma_{\gamma_{x_2 x_1}} , \tag{2.8}$$

with $x_N = y$ and $x_1 = x$. By equations (2.5)-(2.7), $\Gamma_{\gamma_{yx}}$ is independent of the choice of coordinate neighborhoods and coordinates. By the parallel displacement of some $\Phi(x) \varepsilon V_x$ from x to y along $\gamma_{yx}$ we mean the element

$$\Gamma_{\gamma_{yx}} \Phi(x) \quad \text{in} \quad V_y . \tag{2.9}$$

Next, we propose to characterize a fibre bundle with parallel transport in terms of a convenient family of gauge invariant functionals of the connection A. For this purpose we introduce the notion of <u>holonomy groups</u>.

Let x be some point in M, and let $\Omega(x)$ be the manifold of all bounded, continuous, piecewise smooth, oriented paths, $\omega_x$, starting and ending at x, called <u>loops</u>. Given $\omega_x \varepsilon \Omega(x)$, let $\omega_x^{-1}$ denote the same curve as $\omega_x$ but with reversed orientation. On $\Omega(x)$ we define multiplication as the composition of paths, i.e. $(\omega_x, \omega_x') \to \omega_x o \omega_x'$, the composition of $\omega_x$ with $\omega_x'$. With the obvious equivalence relation imposed, $\omega_x o \omega_x^{-1} = \omega_x^{-1} o \omega_x = 1_x$, the identity element in $\Omega(x)$, and $\Omega(x)$ is seen to be an infinite dimensional group.

Given a connection A and some loop $\omega_x \varepsilon \Omega(x)$ we set

$$g_{\omega} = \xi^{-1}_{\Omega_i,x}\, \Gamma_{\omega_x}\, \xi_{\Omega_i,x} \,\varepsilon G, \tag{2.10}$$

where $\Omega_i$ is a coordinate neighborhood containing x, and $\xi_{\Omega_i,x}$ are local coordinates on B. This defines a representation $g\colon \omega_x \varepsilon \Omega(x) \to g_{\omega_x} \varepsilon G$ of $\Omega(x)$ on V. The image of $\Omega(x)$,

$$H_x(A) = \{ g_{\omega_x} : \omega_x \in \Omega(x) \}, \tag{2.11}$$

is called the <u>holonomy group</u> of A.

For continuous $A, H_x(A)$ is a closed subgroup $\subset$ G. If M is connected $H_x(A)$ is independent of x, up to conjugacy, and if M is simply connected $H_x(A)$ is connected. If $H_x(A) = G$ the connection A is <u>irreducible</u>.

Under a gauge transformation, h, $g_{\omega_x}$ transforms according to

$$g_{\omega_x} \to g^h_{\omega_x} = h^{-1}(x) g_{\omega_x} h(x), \tag{2.12}$$

as follows from (2.10) and (2.5)-(2.7). Thus the elements $g_{\omega_x} \in H_x(A)$ depend on the choice of local coordinates (the gauge).

Let $\chi$ be a character of G. Then Y, given by

$$Y(\omega_x) = \chi(g_{\omega_x}) \tag{2.13}$$

is a character of $\Omega(x)$.

By (2.12), $Y(\omega_x)$ is <u>gauge-invariant</u>. We define $a(\omega_x)$, $\omega_x \in \Omega(x)$, to be the infimum of the areas of all smooth surfaces bounded by $\omega_x$.

<u>Theorem 2.1.</u>

Assume M is simply connected. Let Y be an irreducible character of $\Omega(x)$ with the properties:

(1) Y is of positive type on $\Omega(x)$.

(2) $Y(1_x) = n$, for some natural number $n < \infty$.

(3) $|Y(\omega_x) - n| \leq O(a(\omega_x))$, as $a(\omega_x) \searrow 0$.

Then there exist an irreducible, connected subgroup $H \subseteq U(n)$ and a representation $h: \omega_x \in \Omega(x) \to h_{\omega_x} \in H$ of $\Omega(x)$ such that $Y(\omega_x) = tr(h_{\omega_x})$. The representation h of $\Omega(x)$ is unique up to unitary equivalence. Moreover, there exists a connection A with values in the Lie algebra of H such that $h_{\omega_x}$, $\omega_x \in \Omega(x)$, is the parallel transport around $\omega_x$ determined by A. If $H=U\pi(G)U^{-1}$, where $\pi$ is an n-dimensional, faithful representation of some compact, connected Lie group G, and $U \in U(n)$ then $h_{\omega_x} = U\pi(g_{\omega_x})U^{-1}$, and $g_{\omega_x} \in G$ is unique up to (G-valued) gauge transformations. □

For a proof of Theorem 2.1 see |3|. This result says that a principal bundle with structure group G and a connection on it are uniquely determined by the numbers $\{\chi(g_{\omega_x}) : \omega_x \in \Omega(x)\}$ if $\chi$ is a faithful, unitary character of G.

Let $P$ be some principal bundle with structure group G, and $A^I$ the space of all continuous, irreducible connections on $P$. Let $G$ be the group of all gauge transformations modulo those which take values in the center of G. Clearly $G$ acts as a transformation group on $A^I$. We define the orbit space $\mathcal{O}$ as $A^I/_G$. Given a connection $A \in A^I$, the corresponding orbit of A under $G$ is denoted by $[A]$. Unfortunately $\mathcal{O}$ is generally not a linear space, but an infinite dimensional manifold with rather complicated geometrical properties, unless G is abelian. This is an expression of the intrinsic non-linearity of non-abelian gauge fields. Singer has shown that $A^I$ is a principal bundle with base space $\mathcal{O}$, fibre $G$ and projection $\pi$ given by $\pi(A) = [A]$, |4|. (If $M = S^3$ or $S^4$ $A^I$ is not homeomorphic to $\mathcal{O} \times G$, i.e. gauge fixing is impossible; see |4|. If $M = \mathbb{E}^\nu$ this conclusion is however not valid).

Since $\mathcal{O}$ is a manifold one can define a space $C(\mathcal{O})$ of continuous functions on $\mathcal{O}$. The elements of $C(\mathcal{O})$ represent the "observables" of a classical Yang-Mills theory. "Euclidean quantization" consists, in a vague sense, in converting the elements of $C(\mathcal{O})$ into random variables. This procedure requires some more explicit knowledge of the structure of $C(\mathcal{O})$. We thus describe a convenient dense subspace of $C(\mathcal{O})$: Let $A \in A^I$ and let $g_{\omega_x}(A)$ denote the parallel transport around $\omega_x$ determined by A. The functions

$Y(\omega_x;A) := \chi(g_{\omega_x}(A))$, $\omega_x \in \Omega(x)$, $\chi$ a character of G, are

gauge invariant, i.e. depend only on $[A]$ and are continuous in $[A]$. Therefore they belong to $C(\mathcal{O})$.

Theorem 2.2.

Let M be connected and suppose $\chi$ is some faithful, unitary character of G. Then the algebra of functions generated by

$$W: = \{ Y(\omega_x;A): \quad \omega_x \in \Omega(x) \}$$

is dense in $C(\mathcal{O})$.

Proof. By Theorem 2.1 the functions $\{ Y(\omega_x;A) \}$ separate points in $\mathcal{O}$. Moreover $\overline{Y(\omega_x;A)} = Y(\omega_x^{-1};A)$ also belongs to W, and finally $Y(1_x;A) = \text{const.} > 0$. By the Stone-Weierstrass theorem the algebra generated by W is dense in $C(\mathcal{O})$; ($\mathcal{O}$ is supposed to be compact). ∎

Remarks.

1. Theorems 2.1 and 2.2 serve as one motivation for viewing the functions $Y(\omega_x) = \chi(g_{\omega_x})$ as the basic "observables" of a pure Yang-Mills theory.

2. Let $\Omega_d$ be a denumerable set of bounded, closed, piecewise smooth, oriented loops, e.g. the loops of some lattice on M. Let $\overline{\Omega_d}$ be the closure of $\Omega_d$ under inversion of orientation and composition of loops; ($\overline{\Omega}_d$ is a groupoid). Let $C_{\Omega_d}(\mathcal{O})$ be the algebra generated by $\{ Y(\omega): \omega \in \overline{\Omega}_d \}$. If all connections in $A^I$ are continuous then $C_{\Omega_d}(\mathcal{O}) \nearrow C(\mathcal{O})$, (in the supremum topology), as $\overline{\Omega_d} \nearrow \overline{\Omega} \supseteq \Omega(x)$, for some $x \in M$, with M connected.

Let $\Omega_L$ denote all bounded, oriented loops in a lattice $L$. Approximating $C(\mathcal{O})$ by $C_{\Omega_L}(\mathcal{O})$ is the starting point of the lattice approximation to Yang-Mills theory.

It would be interesting to make a systematic study of all convenient, separable approximations to $C(\mathcal{O})$ that could serve to construct gauge-invariant regularizations of (quantized) Yang-Mills theory.

## §3. Random (or Stochastic) Geometry

Throughout these notes we follow the Euclidean (time purely imaginary) approach to quantizing relativistic quantum field theory. In this approach the problem of field quantization is converted into one of constructing random fields and functional integrals, (unless there are Fermi fields in the theory which are ignored in these notes). In the Euclidean approach to quantized, pure Yang-Mills theory the basic random fields turn out to be the variables

$$Y(\omega) = \chi_Y(\omega),$$

studied in §2; $\chi_Y$ is a unitary character of the gauge group G, $\omega$ is a bounded, oriented loop, and $g_\omega$ is a "random holonomy operator" assigned to $\omega$. (If the theory also contains a matter field $\Phi$, assumed here to be spinless, transforming under a representation $U^\Phi$ of G then, in addition to the variables $Y(\omega)$, one must consider the variables

$$(\Phi(y), U^\Phi(g_{\gamma_{yx}})\Phi(x)),$$

where $(\cdot,\cdot)$ is an inner product on the fibre V of the bundle whose sections are given by $\Phi$).

We thus see that in the Euclidean approach to quantized Yang-Mills theory one wants to construct random fields on spaces of geometrical objects, the oriented paths and loops in Euclidean space-time. According to Theorem 2.1, the random fields $Y(\omega)$ are in correspondence with a random connection on a random principal bundle.

The construction of such random fields can thus be viewed as a problem in a hypothetical branch of mathematics attempting to combine geometry and probability theory which one might call <u>random (or stochastic) geometry</u>.

We now give a short list of some problems in random geometry and then discuss a few of them in more detail.

1) Convert geometric objects (loops, surfaces, clusters, holonomy operators, etc.) into random variables, resp. random currents.

2) Construct stochastic processes whose state space is a space of geometrical objects.

3) Construct random fields on spaces of geometrical objects.

4) Construct random holonomy operators on a (random) fibre bundle.

5) Investigate random operators associated with a foliation |6|; etc.

In many situations random geometry is really measure theory on infinite dimensional manifolds, or manifolds modulo the action of some infinite dimensional transformation group, (e.g. a group of gauge transformations, the diffeomorphism group of the circle or a sphere, etc.).

Of concern to us are the following specific problems in random geometry:

(A) Theory of random holonomy operators on random bundles with fixed base space.

(B) Diffusion processes whose state space is a space of loops or a manifold of open paths with fixed endpoints, (modulo the action of the group of reparametrizations).

(C) Theory of random surfaces bounded by some fixed loops.

These problems are relevant for the understanding of quantized gauge fields, as we hope to explain in the remainder of these notes. We emphasize that there are numerous other branches in theoretical physics which pose their own problems in random geometry. In particular, statistical mechanics is rich in such problems.

Unfortunately, it turns out that random geometry in the continuum is very difficult and forces one to study very singular objects. For example, the holonomy operators of a Euclidean quantized Yang-Mills theory on $\nu$-dimensional space-time, with $\nu \geq 3$ cannot be expected to be random fields in the precise sense of the word. To see this one may consider the free electromagnetic field:

Let $A(\cdot) = (A_1(\cdot),\dots,A_\nu(\cdot))$ be the $\mathbb{R}^\nu$-valued Gaussian process with mean 0, i.e. $\langle A(\cdot)\rangle = 0$, and covariance

$$\langle A_j(0)A_m(x)\rangle = \delta_{jm}(2\pi)^{-\nu/2} \int e^{ikx}k^{-2}d^\nu k \tag{3.1}$$

One may attempt to define random holonomy operators (random phase factors), $g_\omega$, by

$$g_\omega := \exp i \oint_\omega A_j(x)dx^j \tag{3.2}$$

Unfortunately $g_\omega$ does not exist as a random field on the space of loops: $g_\omega = 0$, almost surely, when $\nu \geq 3$. To give meaning to $g_\omega$ it needs to be "normal-ordered":

$$g_\omega \to N(g_\omega) = \text{"}\exp\left[\tfrac{1}{2}V_C(0)|\omega|\right] g_\omega\text{"} . \tag{3.3}$$

Here $|\omega|$ is the length of the loop $\omega$, and $V_C$ is the $(\nu-1)$-dimensional Coulomb potential. The r.s. of (3.3) can be defined rigorously as a limit of regularized objects if $\omega$ is sufficiently smooth ($C^2$) and $\nu \leq 4$. Since A is Gaussian, it is easy to calculate $< \underset{=}{N}(g_\omega)N(g_{\omega'}) >$. One checks that if the relative positions and orientations of $\omega$ and $\omega'$ are suitably chosen one gets

$$< N(g_\omega)N(g_{\omega'}) > \sim \exp\left[\text{const. dist. }(\omega,\omega')^{-1}\right], (\text{for } \nu=4). \quad (3.4)$$

Thus, the objects $N(g_\omega)$ are too singular to be random fields in the usual sense of the word. For a (heuristic) theory of normal ordering of holonomy operators in three-dimensional, interacting theories see |7|.

The above discussion suggests that random geometry in the continuum may be plagued with serious difficulties. One way of regularizing the objects studied in random geometry is to pass to random combinatorial geometry by replacing continuum geometry by discrete geometry (combinatorics); see e.g. |8| for some discussion.

We conclude §3 with sketches of three examples of random combinatorial geometry.

I. Let $\mathcal{P} = (P, \mathbb{E}^\nu, G, \pi)$ be a principal bundle over $\mathbb{E}^\nu$. As discussed at the end of §2, we may approximate the space $C(\mathcal{O})$ of continuous functions over the orbit space $\mathcal{O}$ (the "observables") by $C_{\Omega_L}(\mathcal{O})$, where $\Omega_L$ is the set of all bounded, oriented loops in a lattice $L$ which we choose to be $\varepsilon\mathbb{Z}^\nu = \{x: \varepsilon^{-1}x \in \mathbb{Z}^\nu\}$. We assign to each link (nearest neighbor pair) $xy \in L$ an element $g_{xy} \in G$. Given $\omega \in \Omega_L$, let

$$g_\omega = \underset{xy\subset\omega}{\overset{\circlearrowleft}{\pi}} g_{xy}. \quad (3.5)$$

Let $\chi_Y$ be a character of G. We set

$$Y(\omega) = \chi_Y(g_\omega) \quad (3.6)$$

The algebra generated by the Y's is dense in $C_{\Omega_L}(\mathcal{O})$. Thus, in order to convert the elements of $C_{\Omega_L}(\mathcal{O})$ into random variables, it suffices to construct the joint distribution of the "Wilson loop variables," i.e. to construct a measure on $\{g_{xy}\}$. The standard proposal |9|, due to Wilson, is the following: Let $\chi$ be a unitary character of G, let p denote a plaquette (2-cell) of $L = \varepsilon\mathbb{Z}^\nu$, $\partial p$ its boundary.

Let $\Lambda \in \varepsilon\mathbb{Z}^\nu$ be a bounded set. Define an action, $A^{YM}_\Lambda$, by

$$A^{YM}_\Lambda = -\sum_{p\subset\Lambda} \mathrm{Re}\,\chi(g_{\partial p}) \tag{3.7}$$

Let dg denote normalized Haar measure on $G, \beta>0$. We define

$$d\mu^{(\varepsilon)}_\Lambda(g) = Z^{-1}_\Lambda\, e^{-\beta A^{YM}_\Lambda} \prod_{xy} dg_{xy} \tag{3.8}$$

with $Z_\Lambda$ such that $\int d\mu^{(\varepsilon)}_\Lambda(g) = 1$.

By a standard compactness argument one can choose a sequence $\{\Lambda_n\}^\infty_{n=o}$ increasing to $\varepsilon\mathbb{Z}^\nu$ such that

$$d\mu^{(\varepsilon)}(g) = w^*\text{-}\lim_{n\to\infty} d\mu^{(\varepsilon)}_\Lambda(g) \text{ exists.} \tag{3.9}$$

(Conditions for existence and uniqueness of the limit $d\mu$ are given e.g. in |9,10|). The measure $d\mu$ is now interpreted as the joint distribution of the random variables $\{Y(\omega):\omega\in\Omega_L\}$. Of particular interest in the discussion of the resulting theory are the <u>Schwinger functionals</u>

$$S^{(\varepsilon)}_n(Y_1(\omega_1),\ldots,Y_n(\omega_n)) = \int d\mu^{(\varepsilon)}(g) \prod_{j=1}^n \chi_{Y_j}(g_{\omega_j}) \,. \tag{3.10}$$

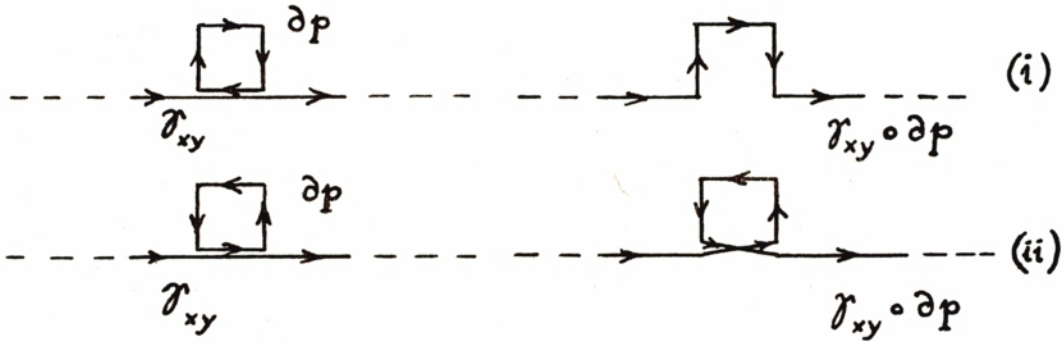
What we have introduced here is the standard lattice approximation to quantized, pure Yang-Mills theory |9|.

II. Let $\Gamma_L(x,y)$ be the set of all finite, oriented curves in $L = \varepsilon\mathbb{Z}^\nu$ starting at x and ending at y. This is clearly a countable set. Let $\ell_{2,r}$ be the Hilbert space of functions $F$ on $\Gamma_L(x,y)$ with the property that

$$\sum_{\gamma_{xy}\in\Gamma_L(x,y)} e^{r|\gamma_{xy}|}\,|F(\gamma_{xy})|^2 < \infty, \tag{3.11}$$

for some $r \geqq 0$. Let p be an oriented plaquette.

If $\partial p \cap \gamma_{xy} \neq \emptyset$ we define $\gamma_{xy} \circ \partial p$ by the following figure:




For $F \in \ell_{2,r}$ define

$$(\delta_p F)(\gamma_{xy}) = \begin{cases} F(\gamma_{xy} \circ \partial p) - F(\gamma_{xy}) \text{ if } \partial p \cap \gamma_{xy} \neq \emptyset \\ \text{and } \gamma_{xy} \circ \partial p \text{ is connected;} \\ 0, \text{ otherwise.} \end{cases} \qquad (3.12)$$

One may now define a functional Laplacean, $\mathcal{D}_1$, as the unique selfadjoint operator determined by the quadratic form

$$F \to \sum_p (\delta_p F, \delta_p F)_{\ell_2}, \qquad (3.13)$$

defined e.g. on $\ell_{2,r}$, $r > 0$; ($\ell_2 = \ell_{2,r=0}$).
Let $V_\alpha$ be the multiplication operator on $\ell_2$ given, for example, by

$$(V_\alpha F)(\gamma_{xy}) = \alpha |\gamma_{xy}| F(\gamma_{xy}), \qquad (3.14)$$

where $|\gamma_{xy}|$ is the number of links in $\gamma_{xy}$.

The operator sum $\mathcal{D}_1 + V_\alpha$ is still selfadjoint, and it follows from a general theorem in |11| that the kernel $(\exp[-t(\mathcal{D}_1 + V_\alpha)])(\gamma_{xy}, \gamma'_{xy})$ is non-negative, for all $\gamma_{xy}, \gamma'_{xy}$ in $\Gamma_L(x,y)$. Thus $\exp{-t(\mathcal{D}_1 + V_\alpha)}$ is the transition function of a stochastic process on $\Gamma_L(x,y)$. This process describes the diffusion of an oriented string with fixed endpoints x and y. It has some significance in the analysis of confinement in lattice gauge theories, |7|. See also §6. (If the deformation (ii) in the definition of $\gamma_{xy} \circ \partial p$ is omitted, the resulting process may be of interest in the study of selfavoiding random walks).

II'. Let $\Omega_L$ be the set of all finite, oriented loops in $L$; $\Omega_L$ is still countable. Therefore one may define spaces $\ell_{2,r} = \ell_{2,r}(\Omega_L)$ and operators $\delta_p$, $\mathcal{D}_1$, $V_\alpha$, etc. in a similar way as above. There results a model for the diffusion of loops in the lattice $L$. For some results concerning a general theory of diffusion of discrete, geometrical objects see e.g. |11|. (They have applications in statistical mechanics).

III. Let $\Omega_L$, $\mathcal{D}_1$, $V_\alpha$,... be as in (II), (II'). we propose to give an example of a random field $\Phi$ on $\Omega_L$. To each $\omega \in \Omega_L$ we assign an $n\times n$ matrix, $\Phi(\omega)$, with a priori distribution $d\Phi(\omega)$ given by the Lebesgue measure on $\mathbb{C}^{n^2}$. There exists a random field $\Phi$ the distribution of which corresponds to the formal measure

$$d\mu(\Phi) = Z^{-1}\exp\Big[-\sum_{\omega} \operatorname{tr}(\Phi(\omega)^*\big[(\mathcal{D}_1+V_\alpha)\Phi\big](\omega))\Big] \cdot \prod_{\omega\in\Omega_L} e^{-2\operatorname{tr}(\Phi(\omega)^*\Phi(\omega))^2} d\Phi(\omega).$$

The measure $d\mu$ can be constructed as a limit of cutoff measures. The field $\Phi$ is conveniently described by its "Schwinger functionals"

$$S_n(\Phi_{\alpha_1\beta_1}(\omega_1)\ldots\Phi_{\alpha_n\beta_n}(\omega_n)) = \int d\mu(\Phi) \prod_{j=1}^{n} \Phi_{\alpha_j\beta_j}(\omega_j) \tag{3.15}$$

which one may interpret as Schwinger functions of a lattice string theory |12|.

If the constraints

$$\prod_{\omega,\omega'} \delta(\Phi^*(\omega\circ\omega')\Phi(\omega)\Phi(\omega')-\mathbb{1}) \tag{3.16}$$

are inserted into $d\mu$ and the couplings are suitably rescaled, the above theory becomes a lattice gauge theory with $G = U(n)$; see |11|. This example is admittedly somewhat naive. It may serve as a challenge for a serious study of more interesting random geometrical models. The most important problem is to find interesting models of this sort for which the continuum limit ($\varepsilon \searrow 0$) exists. This is the subject of the <u>renormalization group</u> ("block spin transformations") and <u>non-perturbative renormalization</u>.

## §4. Schwinger Functionals and Relativistic Quantum Fields

In this paragraph we briefly discuss the question whether Schwinger functionals of the sort defined in (3.10) and (3.15) determine a relativistic quantum field theory. The answer to this question is, for conventional, local field theories, the Osterwalder-Schrader reconstruction theorem |13,14|. We quote here a generalization of that result which accounts for theories of fields defined on spaces of geometrical objects such as the "Wilson loops" of pure Yang-Mills theory. The theorem is first stated for a class of continuum theories and represents a special case of more general results of this type |15|.

The Euclidean space-time manifold is $\mathbb{E}^\nu, \nu=2,3,4$. Let $\Omega^{(d)}$ be the family of all oriented $C^\infty$ d-dimensional surfaces in $\mathbb{E}^\nu$ without self-intersections, (i.e., topologically, d-dimensional spheres), with $d \leq \nu-2$. For $\omega,\omega'$ in $\Omega^{(d)}$, set

$$d(\omega,\omega') = \mathrm{dist}\,(\omega,\omega') \equiv \min_{\substack{x\in\omega \\ y\in\omega'}} |x-y|.$$

Let $$\Omega^{(d)n}_{\neq} = \{ \omega_1,\dots,\omega_n \text{ in } \Omega^{(d)} : d(\omega_i,\omega_j) > 0, \text{ for } i\neq j \}, \tag{4.1}$$

$$\Omega^{(d)n}_{>} = \{(\omega_1,\dots,\omega_n) \in \Omega^{(d)n}_{\neq} : \omega_j \in \{ x=(\vec{x},t): t > 0 \}, j=1,\dots,n \} \tag{4.2}$$

We now assume that we are given a sequence of Schwinger functionals $\{ S_n(Y_1(\omega_1),\dots,Y_n(\omega_n)) \}_{n=0}^\infty$ with the following properties:

(S1) $S_0=1$; $S_n(Y_1(\omega_1),\dots,Y_n(\omega_n))$ is well-defined on $\Omega^{(d)n}_{\neq}$ and continuous under small $C^\infty$ deformations of $\omega_1,\dots,\omega_n$ in $\Omega^{(d)n}_{\neq}$.

Moreover, the growth of $|S_n(Y_1(\omega_1),\dots,Y_n(\omega_n))|$, as $d_n \equiv \min_{i\neq j} d(\omega_i,\omega_j) \searrow 0$, is bounded by $O(\exp[\text{const. } d_n^{-\alpha}])$, for some $\alpha \geq 0$ and constants that depend on n in a suitable way; see |13,15|.

(S2) (Osterwalder-Schrader positivity)
Let r be reflection at $\{ t=0 \}$ and let $Y \to Y_r$ be some reflection map (in the case of Yang-Mills theory $Y_r = \overline{Y}$, the complex conjugation of Y). The N x N matrix C with matrix elements $C_{ij}$ given by

$$S_{n(i)+n(j)}(Y^i_{n(i),r}(\omega^i_{n(i),r}),\dots,Y^i_{1,r}(\omega^i_{1,r}),Y^j_1(\omega^j_1),\dots,Y^j_{n(j)}(\omega^j_{n(j)})),$$

$i,j = 1,\ldots,N$, is positive semi-definite, provided $(\omega_1^k,\ldots,\omega_{n(k)}^k) \in \Omega_>^{(d)n(k)}$, for all $k = 1,\ldots,N$; $N=1,2,3\ldots$ .

(S3) (Symmetry)
$S_n(Y_1(\omega_1),\ldots,Y_n(\omega_n))$ is symmetric under arbitrary permutations of its arguments, for all n.

(S4) (Invariance)
Let $\beta$ be a proper Euclidean motion. Then
$S_n(Y_1(\omega_1),\ldots,Y_n(\omega_n)) = S_n(Y_1(\omega_{1,\beta}),\ldots,Y_n(\omega_{n,\beta}))$, for all n.
(Here $\omega_\beta$ is the image of $\omega$ under $\beta$).

If we consider a lattice theory we replace (S4) by (S4'): Invariance under the symmetries of the lattice.

[(S5) (Clustering; see |13|)]

Heuristically, the Schwinger functionals of a Yang-Mills theory satisfy additional properties, in particular an extended version of (S2) (Osterwalder-Schrader positivity) to which we refer as $(S2^{ext.})$; see |10,15|.

The main theorem about sequences of Schwinger functionals satisfying (S1)-(S4) is

Theorem 4.1.

If $\{ S_n(Y_1(\omega_1),\ldots,Y_n(\omega_n)) \}_{n=0}^{\infty}$ satisfies (S1), (S2) and (S4) then one can reconstruct from those Schwinger functionals a separable physical Hilbert space $\mathcal{H}$, a vacuum vector $\Omega \in \mathcal{H}$, with $<\Omega,\Omega> = 1$, and a unitary representation U of the proper Poincaré group

$$\mathcal{P}_+^\uparrow \text{ on } \mathcal{H} \text{ with } U(a,\Lambda)\,\Omega = \Omega, \tag{4.3}$$

for all $(a,\Lambda) \in \mathcal{P}_+^\uparrow$. The spectrum of the generators $(\vec{P},H)$ of the space-time translations is contained in the forward light cone $\overline{V}_+$.

If, in addition, (S3) holds there exist "local fields"
$y(\omega;Y)$, $\omega \in \Omega^{(d)}$, $\omega \subset \{ x=(\vec{x},t) \in \mathbb{M}^\nu : t = \text{const.} \}$,
with $[y(\omega;Y), y(\omega';Y')] = 0$ if $\omega$ and $\omega'$ are space-like separated.
[If (S1),(S2),(S4) and (S5) hold then the vacuum $\Omega$ is the only vector satisfying (4.3), i.e. the vacuum is unique].

A more precise formulation and a proof of this basic theorem will be given elsewhere, |15|.

Some of the main tools in the proof of Theorem 4.1 not already used in |13,14| are: A result concerning the selfadjoint extensions of symmetric semigroups |16| that serves to construct the representation of the Poincaré group, and the observation that the Schwinger functionals determine a state $< \Omega, \cdot \Omega >$ which satisfies the KMS condition with respect to the Lorentz boosts, |17|. A somewhat novel approach to the results of |17| and to proving locality are consequences of that observation. See |15|.

Next we discuss a few physical properties coded directly into the Schwinger functionals. The first is a consequence of extended Osterwalder-Schrader positivity $|10^1|$ in Yang-Mills theory. In that theory d = 1, and $\Omega^{(d)}$ is the space of loops in $\mathbb{E}^\nu$ diffeomorphic to circles. Let $\omega_{LxT}$ be a (smoothed version of a) rectangular loop with sides of length L and T. Assume that ($S2^{ext.}$), $|10^1,15|$, holds. Then $S_1(Y(\omega_{LxT}))$ is log convex. Therefore

$$V_Y(L) := \lim_{T\to\infty} -\frac{1}{T} \log S_1 \, (Y(\omega_{LxT})) \tag{4.4}$$

exists, and moreover one concludes

<u>Proposition 4.2</u>

$$V_Y(L) \leqq \text{const. } L, \text{ as } L \to \infty. \tag{4.5}$$

For lattice theories this inequality has been established in |18|. Physically, it says that the potential between a static (infinitely heavy) quark and a static anti-quark cannot rise faster than linearly.

It was suggested in |7,11| that $S_1(Y(\omega))$ contains information about the boundstate spectrum of very heavy quarks, and $S_2(Y(\omega),Y(\omega'))$ about the low-lying mass spectrum of pure Yang-Mills theory.

Next, we sketch the notion of "disorder fields" |19-22|. We assume that, in addition to the"random fields" $Y(\omega), \omega \in \Omega^{(d)}$, there are "fields" $B(\gamma)$, $\gamma \in \Omega^{(\nu-2-d)}$, $(\Omega^{(0)} := \mathbb{E}^\nu)$, with joint Schwinger functions

$$\{ S_{n,m}(Y_1(\omega_1),\ldots,Y_n(\omega_n),B(\gamma_1),\ldots,B(\gamma_n)) \}_{n,m=0}^{\infty}. \tag{4.6}$$

These Schwinger functions are supposed to have properties analogous to (S1)-(S4), but in addition they are required to have certain specific discontinuities (which cannot arise in standard field theories of the Wightman type):

Choose $\omega,\gamma \subset \{(\vec{x},t): t = \text{const.}\}$ and let $\nu(\omega,\gamma)$ be the linking number of $\omega,\gamma$. Let $\omega_\varepsilon$ be the translation of $\omega$ in the t-direction. Then $\lim_{\varepsilon \searrow 0} S_{\cdot,\cdot}(\ldots,Y(\omega_\varepsilon),B(\gamma),\ldots)$

$$= z_{Y,B}^{\nu(\omega,\gamma)} \lim_{\varepsilon \searrow 0} S_{\cdot,\cdot}(\ldots,Y(\omega_{-\varepsilon}),B(\gamma),\ldots) \qquad (4.7)$$

In two-dimensional scalar field theories with soliton behavior and three-or four dimensional Yang-Mills theory one can prove that $|z_{Y,B}| = 1$, |19-21|. In fact, in Yang-Mills theory, $z_{YB}$ is an element of the center of the gauge group G which depends on Y and B, |20,22|. An extension of the reconstruction theorem, Theorem 4.1, provides us with fields $y(\omega;Y)$ and $b(\gamma,B)$, which for $\omega,\gamma \subset \{(\vec{x},t): t = 0\}$ satisfy the following formal time 0 commutation relations

$$y(\omega;Y)b(\gamma;B) = z_{YB}^{\nu(\omega;\gamma)} \, b(\gamma;B)y(\omega;Y) \qquad (4.8)$$

(The field b is said to be "dual" to y).

For $\nu = 2$, $d = 0$, $\nu = 3$, $d = 1$, and $\nu = 4$, $d = 2$, (i.e. $\nu-2-d=0$) such commutation relations have been discussed in |19| and representations with $d = 0$ have been constructed for two-dimensional scalar field theories with soliton behavior, (the sine-Gordon and the $\lambda\phi^4$ models). For $\nu = 3$, $d = 1$, and $\nu = 4$, $d = 1$ they have been proposed and interpreted in |20|; see also 't Hooft's contribution to these proceedings. For $\nu = 2,3,4$, $d = 0$, certain "quasi-free" representations have been constructed in a series of remarkable papers by Jimbo, Miwa and Sato |21|. Their work shows how powerful relations like (4.7) may be and has resulted in the calculation of the correlation functions of the two-dimensional Ising model. For lattice theories representations of (4.8) have been constructed for $\nu = 2,3,4$, with $d = 0,1,1$, respectively, |22|.

In |19| properties of the representations of (4.8) when $\nu = 2$, $d = 0$ or $\nu = 3$, $d = 1$ (or $\nu = 4$, $d = 2$) have been related to the structure of super-selection sectors of the corresponding quantum field theories: If $\text{const.}_\omega y(\omega;Y)$ converges on $\mathcal{H}$ to a non-zero element of the center of the observable algebra, as $|\omega| \to \infty$, and $z_{YB} \neq 1$, then $b(x;B)$ intertwines disjoint super-selection sectors of that algebra. (The two-dimensional case has been studied from first principles, whereas in higher dimensions one needs suitable technical assumptions). In |20| 't Hooft has suggested connections between properties of the representations of (4.8) in a gauge theory and quark-resp. monopole confinement. He argues that (4.8) rules out the possiblity that quarks and monopoles are both confined. This has been elaborated and tested in models in |22,23,10²|. See the contributions of 't Hooft and Mack to these proceedings.

§5. Existence of Quantized Gauge Fields

In this paragraph we quote some results concerning the existence of models satisfying the axiomatic scheme of §4. At present, the only models that fit into that scheme are models of quantized interacting gauge fields-and matter fields-on a lattice of arbitrary dimension (see end of §2 and Example (I), §3) and in a continuum space-time ($\mathbb{E}^{\nu}$) of dimension $\nu = 2$, and presumably $\nu = 3$. Of course, the free electromagnetic field in two, three or four dimensions satisfies (S1)-(S5). In the continuum only abelian gauge fields have been constructed so far. If the gauge group is abelian there are, in addition, lattice theories describing abelian gauge fields which are connections on bundles whose base space is e.g. the space of oriented loops in the lattice: To each plaquette p one assigns an element $e^{ia_{\partial p}} \varepsilon G, G \overset{\text{e.g.}}{=} U(1), \mathbb{Z}_n$, with a priori distribution the Haar measure on G. The action is given by $A_{\Lambda} = -\sum_{c\subset\Lambda} \cos \left( \sum_{p\in\partial c} a_{\partial p} \right)$.
These theories have Schwinger functionals of random "holonomy operators" associated with closed lattice surfaces satisfying (S1)-(S3), (S4'), (in the limit $\Lambda = \mathbb{Z}^{\nu}$). The models of the type described in Example (III), §3, (without the constraint (3.16)) are not known to fit into the scheme of §4. (This may be related to the difficulties which are met in string theories |24|). For detailed studies of lattice theories see e.g. |9,10|.

Let $d\mu^{(\varepsilon)}(g)$ denote a limit of the measures $d\mu_{\Lambda}^{(\varepsilon)}(g)$, (Example (I), §3, (3.7)-(3.9)), as $\Lambda \uparrow \mathbb{Z}^{\nu}$.

For small $\beta$, the limit is unique and the Schwinger functionals have exponential cluster properties; detailed properties such as confinement can be investigated by means of high temperature expansions, |9|. Uniqueness of the $\Lambda \nearrow \mathbb{Z}^{\nu}$ limit can also be proven in a class of abelian models, for all $\beta$. A few physical properties of lattice gauge theories are sketched in the next paragraph.

The list of models of quantized, interacting gauge fields in the continuum satisfying (S1)-(S4) is still short:

1) The abelian Higgs model (scalar QED) in two space-time dimensions, |25|.

2) Spinor QED in two space-time dimensions with massless or massive fermions and massless or massive photons, |26|. (For a different approach see |27|).

|3) For spinor QED in three dimensions, a proof of stability of the theory is announced |28|.

4) For some super-renormalizable gauge theories (including non-abelian ones), T. Balaban has announced a proof of stability |29| based on a rigorous form of renormalization group - "block spin" - transformations for lattice theories, extending previous work due to Gallavotti, et al.|30| for the $\lambda\phi^4$ theory in three dimensions |31| |.

## §6. Random Geometrical Methods in Lattice Gauge Theories.

In this paragraph we briefly discuss four examples in lattice gauge theory the analysis of which is based on estimating the joint distribution of random variables labelled by geometrical objects such as closed flux tubes or (interacting) oriented random paths with fixed endpoints. We sketch some typical steps in that analysis thereby providing examples for the uses of random-geometrical arguments in the study of lattice gauge theories.

### Example 1.

We discuss the behavior of the expectation of the Wilson loop in a three-dimensional $\mathbb{Z}_2$ lattice theory, (i.e. $G = \mathbb{Z}_2$ is the gauge group). This model can be thought of as a Kindergarden theory of vortices in a type II superconductor. The Wilson loop - dual to the "vortex field" - is the non-integrable phase factor of the superconducting medium. The action of the model is

$$A = - \sum_p \sigma_{\partial p}, \; \sigma_{\partial p} = \prod_{xy \subset \partial p} \sigma_{xy}, \; \sigma_{xy} = \pm 1. \tag{6.1}$$

The infinite volume expectation in that model at inverse coupling $\beta$ - see Example (I), §3, (3.8), (3.9) - is denoted $\langle \; - \; \rangle_\beta$. (It can be constructed by means of correlation inequalities, for all $\beta$ |32| ).

Let c be an arbitrary 3-cell (unit cube) in $\mathbb{Z}^3$. Then $\prod_{p \subset \partial c} \sigma_{\partial p} = 1$, since $\sigma_{xy}^2 = 1$, for all xy. We now introduce the random phase factors $\sigma_{\partial p}$ as a priori independent variables, inserting the constraint $\prod_{c \subset \mathbb{Z}^3} \delta_{\sigma_{\partial c},1}$, with $\sigma_{\partial c} = \prod_{p \subset \partial c} \sigma_{\partial p}$ (6.2)

We set $\phi_{\partial p} = \begin{cases} 0 \text{ if } \sigma_{\partial p} = 1 \\ 1 \text{ if } \sigma_{\partial p} = -1 . \end{cases}$ (6.3)

Let $\omega$ be a rectangle with sides of length L and T parallel to two coordinate axes. Let $\bar{\omega}$ be the planar surface bounded by $\omega$, i.e. $\partial\bar{\omega} = \omega$ . Since $\sigma_{xy}^{2} = 1$, for all xy,

$$\sigma_{\omega} \equiv \prod_{xy\subset\omega} \sigma_{xy} = \prod_{p\subset\bar{\omega}} \sigma_{\partial p} . \tag{6.4}$$

This is the "non-integrable phase factor" (Wilson loop) observable of the medium.

<u>Theorem 6.1</u>.

For sufficiently small $\beta$

$$< \sigma_{\omega} >_{\beta} \leqq e^{-0(|\bar{\omega}|)}, \; |\bar{\omega}| = \text{area of } \bar{\omega} .$$

For sufficiently large $\beta$,

$$< \sigma_{\omega} >_{\beta} \geqq e^{-0(|\omega|)}, \; |\omega| = \text{perimeter of } \omega.$$

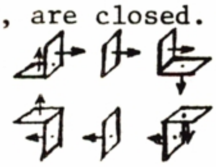
The result for small $\beta$ follows from a standard high temperature expansion |9|. The large $\beta$ result has first been proven in |33|; see also |34|.

We outline a simple proof.

Let $c_{1/0}(\sigma_{\partial p})$ be the characteristic function of $\{\sigma_{\partial p} = 1\}/\{\sigma_{\partial p} = -1\}$. Then

$$<\sigma_{\omega} >_{\beta} = \sum_{\phi} (-1)^{|\phi|} < \prod_{p\subset\bar{\omega}} c_{\phi_p} >_{\beta}, \tag{6.5}$$

with $\phi_p = 1$ or $0$, $c_{\phi_p} \equiv c_{\phi_p}(\sigma_{\partial p})$, and $|\phi| = \sum_{p\subset\bar{\omega}} \phi_p$. The constraint (6.2) implies <u>flux conservation</u>, i.e. the total flux (= # of p's with $\phi_p = 1$) through each closed surface is 0, mod. 2. Thus all flux tubes, $\tau$, are closed.

See Fig. 1: 

Given $\omega$, each flux tube $\tau$ (closed loop in the dual lattice) can be assigned a linking number, $\nu(\omega,\tau)$, (with $\omega$), defined mod. 2. Let

$$0 < \Pr_\omega(n) = \text{prob. } (\{\exists n \text{ flux tubes, } \tau_1,\ldots,\tau_n, \text{ with } \nu(\omega,\tau_i) = 1, \forall i\}).$$

By (6.5) $$< \sigma_\omega >_\beta = \Pr_\omega(0)-\Pr_\omega(1)+\Pr_\omega(2)-\Pr_\omega(3)+\ldots \qquad (6.6)$$

Now to each configuration $\phi = \{\phi_p\}$ contributing to $\Pr_\omega(2n+1)$ there is one contributing to $\Pr_\omega(2n)$ with one flux tube $\tau$, $\nu(\omega,\tau) = 1$, less, (i.e. $\phi_p = 1 \to \phi_p = 0, \forall p\varepsilon\tau$). The statistical weight of one flux tube, $\tau$, is

$$\propto e^{-\beta|\tau|}, \text{where } |\tau| = \# \text{ p's } \varepsilon\,\tau \text{ (with } \phi_p=1). \qquad (6.7)$$

Thus $$\Pr_\omega(2n)-\Pr_\omega(2n+1) \geqq \alpha \Pr_\omega(2n), \text{ with } \alpha \geqq 1-e^{-\text{const.}\beta} > 0. \qquad (6.8)$$

This yields with (6.6)

$$< \sigma_\omega >_\beta \geqq \alpha \Pr_\omega(0).$$

Let $\Pr'_p$ = cond. prob. $(\{\exists\,\tau: p\,\varepsilon\,\tau, \phi_p=1, \nu(\omega,\tau) = 1\})$,

given $\phi_{p'}$, for some $p' \neq p$. A simple argument shows that

$$\Pr_\omega(0) \geqq \prod_{p\varepsilon\bar{\omega}} (1-\Pr'_p), \qquad (6.9)$$

and by (6.7) and standard arguments for counting closed flux tubes through p of a given length one finds

$$\Pr'_p \leqq e^{-\text{const. } \beta \text{ dist.}(p,\omega)} \qquad (6.10)$$

<u>if $\beta$ is large enough.</u> From (6.6)-(6.10) we obtain by a simple calculation

$$< \sigma_\omega >_\beta \geqq \text{const. } e^{-\text{const.}'|\omega|}, \qquad (6.11)$$

for large $\beta$ which proves our contention.

■

Thus if flux tubes have a very small statistical weight, the non-integrable phase factor $\sigma_\omega$ is $\propto e^{-O(|\omega|)}$, in the average.

This situation is analogous to one met in the Ising model: If contours have a very small statistical weight then $\sigma_o\sigma_x \propto$ const., uniformly in x, in the average. Theorem 6.1 has been extended to the four dimensional U(1) model in |35|, (the proof being very different).

More realistic models of superconductors in three (and four) dimensions are discussed e. g. in $|10^2|$,and refs..

Example 2.

We consider pure Yang-Mills lattice theories with gauge group G = U(n) or SU(n), n = 2,3,.... See Example (I), §3, (3.5)-(3.9), and we set $\varepsilon$=1 and choose in (3.7) $\chi$ to be the character of the fundamental representation. Moreover,

$$Y(\omega) = \chi(g_\omega). \tag{6.12}$$

We study the behavior of $S_1(Y(\omega)) = \langle\, Y(\omega)\, \rangle_\beta$ in $\beta$. Let $\omega = \omega_{L\times T}$ be a rectangle in the $(1,\nu)$ plane with sides of length L and T, and let $V_Y(L)$ be the function ($q\bar{q}$ potential) defined in (4.4). It is easy to show that, for $\nu$=2, $V_Y(L) \geqq$ const. L, for all $\beta$; (i.e. permanent confinement by a linear potential). For $\nu$=3, G=U(n), n=1,2,3,...,

$$V_Y(L) \geqq \text{const. } \log(L+1); \text{ see } |36|.$$

There are arguments in support of

$$V_Y(L) \geqq \text{const. } L, \text{ for } G = U(n),\ SU(n),$$

n=2,3..., $\nu$=3. An interesting case is G=SU(2), $\nu$=3 or 4. In |7| the following somewhat remarkable identity has been proven: Let $\Sigma$ be a family of oriented paths, $\{\gamma_u^\Sigma : 1 \leqq u \leqq T\}$, starting at the site (0,...,0,u), ending at (L,0,...,0,u) and lying in the plane $\pi_u = \{x : x^\nu = u\}$. Let $(\gamma_u^\Sigma)^{-1}$ be the path obtained by reversing the orientation of $\gamma_u^\Sigma$. Then

$$S_1(Y(\omega)) \equiv \langle\, Y(\omega)\, \rangle_\beta$$

$$= \sum_{\Sigma:\partial\Sigma=\omega} (2(\nu-1))^{-|\gamma_u^\Sigma|-1} \langle \prod_{u=1}^{T} F(g^h(u-1), g^h(u) | \gamma_u^\Sigma) \cdot \prod_{u=0}^{T} \chi(g_{\gamma_u^\Sigma} \circ (\gamma_{u+1}^\Sigma)^{-1}) \rangle_\beta \tag{6.13}$$

where $\gamma_o^\Sigma$ is the bottom face and $\gamma_{T+1}^\Sigma$ the top face of $\omega$, $g^h(u) = \{ g_{xy} : xy \in \pi_u \}$, and $F(g^h(u-1), g^h(u) | \gamma_u)$ is a gauge invariant function of the "horizontal" gauge fields $g^h(u-1), g^h(u)$ depending on $\gamma_u$. The r.s. of (6.13) can be viewed as a sum over joint correlations of interacting random paths (forming a "random surface" $\Sigma$). In mean

$$F(g^h(u-1), g^h(u) | \gamma_u) \sim e^{-\alpha|\gamma_u|}, \tag{6.14}$$

with $\alpha > -\ln\left[(2\nu-3)(\nu-1)(4\nu-4)^{-1}\beta\right]$.

$$\text{Thus if } \beta < \frac{4}{3}\ (\nu=3), \text{ resp. } \beta < \frac{4}{5}\ (\nu=4) \tag{6.15}$$

$$< Y(\omega)_{L\times T} >_\beta \leqq e^{-0(|\bar{\omega}|)}, \text{ (area decay)}, \tag{6.16}$$

by (6.13) and (6.14). Condition (6.15) is not nearly necessary for area decay, because (6.14) is only a rough estimate and because the factor $\overset{T}{\underset{u=0}{\pi}} \chi(g_{\gamma_u^\Sigma} \circ (\gamma_u^\Sigma)^{-1})$ on the r.s. of (6.13) provides for strong additional damping of $< Y(\omega_{L\times T})>_\beta$,

$$\simeq \overset{T}{\underset{u=0}{\pi}} \exp - 0\ (|\gamma_u^\Sigma \Delta \gamma_{u+1}^\Sigma|). \tag{6.17}$$

We expect that an improvement of the estimates in |7,37| taking into account that factor ought to permit to show that $< Y(\omega_{L\times T}) >_\beta \leqq e^{-0(|\bar{\omega}|)}$, for all $\beta$ when $\nu=3$. (The situation for $\nu=4$ is technically less well understood).

Next, recall Example (II), §3, (3.13), (3.14). Choose $x=0$, $y=(L,0,\dots,0)$, $t \propto T$ and approximate $\exp\left[-t(\mathcal{D}_1+V_\alpha)\right]$ by

$$\{ \exp\left[ - t/T\ \mathcal{D}_1 \right] \exp\left[- t/T\ V_\alpha\right] \}^T \tag{6.18}$$

If we write out (6.18) as a sum over products of matrix elements labelled by paths $\gamma_{xy} \in \Gamma_L(x,y)$ and compare with (6.13), (6.14) and (6.17) we see that for suitably small $\beta$ and a proper choice of t and $\alpha$

$$< Y(\omega_{L\times T}) >_\beta \lessapprox \exp\left[-t(\mathcal{D}_1+V_\alpha)\right] (\gamma_T^\Sigma, \gamma_o^\Sigma).$$

Connections between lattice gauge theories and the diffusion of strings or loops of this sort might have interesting consequences for the heuristic understanding of the string dynamics in Yang-Mills theory.

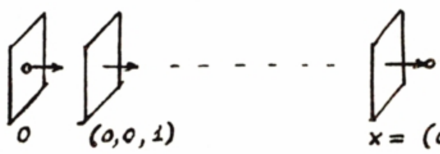
Example 3.

We consider the behavior of the disorder parameter in a three-(or four) dimensional SU(2) lattice gauge theory with distribution $d\tilde{\mu}(g)$ as proposed by Mack and Petkova |22|: $d\tilde{\mu}(g)$ is given by

$$d\tilde{\mu}(g) = \lim_{\Lambda\uparrow \mathbb{Z}^3} \tilde{Z}_\Lambda^{-1} \prod_{c\subset\Lambda} \Theta \Big( \prod_{\partial p\subset c} \chi(g_{\partial p})\Big) d\mu(g), \tag{6.19}$$

with $d\mu(g) \equiv d\mu^{(1)}(g)$ as in (3.8), (3.9).

The expectation in $d\tilde{\mu}$ is denoted $< \text{—} >_{\tilde{\beta}}$. One is interested in the behavior of the expectation of the disorder parameter, $< B(\tau_{ox}) >_{\tilde{\beta}}$, with $\tau_{ox}$ as depicted in Fig. 2:




Let $\phi_p = \frac{1}{2}(1+\operatorname{sgn}\chi(g_{\partial p}))$. The constraint $\prod_c \Theta\big(\prod_{\partial p\subset c} \chi(g_{\partial p})\big)$ enforces that $\sum_{\partial p\subset c} \phi_p = 0$, mod. 2. Thus $\phi_p$ may be interpreted as a $\mathbb{Z}_2$ flux through p, and only closed flux tubes are compatible with the constraint; as in Example 1. The statistical weight of a closed flux tube, $\tau$, is bounded by $e^{-k(\beta)|\tau|}$, with $k(\beta)\nearrow\infty$ as $\beta\nearrow\infty$, as follows from a chessboard estimate |38|. Expanding $< \text{—} >_{\tilde{\beta}}$ in flux tube configurations it is a fairly simple matter of counting flux tubes of given lengths passing through $\tau_{ox}$ to prove that when $e^{-k(\beta)}$ is sufficiently small (i.e. $\beta$ large)

$$< B(\tau_{ox}) >_{\tilde{\beta}} \leqq e^{-O(|x|)}, \tag{6.20}$$

see |22,23|. (In outline we have followed here |23|). One can show, by comparison with the $\mathbb{Z}_2$ model,

$$< \prod_{p\subset\bar{\omega}} \sigma_p >_{\tilde{\beta}} \leqq e^{-O(|\bar{\omega}|)}, \quad \text{where} \quad \sigma_p = \operatorname{sgn}\chi(g_{\partial p}),$$

for $\beta$ small enough, see |22|, and by arguments very similar to those used in Example 1,

$$< \prod_{p \subset \bar{\omega}} \sigma_p >_{\tilde{\beta}} \geqq e^{-O(|\omega|)}, \quad \partial\bar{\omega} = \omega,$$

for large $\beta$.

Example 4.

Let $G=U(N), N= 1,2,3,\ldots$. Let $Y_N(\omega) = \frac{1}{N}\chi_N(g_\omega)$ with $\chi_N$ the character of the fundamental representation of $U(N)$.

One is interested in an expansion of $S_n(Y_N(\omega_1),\ldots,Y_N(\omega_n)) = < \prod_{j=1}^{n} Y_N(\omega_j) >_\beta$ in powers of $\frac{1}{N}$.

To leading order in $\frac{1}{N}$, $< \prod_{j=1}^{n} Y_N(\omega_j) >_\beta$ factorizes, i.e. correlations are suppressed in the $N = \infty$ limit. The problem is to identify and compute the $N = \infty$ limit of $< Y_N(\omega) >_\beta$ and to then determine systematic corrections to $S_n$, in particular to $< Y_N(\omega) >_\beta$, in the form of power series in $\frac{1}{N}$. A somewhat heuristic calculation |39| yields

$$< Y_N(\omega) >_\beta = \sum_{\Sigma:\partial\Sigma=\omega} w(\beta,N,\Sigma), \tag{6.21}$$

where $\{\Sigma: \partial\Sigma = \omega\}$ are all surfaces built of oriented plaquettes (2-cells in $\mathbb{Z}^\nu$) bounded by the loop $\omega$, and $w(\beta,N,\Sigma)$ are the weights of these surfaces. One can argue |39,40| that, to leading order in $\frac{1}{N}$, only simply connected, normal surfaces, $\Sigma$, with $\partial\Sigma = \omega$ contribute to the r.s. of (6.21). The weights of these surfaces are $\propto \exp\left[-d_\beta |\Sigma|\right]$, with $|\Sigma|$ the total area of $\Sigma$. Moreover surfaces of higher genus (with handles) are suppressed by powers of $1/N^2$, |39,40|.

In spite of these preliminary findings a systematic expansion in $\frac{1}{N}$ is missing. To do that one must first find geometrical characterizations of all surfaces contributing to a given order in $1/N$, determine their weights and sum up their contributions. This appears to raise very subtle problems in the combinatorial geometry of lattice surfaces and combinatorics. (An alternate approach based on the techniques sketched in Example 2 has been suggested in |7|).

For a more detailed analysis and refs. see E. Witten's contribution, and for results concerning the $\frac{1}{\nu}$-expansion Parisi's contribution to these proceedings.

Related problems arise in the statistical mechanics of discrete polymers, of crystal growth, etc. A great deal of knowledge in combinatorial geometry required for the solution of such problems seems to be missing, at least among physicists.

## §7. Conclusions and Acknowledgments.

Here are some important open problems which are presumably central to the further development of quantized Yang-Mills theory.

(1) Proof of ultraviolet stability of quantized, non-abelian Yang-Mills theories and connections to renormalization group arguments. Use of "block spin" transformations. (Important progress in this direction in the super-renormalizable case has been announced by Bałaban |29|). See also |25|-|27|, |30|, |31|.

(2) Construction of algorithms permitting rigorous error estimates for the calculation of large scale (low energy) phenomena such as quark confinement, absence of coloured physical states (colour screening), Regge behavior of resonance spectrum, quark bound states, in QCD. (Along these lines one would like e.g. to test the validity of "instanton physics," set up calculable $1/_N$- and $\frac{1}{\nu}$-expansions and prove their asymptotic nature, extend the methods sketched in §6, Example 2, to the continuum limit, find rigorous connections to dual resonance models|see the contribution by J.-L. Gervais and A. Neveu|, etc.).

(3) Investigation of conservation laws and complete integrability (at the classical and quantum mechanical level) of pure, non-abelian Yang-Mills theory. (Existence of Bäcklund transformations, conserved currents?)

(4) Application and extension to theories with non-trivial S-matrix of the methods of Jimbo, Miwa and Sato to Yang-Mills theory. (Their methods are based on using Schwinger-Dyson equations for the Schwinger functionals discussed in §4 and the discontinuity properties (4.7), in conjunction with expressing the fields y in terms of the disorder fields b; see |21|).

In conclusion I wish to thank my collaborators, D. Brydges, B. Durhuus, E. Seiler and T. Spencer for all they have taught me and the joy of collaboration. They should have written these notes. Special thanks are due to H. Epstein, G. Mack and E. Seiler for numerous, very valuable discussions and encouragement. I also thank the organizers of the Cargèse School for inviting me to participate and lecture and for financial support.

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# ON THE CONSTRUCTION OF QUANTIZED GAUGE FIELDS

3.

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ABSTRACT

We give a very elementary introduction to the geometry of classical gauge fields. The "observables" of classical gauge theory are isolated, and discrete approximations are discussed. We then present a general formulation of quantized Yang-Mills theory and state a reconstruction theorem. Subsequently we exemplify the general scheme in terms of lattice theories. Some basic properties - - confinement, phase transitions, etc. - of lattice theories are discussed, and connections to dual resonance models are sketched. We finally outline the main steps in the construction of the two-dimensional, abelian Higgs model in the continuum - and thermodynamic limit.

These lecture notes summarize a small portion of some recent work on the description and construction of quantized gauge fields [1 - 7] . For its major part that work has been done in collaboration with D. Brydges and E. Seiler. There are two excellent reviews [8,9] by E. Seiler which the reader who does not want to read the original publications is advized to consult. Some conceptual and foundational aspects of quantized Yang-Mills theory are discussed in [10,11].

CONTENTS:



Sections I.1. and I.2. have an elementary, introductory character. (The advanced reader should skip them). They are, however, quite useful as a piece of motivation of the basic concepts discussed in Sections I.3. and II.1. The remaining sections are sketchy, and the reader should consult [1-9] .

# I. INTRODUCTION

In this section we try to introduce the main mathematical and physical notions concerning gauge fields.

## I.1. Classical gauge fields

Classical abelian and non-abelian gauge fields have been used implicitly in physics for a long time, namely in the classical mechanics of rigid bodies; ("3 index symbols"). I illustrate this point by means of an example which I learnt from E. Seiler and which serves to explain the concept of a principal bundle.

Consider a spherical ball of radius $\rho$ rolling on a two dimensional Riemannian surface, M, which we may choose for simplicity to be the Euclidean plane. ("Rolling" means that the point of contact with the plane on the ball is at rest at each instant). The orientation of the ball is described by a three-frame attached to the ball, the position of its center of mass by two coordinates $(x^1, x^2)$.

We propose to describe the motion of that three-frame as the ball is rolling along an arbitrary curve $\gamma \subset M$.

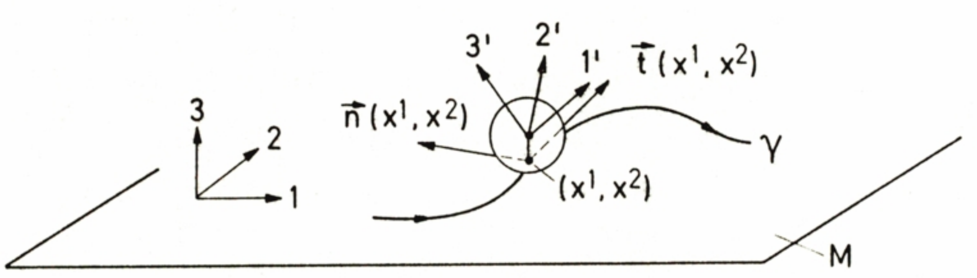


Fig. 1

The components of the vectors 1',2' and 3' in the basis 1,2,3 are given by the column vectors of an orthogonal matrix, $B(x,y)$. At the point $p = (x^1, x^2) \in M$ the ball is rolling in the direction $\vec{t}\ (x^1, x^2)$ tangential to the curve $\gamma$. It thus rotates around the axis $\vec{n}\ (x^1, x^2)$, the unit vector orthogonal to $\vec{t}\ (x^1, x^2)$. If the total displacement of the center of mass is $d\ell$ the rotation angle is $\rho^{-1}\ d\ell$.

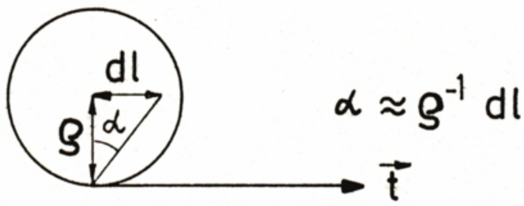


Fig. 2

Let $\vec{L} = (L_1, L_2, L_3)$ be the usual generators of rotations around the 1- , 2-, resp. 3- axis. Then the infinitesimal rotation of the ball is given by

$$B(x^1+dx^1, x^2+dx^2) = (\mathbb{1}+dR(x^1,x^2))\, B(x^1,x^2), \qquad \text{(I.1)}$$

where

$$dx^1 = t^1(x^1,x^2)d\ell \ , \ dx^2 = t^2(x^1,x^2)d\ell,$$

and

$$\begin{aligned} \mathbb{1} + dR(x^1,x^2) &= \mathbb{1} + \vec{n}(x^1,x^2) \cdot \vec{L}\, \rho^{-1} d\ell \\ &= \mathbb{1} + (t^1(x^1,x^2)L_2 - t^2(x^1,x^2)L_1)\, \rho^{-1} d\ell \\ &\equiv \mathbb{1} + \sum_{j=1}^{2} A_j\ (x^1,x^2)dx^j. \end{aligned} \qquad \text{(I.2)}$$

Thus

$$A_j(x^1,x^2) = \rho^{-1} \sum_{i=1}^{2} \varepsilon_{ji}\ L_i, \ j = 1,2. \qquad \text{(I.3)}$$

The 1-form $A=(A_1,A_2)$ with values in so(3), the Lie algebra of SO(3), given in (I.3), is called a <u>connection</u> (on a "principal SO(3) bundle with base space M").

The components $a^\alpha = (a_1^\alpha, a_2^\alpha)$ defined by

$$\left.\begin{array}{l} a_1^1 = 0,\ a_2^1 = \rho^{-1}, \\ \qquad\qquad\qquad a^3 = 0 \\ a_1^2 = -\rho^{-1},\ a_2^2 = 0, \end{array}\right\} \qquad \text{(I.4)}$$

are called <u>vector potential</u>.

Next, imagine the ball is rolling around a small rectangle with sides parallel to the 1- and the 2- axis of length $\varepsilon$, $\delta$ , respectively.

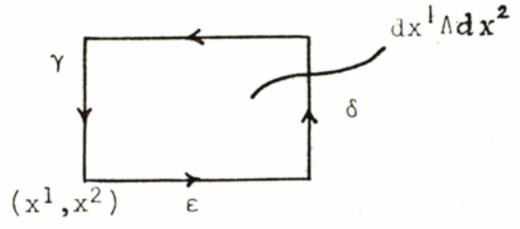


Fig. 3

We propose to determine the total rotation, $\Delta R$, of the ball after one round trip along the curve $\gamma$ depicted in Fig. 3 to second order in $\varepsilon$ and $\delta$ . A simple calculation gives

$$\begin{aligned}\Delta R &= \mathbb{1} - [A_1, A_2]\ \varepsilon \cdot \delta \\ &= \mathbb{1} + \rho^{-2}[L_2, L_1]\ \varepsilon \cdot \delta \\ &= \mathbb{1} - \rho^{-2}\ L_3\ \varepsilon \cdot \delta \\ &\equiv \mathbb{1} - \frac{1}{2} \sum_{i,j=1}^{2} F_{ij}\ dx^i \wedge dx^j,\end{aligned} \tag{I.5}$$

i.e. $F_{ij} = [A_i, A_j] = \rho^{-2} L_3$

If the radius, $\rho$, of the ball depended on $(x^1, x^2)$, i.e. $\rho = \rho(x^1, x^2) \neq$ const., we would find

$$F_{ij}(x^1, x^2) = \frac{\partial A_j}{\partial x^i}(x^1, x^2) - \frac{\partial A_i}{\partial x^j}(x^1, x^2) + [A_i, A_j]\ (x^1, x^2), \tag{I.6}$$

$$\text{with } A_j(x^1, x^2) = \rho(x^1, x^2)^{-1} \sum_{i=1}^{2} \varepsilon_{ji}\ L_i \tag{I.7}$$

The 2- form F is called <u>curvature</u>, its components $F^{\alpha}_{ij}$ in the basis $(L_1, L_2, L_3)$ of so(3) are called <u>field strength</u>.

Suppose now that, at each point $p = (x^1, x^2)$ of the plane M, we introduce a new coordinate system $1"_p, 2"_p, 3"_p$ related to the system

1,2,3 by an orthogonal transformation $O = O(x^1,x^2)$. The orientation of the frame 1',2',3' relative to $1''_p, 2''_p, 3''_p$ is thus given by an orthogonal matrix

$$B^O(x^1,x^2) = O(x^1,x^2)^{-1}\, B(x^1,x^2).$$

By (I.1), (I.2) we have

$$B^O(x^1+dx^1,x^2+dx^2) = (1 + dR^O(x^1,x^2))\, B^O(x^1,x^2), \tag{I.8}$$

where

$$\begin{aligned} 1 + dR^O(x^1,x^2) &= O(x^1+dx^1,x^2+dx^2)^{-1}(1 + dR(x^1,x^2))O(x^1,x^2), \\ &\equiv 1 + \sum_{j=1}^{2} A_j^O\,(x^1,x^2)dx^j. \end{aligned} \tag{I.9}$$

Hence

$$\begin{aligned} A_j^O\,(x^1,x^2) &= O(x^1,x^2)^{-1}\, A_j(x^1,x^2)\; O(x^1,x^2) \\ &\quad - O(x^1,x^2)^{-1}\; \frac{\partial O(x^1,x^2)}{\partial x^j} \end{aligned} \tag{I.10}$$

The mapping $O:M \rightarrow SO(3), (x^1,x^2) \mapsto O(x^1,x^2)$ is called a <u>gauge transformation.</u>

It follows easily from the definition of curvature that

$$F_{ij}^O(x^1,x^2) = O(x^1,x^2)^{-1}\; F_{ij}(x^1,x^2)O(x^1,x^2) \tag{I.11}$$

From the example discussed here the reader can, in principle, abstract most basic notions concerning principal bundles. But see [12].

Next we single out a vector $\vec{k} \in S^2$ (the unit sphere) attached to the ball, i.e. over each point $p \in M$ we have a two-sphere of possible positions of $\vec{k}$. The motion of the vector $\vec{k}$ as the ball is rolled from $(x^1,x^2)$ to $(x^1+dx^1,x^2+dx^2)$ is clearly described by

$$\vec{k}(x^1+dx^1,x^2+dx^2) = (1+A_j(x^1,x^2)dx^j)\vec{k}(x^1,x^2) \tag{I.12}$$

(We have started here to apply the summation convention).

Thus the connection A determines what one calls <u>parallel transport</u> of $\vec{k}$.

Under gauge transformations, $O$, $\vec{k}$ obviously transforms according to the equation

$$\vec{k}^{\,O}(x^1,x^2) = O^{-1}(x^1,x^2)\,\vec{k}\,(x^1,x^2). \tag{I.13}$$

This transformation law leaves (I.12) form-invariant if $A \to A^O$ is given by (1.10).

This example can be generalized as follows: Suppose the mass density of the ball is not rotation invariant. Then the ball will have a moment of inertia, $\Theta$ , which is a symmetric tensor of rank 2 not proportional to a multiple of the identity. With respect to rotations of the ball, $\Theta$ transforms according to a direct sum of the trivial (tr $\Theta$) and a spin 2 $(\Theta - \frac{1}{3}\,\mathrm{Tr}\,\Theta \cdot \mathbb{1})$ representation. More generally, the ball may have some intrinsic properties described by a quantity $\phi$ that transforms according to some representation U of SO(3) when the ball is rotated. It will be no surprise to learn that the parallel transport of $\Phi$ from $(x^1,x^2)$ to $(x^1+dx^1,x^2+dx^2)$ is given by

$$\Phi\,(x^1+dx^1,x^2+dx^2) = (\mathbb{1}+U(A_j(x^1,x^2))dx^j)\,\Phi(x^1,x^2), \tag{I.14}$$

and the gauge transformations by

$$\Phi^{O}(x^1,x^2) = U(O(x^1,x^2)^{-1})\Phi(x^1,x^2). \tag{I.15}$$

What we have discussed here can be extended to the case where M is a general two-dimensional manifold (surface). In this way one can picture many basic notions concerning fibre- and principal bundles with connection.

We end this section by briefly describing how the notions developed in the context of the rolling-ball example apply to classical field theory.

Let M be some manifold, physically the space-(imaginary) time manifold. We consider a classical, physical system described by some field $\Phi$ on M. The field $\Phi$ is supposed to have some internal "degrees of freedom" described as follows: For each point $x \in M$, $\Phi(x)$ is an element of some topological space $V_x$, homeomorphic to some fixed space V. Typically, V is a vector - or a homogeneous space. We also suppose that we are given a topological group G of homeomorphisms of V, physically speaking a group of internal symmetries. We are describing here what the mathematicians call a fibre bundle (with base space M, fibre V and group G), and $\Phi$ is called a cross-section of this bundle. For the moment (and in all examples discussed in subsequent sections) we may imagine that $V_x = V$, for all $x \in M$, and that the bundle is homeomorphic to $M \times V$. (This is however not so e.g. in the theory of the Wu-Yang magnetic monopole or the Yang-Mills instantons on the four sphere).

If one tries to make a dynamical theory of the field $\Phi$ one must be able to couple $\Phi(x)$ to $\Phi(y)$, for $x \neq y$, in other words, one must be able to compare $\Phi(x)$ and $\Phi(y)$, for $x \neq y$. However, a priori, the points in fibres over distinct points of the base space M cannot be compared, unless there is a notion of parallel-transporting $\Phi(x)$ from x to y along a curve $\gamma_{yx}$ joining x to y. (In the example of the ball, parallel-transporting consisted of rolling). If M is a manifold, i.e. continuous, parallel transport can only be defined if G is a Lie group. In that case, suppose we are given a 1-form A on M with values in the Lie algebra $\mathfrak{g}$ of G.

Given $\Phi(x)$, let $\Phi_\gamma(y,x)$ denote the parallel transport (or - displacement) of $\Phi(x)$ from x to y along $\gamma$.

If x and y = x+dx are infinitely proximate the parallel displacement, $\Phi(x+dx,x)$, of $\Phi(x)$ from x to x+dx is defined by the formula

$$\Phi(x+dx,x) = (\mathbb{1}_V + A_j(x)dx^j)\ \Phi(x),\ j= 1,\dots,\nu. \qquad (I.16)$$

The 1-form A is the connection or gauge field. (In the example of the ball "parallel displacement" is the same as rolling). Equation (I.16) permits to calculate $\Phi_\gamma(y,x)$, without y being infinitely proximate to x, see Section II.2, and to define the covariant gradient: Let t be some vector in the tangent space at x. We set

$$\begin{aligned} t\cdot(\nabla_A\Phi)(x) &= \lim_{h\to 0} h^{-1}(\Phi(x+ht,x) - \Phi(x)). \\ &= (\nabla\Phi)(x) + A(x)\,\Phi(x). \end{aligned} \qquad (I.17)$$

(If M is not flat the expression in the middle requires some obvious changes).

It is easy to see that, (with $\partial_i \equiv \partial/\partial x^i$),

$$F = [\nabla_A, \nabla_A],\ F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j] \qquad (I.18)$$

corresponds to what was called curvature in the rolling-ball example. It is called curvature (2-form) or field strength.

Gauge transformations are homeomorphisms

$$h(x) : V_x \longrightarrow V_x;$$

$\Phi$ and A transform according to

$$\Phi(x) \longmapsto \Phi^h(x) = h(x)^{-1}\Phi(x),$$
$$A(h) \longmapsto A^h(x) = h(x)^{-1}A(x)h(x) - h(x)^{-1}(\nabla h)(x) \tag{I.19}$$

The dynamics of $\Phi$ can be specified with the help of a field equation, e.g.

$$\nabla_A(\nabla_A \Phi)(x) = m^2 \Phi(x), \tag{I.20}$$

(the covariant Klein-Gordon equation).

One may wish to introduce dynamics for the connection A, itself. A prominent example of field equations for A is the Yang-Mills equations

$$\nabla_A \cdot F = 0. \tag{I.21}$$

(If $F = [\nabla_A, \nabla_A]$ the equations

$$\nabla_A \cdot (*F) = 0 \tag{I.22}$$

are automatic. They are called Bianchi identities).

For discussions of classical field equations, see e.g. [13] and refs. given there. They will not be studied in the present notes.

The basic ansatz of present day elementary particle physics (without gravitational interactions) is to describe matter in terms of quantized versions of fields that are cross-sections of fibre bundles with connection and the fundamental interactions of matter in terms of quantized versions of those connections. The present choices for G are such that it equals or contains as a subgroup

$$SU(3)_{colour} \times SU(2)_{weak} \times U(1)_{electromag.}. \tag{I.23}$$

Although it is appealing that present day physics of matter and its fundamental interactions has become intrinsically geometrical it remains unsatisfactory that two kinds of geometries are involved, Riemannian (or affine) geometry in gravity, the geometry of fibre bundles in strong and electroweak interactions. Moreover, there is no convincing theoretical argument as to what the right fibre bundle (the right gauge group G) of elementary particle physics is.

We shall henceforth ignore these problems and proceed to sketch some rigorous results concerning quantum field theories that are based on fibre bundle geometry.

## I.2. Some facts about the geometry of fibre bundles

The intuitive concept of fibre- and principal bundles has been developed in Section I.1. What the mathematicians understand by these words can be looked up e.g. in [12]. For our purposes the following may suffice:

Let M be the physical space-(imaginary)time manifold. Let V be a topological space with a topological group G of homeomorphisms of V into itself. Throughout these notes G will be a compact Lie group. Points in M are denoted $x, y, \ldots$, $\Phi$ denotes a point in V, and h, g, ... elements of G. A fibre bundle $\mathcal{F}$ over M with fibre V and group G consists roughly of a bundle space F with projection $\pi$ such that, for all $p \in F$, $\pi(p) \in M$, and for all $x \in M$ $V_x := \pi^{-1}(x)$, the fibre over x, is homeormorphic to V.

For each $x \in M$, there is an open neighborhood $\Omega \subset M$ of x and a homeomorphism $\xi_\Omega : \Omega \times V \longrightarrow \pi^{-1}(\Omega)$ such that $\pi \xi_\Omega(x, \Phi) = x$, and $\xi_{\Omega, x}(\Phi) := \xi_\Omega(x, \Phi)$ is a homeomorphism from V to $V_x$. If $\xi_y$, $\xi_y'$, $y \in \Omega$, are two homeomorphisms from V to $V_y$ then $h(y) = \xi_y^{-1} \xi_y'$ is supposed to be a continuous function of $y \in \Omega$ with values in G. The functions h are called gauge transformations. Finally, for

$$y \in \Omega \cap \Omega', \quad g_{\Omega\Omega'}(y) := \xi_{\Omega, y}^{-1} \, \xi_{\Omega', y}$$

is supposed to be a continuous, G-valued function of y. It is called transition function.

If V happens to be the group G, we speak of a principal bundle (with base space M). The group G is called gauge group.

It follows from these definitions that bundles can be characterized by means of their transition functions:

Let $\{\Omega_i\}_{i \in I}$ be a cover of M by open neighborhoods with the property that for all $i \in I$ there exists a homeomorphism

$$\xi_{\Omega_i} : \Omega_i \times V \longrightarrow \pi^{-1}(\Omega_i)$$

with all the properties specified in the above definition. For

$$\Omega_i \cap \Omega_j \neq \emptyset, \text{ let } g_{ij} := g_{\Omega_i \Omega_j}$$

denote the transition function. Two sets of transition functions

$$\{g_{ij}\}, \ \{g_{ij}'\}$$

determine equivalent bundles iff

$$g'_{ij} = h_i^{-1}\, g_{ij}\, h_j, \tag{I.24}$$

with $h_j(x)$ a G-valued function on $\Omega_j$.

This permits to associate with each fibre bundle $\mathcal{F} = (F, M, V, G, \pi)$ a principal bundle $\mathcal{P} = (P, M, V = G, G, \pi)$: $\mathcal{P}$ is the principal bundle with the same transition functions as $\mathcal{F}$. See [12] for details.

Examples: (1) Möbius strip (base space $S^1$ = circle, fibre $[-1,1]$, group $\mathbb{Z}_2$); (2) $F = M \times V$, $G = \{\mathbb{1}\}$, $\pi(\{x,\phi\}) = x$; this is called the product bundle; (3) The 3-sphere $S^3$ is a principle bundle with base space $S^2$, fibre $S^1 \cong U(1)$ and group $U(1)$. Incidentally, this is the bundle space of the instanton of the two-dimensional $\mathbb{C}P^1\sigma$-model and of the Wu-Yang monopole.
(4) Interesting examples arise in the theory of functions of complex variables.

Next, we consider fibre bundles with connections, i.e. we reconsider the notion of parallel transport (or - displacement).

Let $\mathcal{F} = (F, M, V, G, \pi)$ and $\{\Omega_i\}_{i\in I}$ be as above.

We suppose G is a (compact) Lie group with Lie algebra $\mathfrak{g}$. We assume that all transition functions are continuously differentiable on their domain of definition.

A connection, A, on $\mathcal{F}$ is a family of 1-forms
$\{A^{(i)}\}_{i\in I}$
with values in $\mathfrak{g}$ such that $A^{(i)}$ is defined on $\Omega_i$, $i\in I$, and for $x\in\Omega_i\cap\Omega_j \neq \emptyset$,

$$\begin{aligned} A^{(i)}(x) &= g_{ij}(x)A^{(j)}(x)g_{ij}^{-1}(x) - g_{ij}(x)(dg_{ij}^{-1})(x) \\ &= g_{ji}^{-1}(x)A^{(j)}(x)g_{ji}(x) - g_{ji}^{-1}(x)(dg_{ji})(x) \end{aligned} \tag{I.25}$$

Moreover, if h is a gauge transformation defined on $\Omega_i$, $A^{(i)}$ transforms according to

$$A^{(i)} \longmapsto A^{h(i)} = h^{-1}A^{(i)}h - h^{-1}dh. \tag{I.26}$$

We have started, here, to use the notation

$$A = \sum_{j=1}^{\nu} A_j\, dx^j \tag{I.27}$$

We shall see that a connection is precisely what we need to define parallel transport on F.

Next, let $\Omega \subseteq \Omega_i$, for some $i \in I$, be some open subset of M.

Restricted to $\Omega$, F is homeomorphic to

$$F'_\Omega = \Omega \times V.$$

First we define parallel transport on $F'_\Omega$.

Let $x \in \Omega$, $\Phi(x) \in V$. We want to define the parallel transport,

$g_{\gamma_{yx}} \Phi(x)$, of $\Phi(x)$ from x to $y \in \Omega$

along a curve $\gamma_{yx} \subset \Omega$, with $g_{\gamma_{yx}} \in G$ a homeomorphism from V onto V, given the connection $A = A^{(i)}$. Suppose $y = x+dy$ is infinitely proximate to x. Then

$$g_{\gamma_{yx}} \Phi(x) \equiv g_{x+dx,x} \Phi(x) := (\mathbb{1}_V + A(x))\ \Phi(x), \tag{I.28}$$

with

$$A(x) = \sum_{j=1}^{\nu} A_j(x)\ dx^j$$

This equation can be integrated along any oriented, continuous, piecewise smooth curve $\gamma_{yx} \subset \Omega$ connecting $x \in \Omega$ with $y \in \Omega$. To see this we may temporarily assume that $\Omega$ is flat, i.e. $\Omega$ is a subset of $\mathbb{R}^\nu$.

Let $\{x_i^k\}_{i=1}^{N_k} \subset \gamma_{yx}$ be a family of ordered sequences of points on $\gamma_{yx}$ with the property that

$x_1^k = x$, $x_{N_k}^k = y$, for all $k = 1,2,3,\ldots$, and

$\mathrm{dist}\,(x_i^k, x_{i+1}^k) \longrightarrow 0$, as $k \longrightarrow \infty$, for all $i=1,\ldots,N_k-1$.

Then

$$g_{\gamma_{yx}} = \lim_{k\to\infty} \prod_{k=N_k}^{1} \{\mathbb{1}_V + A_j(x_i^k)(x_{i+1}^k - x_i^k)^j\} \tag{I.29}$$

The physicists like the following compact formula

$$g_{\gamma_{yx}} = P\ \{\exp \int_{\gamma_{yx}} A_j(z)dz^j\} \tag{I.30}$$

as an abreviation for the r.s. of (I.29); (P:= "path ordering").
It follows from (I.26) and (I.29) that, under a gauge transformation $h$, $g$ transforms according to

$$g_{\gamma_{yx}} \longmapsto g^{h}_{\gamma_{yx}} = h^{-1}(y)\, g_{\gamma_{yx}}\, h(x). \tag{I.31}$$

This is the basic property that permits us to define parallel transport on $F$ in terms of $g$. It is given by a homeomorphism

$\Gamma_{\gamma_{yx}} : V_x \longrightarrow V_y$, defined by

$$\Gamma_{\gamma_{yx}} = \xi_{\Omega,y}\; g_{\gamma_{yx}}\; \xi^{-1}_{\Omega,x} \tag{I.32}$$

Note that if $\xi_\Omega$ and $\xi'_\Omega$ are two homeomorphisms related by a gauge transformation, i.e. $h(y) \equiv \xi^{-1}_{\Omega,y}\, \xi'_{\Omega,y} \in G$ then

$$\Gamma_{\gamma_{y,x}} = \xi_{\Omega,y}\; g_{\gamma_{y,x}}\; \xi^{-1}_{\Omega,x} = \xi'_{\Omega,y}\, h^{-1}(y) g_{\gamma_{y,x}}\, h(x)\, \xi'_{\Omega,x}$$

$$= \xi'_{\Omega,y}\; g^{h}_{\gamma_{y,x}}\; \xi'_{\Omega,x}\,, \tag{I.33}$$

i.e. $\Gamma_{\gamma_{y,x}}$ is independent of the choice of coordinates (the gauge). Equations (I.25) and (I.33) permit us to define $\Gamma_{\gamma_{yx}}$ for curves $\gamma_{yx}$ that are not contained in a single coordinate neighborhood $\Omega_i$: One cuts up $\gamma_{yx}$ into curves

$$\gamma_{x_{\alpha+1}\, x_\alpha}$$

contained in $\Omega_{i(\alpha)}$, with

$\Omega_{i(\alpha)} \cap \Omega_{i(\alpha+1)} \neq \emptyset$ and sets

$$\Gamma_{\gamma_{yx}} = \Gamma_{\gamma_{x_N x_{N-1}}}\; \Gamma_{\gamma_{x_{N-1} x_{N-2}}} \;\cdots\; \Gamma_{\gamma_{x_2 x_1}} \tag{I.34}$$

with $x_N = y$, $x_1 = x$. By (I.25), (I.31) and (I.33) this is a consistent definition.

One may now ask the question under what conditions does

$$\Gamma_{\gamma_{yx}} \quad (\text{resp.}\, g_{\gamma_{yx}})$$

depend on the path $\gamma_{yx}$ only through the endpoints x and y. We first discuss this question locally, for $\gamma_{yx}$ in a simply connected, open set $\Omega \subseteq \Omega_i$, for some $i \in I$.
In this case the answer is very simple: If and only if $g_{\gamma_{yx}}$ is of the form

$$g_{\gamma_{yx}} = h(y)\ h(x)^{-1} \tag{I.35}$$

Clearly (I.35) is sufficient. To see that it is necessary one chooses a point $x_o \in \Omega$ and sets $h(x_o) = \mathbb{1}_V$. One then chooses a family, $L_{x_o}$, of piecewise smooth, oriented curves, $\hat{\gamma}$, starting at $x_o$ with the property that each $x \in \Omega$ is contained in precisely one line $\hat{\gamma} \in L_{x_o}$. Let $\hat{\gamma}_{xx_o}$ be the portion of $\hat{\gamma}$ with endpoints $x_o$ and x. We set

$$h(x) = g_{\hat{\gamma}_{xx_o}} \quad .$$

Let x and y be arbitrary points in $\Omega$ and $\gamma_{yx}$ a path connecting them. Since $g_\gamma$ only depends on the endpoints of $\gamma$

$$g_{\hat{\gamma}_{yx_o}} = g_{\gamma_{yx}}\ g_{\hat{\gamma}_{xx_o}} , \qquad \text{i.e.}$$

$$g_{\gamma_{yx}} = g_{\hat{\gamma}_{yx_o}}\ g_{\hat{\gamma}_{xx_o}}^{-1} = h(y)\ h(x)^{-1}$$

which proves (I.35).

The <u>curvature</u>, F, of a connection A is defined by

$$F = dA + A \wedge A, \tag{I.36}$$

i.e. $F_{ij} = \partial_i A_j - \partial_j A_i + [A_i, A_j]$ ; see also (I.18).

We now claim that <u>for</u> (I.35) <u>to hold it is necessary and sufficient that F vanishes on $\Omega$</u>.
(In the example of the rolling ball this can only happen in the limit $\rho = \rho(x^1,x^2) \longrightarrow \infty$, for all $(x^1,x^2) \in \Omega$).
A proof of this last assertion can be obtained from the following consideration that is of independent interest: Pick a curve $\gamma_{yx} \subset \Omega$. Parametrize $\gamma_{yx}$ by a function

$$x(s) = (x^1(s), \ldots, x^\nu(s)), \quad o \leq s \leq 1,$$
with $x(o) = x$, $x(1) = y$.

Now consider a function

$$x : [-1,1] \times [-1,1] \longrightarrow \Omega, \quad (s,t) \longmapsto x(s,t)$$

which is smooth in t and such that $x(o,t) = x, x(1,t) = y$, for all t, and $x(s,o) = x(s)$. Let $\gamma_{yx}(t)$ be the curve parametrized by $x(s,t)$. (It is a deformation of $\gamma_{yx}$ leaving the endpoints fixed). We propose to calculate

$\frac{d}{dt} g_{\gamma_{yx}}(t)$. Let $\gamma_{x(s',t)x(s,t)}$ be the portion of $\gamma_{yx}(t)$ starting at $x(s,t)$ and ending at $x(s',t)$. Using (I.29) it is easy to see that

$$\frac{d}{dt} g_{\gamma_{yx}}(t) = \int_o^1 ds\, g_{\gamma_{yx(s,t)}} \left[ \frac{\partial A_i}{\partial x^j}(x(s,t)) \, \frac{\partial x^i(s,t)}{\partial s} \, \frac{\partial x^j(s,t)}{\partial t} \right] g_{\gamma_{x(s,t)x}} \tag{I.37}$$

It is a simple exercise in integration by parts to show that

$$\int_o^1 ds \;\; g_{\gamma_{yx(s,t)}} \left\{ \frac{\partial \Psi}{\partial x^i} (x(s,t)) - \left[ \Psi, A_i \right] (x(s,t)) \right\} \times \frac{\partial x^i(s,t)}{\partial s} \; g_{\gamma_{x(s,t)x}} = 0 \tag{I.38}$$

for any differentiable function $\Psi$ on $\Omega$ . If we set

$$\Psi(x(s,t)) = A_j(x(s,t)) \frac{\partial x^j(s,t)}{\partial t}$$

in (I.38) and use (I.36),(I.37), we find

$$\frac{d}{dt} \; g_{\gamma_{yx}}(t) = \int_o^1 ds \;\; g_{\gamma_{yx(s,t)}} \left[ F_{ji}(x(s,t)) \, \frac{\partial x^i(s,t)}{\partial s} \cdot \frac{\partial x^j(s,t)}{\partial t} \right] \times g_{\gamma_{x(s,t)x}} \tag{I.39}$$

Incidentally, by differentiating both sides of (I.39) with respect to t, applying (I.38) with

$$\Psi \;(x(s,t)) = F_{jk}(x(s,t)) \; \frac{\partial x^j(s,t)}{\partial t} \cdot \frac{\partial x^k(s,t)}{\partial t} \,,$$

and using the Bianchi identity

$$dF + [A,F] = 0 \tag{I.40}$$

one may calculate

$$\frac{d^2}{dt^2} g_{\gamma_{yx}}(t) .$$

(For a slightly cumbersome way of calculating this see e.g.[14]. The physicists have been interested in equations for

$$\sum_\ell \frac{\partial^2}{\partial t_\ell^2} g_{\gamma_{yx}(t_1,\ldots,t_\ell,\ldots)} ,$$

because they suggest formal connections between Yang-Mills theory and dual resonance models [15]. To the author these connections appear, however, somewhat superficial).

As a simple corollary of equation (I.39) we have:

$$g_{\gamma_{yx}} \text{ \underline{depends only on} } x \text{ \underline{and} } y \text{ \underline{if and only if} } F = 0. \tag{I.41}$$

Suppose now we are given two connections $A, A'$ on $\Omega$ such that $F \equiv F(A) = F(A') \equiv F'$.

Question: Are $A$ and $A'$ gauge-equivalent, in the sense of eqn.(I.26)? Unless $G$ is abelian, the answer is: In general they are not gauge-equivalent. (The reader can find a simple example of this by studying the rolling-ball example!) This is an aspect of the intrinsic non-linearity of non-abelian gauge fields.

The "globalization" of the above considerations is only straightforward if the base space $M$ is simply connected.(Recall the Bohm-Aharonov effect).

**We now skip some material roughly identical to the one in 2 , § 2, (2.9) through end of § 2. The correspond nce is given by**

3 (I.42) $\longleftrightarrow$ 2 (2.10)
3 (I.43) $\longleftrightarrow$ 2 (2.11)
3 (I.44) $\longleftrightarrow$ 2 (2.13)
3 Theorem (I.45) $\longleftrightarrow$ 2 Theorem 2.1
3 Theorem (I.47) $\longleftrightarrow$ 2 Theorem 2.2
3 Remarks 1), 2) $\longleftrightarrow$ 2 Remarks 1,2 .

## I.3 A tentative general formulation of quantized Yang-Mills theory

Let $M = \mathbb{R}^\nu$ be Euclidean space-time. Let G denote the gauge group. Every principal bundle with base space $M=\mathbb{R}^\nu$ is equivalent to a product bundle, i.e. we may consider $M \times G$ to be the bundle space. Motivated by Theorem (I.47), Section I.2, we regard the functions $Y(\mathcal{L}) = \chi(g_{\mathcal{L}})$, where $\chi$ is an arbitrary unitary character of G as the basic "fields" of a Euclidean gauge theory on M with gauge group G. The purpose of this section is to propose a scheme for quantization of such a theory by which the $Y(\mathcal{L})$'s are converted into random fields on the space L of all oriented, smooth loops on M.

If one studies the example of free electromagnetism in $\nu=4$ dimensions as a theory of loop observables, the so called Wilson loops it becomes clear that one should require all loops in L to be at least twice continuously differentiable, oriented closed loops which are free of self-intersections, ("selfavoiding loops"). From now on L will be understood to be the space of all loops which have this property. (Classically, for a space $\mathfrak{A}^I$ of continuous, irreducible gauge fields on M, the algebra generated by $\{Y(\mathcal{L}) = \chi(g_{\mathcal{L}}) : \mathcal{L} \in L\}$ is still dense in $C(\mathcal{O})$, if $\chi$ is faithful).

We propose to discuss quantized gauge theories in terms of Euclidean Green's - or Schwinger functionals

$$S_n(Y_1(\mathcal{L}_1), \ldots, Y_n(\mathcal{L}_n)) \quad (Y_j(\mathcal{L}_j) = \chi_j(g_{\mathcal{L}_j})),$$

corresponding to "quantized versions" $y_j(\mathcal{L}_j)$ of the functions $Y_j(\mathcal{L}_j)$.

First, we describe this program heuristically.

Let $U = \int d^\nu x \operatorname{tr}(F_{\mu\nu}(x)\, F^{\mu\nu}(x))$ (I.48)

denote the classical Euclidean Yang-Mills action. (It is assumed here that G is a subgroup of some unitary matrix group. Then the r.s. of (I.48) is well defined). Let d [A] denote a formal "Lebesgue measure" on the orbit space $\mathcal{O}$ of (very rough) gauge fields modulo gauge transformations. Consider the formal probability measure on $\mathcal{O}$ given by

$$d\mu([A]) = Z^{-1}\, e^{-\beta U([A])} d[A], \tag{I.49}$$

with $Z = \int_{\mathcal{O}} e^{-\beta U([A])} d[A]$,

which, mathematically, is perfectly meaningless.

Heuristically, the Schwinger functionals are given by the Euclidean Gell'Mann-Low formula

$$S_n(Y_1(\mathcal{L}_1), \ldots, Y_n(\mathcal{L}_n)) = \int_{\mathcal{O}} d\mu([A]) \prod_{j=1}^{n} N(Y_j(\mathcal{L}_j)) . \tag{I.50}$$

In the case of free electromagnetism, (I.49) and (I.50) can be given a rigorous, mathematical meaning if one defines N(Y( )) by

$$N(Y(\mathcal{L})) = \exp\left[i\oint_{\mathcal{O}} A_\mu(x)\, dx^\mu\right] e^{K|\mathcal{L}|} \tag{I.51}$$

where $K = \beta^{-1}$" $\lim\limits_{|x|\to 0} 1/4\pi|x|$

is a divergent (normal ordering) constant, and $|\mathcal{L}|$ is the length of $\mathcal{L}$. (We have chosen $A_\mu$ to be real-valued here. (I.50) and (I.51) have to be understood as limits of regularized objects. See [ 5] for some general considerations concerning (I.51) ). In this example one can see explicitly that $S_n(Y_1(\mathcal{L}_1), \ldots, Y_n(\mathcal{L}_n))$ diverges when

$$d(\mathcal{L}_i, \mathcal{L}_j) \quad \min_{\substack{x \in \mathcal{L}_i \\ y \in \mathcal{L}_j}} |x-y| \text{ tends to } 0, \text{ unless } \nu=2.$$

The remainder of this section is contained in 2 , § 4 . The notations of 2 are unfortunately somewhat different. Here is a key :

| | | | | |
|---|---|---|---|---|
| 3 | $\mathcal{L}$ | | 2 | $\omega$ |
| | $L$ | | | $\Omega^{(1)}$ |
| | $L^n_{\neq}$ | $\Leftrightarrow$ | | $\Omega^{(1)n}_{\neq}$ |
| | $L^n_{>}$ | | | $\Omega^{(1)n}_{>}$ |

| | | | | |
|---|---|---|---|---|
| 3 | (YM1) | $\Leftrightarrow$ | 2 | (S1) |
| | (YM2) | $\Leftrightarrow$ | | (S4) |
| | (YM3) | $\Leftrightarrow$ | | (S2) |
| | (YM3') | $\Leftrightarrow$ | | ($S2^{ext.}$) |
| | (YM4) | $\Leftrightarrow$ | | (S3) |
| | (YM5) | $\Leftrightarrow$ | | (S5) |

| | | |
|---|---|---|
| Theorem (I.55) | $\Leftrightarrow$ | Theorem 4.1 |
| (I.56) | $\Leftrightarrow$ | (4.4) |
| (I.57) | $\Leftrightarrow$ | (4.5) |

## II. LATTICE GAUGE THEORIES

One of the main virtues of lattice gauge theories is that they represent a gauge-invariant regularization of continuum gauge theories for which Schwinger functionals exist satisfying properties (YM1), (YM3), (YM3'), (YM4), (YM6), see Section I.3, and (YM$2_\ell$) Invariance under all those Euclidean motions which leave a lattice $\ell$, typically

$$\varepsilon \mathbb{Z}^\nu \equiv \{x : \varepsilon^{-1} x \in \mathbb{Z}^\nu\}, \ \varepsilon > 0, \ \text{invariant.}$$

This is still enough for the reconstruction of a quantum mechanical system, as described in Theorem (I.55), with the exception of full Poincaré covariance of the resulting theory. As a consequence, only a weak form of locality is verified for lattice theories. See [1,21-23].

In order to understand the basic structure and intrinsic properties of lattice gauge theories one is advised to go back to Remark 2), following Theorem (I.47), Section I.2: Let

$$\Lambda \equiv L_\ell = \{\mathcal{L}_1, \mathcal{L}_2, \ldots\}$$

be all finite, oriented closed loops composed of links of a lattice $\ell$, e.g. $\varepsilon \mathbb{Z}^\nu$. Let

$$C_{L_\ell}(\mathcal{O})$$

denote the algebra of functions on the orbit space, $\mathcal{O}$, of continuous, classical connections (gauge potentials) on Euclidean space-time, $\mathbb{R}^\nu$, modulo gauge transformations, generated by

$$\{Y_j(\mathcal{L}) = \chi_{Y_j}(g_{\mathcal{L}}) : \mathcal{L} \in L_\ell\},$$

with $\chi_Y$ arbitrary irreducible, unitary characters of the gauge group G.

Clearly, $C_{L_\ell}(\mathcal{O})$ is a separable approximation to the space $C(\mathcal{O})$ of all "observables" of a classical gauge theory.

The idea is now to convert the elements of

$$C_{L_\ell}(\mathcal{O})$$

into random variables distributed according to a probability measure $d\mu$ (a positive, normalized, continuous linear functional on $C_{L_\ell}(\mathcal{O})$) with the property that the Schwinger functionals

$$S_n(Y_1(\mathscr{L}_1),\ldots, Y_n(\mathscr{L}_n)) = \int d\mu \prod_{j=1}^{n} Y_j(\mathscr{L}_j) \qquad \text{(II.1)}$$

satisfy properties (YM1), ($YM2_\ell$),(YM3)-(YM6), (with the possible exception of (YM5) = clustering = uniqueness of the vacuum).

Thus, we are really trying to construct random fields $Y_j$ on the loop space $L_\ell$ of a lattice $\ell$ having the mathematical structure determined by ($YM1_\ell$)-($YM6_\ell$), (with ($YMn_\ell$)=(YMn), except for n=2). Given the values of all $Y_j(\mathscr{L})$, $\mathscr{L}\in L_\ell$, a simple variant of Theorem (I.45) shows that they determine a "lattice gauge field"

$$g = \{g_{xy} \in G : xy \in B_\ell\}$$

which is unique up to gauge transformations. Here $B_\ell$ is the set of all line segments, b(xy), whose endpoints, x and y, are "nearest neighbors" in $\ell$. The b's are called bonds or links. A variant of Theorem (I.47) shows that the closure of

$$C_{L_\ell}(\mathcal{O})$$

is the space of all gauge-invariant functions of

$$g = \{g_{xy} : xy \in B_\ell\}.$$

(We know from Section II.2, (I.31), that a gauge transformation $g \longrightarrow g^h$ is given by a function, h, on $\ell$ with values in the gauge group G, and

$$g^h_{yx} = h(y)^{-1} g_{yx}\, h(x) \ ).$$

Let

$$\overset{\curvearrowleft}{\prod_{xy\subset\mathscr{L}}}$$

denote an ordered product along an oriented loop (or curve) $\mathscr{L}$. From the above discussion we infer that

$$Y_j(\mathscr{L}) = \chi_{Y_j}(g_{\mathscr{L}}), \text{ where } g_{\mathscr{L}} = \overset{\curvearrowleft}{\prod_{xy\subset\mathscr{L}}} g_{xy}, \qquad \text{(II.2)}$$

and g is the lattice gauge field determined (up to gauge transformations) by the values of the $Y_j(\mathscr{L})$'s, $\mathscr{L}\in L_\ell$. From now on the random variables $Y_j(\mathscr{L})$ are called Wilson loops.

Other examples of random fields on a loop-space, $L_\ell$, are supplied by the lattice approximation to (Euclidean) dual resonance - or string models [26]. Among the main goals in the study of lattice gauge theories are

(A) Let $\mathscr{L}$ = L×T be a rectangular loop with sides parallel to two axes of $\ell$ of length L, resp. T.

Let

$$V_j(L) = \lim_{T\to\infty} -\frac{1}{T}\log S_1(Y_j(L\times T)); \tag{II.3}$$

see (I.56). Investigate the properties of $V_j(L)$, as $L\to\infty$, in particular for those characters $\chi_{Y_j}$ of G which are <u>non-trivial on the center</u> $\mathfrak{Z}_G$ of G.

This is the famous problem of <u>static quark confinement</u> (resp.- liberation). See [21,22,23,25,1,27,5,6,7] .

(B) Investigate the "excitation spectrum" (energy-momentum spectrum of low-lying "particles") in

$$S_1(Y_j(L\times T)) \text{ and } S_2(Y_j(\mathcal{L}),\ Y_j(\mathcal{L}'));$$

see e.g. [5,24,26]. This will supply information on the particle- and bound state content of lattice Yang-Mills theory.

(C) Improve the analysis in (A) and (B) in such a way that the results are uniform in the lattice spacing $\varepsilon$, $(\ell=\varepsilon\mathbb{Z}^\nu)$.

(D) For $\ell=\varepsilon\mathbb{Z}^\nu$ (and $\nu = 2,3,(4?)$), exhibit lattice gauge theories (other than free electromagnetism) with the property that the limits as $\varepsilon\searrow 0$, of the Schwinger functionals $S_n(Y_1(\mathcal{L}_1),\dots,Y_n(\mathcal{L}_n))$, $n = 1,2,\dots$, exist and satisfy (YM1)-(YM6) if the measures

$$\{d\mu = d\mu_\varepsilon\}_{\varepsilon>0}$$

are correctly renormalized and the Wilson loops, $Y_j(\mathcal{L})$, are correctly normal ordered. See [1,3,4] and [20] for results or progress in this direction.

<u>Remark concerning matter fields.</u>

For pedagogical reasons we shall only consider bosonic matter in these notes; but see [1,22,18,19] . Given a gauge group G, a lattice matter field $\Phi$ is a random field on the lattice $\ell$ with values in a Hilbert space V (usually finite dimensional) that carries a unitary representation $U^\Phi$, of G (as an endomorphism group). Thus

$$\Phi : x\in\ell \longmapsto \Phi(x)\in V.$$

Gauge transformations of $\Phi$ are of course defined by

$$\Phi \longrightarrow \Phi^h,\ \Phi^h(x) = U^\Phi(h(x))^*\ \Phi(x), \tag{II.4}$$

where h takes values in G. The random variables

$$Y^{\Phi}(\gamma_{xy}) \equiv (\Phi(x), U^{\Phi}(\gamma_{xy})\,\Phi(y)), \text{ with } g_{\gamma_{xy}} = \prod_{uv \in \gamma_{xy}}^{\circlearrowright} g_{uv}, \tag{II.5}$$

$\gamma_{xy} \subset B_\ell$ a connected, oriented curve starting at y and ending at x, are gauge-invariant. (These notions correspond to what is developed at the end of Section I.1 and in Section I.2).

### II.1 Some of the basics about lattice gauge theories

General results may be found in [1,2,21,22,23,25]. The <u>general ansatz</u> for the measures $d\mu = d\mu_\varepsilon$ is the lattice version of the Euclidean Gell'Mann-Low formula (I.49) (including a matter field $\Phi$):

$$d\mu_\varepsilon(\Phi,g) = Z_\varepsilon^{-1} e^{-\beta U_\varepsilon(\Phi,g)} Dg\, D\Phi_\varepsilon, \tag{II.6}$$

where

$$Z_\varepsilon = \int e^{-\beta U_\varepsilon(\Phi,g)} Dg\, D\Phi_\varepsilon, \quad Dg = \prod_{xy \in B_\varepsilon} dg_{xy},$$

with $B_\varepsilon = B_{\varepsilon \mathbb{Z}^\nu}$ and dg Haar measure on G,

$$D\Phi_\varepsilon = \prod_{x \in \varepsilon \mathbb{Z}^\nu} d\rho_\varepsilon(\Phi_x),$$

with $d\rho_\varepsilon$ a G-invariant probability measure on V, and

$$U_\varepsilon(\Phi,g) = U_\varepsilon^{YM}(g) + U_\varepsilon^{M}(\Phi,g) \tag{II.7}$$

a <u>lattice action</u>.

Wilson [21] was the first to propose lattice gauge theories and explicit expressions for (II.6) and (II.7). In the introduction to Section II we have proposed to view the lattice gauge field $g_{xy}$ as arising from a "nice" continuum gauge field (connection)

$$A, \text{ via } g_{xy} \equiv g_{b(x,y)} = P\{\exp \int_{b(x,y)} A_j(z)\,dz^j\}, \tag{II.8}$$

see (I.30), Section I.2.

This is particularly useful if the continuum gauge field A is known to exist as a random field with the desired properties, as is the case for free electromagnetism (G = U(1) ). In this case we may e.g. choose A to be a Gaussian random field with mean $0, \langle A_i(x) \rangle_t = 0$, and covariance

$$\langle A_i(x)\, A_j(y) \rangle_t = D_{ij}^{t}(x-y), \tag{II.9}$$

where $D^t_{ij}(x)$, $t \geq o$, is the Fourier transform of

$$(\delta_{ij} - {}^{p_i p_j}/(p^2 + \mu^2))\,(p^2+\mu^2)^{-1} e^{-t \sum_{j=1}^{\nu-1} p_j^2}. \qquad \text{(II.10)}$$

Here $t \geq o$ labels an ultraviolet cutoff, and $\mu \geq o$ is a bare mass introducing an infrared cutoff. As long as $t > o$, $g_{xy}$ given by (II.8) is well defined for A as in (II.9),(II.10), in arbitrary dimenion $\nu$. For $\mu > o$, the resulting lattice U(1) theory is not gauge-invariant, but when $\mu = o$, gauge invariance is restored even for $t > o$; see [1,3] . These observations are useful in the construction of the two-dimensional abelian Higgs model in the continuum limit [1,3,4 ] which we sketch in Section III. For G non-abelian and $\nu > 2$, no such construction of a lattice approximation is known. Instead one recurs to (II.6) and (II.7) with

$U^{YM}_\varepsilon$ and $U^M_\varepsilon$

conventionally given by

$$U^{YM}_\varepsilon = - \sum_p \varepsilon^{\nu-4} \chi(g_{\partial p}),\; g_{\partial p} = \prod_{xy \subset \partial p}^{\circlearrowright} g_{xy} \qquad \text{(II.11)}$$

where $\chi$ is a faithful, unitary character of G, (e.g. the character of the fundamental representation if G is a unitary matrix group), and p denotes the unit squares (plaquettes, with boundary p = four links) in $\varepsilon \mathbb{Z}^\nu$;

$$U^M_\varepsilon = (1/2\beta) \sum_{yx \in B_\varepsilon} \varepsilon^{\nu-2} \left\| \Phi(x) - U^\Phi(g_{xy})\,\Phi(y) \right\|^2, \qquad \text{(II.12)}$$

and

$$d\rho_\varepsilon(\Phi) \overset{e.g.}{=} \exp\left[-\varepsilon^\nu\left(\left(\frac{m^2}{2}\right)\|\Phi\|^2 - \lambda :\|\Phi\|^4:_\varepsilon\right)\right] d\Phi, \qquad \text{(II.13)}$$

where $:-:_\varepsilon$ denotes a Wick order, and $d\Phi$ is the Lebesgue measure on V.

In order to start with a well defined expression, one first restricts the summation on the r.s. of (II.11) to plaquettes p contained in some bounded set $\Lambda \subset \varepsilon \mathbb{Z}^\nu$ and the one on the r.s. of (II.12) to links $xy \subset \Lambda$ . By (II.6) this yields a cutoff measure $d\mu_{\varepsilon,\Lambda}(\Phi,g)$. If $\Lambda$ belongs to a sequence of hypercubes and periodic boundary conditions are imposed at $\partial\Lambda$ then a weak limit, $d\mu_\varepsilon(\Phi,g)$, (the thermodynamik limit), of the measures $d\mu_{\varepsilon,\Lambda}(\Phi,g)$ as $\Lambda \uparrow \varepsilon \mathbb{Z}^\nu$ , can be constructed by a standard compactness argument. The lattice Schwinger functions

$$S^{(\varepsilon)}_{n,m}\,(Y_1(\mathcal{L}_1),\ \ldots\ldots, Y^{\Phi}\,(\gamma^1_{x_1,y_1}),\ldots\ldots)$$

$$= \int d\mu_{\varepsilon,\Lambda}\,(\Phi,g)\prod_{j=1}^{n} Y_i(\mathcal{L}_j)\prod_{k=1}^{m} Y^{\Phi}(\gamma^k_{x_k y_k}) \tag{II.14}$$

obey properties (YM1), (YM2$_\ell$), (YM3), (YM4) and (YM6) (modified in the obvious manner to account for the

$Y^{\Phi}\,(\gamma^k_{x_k,y_k})$-variables).

Clustering (YM5) may fail in general, but is known to hold e.g. for small $\beta$[22] . Thus, lattice gauge fields exist, for arbitrary G and arbitrary space-time dimension $\nu$.

Among numerous, very general results we mention the following two which turn out to be important.

(1) Universality of diamagnetism [1,2] :

Define

$$Z_{\varepsilon,\Lambda}\,(g) = \int e^{-\beta U^{M}_{\varepsilon,\Lambda}(\Phi,g)}\,D\Phi_{\varepsilon} \tag{II.15}$$

Let $\Lambda$ be a rectangle and impose periodic b.c. at $\partial\Lambda$. Then

$$|\,Z_{\varepsilon,\Lambda}\,(g)\,| \le Z_{\varepsilon,\Lambda}(1), \tag{II.16}$$

($g = 1$ means $g_{xy}$ = identity in G, for all xy).

Inequality (II.16) holds no matter what gauge group G is chosen and even if Fermionic matter (leptons or quarks) is coupled to the gauge field. It expresses the fact that matter behaves diamagnetically under coupling to gauge fields. Inequality (II.16) does generally not survive ultraviolet renormalizations necessary for taking $\varepsilon \searrow 0$ , unless the vacuum polarization is finite (i.e. $\nu \le 3$). (There are related inequalities for pure Yang-Mills theories mentioned in [5] which appear to be renormalization-independent).

Next, suppose that G is abelian. Without loss of generality we may assume that $G = \mathbb{Z}_n$, n=2,3,4..., or G = U(1). Then we may introduce polar coordinates

$$g_{xy} = e^{ia_{xy}}\ ,\quad a_{xy} \in \mathbb{R}$$

$$\Phi(x) = r_x\, e^{i\theta_x},\qquad 0 \le \theta_x < 2\pi .$$

Let

$$dm_\varepsilon(a) = \lim_{\Lambda\uparrow\varepsilon\mathbb{Z}^\nu} Z_{\varepsilon,\Lambda}^{-1}\, e^{\beta\Sigma_{p\subset\Lambda}\cos(\delta\Sigma_{xy\subset\partial p}a_{xy})}\, \Pi\, da_{xy},$$

with $da$ the Lebesgue measure on $\left[o,\frac{2\pi}{\delta}\right]$, or let

$dm_\varepsilon(a) = dm_{\varepsilon,t,\mu}(a)$ be the restriction of the Gaussian measure introduced in (II.9), (II.10) to the variables $a_{xy} = \int_{b(x,y)} A_j(z)\, dz^j$. (II.17)

(2) <u>Correlation Inequalities</u> [1,2,27,6] :

Let $G = \mathbb{Z}_n$ or $U(1)$, $dm_\varepsilon$ as in (II.17), and $\langle - \rangle$ the expectation given by the probability measure

$$d\mu'_{\varepsilon,\Lambda}(\Phi,a) = (Z'_{\varepsilon,\Lambda})^{-1}\, e^{-\beta U^M_{\varepsilon,\Lambda}(\Phi,a)}\, dm_\varepsilon(a)\, D\Phi_\varepsilon\ , \qquad \text{(II.18)}$$

with $\Lambda \subseteq \varepsilon\mathbb{Z}^\nu$. Let F and G be in the multiplicative cone generated by $r(f)$, $f(x) \geq o$, $\cos(a(g) + \Theta(h))$.

Then

$$\langle F\, G \rangle - \langle F \rangle \langle G \rangle \geqq o \qquad \text{(II.19)}$$

For applications, see e.g. [2,4] .

Next, we consider a general lattice theory described by a measure as in (II.6) with action as in (II.7),(II.11) and (II.12). Suppose that the representation $U^\Phi$ of G on V is trivial on the center $\mathfrak{Z}_G$ of the gauge group. Let $\langle - \rangle_G$ denote the expectation determined by the measure $d\mu_\varepsilon$ given in (II.6). Let $\langle - \rangle_{\mathfrak{Z}_G}$ denote the expectation in the pure $\mathfrak{Z}_G$ lattice gauge theory with measure

$$d\mu_\varepsilon(\tau) = \lim_{\Lambda\uparrow\varepsilon\mathbb{Z}^\nu} (Z''_{\varepsilon,\Lambda})^{-1}\, e^{\beta\Sigma_p\varepsilon^{\nu-4}\chi(\tau_{\partial p})}\, D\tau, \qquad \text{(II.20)}$$

where $\tau_{xy} \in \mathfrak{Z}_G$, for all xy, and $d\tau$ is Haar measure on $\mathfrak{Z}_G$. Then

$$\langle \Pi\, Y_j(\mathcal{L}_j) \rangle_G \leq \langle \Pi\, Y_j(\mathcal{L}_j) \rangle_{\mathfrak{Z}_G} . \qquad \text{(II.21)}$$

Proof and applications are given in [6]. A special case of (II.21) was first proven in [27] .

The arguments used in the proofs of such inequalities are patterned on Ginibre's methods [28] .

II.2 On the phase diagram of some lattice gauge theories

Rigorous results on "high temperature"expansions ($\beta$ small) in lattice gauge theories are established in [22]. It is proven there that if $\beta$ is small enough and $\chi_{y_j}$ is non-trivial on the center $\mathfrak{Z}_G$ of the gauge group G the "quark-antiquark potential" $V_j$ defined in (II.3) satisfies

$$V_j\ (L) \geq \text{const.}\ \ L \tag{II.22}$$

Moreover, the Higgs mechanism is for lattice theories analyzed in that reference, too. In [5,23] there are general arguments suggesting that $V_j(L) \leq$ const., uniformly in L if $Y_j$ is trivial on the center. A. Guth has announced that the four-dimensional pure U(1) lattice theory (in the so called Villain form) has a phase transition as $\beta$ is varied: For $\beta$ small (II.22) is valid, for $\beta$ large $V_j(L) \leq$ const.. The proof is based on a combination of correlation inequalities (of the type proven in [1,2] ) and a high temperature expansion. Similar results were previously proven for the $\mathbb{Z}_n$ theories in three and four dimensions and are discussed in Guerra's contribution where the reader can also find references to the original articles of Guerra et al.

In [6] the author has applied inequality (II.21) to prove that in all two-dimensional Yang-Mills theories $V_j$ (L) $\geq$ const. L for all characters $Y_j$ which are non-trivial on the kernel of the representation $U^{\phi}$ used in the matter action (II.12).

This extends results of [2,30] . It is also shown in [6] that for three-dimensional U(n) theories, with $U^{\phi}$ trivial on $U(1) \subset \mathfrak{Z}_{U(n)}$,

$$V_j(L) > \text{const}\ \log\ (\ L + 1), \tag{II.23}$$

if $\chi_{y_j}$ is non-trivial on U(1).

In [2,5] connections between lattice gauge theories on $\varepsilon\mathbb{Z}^{\nu}$ and non-linear $\sigma$-models on $\varepsilon\mathbb{Z}^{\nu-1}$ have been found. The following models are investigated there:

(i) Classical, two-component, neutral Coulomb gases and abelian $\sigma$-models (Ising, $\mathbb{Z}_n$- and classical XY models).

(ii) Abelian lattice Higgs theories, in particular Landau-Ginzburg type theories.

(iii) Non-linear lattice-$\sigma$-models (e.g. the classical O(4) lattice model).

(iv) Pure, non-abelian lattice gauge theories.

The results are of the following kind:

(a) Rigorous connections between (ii) in $\nu$ dimensions and (i) in $\nu$-1 dimensions, and between (iv) in $\nu$ dimensions and (iii) in $\nu$-1 dimensions. E.g., $S_1(Y_j(\quad))$ of a $\nu$-dimensional gauge theory can generally be bounded above by (an integral of) a product of two-point functions of a ($\nu$-1)-dimensional $\sigma$-model. As examples we mention:

If the two-dimensional Coulomb gas has a transition from a high temperature plasma phase with Debye screening [31] to a low temperature, dipolar phase with power low decay, as expected, then the three-dimensional Landau-Ginzburg (abelian Higgs) lattice theory has a transition from a superconducting phase without confinement of fractional charges, massive photons and vortices, at small electric charge, to a QED phase with massless photons and confined fractional charges, at large electric charge. This is shown in [2]. It is also shown there that Guth's result for U(1) implies the existence of a superconductor $\rightarrow$ QED transition in a four-dimensional Landau-Ginzburg lattice theory, with liberated magnetic monopoles in the QED phase.

For further results on phase transitions in lattice gauge theories see [5,27,32] and Guerra's contribution to these proceedings. Some other, general consequences of correlation inequalities in lattice gauge theories (confinement, Higgs mechanism,...) are given in [2,4].

## II.3 Connections to dual resonance models

Recently many connections between (lattice) Yang-Mills theories and string (dual resonance) models have been proposed [33,14,15,26,34] . It has been suggested that lattice Yang-Mills theory is a theory of random surfaces [33,5,15,34] related to the lattice theory of dual strings (e.g. [11,34] ). Such a connection would be useful as a starting point for an investigation of the particle spectrum of pure Yang-Mills theory.

In [5] an expansion of the n-loop Schwinger functionals $S_n^{(\varepsilon)}(Y_1(\mathcal{L}_1),\dots,Y_n(\mathcal{L}_n))$ of pure lattice Yang-Mills theories in terms of random surfaces bounded by the loops $\mathcal{L}_1,\dots,\mathcal{L}_n$ has been derived when the gauge group G is U(n) or O(n), n = 1,2,3,...,or SU(2).

For G = SU(2) this has provided rather powerful lower bounds on the potential $V_j(L)$ ($\chi_{y_j}$ = spin 1/2 character) and has revealed an interesting connection with the theory of interacting random paths and non-relativistic strings. A method for obtaining upper-bounds on $V_j(L)$ has also been suggested there.

## III. REMARKS ON THE CONTINUUM LIMIT OF THE ABELIAN HIGGS MODEL IN TWO SPACE-TIME DIMENSIONS

The only continuum gauge theories satisfying properties (YM1) - (YM6) (except possibly (YM5) ) of Section I.3 are

- free electromagnetism in arbitrary dimension
- massive spinor QED [18,19] } in two space-
- the abelian Higgs model [1,3,4] } time dimensions.

The situation concerning two- and three-dimensional, super-renormalizable (abelian and non-abelian) gauge theories looks fairly promising; see the contributions of Balaban and Magnen-Sénéor to [20] .

This situation is thus not overly encouraging. We present a few remarks on Higgs models. For some general information about constructive quantum field theory see [CQFT].

### III.1 External (c-number) Yang-Mills fields

In [3] weak convergence of the measures

$$(Z_{\varepsilon,\Lambda}(g))^{-1}\, e^{-\beta U^{M}_{\varepsilon,\Lambda}(\Phi,g)}\, D\Phi_{\varepsilon}\ , \text{ as } \varepsilon \searrow o, \tag{II.24}$$

has been shown for $\nu = 2$ and

$$g_{xy} = P\,\{\exp \int_{b(x,y)} A_j\,(z)\, dz^j\,\},$$

with A(z) Hölder continuous in z, and

G = U(1), SU(2), etc.

The proof of convergence (for various boundary conditions) is rather complicated. In principle, it can be extended to $\nu = 3$, but this has only been done if the self-interaction of $\Phi$ vanishes, i.e. $\lambda = o$ in (II.13).

The following elements are crucial in the proof: Let $\Delta^{(\varepsilon)}_{\Lambda}$ be the finite difference covariant Laplacean on $\ell_2(\Lambda) \otimes V$ with periodic or O-Dirichlet boundary conditions at $\partial\Lambda$.

(a) $\| (-\Delta_A^{(\varepsilon)} + m^2)^{-1}(x,y) \|_V \leq (-\Delta^{(\varepsilon)} + m^2)^{-1}(x-y)$,

where $\Delta^{(\varepsilon)}$ is the usual finite difference Laplacean on $\ell_2(\Lambda)$ with the same b.c.. A proof of this "diamagnetic inequality" can be found in [1] and refs. given there.

(b) Convergence of

$$(-\Delta_A^{(\varepsilon)} + m^2)^{-1}$$

in various trace ideals, and $L_p$ convergence of

$$(-\Delta_A^{(\varepsilon)} + m^2)^{-1}(x,y) \text{ for } p < \frac{\nu}{\nu-2}.$$

The proof involves showing real analyticity in A and using a Neumann series expansion in A for "small" A; see [3] .

(c) $\det \left( (-\Delta_A^{(\varepsilon)} + m^2)^{-1} (\Delta^{(\varepsilon)} + m^2) \right) \leq 1$;

this is a special case of the diamagnetic inequality (II.16) due originally to R. Schrader and R. Seiler.

(d)

$$\det \left( (-\Delta_A + m^2)^{-1} (\Delta + m^2) \right)$$

$$\equiv \lim_{\varepsilon \searrow 0} \det \left( (-\Delta_A^{(\varepsilon)} + m^2)^{-1} (\Delta^{(\varepsilon)} + m^2) \right)$$

exists for Hölder-continuous A, $\nu=2$; see [3].

These elements somewhat cleverly combined with the diamagnetic inequality (II.16), the original Nelson-Glimm method (proving stability of $P(\Phi)_2$ theories, see [CQFT] and refs.given there) and numerous,lengthy estimates yield a proof of (II.24).
For

$$\nu = 2,\ G = U(1),\ dm_\varepsilon(A) = d\, m_{\varepsilon,t,\mu}(A)$$

the Gaussian measure defined in (II.9),(II.10),one derives from (II.24) that the weak limit of the measures

$$d\,\mu_{\varepsilon,\Lambda,t,\mu}(\Phi,a)$$

exists, as $\varepsilon \searrow 0$, for $t > 0, \mu \geq 0$. This follows from the diamagnetic inequality (II.16) and (II.24) by Lebesgue dominated convergence.

III.2 Removal of cutoffs

In order to show that the weak limit of the measures

$$d\mu_{\Lambda,t,\mu} = W - \lim_{\varepsilon \searrow o} d\mu_{\varepsilon,\Lambda,t,\mu}, \text{ as } t \searrow o,$$

exists one must do an ultraviolet expansion [4], involving a truncated (high-momentum) perturbation expansion which exhibits cancellations of divergent Feynman diagrams with counterterms.

The ultraviolet expansion is applied to unnormalized expectations

$$Z_{\Lambda,t,\mu} < F >_{\Lambda,t,\mu},$$

where $< \text{—} >_{\Lambda,t,\mu}$ is the expectation obtained from $d\mu_{\Lambda,t,\mu}$, and $Z_{\Lambda,t,\mu}$ is the natural continuum partition function.

In the following $\Lambda$ and $\mu$ are suppressed temporarily. The initial form of the expansion is roughly

$$Z_{t_N} < F >_{t_N} = \sum_{n=1}^{N} (Z_{t_n} < F >_{t_n} - Z_{t_{n-1}} < F >_{t_{n-1}}), \qquad \text{(II.25)}$$

with $t_o > o$ some suitable constant.

The differences,

$$Z_{t_n} < F >_{t_n} - Z_{t_{n-1}} < F >_{t_{n-1}},$$

are then interpolated in a somewhat sophisticated way that depends on n and involves "changes of A-covariance" and "integrations by part on function space" with subsequent cancellations of divergent diagrams; see [3,4] .

One obtains an upper bound on

$$\left| Z_{t_n} < F >_{t_n} - Z_{t_{n-1}} < F >_{t_{n-1}} \right|$$

of the form:

$$c^n \prod_{i=1}^{n} t_i^{\delta} e^{c(\log t_n)^2} (n!)^r (\log t_n)^n$$

This proves convergence of (II.25), for $t_n \overset{\text{e.g.}}{\propto} \exp(-n^{\gamma}), o<\gamma<1$.

A technically subtle part in the proof of the upper bounds is the estimation of large Feynman diagrams. (There one makes use, among many other things, of (a) ). Convergence of the ultraviolet expansion suffices to show that

$$< F >_{\Lambda,\mu} = \lim_{t \searrow o} < F >_{\Lambda,t,\mu}$$

exists, for $\Lambda$ bounded and $\mu^2 > o$.

Subsequently one uses fairly standard methods to establish upper bounds on

$$< F >_{\Lambda,\mu}$$

that are uniform in $\Lambda$ and $\mu$. Thanks to the correlation inequalities (I.19) one has monotonicity in $\Lambda$ and $\mu$, for a total set of random variables, F. Thus, the limits $\Lambda \uparrow \mathbb{R}^2$ and $\mu \downarrow o$ exist. The existence of the O-bare-mass limit, $\mu \downarrow o$, is yet another manifestation of the well established experience that constructive field theory methods never create artificial infrared problems (which might be regarded as one of its modest triumphs).

To date it is only known that the Schwinger functionals

$$S_{n,m} (Y_1(\mathcal{L}_1), \ldots, Y^{\Phi}(\gamma^1_{x_1 y_1}), \ldots)$$

$$= < \prod_{j=1}^{n} Y_i(\mathcal{L}_j) \prod_{k=1}^{m} Y^{\Phi}(\gamma^k_{x_k y_k}) >$$

of the limiting expectation $< \text{---} >$ satisfy properties (YM1) - (YM4), (YM6) (without normal ordering of $Y_j$'s, $Y^{\Phi}$'s) so that they determine a relativistic quantum field theory (Theorem (I.55), Section I.3), but detailed, physical information is lacking, (e.g. Higgs mechanism ?).

## IV. A LOOK INTO THE FUTURE OF THE SUBJECT

In the Euclidean approach to quantized Yang-Mills theory one proposes to convert (the traces of) holonomy operators on a principal bundle into random fields on a loop space over physical space-time. Thus, one attempts, in fact, to construct stochastic processes and random fields on spaces of geometrical objects, the closed loops in physical space-time. This is an instance of combining geometry and probability theory, i.e. a problem in random geometry. Random geometry still appears to be an underdeveloped branch of mathematics. (For other examples in random geometry see e.g. [10,11] and refs. given there).

A number of conceptual problems arises: E.g. is there a reasonable notion of "distribution-valued connections", or, in other words, is there a geometric interpretation of "normal-ordered" holonomy operators, (see Section I.3), etc..

Gauge fields (i.e. the gauge orbits, [A], of connections, A, or the traces of holonomy operators) are intrinsically non-linear fields (at least in the non-abelian case). Constructive quantum field theory methods have so far not had much success as a means of studying non-linear fields. One of the main reasons might be that non-linear fields cannot be localized on classical phase space, a technical device that has so far appeared to be crucial for non-perturbative renormalization, [35]. In the analysis of [1,3,4] outlined in Section III and in [8,9] the non-linearily of gauge fields has been circumvented in a somewhat unnatural way. Presumably, this is only possible if the gauge group is abelian, and the gauge field couples to a conserved current. Even then the price to be paid is a fairly clumsy and tedious analysis.

We have tried to explain the underlying geometric reasons why the lattice approximation is a natural gauge-invariant regularization of continuum Yang-Mills theory (End of Section I.2, introduction to Section II). What remains to be seen is how one can do hard analysis (non-perturbative renormalization) starting from lattice theories. The popular magic word is: Renormalization group ("block spin") transformations. This has first been advertized by Kadanoff and Wilson. A rigorous program of this sort has been described by Balaban in [20]. The program can only be regarded as really successful if one eventually achieves a non-perturbation renormalization of a four-dimensional, non-super-renormalizable, asymptotically free gauge theory.

Another approach, due to Jimbo, Miwa and Sato [17] is based on analyzing the monodromy structure of the Schwinger functionals of the loop variables, $Y_j(\mathcal{L})$, and the dual ("disorder") variables. The general monodromy properties of the Schwinger functionals follow from "topological commutation relations". One then studies monodromy preserving deformations and uses the Schwinger-Dyson equations for the Schwinger functionals.

In some examples (e.g. the two-dim. Ising model), with free Schwinger-Dyson equations, Jimbo, Miwa and Sato have carried out their program, with impressive success. One might hope that there exist "non-local" conserved currents in Yang-Mills theory yielding relations between Schwinger functionals which reinforce the J-M-S program in a suitable way.

But this is mere speculation.

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