Confinement in $\mathbb{Z}_n$ Lattice Gauge Theories Implies Confinement in SU(n) Lattice Higgs Theories

A New Look at Generalized, Non-Linear σ-Models and Yang-Mills Theory

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Abstract.

Let G be an arbitrary compact Lie group with center $Z(G)$. It is proven that if static quarks transforming under a non-trivial representation of $Z(G)$ are confined in a pure $Z(G)$ lattice gauge theory with gauge coupling constant g' they are confined in a lattice Higgs - (in particular a pure Yang-Mills) theory with gauge group G, Higgs scalars in a representation that is trivial on $Z(G)$, and coupling constant $g = \text{const}.g'$. Permanent confinement of "fractionally charged" quarks in any two dimensional lattice gauge theory and in three dimensional $U(n)$-theories are consequences.

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We compare the standard lattice Higgs theories, including the pure Yang-Mills theories, with compact gauge group G with pure lattice gauge theories the gauge group of which is the center, $Z(G)$, of G. The main result is the one described in the abstract, special cases of which were first proven in two beautiful papers of Mack[1] and Mack and Petkova[2] and in ref.[3]. Our proof, originally stimulated by[3], extends and simplifies the arguments of[1,2], but involves closely related ideas. It establishes the general conjecture 3 of[3]. Since our proof yields results more general than the ones of [2] and seems somewhat more transparent than the one given there, and since it was developed in part before we learnt of [1,2], it may be useful to make it public.

The lattice is chosen e.g. to be $\mathbb{Z}^\nu$ (ν-dimensional, simple, cubic lattice). Since we do not attempt to be uniform in the lattice spacing,we choose it to be unity. The gauge groups of main interest are $G = SU(n)$, $n = 2,3,\dots$, in which case $Z(G) = \mathbb{Z}_n$, but our arguments cover the general case, in particular $G = U(n)$. (This is of interest, e.g. because $U(n)$ is the gauge group of the $G_{N,n}(\mathbb{C})$ non-linear σ-models of refs.[4,5] which are generalizations of the $\mathbb{C}P^{N-1}$ σ-models of [6]. They are discussed at the end of this letter).

Next, we state our main results in more detail. We consider a general lattice Higgs theory with gauge group some compact Lie group G, Higgs scalars in a representation of $G/Z(G)$ (e.g. the trivial one, the case of pure Yang-Mills) and gauge coupling constant g. Let χ be the irreducible character of G used in the definition of the pure Yang-Mills action, and d the dimension of the corresponding representation. Moreover, we consider a pure, abelian lattice gauge theory with gauge group $Z_\chi \equiv \chi(Z(G))$ and coupling constant $(2d)^{-\frac{1}{2}} g$.

A representation of G (resp. its character) is called "fractionally charged" if it determines a non-trivial representation of $Z(G)$. (In the case $G = U(1)$ we adopt the definition of [3] : A charge is fractional iff it is a fraction of the electric charge of the Higgs scalar).

As confinement criterion we may use in this letter both, the one of Wilson[7], or the more refined one used in [3].

Theorem 1.

If in the pure Z_χ-lattice gauge theory defined above fractionally charged, static quarks are confined then so they are in the G-lattice Higgs theory. □

Application of the inequalities of ref.[3] (see Theorem 6[3]) yields

Theorem 2.

The ν-dimensional, pure Z_χ-lattice gauge theory confines fractionally charged, static quarks if the $(\nu-1)$ dimensional, nearest neighbor Z_χ-Ising (or Potts) model has exponential clustering. □

One consequence of Theorems 1 and 2 (and of Theorem 6,2) of [3]) is

Corollary 1.

Every abelian [3] or non-abelian[1] two (space-time) dimensional lattice Higgs theory permanently confines fractionally charged, static quarks.

As another (more interesting) consequence we mention

Corollary 2.

Every $U(n)$, $n = 1,2,3,\ldots,$ three (space-time) dimensional lattice Yang-Mills theory permanently confines static quarks. □

Further corollaries are mentioned below. We also suggest a connection between the breakdown of confinement and the spontaneous breaking of the internal ("flavour") symmetry group in the non-linear σ-models of refs.[4,5].

Next, we give an analytical definition of the models considered in this letter. The action of the standard lattice Higgs theory defined over a bounded ("space-time") region $\Lambda \subset \mathbb{Z}^\nu$ is given by

$$A_\Lambda = A_\Lambda^{YM} + A_\Lambda^{M}, \quad \text{where} \tag{1}$$

$$\left.\begin{aligned} A_\Lambda^{YM} &= -\beta \,\mathrm{Re} \sum_{p\subset\Lambda} \chi(g_{\partial p}), \text{ and} \\ A_\Lambda^{M} &= -f \sum_{xy\subset\Lambda} (\phi_x, U^\phi(g_{xy})\phi_y) . \end{aligned}\right\} \tag{2}$$

Here $\beta = g^{-2}$ and f are positive coupling constants, xy are arbitrary nearest neighbors (bonds) and p an arbitrary plaquette, with boundary ∂p, in $\mathbb{Z}^\nu$, $g_{xy} = g_{yx}^{-1} \in G$, for all xy, $g_C = \prod_{xy\subset C}^{\circlearrowleft} g_{xy}$, for any closed loop C of nearest neighbors in $\mathbb{Z}^\nu$, χ is an irreducible character of G (which is non-trivial on $Z(G)$), U^ϕ is some representation of $G/Z(G)$, and ϕ is the Higgs scalar. The a priori distribution of g_{xy} is given by the Haar measure, dg_{xy}, on G, the one of ϕ_x by a G-invariant probability measure $d\rho(\phi_x)$ on the representation space of U^ϕ. We set

$$dg_\Lambda = \prod_{xy\subset\Lambda} dg_{xy}, \quad d\rho(\phi_\Lambda) = \prod_{x\in\Lambda} d\rho(\phi_x) . \tag{3}$$

The "Euclidean vacuum expectation", $< - >_G(\beta)$, of this model is given by the measure

$$d\mu_\Lambda(\phi,g) = Z_\Lambda^{-1}\, e^{-A_\Lambda}\, dg_\Lambda\, d\rho(\phi_\Lambda) , \tag{4}$$

with Z_Λ chosen such that $\int d\mu_\Lambda = 1$.

The boundary conditions at $\partial\Lambda$ may be chosen to be periodic, or free, but many others can be used in the following arguments. All our estimates will be uniform in Λ, so that this subscript is omitted hence forth.

Let U^{χ} be the representation of G with character χ. Since it is irreducible,

$$U^{\chi}(\tau) = \chi(\tau)\cdot \mathbb{1} \quad , \text{ for all } \tau \in Z(G) \ . \tag{5}$$

The image, $Z_{\chi} = \chi(Z(G))$, of $Z(G)$ under χ is a compact, abelian group contained in a torus. Without loss of generality we may assume that the torus is one-dimensional, i.e. a circle. The elements of Z_{χ} can then be labelled by an angle θ which is distributed according to a probability measure $d\lambda$ on the circle, the Haar measure of Z_{χ} . All subsequent arguments hold in general, but our assumption somewhat simplifies notations.

The action of the pure Z_{χ} -lattice gauge theory is given by

$$A' = -\beta' \sum_{p} \cos(\theta_{\partial p}) \ , \tag{6}$$

where $\theta_C = \sum_{xy \subset C} \theta_{xy}$, and $\theta_{xy} = -\theta_{yx}$. Its Euclidean vacuum expectation, $< - >_{Z_{\chi}}(\beta')$, is defined by the probability measure

$$d\mu'(\theta) = (Z')^{-1}\, e^{-A'}\, d\lambda(\theta) \ , \tag{7}$$

with $d\lambda(\theta) = \prod_{xy \subset \Lambda} d\lambda(\theta_{xy})$.

Let χ^{q} be some irreducible character of G , such that, for all $\tau \in Z(G)$,

$$\chi^{q}(\tau) = \chi(\tau)^{q} = e^{iq\cdot\theta} \neq 1 \ , \tag{8}$$

for some angle θ (depending on τ) in the support of $d\lambda$ and some fixed integer q . (If (8) is violated, Theorem 1 is either empty or trivial !).

We now proceed to the proof of Theorem 1 for the models introduced above. First, we integrate out the Higgs field ϕ . We define

$$Z^M(g) = \int e^{-A^M} d\rho(\phi) \ . \tag{9}$$

Since U^ϕ is a representation of $G/Z(G)$,

$$Z^M(g') = Z^M(g) \quad \text{if} \quad g'_{xy} = g_{xy} \cdot \tau_{xy} \ , \tag{10}$$

for some $\tau_{xy} \in Z(G)$ and all $xy \subset \Lambda$.

Equation (10) is the only property required of $Z^M(g)$, so that Z^M could come from integrating out more general matter fields ! Moreover, the reader will notice that, in the proof, it suffices that Z^M be invariant under the minimal subgroup $Z_{min.} \subseteq Z(G)$ for which $\chi^q(Z_{min.}) \neq 1$, provided Z_χ is replaced everywhere by $Z_{min.}$. Physically, this means that the colour of the quarks cannot be shielded by the colour of the Higgs fields. (More refined hypotheses on $Z^M(g)$ are possible).

The basic identities (already used in [1]) are : If Z^o is any subgroup contained in or equal to $Z(G)$

$$\int_G dg \ F(g) = \int_G dg \int_{Z^o} d\tau \ F(g \cdot \tau) \ , \tag{11}$$

for any bounded function F on G , and $d\tau$ the normalized Haar measure on Z^o ,

$$\chi((g \cdot \tau)_C) = \chi(g_C \cdot \tau_C) = \chi(g_C) \, e^{i\theta_C} \ , \tag{12}$$

$$\chi^q((g \cdot \tau)_C) = \chi^q(g_C \cdot \tau_C) = \chi^q(g_C) \, e^{iq\theta_C} \ , \tag{13}$$

for any loop $C \subset \Lambda$, $\tau_{xy} \in Z^o$, for all $xy \subset C$, and $e^{i\theta_{xy}} = \chi(\tau_{xy})$.

Using (10) - (13) we obtain

$$< \chi^q(g_C) >_G(\beta) = Z^{-1}\int \chi^q(g_C)\, Z^M(g)\, e^{-A^{YM}} dg$$

$$= Z^{-1}\int dg\, Z^M(g)\int d\lambda(\theta)\chi^q((g\cdot\tau)_C)\, e^{-A^{YM}(g\tau)}$$

$$= Z^{-1}\int dg\, Z^M(g)\chi^q(g_C)\int d\lambda(\theta)\, e^{iq\theta_C} \tag{14}$$

$$\cdot\, e^{\beta \sum_{p\subset\Lambda} \mathrm{Re}(\chi(g_{\partial p})\, e^{i\theta_{\partial p}})}$$

Clearly

$$\mathrm{Re}(\chi(g_{\partial p})e^{i\theta_{\partial p}}) = \mathrm{Re}\ \chi(g_{\partial p})\cos(\theta_{\partial p}) - \mathrm{Im}\ \chi(g_{\partial p})\sin(\theta_{\partial p})$$

$$\equiv J_p\cos(\theta_{\partial p}) + K_p\cos(\theta_{\partial p} + \frac{\pi}{2}) . \tag{15}$$

Define

$$Z'(g) = \int d\lambda(\theta)\exp\Big[\beta \sum_{p\subset\Lambda} \mathrm{Re}(\chi(g_{\partial p})\, e^{i\theta_{\partial p}})\Big] , \tag{16}$$

and let the expectation $< - >_{J,K}$ be given by the measure

$$Z'(g)^{-1}\exp\Big[\beta \sum_{p\subset\Lambda} J_p\cos(\theta_{\partial p}) + K_p\cos(\theta_{\partial p} + \frac{\pi}{2})\Big]d\lambda(\theta) . \tag{17}$$

Then, by (14) - (17) ,

$$< \chi^q(g_C) >_G (\beta) = Z^{-1}\int dg\, \chi^q(g_C)\, Z^M(g)\, Z'(g) < e^{iq\theta_C} >_{J,K} , \tag{18}$$

where

$$Z^M(g) \geq 0^{*)},\ Z'(g) \geq 0 , \text{ and } Z^{-1}\int dg\, Z^M(g)\, Z'(g) = 1 . \tag{19}$$

Next, we note that the measure $d\lambda(\theta_{xy})$ is the weak limit of

$$(\int e^{\mu\cos(m\theta_{xy})} d\theta_{xy})^{-1} e^{\mu\cos(m\theta_{xy})} d\theta_{xy} ,$$

as $\mu \to \infty$, for some $m = 0,1,2,...$ determined by Z_χ . Thus the measure in (17) is the weak limit of the probability measures

*) This can be replaced by $Z^{-1}|\int dg\ Z^M(g)\, Z'(g)\, \chi^q(g_C)| \leq$ const.

$$Z_{\mu}^{-1} \exp[\beta \sum_{p\subset\Lambda} J_p \cos(\theta_{\partial p}) + K_p \cos(\theta_{\partial p} + \frac{\pi}{2})]$$
$$\cdot \prod_{xy\subset\Lambda} e^{\mu \cos(m\theta_{xy})} d\theta_{xy} \text{ , as } \mu \to \infty \text{ .} \tag{20}$$

Therefore Ginibre's inequalities [8] in the form proven in Proposition 1 of ref.[9] can be applied and give

$$\pm < \cos(q\theta_C + \alpha) >_{J,K} \leq < \cos(q\theta_C) >_{Z_\chi} (\beta') \tag{21}$$

(see (6), (7)) , for an arbitrary integer q and real phase α , provided

$$\beta[|J_p| + |K_p|] \leq \beta' \text{ , for all } p \subset \Lambda \text{ .}^{9} \tag{22}$$

For J_p and K_p as in (15), (22) is valid for $\beta' \geq 2d\beta$, (since $|\mathrm{Re}\ \chi(g)| \leq d$, $|\mathrm{Im}\ \chi(g)| \leq d)$, so that

$$| < e^{iq\theta_C} >_{J,K}| \leq 2 < \cos(q\theta_C) >_{Z_\chi} (2d\beta) \text{ .} \tag{23}$$

But (18), (19) and (23) together with $|\chi^q(g_C)| \leq \chi^q(1)$ yield

$$| < \chi^q(g_C) >_G (\beta)| \leq 2\chi^q(1) < \cos(q\theta_C) >_{Z_\chi} (2d\beta) \text{ .} \tag{24}$$

From this Theorem 1 follows by the usual arguments [2,3,7] .

Remarks.

Related inequalities (see [2] for examples and interpretation) can be proven by the same methods. Moreover, they can be used to e.g. prove that the two point function of the G-non-linear σ-model on the lattice $\mathbb{Z}^\nu$ is dominated by the one of the ν-dimensional $Z(G)$-σ-model. This obviously implies that the Mc Bryan - Spencer upper bound [10] extends to the $U(n)$-σ-models , $n = 2,3,\ldots$.

To prove Theorem 2 we modify the ν-dimensional Z_χ-lattice gauge theory by adding a non-gauge-invariant term

$$\delta A = z \sum_{xy \perp 1\text{-direction}} \cos(\theta_{xy})$$

to the action. As $z \to +\infty$, the corresponding expectation converges weakly to a product of <u>independent</u>, $(\nu-1)$ dimensional Z_χ-Ising (or Potts) models with expectation, $\langle - \rangle_{x^1}(\beta')$, given by

$$Z^{-1}_{\Lambda(x^1)}\, e^{\beta' \sum_{xy\subset\Lambda(x^1)} \cos(\theta_x-\theta_y)} \prod_{x\in\Lambda(x^1)} d\lambda(\theta_x) \quad ,$$

where $\Lambda(x^1) = \Lambda \cap \{y = (y^1,\ldots,y^\nu) : y^1 = x^1\}$, and $\theta_z = \theta_{z+e_1}$ (e_1 the unit vector in the 1-direction), for $z \in \Lambda(x^1)$. This is shown in [3]. In the thermodynamic limit $(\Lambda \uparrow \mathbb{Z}^\nu)$, these expectations become independent of x^1 (for e.g. periodic or free boundary conditions), so that that subscript can be dropped.

Let C be e.g. a rectangular loop in the (1,2)-lattice plane with sides of length T and L, respectively, parallel to the 1 - and 2-axis. The inequalities of [8] give monotonicity in z for the expectation of $\cos(q\theta_C)$, so that, as $z \to \infty$,

$$\begin{aligned} \langle e^{iq\theta_C} \rangle_{Z_\chi}(\beta') &= \langle \cos(q\theta_C) \rangle_{Z_\chi}(\beta') \\ &\leq \{ \langle \cos(q(\theta_o-\theta_{x(L)}) \rangle (\beta')\}^T \quad , \end{aligned} \tag{25}$$

where $x(L)$ is the site $(0,L,0,\ldots)$. See inequalities (18) - (20) of ref.[3]. This proves Theorem 2.

Corollary 1 follows easily from Theorems 1 and 2 (and Theorem 6 of [3]) by setting $\nu = 2$ and noting that, for a one-dimensional Z_χ-Ising model, the r.s. of (25) is bounded by $\exp[-O(1)L\cdot T]$, for all β'. (For an alternate argument see [1]). We conjecture that inequality (25) is also valid for the original ν-dimensional gauge theory (with gauge group $G = SU(n)$) and a two point function of the $(\nu-1)$ dimensional G-non-linear σ-model on the r.s. of (25). A related result will be proven elsewhere, for $G = SU(2)$.

Corollary 2 follows from Theorems 1 and 2 by extracting from $U(n)$ a $U(1)$ subgroup, i.e. choosing $Z_\gamma = U(1)$. To the resulting $U(1)$-theory one may then apply Theorem 6,1) of [3] or [11] . We note that, by the inequalities of [8,9],

$$< \cos(q\theta_C) >_{\mathbb{Z}_{n\cdot k}} (\beta') \leq < \cos(q\theta_C) >_{\mathbb{Z}_n} (\beta') \ , \tag{26}$$

for arbitrary, positive integers n and k , and $\mathbb{Z}_\infty = U(1)$, if $k = \infty$; see also [3]. We conjecture that (26) remains true if $\mathbb{Z}_m$ is replaced by $(S)U(m)$, on both sides.

As an example of further applications of Theorem 1 and its proof we mention that they can be used to compare e.g. a lattice Weinberg-Salam theory with a $U(1)$ (purely electromagnetic) lattice gauge theory.

Finally, we consider the non-linear σ-models of refs.[4,5,6]. A possible lattice action is

$$A_\sigma = -\beta \sum \mathrm{tr}(\xi_x \, U(g_{xy})\xi_y^*) \ , \tag{28}$$

where, for all $x \in \mathbb{Z}^\nu$, $\xi_x = (\xi_x^1, \ldots, \xi_x^n)$ is an $N \times n$ matrix of <u>orthonormal</u> vectors, $\xi_x^1, \ldots, \xi_x^n$, in $\mathbb{C}^N$, $g_{xy} \in U(n)$, and U is the defining representation of $U(n)$. (See [4] for yet more general models and alternate lattice actions). The a priori distribution of g_{xy} is Haar measure, dg_{xy} , the one of ξ_x is the uniform measure on orthonormal n-frames in $\mathbb{C}^N$. By integrating out the ξ-field one obtains a $U(n)$-lattice gauge theory. According to Theorem 1, it can be dominated by a $U(1)$- gauge theory, in the sense of inequality (24). Extensions of Theorem 2 can be used to discuss confinement of quarks for $\nu = 2$. The action A_σ has a global $U(N)$-symmetry group. A connection between the breakdown of confinement of static quarks in these models and the spontaneous breaking of the global ("parton-flavour") $U(N)$-symmetry, for $\nu \gtrsim 3$, is suggested in [4].

If the normalization condition $\xi^*\xi = \mathbb{1}_n$ is replaced by $\xi^*\xi = c \cdot N^{3/4} \cdot \mathbb{1}_n$, the ξ-field is then integrated out, and the limit $N \to \infty$ is taken, the resulting $U(n)$-gauge theory is the pure $U(n)$-lattice Yang-Mills theory, for arbitrary ν.

Details of these results will be discussed elsewhere.

References.

1. G. Mack, Confinement of Static Quarks in Two Dimensional Lattice Gauge Theories, to appear in Commun. math. Phys.

2. G. Mack and V.B. Petkova, Comparison of Lattice Gauge Theories with Gauge Groups $\mathbb{Z}_2$ and SU(2), Preprint, Hamburg, Dec. 78.

3. D. Brydges, J. Frøhlich and E. Seiler, Diamagnetic and Critical Properties of Higgs Lattice Gauge Theories, to appear in Nucl. Phys. B.

4. J. Frøhlich, A New Look at Generalized, Non-Linear σ-Models and Yang-Mills Theory, to appear in the Proceedings of the Bielefeld Symposium, Dec. 78, edited by L. Streit.

5. M. Dubois-Violette and Y. Georgelin, Gauge Theory in Terms of Projector Valued Fields, Preprint, Orsay, Dec. 78.

6. H. Eichenherr, Ph.D. Thesis, Heidelberg, 1978.
A. D'Adda, P. Di Vecchia and M. Luscher, Preprint NBI-HE-78-26, to appear in Nucl. Phys. B.

7. K. Wilson, Phys. Rev. D 10, 2445, (1974).

8. J. Ginibre, Commun. math. Phys. 16, 310, (1970).

9. A. Messager, S. Miracle-Sole and C. Pfister, Commun. math. Phys. 58, 19, (1978).

10. O. Mc. Bryan and T. Spencer, Commun. math. Phys. 53, 299, (1977).

11. J. Glimm and A. Jaffe, Physics Letters 66 B, 67, (1977).

A New Look at Generalized, Non-linear σ-Models and Yang-Mills Theory

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Abstract :

First a list of recent papers on (lattice) gauge theories and non-linear σ-models is presented which serves as an introduction to the subject.

Subsequently, a new, quantum mechanical interpretation of the formalism used by Atiyah et al. and Corrigan et al. for the construction of self-dual Yang-Mills fields is attempted and criticized. Yang-Mills theory turns out to be a natural generalization of non-linear σ-models which has many conserved (Noether) currents. Confinement is linked to the presence of an "intrinsic (or parton) flavour" symmetry, at least in the case of the σ-models.

For the Proceedings of the Bielefeld Symposium, Dec. 1978, to be edited by L. Streit.

0. A short Guide to the Literature.

Before we come to the main part of this contribution, describing some new ideas and results concerning non-linear σ-models and Yang-Mills theories, we present a short list of papers which is a tiny selection out of a huge number of publications on non-linear σ-models and gauge theory. It replaces an introduction to the subject. Our selection does not represent a value judgment. Many (in certain respects perhaps most) important papers are missing in our list. The author has been guided largely by his lack of comprehension and time, ignorance and taste.

First, we quote some papers on classical, non-linear σ-models :

- Refs. 9 and 13 of the bibliography.
- K. Pohlmeyer, Commun. math. Phys. 46, 207, (1976).
- K. Pohlmeyer, in "New Developments in Quantum Field Theory and Statistical Mechanics", M. Lévy an P. Mitter (eds.) , Plenum Press, New York 1977.
- A. D'Adda, P. Di Vecchia and M. Lüscher, in ref. 6 of the bibliography, and Preprint, Niels Bohr Institute, 1978.
- V.L. Golo and A.M. Perelomov, Phys. Lett. 79 B, 112, (1978).
- E. Brézin, C. Itzykson, J. Zinn -Justin and J.-B. Zuber, "Remarks about the Existence of Non-Local Charges in Two-Dimensional Models", to appear in Phys. Lett. B, (1979).
  (This paper contains an explicit construction of infinitely many conserved currents for all two-dimensional σ-models considered to be interesting).

Questions of complete integrability of two-dimensional, non-linear σ-models are discussed by the Russian Inverse-Scattering school.

Some important, recent papers on the quantum field theory of two-dimensional, non-linear σ-models are :

- Refs. 4 , 14 , 16 , 17 , 18 of the bibliography.
- A.M. Polyakov, Preprint ICTP 77/122.
- E. Witten, Instantons, the Quark Model, and the $1/N$ Expansion, Harvard Preprint HUTP-78/A042.
- K. Symanzik, in ref. 6 and these proceedings, and refs. given there.

Further very interesting, recent papers have been written by the Leningrad - , Berlin - and Sato (Japan) groups.

Next, we collect some convenient references concerning classical Yang-Mills theory :

- Refs. 1 , 2 , 3 , 5 , 6 of the bibliography.

A readable survey of classical Yang-Mills theory, describing the developments and including all important references prior to fall 1977 is :

- R. Stora, in "Invariant Wave Equations", G. Velo and A.S. Wightman (eds.), Lecture Notes in Physics 73 , Springer-Verlag, Berlin-Heidelberg-New York, 1978.

Moreover,

- M.F. Atiyah, in "Mathematical Problems in Theoretical Physics", G.F. Dell' Antonio, S. Doplicher and G. Jona-Lasinio (eds.), Lecture Notes in Physics 80, Springer-Verlag, Berlin-Heidelberg-New York, 1978.
  More recent reviews, by R. Stora and E. Corrigan, may be found in ref. 6.

Additional, recent papers (among numerous others) are :

M.F. Atiyah, N. Hitchin and I.M. Singer, Proc. Nat. Acad. Sci. 74, 2662, (1977), and Proc. Royal Soc., to appear.

I.M. Singer, Commun. math. Phys. 60, 7, (1978).

M.F. Atiyah and J.D.S. Jones, Commun. math. Phys. 61, 97, (1978).

An extensive review, "Gauge Theories and Differential Geometry", by T. Eguchi, P.B. Gilkey and A.J. Hanson is to appear in Physics Reports (1979).

A list of references to work on lattice gauge theories, with emphasis on recent papers, follows : (among) the classic papers are

- Ref. 19, bibliography.
- A.M. Polyakov, Phys. Letts. 59 B, 79 , 82 , (1975).
- J. Kogut and L. Susskind, Phys. Rev. D11 , 395, (1975).

Reviews are e.g.

- K. Osterwalder, - K. Wilson } in : "New Developments in Quantum Field Theory and Statistical Mechanics", loc. cit.
- L. Kadanoff, Rev. Mod. Phys. 49, 267, (1977) , and refs. given there.
- A. Jaffe, - F. Guerra, - E. Seiler } in : "Mathematical Problems in Theoretical Physics", loc.cit.

More recent, useful papers are :

- K. Osterwalder and E. Seiler, Ann. Phys. (N.Y.) 110, 440, (1978).
- E. Seiler, Phys. Rev. D18, 482, (1978).
- M. Lüschner, Commun. math. Phys. 54, 283, (1977).
- J. Glimm and A. Jaffe, Phys. Letts. 66B, 67 , (1977)
- K. Gawedçki, Commun. math. Phys. 63, 31, (1978).
- J. Challifour and E. Weingarten, University of Indiana, Preprint 1978.
- G. 't Hooft, Nucl. Phys. B138, 1 , (1978).

Very recent papers are

- D. Brydges, J. Fröhlich and E. Seiler, "On the Construction of Quantized Gauge Fields, I : General Results", to appear in Ann. Phys. (N.Y.) (1979), and preprint in preparation.
- R. Israel and C. Nappi, "Quark Confinement in the Two-Dimensional Lattice Higgs-Villain Model", to appear in Commun. math. Phys. (1979).

(For earlier, related results see also : J. Fröhlich, in "Math. Problems of Theor. Phys." loc. cit.).

- D. Brydges, J. Fröhlich and E. Seiler, "Diamagnetic and Critical Properties of Higgs Lattice Gauge Theories", to appear in Nucl. Phys. B , (1979).
- G. Mack, "Confinement of Static Quarks in Two Dimensional Lattice Gauge Theories", to appear in Commun. math. Phys. (1979).
- T. Yoneya, "Topological Excitations in Yang-Mills Theories : Duality and Confinement", Preprint, City College, 1978.
- J. Glimm and A. Jaffe, "Charges, Vortices and Confinement", Harvard Preprint, 1978.
- G. Mack, and V.B. Petkova, "Comparison of Lattice Gauge Theories with Gauge Groups $\mathbb{Z}_2$ and SU(2)", Preprint, Hamburg, Dec. 78.
- G. Mack and V.B. Petkova, "Sufficient Condition for Confinement of Static Quarks by a Vortex Condensation Mechanism", Preprint, Hamburg, Dec. 78.
- A.M. Polyakov, "String Representation and Hidden Symmetries for the Gauge Fields", Preprint 1978.

(In this paper the existence of infinitely many, non-local conserved charges for three dimensional Yang-Mills theory is suggested).

- J. Fröhlich, "Confinement in $\mathbb{Z}_n$-Lattice Gauge Theories Implies Confinement in SU(n) Lattice Higgs Theories", Preprint, IHES, Feb. 1979.

(Some extensions of this paper, due to B. Durhuus and J. Fröhlich, are in preparation).

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For a recent survey of quantized Yang-Mills theory in the continuum limit see

- S. Coleman "The Uses of Instanto s" Erice Lectures 1977 to be published, A. Zichichi, (ed.), and refs. to the original papers given there.

Work by D. Brydges, E. Seiler and the author concerning the quantized Higgs theory in two space-time dimensions is in preparation. (In that work, some earlier results by R. Schrader and R. Seiler, "A Uniform Lower Bound on the Renormalized Functional Determinant", to appear in Commun. math. Phys. (1979), and by B. Simon, "Kato's Inequality and the Comparison of Semi Groups, to appear in J. Funct. Anal. (1979), and refs. given there, were very useful).

We omit references to "infinitely many" papers on perturbative renormalization of Yang-Mills theory, applications of the renormalization group (e.g. asymptotic freedom), model building or axiomatic investigations of gauge theories, although much of the present faith in gauge theories is certainly founded on the results of those earliear papers. Many of the out-standing references are by now so well known that we need not give them here again.

We now proceed to the main part of this contribution, where we suggest a somewhat novel way of looking at σ-models and Yang-Mills theory. (It is quite doubtful, though, whether it will provide a useful point of view for the quantization of Yang-Mills theories).

1. Introduction

In the following I propose and discuss a quantum mechanical interpretation of the construction - due to Atiyah, Drinfeld, Hitchin and Manin [1,2] - of all self-dual, Euclidean Yang-Mills fields as described and elaborated on by Corrigan et al. [3]. My program received stimulation from the work of D'Adda et al.[4] on the $\mathbb{C}P^{N-1}$ non-linear σ-models *). After the main results described below had been found a preprint of Dubois-Violette and Georgelin [5] appeared in which a program related to mine is announced. However, the main ideas presented below and the way the emphasis is placed differ much from theirs. Those ideas may briefly be summarized as follows : It is known from [1,2,3] that every self-dual $U(n)$ Yang-Mills connection, A, can be written as

$$A_\mu(x) = \phi(x)^*(\partial\phi/\partial x^\mu)(x) \equiv \phi(x)^* \phi_\mu(x) \quad . \tag{1}$$

where $\phi : S^4 \ni x \mapsto \phi(x)$ is a mapping from S^4 into orthonomal n-frames in $\mathbb{C}^N$,

$$\left.\begin{aligned} &\phi(x) = (\varphi_1(x),\ldots,\varphi_n(x)) \ , \ \varphi_j(x) \in \mathbb{C}^N \ , \\ &\text{for some} \quad N > n \ , \text{ and } \quad (\varphi_i(x), \varphi_j(x)) = \delta_{ij} \ , \end{aligned}\right\} \tag{2}$$

for all $i,j = 1,\ldots,n$.

It is natural to try to abstract from self-duality, in such a way that one views ϕ , resp. the gauge-invariant, projection-valued field $P = \phi\phi^*$ of ref.[3] , as the fundamental fields of the theory, and $A = A_\mu(\cdot)$ as derived, via equ. (1). For this purpose, one reexpresses the Yang-Mills action, A_{YM} , as a functional of P (or ϕ) ,

$$A_{YM} = \int d^4x \ \mathrm{tr}[(P_{[\mu}(1-P)P_{\nu]})^2 \ (x)] \ , \tag{3}$$

where $f_\mu(x) \equiv (\partial f/\partial x^\mu)(x)$, for any f ; see [3]. With this action one now tries to associate a Euclidean field theory [7] for P satisfying a suitable form of Osterwalder-Schrader axioms [8] .

*) This program started taking shape after talks of E. Corrigan and M. Lüscher on refs. [3,4] , resp., at the Les Houches work shop on gauge theories, in spring 1978 [6].

The natural analogue of the Yang-Mills action, A_{YM}, in a two dimensional space-time is

$$A_\sigma = \int d^2x \; \mathrm{tr}[(P_\mu(1-P)P_\mu)(x)] \tag{4}$$

which defines a generalized, non-linear σ-model. For $n = 1$, it coincides with the $\mathbb{C}P^{N-1}$-model of [9,4]. This observation suggests that there are close, mathematical connections between four dimensional Yang-Mills theories and two dimensional, generalized, non-linear σ-models. One of the main purposes of this contribution is to exhibit such connections. The main results can be found in §§ 3 and 5. Details of these and other results and proofs will be given elsewhere.

I thank P. Collet, H. Epstein, M. Luscher and K. Osterwalder for valuable discussions and M. Dubois-Violette and Y. Georglin for informing me of their independent results [5] prior to publication.

2. Mathematical preliminaries.

Euclidean space-time is denoted $\mathbb{E}^\nu$, (related to the sphere S^ν by stereographic projection). Let $\mathbb{D}$ be the real ($\mathbb{R}$), complex ($\mathbb{C}$) or quaternionic ($\mathbb{H}$) numbers, and $\mathbb{D}^N$ the linear space of N-tuples of elements in $\mathbb{D}$. If A is an $i \times j$ matrix with entries in $\mathbb{D}$ then A^* denotes the $j \times i$ matrix defined by $(A^*)_{ij} = \overline{A_{ji}}$, with $\bar{a}$ the conjugate of a in $\mathbb{D}$. If B is a $j \times k$ matrix, AB is the matrix product of A and B which is an $i \times k$ matrix.

An orthonormal n-frame in $\mathbb{D}^N$ is given by an $N \times n$ matrix, $\phi = (\varphi_1, \ldots, \varphi_n)$, with $\varphi_j \in \mathbb{D}^N$ and

$$\phi^*\phi = 1_n \text{ , i.e. } (\varphi_i, \varphi_j) = \delta_{ij} \text{ , for all } i,j = 1,\ldots,n \text{ .} \tag{5}$$

The manifold of all n-frames satisfying (5) is denoted $S_{N,n}(\mathbb{D})$. Clearly

$$P \equiv \phi\phi^* \tag{6}$$

is the orthogonal projection onto the hyperplane spanned by $\varphi_1, \ldots, \varphi_n$ [3]. The manifold of all such hyperplanes is denoted $G_{N,n}(\mathbb{D})$, the "Grassmannian of n-planes", and can be written as a coset space [10].

Let H be the group of all linear transformations, h, of $\mathbb{D}^N$ with $h^*h = 1_N$, and $G_P \cong G$ the group of linear transformations, g, of an n-plane, $P\,\mathbb{D}^N$ into itself, with $g^*g = 1_n$; H is called "intrinsic flavour group", G gauge group.

$$\left.\begin{array}{l} \text{For } \mathbb{D} = \mathbb{R} : H = O(N)\ ,\ G = O(n)\ , \\ \text{for } \mathbb{D} = \mathbb{C} : H = U(N)\ ,\ G = U(n)\ , \\ \text{for } \mathbb{D} = \mathbb{H} : H = Sp(N)\ ,\ G = Sp(n) \end{array}\right\} \tag{7}$$

Given ϕ, one sets

$$h_\phi g = h\phi g\ ,\ h \in H\ ,\ g \in G_P\ . \tag{8}$$

Next, consider arbitrary mappings

$$\begin{array}{l} \phi : \mathbb{E}^\nu \ni x \mapsto \phi(x) \in S_{N,n}(\mathbb{D})\ ,\ \text{resp.} \\ P : \mathbb{E}^\nu \ni x \mapsto P(x) = \phi(x)\ \phi(x)^* \in G_{n,N}(\mathbb{D})\ , \end{array} \tag{9}$$

and define a field $A = A_\mu(\cdot)$ by equ. (1). By (1) and (6),

$$A_\mu(x) = \phi(x)^*\ \phi_\mu(x) = \tfrac{1}{2}(\phi(x)^*\ \phi_\mu(x) - \phi_\mu(x)^*\ \phi(x))\ , \tag{10}$$

is in $\mathfrak{G}$, the Lie algebra of G. Let $g(\cdot)$ be an arbitrary G-valued function on $\mathbb{E}^\nu$ (a gauge transformation). Then

$$A^g_\mu(x) = {}^1\phi^g(x)^*\ {}^1\phi^g_\mu(x) = g(x)^*\ A_\mu(x)g(x) + g(x)^*g_\mu(x)\ . \tag{11}$$

Thus $A = A_\mu(\cdot)$ is the connection form of a principal G-bundle with base space $\mathbb{E}^\nu$ and fiber G.

<u>Remark</u>. In an interesting preprint [5], Dubois-Violette and Georgelin quote an important theorem (see ref. [2] of [5]) saying that for $\mathbb{D} = \mathbb{C}$, there exists a finite integer, N, depending on ν and n ($N \geq \frac{\nu+1}{2}\, n$) such that, to each given A, there exists a ϕ such that $A_\mu = \phi^* \cdot \phi_\mu$.

Next, let $h(\cdot)$ be a $U(N)$-valued function on $\mathbb{E}^\nu$. Then

$$^{h}A_{\mu} \equiv {}^{h}\phi^{1*}\, {}^{h}\phi^{1}_{\mu} = A_{\mu} \quad \text{if and only if}$$

$$\phi^{*}h^{*}h_{\mu}\phi = 0 \text{ , i.e. } Ph^{*}\,h_{\mu}P = 0 \quad . \tag{12}$$

An h satisfying (12) is called an "intrinsic flavour transformation". Clearly a constant $h \equiv h_o \in U(N)$ obeys (12).

The curvature of A is given in terms of ϕ by

$$F_{\mu\nu} = \phi^{*}{}_{[\mu}\phi_{\nu]} + \phi^{*}\phi_{[\mu}\phi^{*}\phi_{\nu]} \quad , \tag{13}$$

and the gauge-invariant form of $F_{\mu\nu}$ by

$$F^{I}_{\mu\nu} \equiv \phi F_{\mu\nu}\phi^{*} = P_{[\mu}(1-P)P_{\nu]} = P[P_{\mu},P_{\nu}] = [P_{\mu},P_{\nu}]\,P \quad , \tag{14}$$

where the parantheses denote anti-symmetrization, and we have used $(1-P)P = P(1-P) = 0$, i.e.

$$P_{\mu}P = (1-P)P_{\mu} \quad \text{and} \quad PP_{\mu} = P_{\mu}(1-P) \ . \tag{15}$$

With the help of (14) the Yang-Mills equations and the action (see (3)) can be rewritten in terms of P [3]. Let γ be a closed curve in $\mathbb{E}^{\nu}$, and $\{x^{m}_{j}\}^{m+1}_{j=1}$, $x^{m}_{m+1} = x^{m}_{1} = x_o$ a family of points on γ , with $\sup_j |x^{m}_{j+1} - x^{m}_{j}| \to 0$, as $m \to \infty$. Let

$$P_{x_o}(\gamma) = \lim_{m\to\infty} \prod_{j=1}^{m} P(x^{m}_{j}) \tag{16}$$

Then $W(\gamma) = \mathrm{tr}[P_{x_o}(\gamma)]$ is the Wilson loop[3] .

3. Actions, conserved currents and infrared critical dimension.

Let $D^{\phi}_{\mu} = \partial/\partial x^{\mu} - A^{L}_{\mu}$, where A^{L}_{μ} is left multiplication by $A_{\mu} = \phi^{*}\phi_{\mu}$. Then

$$(D^{\phi}_{\mu}\phi)\phi^{*} = (1-P)P_{\mu} = P_{\mu}P \ . \tag{17}$$

Using (5) and (15) one sees that

$$A_\sigma = \int d^2x\ \mathrm{tr}[|D_\mu^\phi \phi|^2(x)] = \int d^2x\ \mathrm{tr}[|(D_\mu^\phi \phi)\phi^*|^2(x)]$$
$$= \tfrac{1}{2} \int d^2x\ \mathrm{tr}[P_\mu(x)^2] . \tag{18}$$

Thus, the models with action A_σ are natural generalizations of the $\mathbb{C}P^{N-1}$ models [9,4] with which they coincide when $\mathbb{D} = \mathbb{C}$, $n = 1$. The following further isomorphisms are noteworthy : The $\mathbb{R}P^1$ model is isomorphic to the S^1(XY - or rotator) model, the $\mathbb{C}P^1$ model to the S^2- (classical Heisenberg) model, and the $\mathbb{H}P^1$ (or $G_{2,1}(\mathbb{H})$-) model to the S^4- (5 vector) model. The last two isomorphisms are obtained by expressing P in terms of Pauli-, resp. γ-matrices. The last one can also be reduced to the simpler isomorphism between the S^3- (4 vector) model and the $SU(2)$ model with action

$$A_\sigma = \int d^2x\ \mathrm{tr}[g_\mu(x)^* g_\mu(x)] ,\ g(x) \in SU(2) \cong S^3 .$$

For the lattice σ-models defined in (27), §5, the proofs of these isomorphisms are simple.

Finally, we note that equations (17) and (18) remain meaningful for $\mathbb{D} = \mathbb{O}$, the octonions or Cayley numbers [11], with $G_{N,n}(\mathbb{D}) = G_{3,1}(\mathbb{O})$. In this case the projections $P(x)$ label the points of the Moufang projective plane [11,12], and one obtains an octonionic non-linear σ-model whose symmetry group is the exceptional Lie group F_4. These models and Yang-Mills versions thereof may be of interest to strong interaction physics [12].

On a classical level, the discussion of the $G_{N,n}(\mathbb{D})$ - σ-models proceeds as in [4,13 *)]. In two dimensional, Euclidean space-time the $\mathbb{C}P^{N-1}$ models are presumably the most interesting ones, since, for $n > 1$, the $G_{N,n}(\mathbb{D})$ models do generally not admit new classes of instanton solutions with non-trivial homotopy [10]. (E.g. the $\mathbb{H}P^1$ model does not have such solutions). We conjecture, however, that the results of [14] extend to the quantized version of these models. Finally, we note that the infrared critical dimension of the σ-models is $\nu_{crit.} = 2$. This is related to the fact that the naive dimension of the conserved currents is unity; see (25) and §5.

The Yang-Mills action is obtained from $F_{\mu\nu} = [D_\mu^\phi, D_\nu^\phi]$, by setting

$$A_{YM} = \int d^4x\ \mathrm{tr}[F_{\mu\nu}(x)^2] = \int d^4x\ \mathrm{tr}[F^I_{\mu\nu}(x)^2] , \tag{19}$$

*) After completion of this paper, a preprint by E. Brézin, C. Itzykson, J. Zinn-Justin and J.-B. Zuber appeared, where, in addition to one local and one non-local, infinitely many other non-local, conserved currents are constructed explicitly : see Section 0.

which by (14) coincides with (3) .

Next, one tries to form higher tensors, $T_{\mu_1\cdots\mu_d}$, such that $A_d = \int d^{2d}x \, tr[T_{\mu_1\cdots\mu_d}(x)^2]$ can serve as an action[1] for a conformally invariant theory in $\nu = 2d$ dimensions. If one insists on formal Osterwalder-Schrader positivity [8] and conformal invariance, T is necessarily totally anti-symmetric and of rank d . (Unfortunately, the tensors $T_{\mu_1\cdots\mu_d} = P_{[\mu_1}(1-P)P_{\mu_2}(1-P)\ldots P_{\mu_d]}$, a natural generalization of $F^I_{\mu\nu}$, vanish identically, for $d > 2$, because of the identity

$$\ldots (1-P)P_{\mu_j}(1-P)\ldots = \ldots P_{\mu_j}P(1-P)\ldots = 0 \text{ ; see (15))} .$$

In general, the correct expression for $tr[T_{\mu_1\cdots\mu_d}(x)^2]$ can be obtained from the following limiting procedure : Choose a hypercube, Δ_ϵ , in the $(\mu_1,\ldots,\mu_d)$-hyperplane with sides of length ϵ , centered at x . Let $\partial\Delta_\epsilon$ be its boundary. Then

$$- tr[T_{\mu_1\cdots\mu_d}(x)^2] = \lim_{\epsilon\downarrow 0} \epsilon^{-d}\{tr[P(\partial\Delta_\epsilon)] - c(n,\mathbb{D})\} , \tag{20}$$

where $P(\partial\Delta_\epsilon) = \prod_{x\in\partial\Delta_\epsilon} P(x)$, see (16), and $c(n,\mathbb{D})$ is a constant. These expressions make sense and are the desired ones if and only if $d = 1$ (σ-model) or $d = 2$ (Yang-Mills theory). When $d > 2$, $P(\partial\Delta_\epsilon)$ is ill defined because of ordering problems. This rules out the existence of admissible actions, A_d , for $d > 2$, (i.e. $\nu > 4$). (If one does not insist on Osterwalder-Schrader positivity then, of course, there are plenty of actions for $d > 2$).

Clearly, the actions A_σ and A_{YM} have a global H-symmetry. In addition, A_{YM} has a <u>local symmetry</u> : By (12) A_{YM} is invariant under

$$P(x) \mapsto P^h(x) \equiv h(x)P(x)h(x)^* \text{ , with}$$

$$Ph^*h_\mu P = 0 \tag{21}$$

This equation has non-trivial solutions. Associated with these symmetries are Noether currents, J^δ_μ . They are given by

$$J^\delta_\mu(x) = \partial\mathcal{L}_{YM}(x)/\partial P_\mu(x)_{ij} \, (\delta P)_{ij}(x) , \tag{22}$$

where $\mathcal{L}$ is the Lagrange density corresponding to A . (In Euclidean field theory, $\mathcal{L}$ is replaced by the action density [15]). Furthermore, δ is the derivation associated with a symmetry transformation, i.e.

$$\delta P = [B,P] \text{ , with } PB_{\mu}P = 0 \text{ ,} \tag{23}$$

where $B(x) \in \mathfrak{H}$, the Lie algebra of H, for all x. Apart from $B(x) = B = \text{const.}$ there will be other solutions of (23). They form a linear space. By solving (23) in terms of P one gets a linear space of conserved currents. (After quantization they ought to determine a Lie algebra of conserved charges properly containing $\mathfrak{H}$).

Heuristically, quantization consists in associating with an action A the formal measure on the space of $G_{N,n}(\mathbb{D})$-valued distributions

$$d\mu(P) = e^{-g^{-2}A} \prod_{x \in \mathbb{E}^{\nu}} \mathcal{D}P(x) \text{ ,} \tag{24}$$

where $\mathcal{D}P$ is, heuristically, the uniform measure on $G_{N,n}(\mathbb{D})$, and g is a coupling constant (dimensionless for $\nu = 2$, $A = A_{\sigma}$ and $\nu = 4$, $A = A_{YM}$). In the process of renormalization, $P^2 = P$ will have to be replaced by $P^2 = fP$, f divergent [4], g is renormalized, and the conserved currents yield many Ward identities [16]. Formally, (i.e. disregarding from the existence problem), $d\mu$ satisfies Osterwalder-Schrader positivity for the fields P, when $A = A_{\sigma}$, resp. $W(\gamma)$, see (16), when $A = A_{YM}$. This is shown by approximating A by an action A_{ϵ} constructed in terms of $P(\partial\Delta_{\epsilon})$; see (20), (16). Note that the naive dimension of P is zero. In four dimensions there are no dimensionless fields satisfying positivity. Hence, for $A = A_{YM}$ ($\nu = 4$), presumably only loop observables, $W(\gamma)$, and functionals thereof, survive the $\epsilon \downarrow 0$ limit.*) Thus, whereas for $A = A_{\sigma}$, $\nu = 2$, spin wave theory about $P = P_o = \text{const.}$ makes sense, providing an expansion in g - at least after adding a term $m^2 \int d^2x \, \mathrm{tr}[P(x)P_o]$ to A_{σ} - this is not clear, at all, for $A = A_{YM}$. It is further complicated by the symmetries (21), (23). (Moreover, it is doubtful whether one can add a term $m^4 \int d^4x \, \mathrm{tr}[P(x)P_o]$ to A_{YM}, destroying those symmetries, to eliminate infrared divergences). A natural question is whether the symmetries of A can be broken spontaneously. The arguments of [15] suggest that the symmetry associated with a current J_{μ} cannot be broken spontaneously in dimension

$$\nu \leq \nu_{\text{crit.}} = [J_{\mu}] + 1 \text{ ,} \tag{25}$$

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*) If, for $A = A_{YM}$, $\nu = 4$, P exists as a quantized field its ultraviolet dimension must be $\gtrsim 1$. (Källen-Lehmann representation. Since its naive dimension is 0, P may, in fact, not survive quantization. See also Section 5).

where $[J_\mu]$ is the infrared dimension of J_μ. For $A = A_{YM}$, we obtain from (22), by naive dimensional analysis, $[J_\mu] \geq 3$, i.e. $\nu_{crit.} \geq 4$! (For $A = A_\sigma$ one obtains of course $\nu_{crit.} = 2$). In order for these arguments to be convincing they should be reformulated in terms of the loop observables, $W(\gamma)$.

4. Couplings to quark fields.

Let $\bar{\psi}_\alpha$, ψ_α be Dirac spinors transforming under the same representation of G as Φ, Φ^*, resp.; (α is the G-(colour) index). In addition $\bar{\psi}$ and ψ may transform under some flavour group, i.e. carry a flavour index $j = 1,\ldots,F$. The components $\psi_{\alpha j}$ are the matrix elements of an $n \times F$ matrix, denoted ψ, and $\bar{\psi}$ is the corresponding, conjugate $F \times n$ matrix. One defines the gauge-invariant fields [3] $\bar{\psi}^1 = \bar{\psi}\,\Phi^*$, $\psi^1 = \Phi\psi$. When $\mathbb{D} = \mathbb{H}$, there are further invariants, $\psi^2 = \psi^T \epsilon\, \Phi^*$ and $\bar{\psi}^2 = \Phi\, \epsilon\, \bar{\psi}^T$, where

$$\epsilon = \begin{pmatrix} \epsilon_1 & & 0 \\ & \ddots & \\ 0 & & \epsilon_n \end{pmatrix}, \quad \epsilon_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad j = 1,\ldots,n .$$

One observes that N and F play symmetric rôles. Thus one may speculate that $N = F$, i.e. "intrinsic flavour" = flavour. The result quoted in [5] then suggests that $F = N \geq (\nu+1)n/2$, i.e. $F \geq 9$, for $\nu = 4$, $n = 3$.

The minimal coupling matter actions are

$$A_j^\mu = \int d^\nu x \; \mathrm{tr}[\bar{\psi}^j(x)(\not{\partial}\psi^j)(x)] , \tag{26}$$

$j = 1,2$. For $N = F$, there are the following flavour invariants : $\psi_I^j = \mathrm{tr}[\psi^j]$, $\bar{\psi}_I^j = \mathrm{tr}[\bar{\psi}_j]$. One can then form the action $A^\mu_{j,I} = \int d^\nu x\; \bar{\psi}_I^j(x)\, (\not{\partial}\psi_I^j)(x)$. However, they give rise to trivial interactions, and the Green function for $\bar{\psi}$ and ψ can be calculated in an arbitrary Φ-field.

5. Lattice models.

The $G_{N,n}(\mathbb{D})$ non-linear σ-models can be put onto the lattice, $\mathbb{Z}^\nu$, in two different ways :

$$A'_\sigma = \sum_{n.n.} \mathrm{tr}[P_x P_y] , \tag{27}$$

$$A''_\sigma = \sum_{n.n.} \mathrm{tr}[\phi_x g_{xy} \phi^*_y] , \tag{28}$$

where $g_{xy} \in G$, for all nearest neighbors $(n.n.)x,y$. The gauge field, $g = \{g_{xy}\}$, plays the rôle of a Lagrange multiplier field [4] .

As a priori distributions one chooses the uniform measure on $G_{N,n}(\mathbb{D})$ for P_x , resp. the Haar measure on G for g_{xy} , and for ϕ and ϕ^* the measure

$$d\lambda(\phi) = \prod_{\alpha\leq\beta} \delta([\phi^*\phi]_{\alpha\beta} - \delta_{\alpha\beta}) \prod_{i\alpha} d\phi_{i\alpha}\, d\overline{\phi_{i\alpha}}$$

This yields two different lattice models with identical, formal continuum limit, (18). The Mermin-Wagner argument [15] excludes spontaneous breaking of the intrinsic flavour symmetry group H , when $\nu = 2$. For the discussion of the breaking of H in $\nu \geq 3$ dimensions, the methods of [17] can be applied to the model with action A'_σ . Good results concerning the symmetric phase are achieved by applying the methods of [18] to the model with action A''_σ . Those methods vaguely correspond to partially resummed $1/N$ expansions and are rigorous. It would be of interest to develop a double expansion in $1/N$ and $1/n$, or, for $A = A_{YM}$, one in $\frac{N-n}{N}$ (about pure gauges). The following result may be useful.

<u>Theorem : If one replaces</u> $d\lambda$ <u>by</u>

$$d\lambda^{(N)}(\phi) = \prod_{\alpha\leq\beta} \delta([\phi^*\phi]_{\alpha\beta} - N^{3/4}\, \delta_{\alpha\beta}) \prod_{i\alpha} d\phi_{i\alpha}\, d\overline{\phi_{i\alpha}} ,$$

<u>and defines</u>

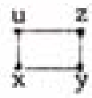
$$d\mu^{(N)}(g) / \prod_{n.n.} dg_{xy} \equiv Z^{(N)^{-1}} \int e^{-\frac{1}{g^2} A''_\sigma} \prod_{x\in\mathbb{Z}^\nu} d\lambda^{(N)}(\phi(x))$$

<u>then</u> $w^*\text{-}\lim_{N\to\infty} d\mu^{(n)}(g) \equiv d\mu^W(g)$ <u>exists and is the measure of Wilson's pure Yang-Mills lattice theory [19] with group</u> G

Analogous limit theorems hold for Gaussian $d\lambda^{(N)}$ and $|\phi|^4$-type $d\lambda^{(N)}$. In the last case, one obtains a formal $|\phi|^4$-theory which is conformal invariant, for $\nu = 4$, and whose $N \to \infty$ limit is pure Yang-Mills. One may speculate that that theory is asymptotically free.

On the lattice one can try to mimick A_{YM} by the action

$$A_S = -\sum_p \left(\mathrm{tr}[P_x P_y P_z P_u] + \mathrm{tr}[P_x P_u P_z P_y]\right) ,$$

where p is the plaquette 

. We note that $\mathrm{tr}[P_x P_y P_z P_u]$ is a lattice approximation to $W(\partial p)$; see (16), (20). In spite of this formal relationship with A_{YM} , the naive continuum limit of A_S is not A_{YM} . (The model with action A_S is related to an Ising type model of Slawny [20]). A better approximation to A_{YM} is

$$A'_{YM} = -\sum_{\Delta} \left(\mathrm{tr}[\prod_{x\in\partial\Delta}^{\circlearrowleft} P_x] + \mathrm{tr}[\prod_{x\in\partial\Delta}^{\circlearrowright} P_x]\right) , \tag{29}$$

where Δ is an arbitrary lattice square parallel to two axes of $\mathbb{Z}^\nu$ each side of which contains <u>three</u> sites. Both actions, A_S and A'_{YM} , admit a transfer matrix formalism with selfadjoint, generalized transfer matrix [17] . This guarantees Osterwalder-Schrader positivity. Wilson's lattice gauge theory [19] is recovered by choosing $A_W = -\sum_p W(\partial p)$, where $W(\partial p)$ is given by (16), and as a priori distribution "$\prod_{x\in\partial p} dP(x)$" the product of the Haar measures for G on the sides of ∂p .

If one discusses confinement for the models with actions A'_σ , or A'_{YM} in terms of a lattice version of the Wilson loop, $W(\gamma)$, one arrives at the heuristic picture that confinement breaks down if the intrinsic flavour symmetry, H , is spontaneously broken. Such breaking is expected for $\nu > \nu_{crit.}$ (i.e. $\nu \geq 5$ for A'_{YM}) and small g .

One can show that if one couples quark fields to P with a large number, F , of flavours and a mass $\propto F^{1/2}$, $F \gg 1$, by means of a lattice version of A^M_j , this enhances the spontaneous breaking of H which appears to become possible in $\nu \geq 3$ dimensions.

Rigorous proofs are so far restricted to the σ-models, because neither the techniques of [17] nor the ones of [18] , in their present form, apply to Yang-Mills.

References :

1. M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Yu.I. Manin, Phys. Lett. 65 A, 185, (1978).

2. V.G. Drinfeld and Yu.I. Manin, Commun. math. Phys. 63 , 177, (1978).

3. E.F. Corrigan, D.B. Fairlie, S. Templeton and P. Goddard, Nucl. Phys. B 140, 31, (1978).

4. A. D'Adda, P. Di Vecchia and M. Luscher, Preprint, Niels Bohr Institute, 1978; and in ref.[6].

5. M. Dubois-Violette, and Y. Georgelin, Preprint, Orsay 1978.

6. E. Brézin and J.-L. Gervais (eds.), Physics Reports, to appear.

7. "Constructive Quantum Field Theory", G. Velo and A.S. Wightman (eds.) Lecture Notes in Physics 25, Springer-Verlag, Berlin-Heidelberg-New York, 1973.

8. K. Osterwalder and R. Schrader, Commun. math. Phys, 42, 281, (1975).

9. H. Eichenherr, Ph. D. thesis, Heidelberg, 1978.

10. N. Steenrod, "The Topology of Fibre Bundles", Princeton University Press, Princeton, 1951.

11. N. Jacobson, "Lie Algebras", Wiley, New York-London, 1962.

12. M. Günaydin, C. Piron and H. Ruegg, Commun. math. Phys. 61, 69, (1978); and refs. given there.

13. M. Luscher and K. Pohlmeyer, Nucl. Phys. B 137, 46, (1978).

14. M. Luscher, Nucl. Phys. B 135, 1, (1978); see also A.B. Zamolodchikov and A.B. Zamolodchikov, Nucl. Phys. B 133, 525, (1978).

15. Predictions that $\nu_{\text{crit.}} \geq 4$, for Yang-Mills, were previously made by A. Migdal and G. Parisi. Our argument extends that in : K. Symanzik, Commun. math. Phys. 6, 288, (1967). also J. Fröhlich and T. Spencer, in "New Developments in QFT and Stat. Mech.", M. Lévy and P. Mitter (eds.) Plenum, New York, 1977.

16. E. Brézin, J. Zinn-Justin and J.C. Le Guillou, Phys. Rev. D 14, 2615, (1976).

17. J. Fröhlich, B. Simon and T. Spencer, Commun. math. Phys. 50, 79, (1976); J. Fröhlich, R. Israel, E. Lieb and B. Simon, Commun. math. Phys. 62, 1, (1978).

18. K. Symanzik, in "Local Quantum Theory", R. Jost (ed.), Academic Press, New York, 1969; D. Brydges and P. Federbush, Commun. math. Phys. 62, 79, (1978); J. Fröhlich and T. Spencer, unpublished.

19. K. Wilson, Phys. Rev. D 10, 2445, (1974); R. Balian, J.M. Drouffe and C. Itzykson, Phys. Rev. D 10, 3376, (1974), D 11, 2098, 2104, (1975).

20. J. Slawny, Commun. math. Phys. 46, 75, (1976).