On the Construction of Quantized Gauge Fields, I

General Results

David Brydges[*+]
Department of Mathematics
The Rockefeller University
New York, New York 10021

Jurg Fröhlich[**]
Institut des Hautes Etudes Scientifiques
35, Route de Chartres
F-91440 Bures-sur-Yvette

Erhard Seiler[*]
Department of Physics
Princeton University
Princeton, New Jersey 08540

Institut des Hautes Etudes Scientifiques
35, rt de Chartres
91440 Bures-sur-Yvette (France)

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[+] Present address : Dept. of Math., University of Virginia, Charlottesville, Virginia 22903.

[**] A. Sloan Foundation Fellow.

Abstract :

In this paper gauge theories are analyzed from the point of view of constructive quantum field theory. Diamagnetic inequalities for general lattice gauge theories are proven. They say that in a local field theory the ground state energy density rises when the fields (scalars, Dirac fields, etc.) are minimally coupled to an external gauge field. A mass generation mechanism is described. Abelian Higgs models (scalar QED) on the lattice are investigated in more detail : Strong diamagnetic inequalities; infrared bounds; correlation inequalities yielding monotonicity in the space-time cutoff, in the mass of the photon and in the electric charge; gauge invariance and Osterwalder-Schrader positivity, etc. are shown to hold uniformly in the lattice spacing. For the two dimensional Higgs model on the lattice the θ-vacua are constructed, a first order phase transition at $\theta = \pi$ and the confinement of fractionally charged quarks are described. Details and applications to the two dimensional Higgs model in the continuum limit appear in subsequent papers.

1. Introduction : Summary of Results and Table of Contents

1.1. The subject of this paper

This is the first in a series of papers devoted to the construction and analysis of quantized gauge fields interacting with scalar and/or Dirac fields through minimal coupling. Many of our results concern lattice theories, but our main goal is the construction of the continuum Higgs model in two space-time dimensions. We propose to show thereby that present techniques of constructive quantum field theory combined with some new inequalities appear to suffice for the construction of super-renormalizable, abelian gauge theories in the continuum limit and to investigate some of their physical properties, e.g. the structure of the physical vacuum.

So far only one very simple continuum gauge theory has been shown to exist : QED of massive (or massless) Fermions minimally coupled to massive photons in two space-time dimensions [1]. The limit as the mass of the photon tends to 0 , yielding the massive Schwinger (-Thirring) model, was analyzed in detail in [2]. In comparison, the Higgs model poses much more challenging problems and has a more interesting structure.

Apart from preparing the grounds for the construction of the continuum Higgs_2 model we present a general analysis of lattice gauge theories, including non-abelian ones, emphasizing those properties which may be important for taking the lattice spacing to 0 and may survive the continuum limit.

So far the mathematical analysis of gauge theories has proceeded along several different lines (with almost empty intersection) :

(1) Analysis of the classical, pure Yang Mills equations in Minkowski space [3]. This is a hyperbolic problem.

(2) Analysis of the classical, pure Yang-Mills equations (in particular the self-dual equations) at imaginary time [4] , an elliptic problem. It is hoped that its solution may be useful for the Euclidean description of quantized Yang-Mills fields in terms of functional integrals.

This line is evolving into a general analysis of the geometry of Yang-Mills fields at imaginary time [5].

(3) Analysis of quantized scalar and Dirac fields in an external c-number gauge field at imaginary time [6, 7, 8].

(4) Rigorous study of gauge field theories on the lattice [9,10,11].

Apart from these mathematical approaches towards understanding gauge fields direct attacks on the physics of gauge quantum field theories, in particular QCD models, have been made [12,13,14,15].

Our point of view is that detailed knowledge of line (3) can perhaps be put together, eventually, with detailed knowledge of quantized, pure Yang-Mills fields to achieve a clear insight into interesting gauge quantum field theories. This may serve as a partial motivation of our study : Our results concerning general lattice gauge theories, including non-abelian ones, belong to lines (3) and (4).

We describe these results below, but some of the detailed statements and proofs appear in a subsequent article.

The main issue of the present paper is to prove some results which appear to be useful for

(5) Construction of super-renormalizable, abelian gauge field theories in the continuum - and infinite volume limit.

In a forthcoming paper this program is carried out for the abelian Higgs model in two dimensions.

1.2 Summary of main results

In the following we summarize some of our main results in a somewhat cavalier formulation. First we consider a general field theory on a simple, cubic lattice, describing scalar fields, Dirac fields, etc., but without gauge fields. We assume that the couplings (interactions) are such that the theory satisfies Osterwalder-Schrader positivity [16,17] (on the lattice also called reflection positivity [18]) . The vacuum energy density of this theory is denoted $\varepsilon(\underset{\sim}{1})$; (the normalization being such that $\varepsilon = 0$ in the case of free fields).

We propose to study the effect of coupling this theory to an arbitrary but fixed external lattice gauge field $\underset{\sim}{g} = \{g_b\}_{b\in\mathcal{B}}$ (where $\mathcal{B}$ are all directed nearest neighbor bonds of the lattice). The coupling is assumed to be the standard, gauge-invariant minimal one, [9,10,17].

The vacuum energy density of the theory in an external Yang-Mills field $\underset{\sim}{g}$ is denoted $\varepsilon(\underset{\sim}{g})$ (the normalization being kept fixed). Our first main result is

Theorem A (See Section 2)

$$\varepsilon(\underset{\sim}{g}) \geq \varepsilon(\underset{\sim}{1})$$

(Universality of "diamagnetism" in local field theory).

□

Remarks :

(1) This theorem says that coupling a field theory to an external Yang-Mills field tends to make it more stable. The result belongs to line (3) in the above classification. It extends earlier work of R. Schrader and R. Seiler [19] concerning the special case of the free, scalar field in an external Yang-Mills field. (An analogous result for spinor QED is due to Schwinger [20]).

(2) We have proven other general results of the type of Theorem A (see Sections 2 and 4), in particular chessboard estimates and infrared bounds for gauge fields (patterned on [18,21]). They are useful to analyze cluster properties (Theorem C) or the decay of the Wilson loop (Theorem E) in lattice gauge theories.

The following result also represents line (3).

Theorem B.

(1) (See Section 3) Consider a free (Gaussian), scalar, multi-component field ϕ_α , $\alpha = 1,\ldots,N$, on the lattice in an external Yang-Mills field $\underset{\sim}{g}$. Denote its Euclidean propagator (two point function) by $G^E_{\alpha\beta}(x,y;\underset{\sim}{g})$. Then ,for an arbitrary sequence $\{z_\alpha\}_{\alpha=1}^N$ of complex number

$$\Big|\sum_{\alpha,\beta} \bar{z}_\alpha z_\beta G^E_{\alpha\beta}(x,y;\underset{\sim}{g})\Big| \leq \sum_{\alpha,\beta} \bar{z}_\alpha z_\beta G^E_{\alpha\beta}(x,y;\underset{\sim}{1})$$

(2) (Theorem 4.1, Section 4) Suppose the gauge group is $U(1)$. Let $\phi(x)$ be a complex, scalar field with arbitrary, gauge-invariant self-couplings in an external electromagnetic field. Then the unnormalized Schwinger functions (in finite volume) are bounded above by the ones in zero external field.

□

We remark that $G^E_{\alpha\beta}(x,y;\underset{\sim}{g})$ is the kernel of $(-\Delta_{\underset{\sim}{g}} + m^2)^{-1}$, where $\Delta_{\underset{\sim}{g}}$ is the covariant, finite difference Laplacian. The inequality of Theorem B, (1) and other inequalities, all concerning the covariant, finite difference Laplacean, can be extended to general lattices. Their proofs are based on lattice Feynman-Kac formulas. As earlier shown by Schrader et al. [6] and Simon [7] (by arguments very different from ours) such inequalities also hold in the continuum limit ("Kato-inequalities"). Our methods are more direct than theirs.

Theorem B, (2) and other results (e.g. upper bounds on normalized, gauge-invariant Schwinger functions in infinite volume, etc.) concerning the abelian case are proven in Sections 4-6. They are important for the construction of abelian Higgs models in the continuum limit. In particular, Theorems A and B are applied in our construction of the two dimensional, abelian continuum Higgs model. Previously, Schrader has applied the continuum version of Theorem B, (1) to study the $P(\Phi)_2$ models in an external Yang-Mills field [22].

The following result belongs to line (4), (lattice gauge theories) : Consider a general abelian or non-abelian gauge theory on the simple cubic lattice describing two scalar fields, Φ and χ, interacting with a gauge field through minimal coupling. We assume that Φ and χ transform under the same representation of the gauge group; Φ may have arbitrary, local gauge-invariant self-interactions, the bare mass of χ is $= 0$. We choose the gauge in which the "angular components" of Φ are eaten up by the gauge field.

Theorem C. (See [23], hereafter referred to as III)

In dimension $\nu \geq 3$, the two point-function of χ has exponential clustering.

□

This is interpreted as the dynamical generation of a mass for the χ-field through minimal coupling. (The methods used to prove Theorem C can also be applied to estimate the radiative corrections to the physical mass of a

Dirac field Ψ (in place of χ) coupled to the Higgs system, in the large bare mass limit). Results related to Theorem C have previously been proven in statistical mechanics models [24]. They are based on inequalities such as the ones of Theorem B, (1) (resp. the Feynman-Kac formulas of Section 3) and chessboard estimates. For a precise statement and proof of Theorem C we refer the reader to III.

Next, we discuss abelian Higgs models on the lattice.Abelian gauge theories can be put onto the lattice in a gauge-invariant manner in many different ways. For example, we can always work with a (transverse) Gaussian expectation for the pure gauge field (free, electromagnetic field on the lattice) which is gauge-invariant. Furthermore, if the gauge field only couples to conserved currents we can change the gauge freely and we can give the gauge field a positive bare mass. These observations are important in our construction of the continuum Higgs model in two space-time dimensions which is completed in [25] (hereafter referred to as II).

Other lattice approximations for the abelian gauge field are the one proposed by Wilson [9] and the one used by Polyakov [9] (which differs from Wilson's one as the rotator model differs from its Villain approximation. Polyakov's approximation is hence forth called PV-approximation. It was previously used in [26,27,28]See Section 4 for definitions). Most of the following results can be proven in many of these lattice approximations. In two dimensions we usually find the Gaussian lattice approximation for the pure gauge field and the PV-approximation for the action of the Higgs field in an electromagnetic field most attractive, in higher dimensions a class of PV-approximations (with different, but discrete frequency spectra).

One general idea behind our analysis of abelian Higgs models on the lattice is to use Fourier transformation ("duality transformation") in the electromagnetic field, see [26,28,29] and III. This converts the Higgs models into models familiar in classical statistical mechanics. (E.g., the two dimensional abelian Higgs model is, by duality, isomorphic to a classical spin system with spins taking values in $\mathbb{Z}$ and nearest neighbor coupling. This model is isomorphic to a Yukawa type lattice gas; see e.g. [28,29]). The isomorphism described here permits us to apply high - and low temperature expansions similar to the ones applied in Ising models which are considerably simpler than the ones developped for continuum field theories in [30,31,32].

As a consequence we obtain a domain in the coupling constant space for which the Higgs mechanism is known to occur which is much larger than the one found for general Higgs models in [17]. Since these expansion methods are quite standard we shall not present detailed proofs; but see III.

Furthermore, for the <u>abelian</u> Higgs models (and their duals) we have found various new correlation inequalities of the type of Ginibre's inequalities [33] and [34,35] which can be used to extend some expansion results beyond the region of convergence of the expansions. The most important application of these inequalities (see Section 6) is the following : They are <u>stable</u> undertaking the continuum limit, whenever the latter exists. Therefore they permit us to construct the infinite volume limit for the continuum Higgs model in two dimensions and to prove that, in this limit, the Schwinger functions are <u>Euclidean invariant</u>.

Rather than summarizing these results in the form of theorems we refer the reader to Section 6 and to II and III.

We conclude this introduction with summarizing some results for the two dimensional, abelian Higgs model on the lattice which we find amusing.

(The reader should think of a mixed Gaussian-PV-lattice approximation but our results can also be formulated for other lattice approximations).

Theorem D. (See III)

The two dimensional, abelian lattice Higgs model has a family of physically different "equilibrium (= Euclidean vacuum) expectations", denoted $< - >_\theta$, which are labelled by an angle $\theta \in [0,2\pi)$.

Different values of the angle θ correspond to different boundary, conditions at ∞, interpreted as classical charges at spatial ∞.

At $\theta = \pi$, there is a first order phase transition accompanied by spontaneous breaking of parity, and there are at least two different pure equilibrium expectations corresponding to a non-zero, universal electric field pointing to spatial $+\infty$, resp. $-\infty$, provided the self-interaction of the Higgs field increases sufficiently rapidly at infinity, and the electric charge is large enough.

□

Remarks : This effect can also be seen in the dilute gas approximation [13,14,36] to the two dimensional, abelian Higgs model; see [28]. θ-vacua in gauge theories with instantons were discovered in [37,38]. In the case of the Schwinger model with massive Fermions they were earlier found in [39] and further analyzed, more rigorously, in [2]. The situation described in Theorem D is closely related to the one met in the massive Schwinger model.

We have checked that in a dilute gas approximation to a four dimensional SU(2) Higgs model (with total "symmetry breakdown" and without quarks), a theory which has θ-vacua, a phase transition and parity breaking occur at $\theta = \pi$, too. (This is a problem in the statistical mechanics of classical dipole gases [14] with short range interactions and complex, chemical potential).

Finally we mention a result concerning the confinement of very heavy, fractionally charged quarks coupled to the two (or more) dimensional, abelian lattice Higgs model, (resp. the three dimensional $U(1)$ lattice gauge theory in the Villain approximation [27,28]).

Let $\epsilon(\theta)$ denote the vacuum energy density (in some fixed normalization) of the two dimensional, abelian lattice Higgs model for a given value, θ, of the angle.

Theorem E. (See III)

Let q be a fraction of the electric charge of the Higgs scalar in the two dimensional lattice Higgs model. Then

$$< \exp[iq \sum_{<i,j>\in\Gamma} A_{ij}] >_{\theta=0} \leq e^{[\epsilon(2\pi q)-\epsilon(0)]|\Gamma|} \quad , \tag{1}$$

where Γ is a closed loop, $|\Gamma|$ the area enclosed by it, and A_{ij} is the abelian gauge field.

For all $q < 1$, $\epsilon(2\pi q) \leq \epsilon(0)$, and $\epsilon(2\pi q) < \epsilon(0)$ if e.g. the electric charge is large enough. Then the Wilson loop (1) has area decay.

Remarks :

(1) The proof of Theorem E is not based on expansion methods. It follows from chessboard (see Section 2) and "thermodynamic" estimates, so that (1) is true in general. For values of θ different from 0, Wilson's confinement criterion does not seem to be applicable, due to the presence of non-zero charges at spatial infinity.

Some new results concerning the Villain model and the pure U(1) lattice gauge theories in the PV-lattice approximation, similar in spirit to the ones reported in Theorem E are given elsewhere. See also [40,27].

(2) The circle of results described here can be regarded as a partial confirmation of <u>Wilson's conjecture</u> [41] : In a theory where the gauge field acquires a mass (in the model of Theorem E via a Higgs mechanism), the Fermions ("quarks") are confined, unless their charge can be shielded by the Higgs scalar, i.e. $q = 1$. We are grateful to M. Lüscher for pointing out to us Wilson's conjecture.

(3) In many circumstances the Wilson loop does not seem to be applicable as a criterion for confinement; (e.g. when the mass of the Fermions is small, see [42,43], or for the two dimensional Higgs model with $\theta \neq 0$. See also III). It is therefore of interest to note that confinement of fractionally charged Fermions can also be verified in the dilute gas approximation to the two dimensional, abelian Higgs model; see [36] and [28] (where <u>no</u> use of the Wilson loop is made).

For results useful for the construction of the continuum Higgs model (e.g. correlation inequalities, etc.) see Sections 4-6.

<u>Notes</u> :

1. After completion of the manuscript of this paper we have received a Harvard Preprint, by R. Israel and C. Nappi, which contains a proof of Theorem E for the special case of a two dimensional Higgs model where the radial degrees of freedom of the scalar field are "frozen" ("Stückelberg model"), in some region of coupling constants.

We thank R. Israel and C. Nappi for sending us their paper prior to publication and correspondence.

2. a) We have recently shown that, in some special cases, diamagnetic inequalities of the type summarized in Theorem A have the following physical consequences :

- The interactions between gauge field and matter fields tend to increase the physical mass of the gauge field.

- The interactions between gauge field and matter fields produce an attractive, effective "gluon-gluon" interaction.

In view of Wilson's conjecture [41] and recent speculations due to Mack [15] these diamagnetic effects appear to be of interest.

So far, rigorous proofs are limited to abelian lattice gauge theories, but similar diamagnetic effects are expected in the non-abelian case, too, among them a generalized Meissner effect !

2. b) For a class of abelian lattice gauge theories (in particular Higgs models) in arbitrary dimension ν, one can show that the potential between two heavy charges of opposite sign, defined in terms of the Wilson loop, is never stronger than the $(\nu - 1)$ dimensional Coulomb potential. In particular, in $\nu \geq 4$ dimensions these models do not confine the charge (in the sense of the Wilson loop criterion).

2. c) The main method developped in Section 4 has been extended to general, non abelian Higgs models.

Detailed statements, proofs and further discussion will be given elsewhere.

3. After completion of the manuscript of this paper we became aware of some papers by B. Simon (besides ref. [7], see also : B. Simon, Math. Z. 131, 361, (1973), Phys. Rev. Lett. 36, 1083, (1976), Ind. Univ. Math. J. 26, 1067, (1977)) which have played an important role in developping a mathematically rigorous theory of diamagnetism and are relevant for the material presented in Section 3.

Part 1

2. Lattice Gauge Theories

In this section we briefly review the general formalism of lattice gauge theories [10,17] as proposed by Wilson [9], emphasizing some slightly novel points of view. We then prove the general diamagnetic inequalities (Theorem A), chessboard estimates and infrared bounds. In our presentation of the general formalism and some basic results we follow closely [17]. Some techniques were inspired by statistical mechanics [18].

2.1. The general formalism of lattice gauge theories

For convenience we only study gauge theories on the simple, cubic lattice $\mathbb{Z}^{\nu}_{\frac{1}{2}} = \mathbb{Z}^{\nu} + (\frac{1}{2},\ldots,\frac{1}{2})$.*

a) First we define a multi-component, scalar field ("Higgs field") on the lattice : Let V^{Φ} be a finite dimensional, real or complex Hilbert space, and G a compact Lie group, the gauge group. The space V^{Φ} carries a unitary representation U^{Φ} of G. (Generally $U^{\Phi} = 1 \otimes U^{\Phi}_{C}$ where C denotes "colour").

A field configuration on a bounded subset Λ of $\mathbb{Z}^{\nu}_{\frac{1}{2}}$ is a map

$$\underset{\sim}{\Phi} : \Lambda \to V^{\Phi} \tag{2.1}$$

The class of field configurations on Λ can obviously be identified with

$$\bigotimes_{x \in \Lambda} V^{\Phi}_{x} = V^{\Phi}_{\Lambda} \quad , \tag{2.2}$$

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* At a later stage it might be advantageous to consider, instead of $\mathbb{Z}^{\nu}_{\frac{1}{2}}$, the vertices and edges of more general (e.g. triangular) lattices, as this might do more justice to the geometry of gauge fields. Some of our results (see e.g. section 4) can be extended to this more general situation.

where V_x^{Φ} is a copy of V^{Φ}, for all x .

The set of all bounded, continuous complex-valued functions on V_Λ^{Φ} is denoted by $\mathfrak{A}_\Lambda^{\Phi}$, the "field algebra". Let $d\rho$ be a regular Borel probability measure on V^{Φ} which is invariant under the action U^{Φ} of G on V^{Φ} . For arbitrary $F \in \mathfrak{A}_\Lambda^{\Phi}$, we define its <u>expectation</u> by

$$< F >_o^{\Phi} = \int \prod_{x\in\Lambda} d\rho(\Phi(x)) \, F(\underset{\sim}{\Phi}) \ , \tag{2.3}$$

where $\underset{\sim}{\Phi} = \{\Phi(x)\}_{x\in\Lambda}$.

b) Next, we introduce <u>Dirac Fermi fields</u> on the lattice, [17] : With each site $x \in \Lambda$ we associate two copies, $V_x^{\psi^1}$, $V_x^{\psi^2}$, of a fixed, finite dimensional complex Hilbert space V^{ψ} . Furthermore, we are given a unitary representation U^{ψ} of the gauge group G on V^{ψ} . The action of G on $V_x^{\psi^2}$ is then given by U^{ψ} , and on $V_x^{\psi^1}$ by $\overline{U^{\psi}}$ (the representation conjugate to U^{ψ}) .

The space V^{ψ} is a tensor product

$$V^{\psi} = V_S^{\psi} \otimes V_F^{\psi} \otimes V_C^{\psi} \ , \tag{2.4}$$

with S for "spin", F for "flavor and C for "colour". In the sequel we will often suppress flavour, (e.g. replace V_F^{ψ} by $\mathbb{C}$) .

In accordance with (2.4) we assume U^{ψ} to be of the form

$$U^{\psi} = 1_S \otimes 1_F \otimes U_C^{\psi} \ . \tag{2.5}$$

On V_S^{ψ} we have a representation of the Dirac-Clifford algebra

$$\gamma_i\gamma_j + \gamma_j\gamma_i = 2\delta_{ij} \ , \tag{2.6}$$

by hermitean matrices of determinant 1. (For $\nu = 4$ one may choose e.g. the representation

$$\gamma_o = \begin{pmatrix} \sigma_o & 0 \\ 0 & -\sigma_o \end{pmatrix}, \quad \gamma_k = i\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \quad k = 1,2,3, \tag{2.7}$$

where the σ's are the Pauli matrices. This representation has the required properties. Note that the γ's are the Euclidean Dirac matrices).

Let

$$V^{\psi}_{\Lambda} = \bigoplus_{x\in\Lambda} (V^{\psi^1}_{x} \oplus V^{\psi^2}_{x}) , \tag{2.8}$$

and $\Lambda(V^{\psi}_{\Lambda})$ the exterior algebra over V^{ψ}_{Λ}.

A field configuration is a choice of an orthonormal basis $\{\psi^k_{\alpha a}(x)\}_{\alpha a}$, for each $V^{\psi^k}_{x}$. (Here α labels spin and flavour, and a colour). In $\Lambda(V^{\psi}_{\Lambda})$ we may consider the polynomials in the basis vectors, i.e. the totally antisymmetric tensors. We define a "field algebra"

$$\mathfrak{A}^{\psi}_{\Lambda} = \{\text{all polynomials in the basis vectors}\} \tag{2.9}$$

Note that $\mathfrak{A}^{\psi}_{\Lambda}$ thus consists of the functions from the field configurations into $\Lambda(V^{\psi}_{\Lambda})$, in analogy to our definition of $\mathfrak{A}^{\Phi}_{\Lambda}$ for the Bose field $\underset{\sim}{\Phi}$.

Following Berezin [44] we define a (by now well known) expectation on $\mathfrak{A}^{\psi}_{\Lambda}$ as the unique linear functional $< - >^{\psi}_{o}$ obeying

$$< \bigwedge_{\substack{\alpha, a \\ x\in\Lambda}} (\psi^1_{\alpha a}(x) \wedge \psi^2_{\alpha a}(x) >^{\psi}_{o} = 1 \text{ , and} \tag{2.10}$$

$$< P >^{\psi}_{o} = 0 \text{ , for all polynomials } P \tag{2.11}$$

of less than maximal degree. (In our notation the dependence of $< - >^{\psi}_{o}$ on Λ

is usually suppressed!).

We now define the ("Euclidean field") algebra for the matter fields $\underset{\sim}{\Phi}$ and $\underset{\sim}{\psi}^k$ by

$$\mathfrak{A}_\Lambda^M = \mathfrak{A}_\Lambda^\Phi \otimes \mathfrak{A}_\Lambda^\psi , \quad \text{and} \tag{2.12}$$

$$< - >_o^M = < - >_o^\Phi \otimes < - >_o^\psi \tag{2.13}$$

as an expectation on $\mathfrak{A}_\Lambda^\psi$; (M stands for "matter"). In the following $\mathfrak{A}_\Lambda^{\psi,e}$ denotes the even subalgebra of $\mathfrak{A}_\Lambda^\psi$ and $\mathfrak{A}_\Lambda^{M,e} = \mathfrak{A}_\Lambda^\Phi \otimes \mathfrak{A}_\Lambda^{\psi,e}$. (Note that $\mathfrak{A}_\Lambda^\Phi$ is normed in the obvious way, and $\mathfrak{A}_\Lambda^\psi$ can be normed, using the Hilbert space structure of $\Lambda(V_\Lambda^\psi)$. So, $\mathfrak{A}_\Lambda^M$ is a normed algebra).

c) Finally, we introduce <u>gauge fields</u> on the lattice, and here we deviate slightly from the standard presentation. Our definitions are somewhat more complicated, but have some advantages which will become clear later.

Given a site $x \in \Lambda$, $\mathcal{B}_x$ denotes the family of all directed bonds joining x to one of its nearest neighbors; (clearly, there are 2ν directed bonds in $\mathcal{B}_x$). Elements of $\mathcal{B}_x$ are denoted by $b \equiv b(x) \equiv < x,y >$, with y the nearest neighbor of x joined to x by b . With each pair (x,b) , $b \in \mathcal{B}_x$, we associate a copy $G_{x,b}$ of the gauge group G . Elements of $G_{x,b}$ are denoted $w_{x,b}$; $\mathfrak{A}_{x,b}^G$ is the algebra of all complex-valued, continuous functions on $G_{x,b}$.

Given a bond $< x,y >$ we define $\mathfrak{A}_{<x,y>}^G$ to be that subalgebra of $\mathfrak{A}_{x,<x,y>}^G \otimes \mathfrak{A}_{y,<y,x>}^G$ consisting of all complex-valued, continuous functions on $G \times G$ which are of the form

$$F(w_{x,<x,y>}^{-1} \cdot w_{y,<y,x>}) \tag{2.14}$$

We denote $w^{-1}_{x,<x,y>} \cdot w_{y,<y,x>}$ by g_{xy} , with the convention

$$g_{yx} = g^{-1}_{xy} \tag{2.15}$$

in accordance with (2.14). The group element g_{xy} is the value of the gauge field, $\underset{\sim}{g}$, on the bond $< x,y >$

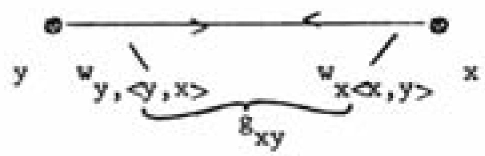


Fig 1

We define

$$\overline{\mathfrak{A}}^G_\Lambda = \underset{x\in\Lambda}{\otimes} \Big(\underset{b\in\mathcal{B}_x}{\otimes} \mathfrak{A}^G_{x,b} \Big) \quad , \quad \text{and} \tag{2.16}$$

$$\mathfrak{A}^G_\Lambda = \underset{<x,y>\subset\Lambda}{\otimes} \mathfrak{A}^G_{<x,y>} \subset \overline{\mathfrak{A}}^G_\Lambda \quad . \tag{2.17}$$

(These algebras are obviously normed).

Next, we define an expectation $< - >^G_o$ on $\overline{\mathfrak{A}}^G_\Lambda$:

$$< A > = \int \prod_{x\in\Lambda} \prod_{b\in\mathcal{B}_x} dw_{x,b} \; A(\underset{\sim}{w}) \quad , \tag{2.18}$$

with $\underset{\sim}{w} = \{w_{x,b}\}_{b\in\mathcal{B}_x, x\in\Lambda}$, $A \in \overline{\mathfrak{A}}^G_\Lambda$.

Here $dw_{x,b}$ is the normalized Haar measure on $G_{x,b}$. Since this measure is right and left invariant and of total mass 1, we obtain from (2.18)

$$< B >^G_o = \int \prod_{<x,y>\subset\Lambda} dg_{xy} \; B(\underset{\sim}{g}) \tag{2.19}$$

with $\underset{\sim}{g} = \{g_{xy}\}_{<x,y>\subset\Lambda}$, $B \in \mathfrak{A}^G_\Lambda$ and dg_{xy} the normalized Haar measure on G .

We also introduce the algebras

$$\overset{(-)}{\mathfrak{A}}_{\Lambda} = \overset{(-)}{\mathfrak{A}}{}^{G}_{\Lambda} \otimes \mathfrak{A}^{M}_{\Lambda} \, , \quad \overset{(-)}{\mathfrak{A}}{}^{e}_{\Lambda} = \overset{(-)}{\mathfrak{A}}{}^{G}_{\Lambda} \otimes \mathfrak{A}^{M,e}_{\Lambda} \, , \tag{2.20}$$

and the expectation

$$< - >_{o} = < - >_{o}^{G} \otimes < - >_{o}^{M} \, . \tag{2.21}$$

Finally, we define <u>local gauge transformations</u> : These are maps $\underset{\sim}{h}$ from $\mathbb{Z}^{\nu}_{\frac{1}{2}}$ to the gauge group G,

$$\underset{\sim}{h} : x \in \mathbb{Z}^{\nu}_{\frac{1}{2}} \to h_x \in G \quad , \tag{2.22}$$

with $h_x \neq 1$ (the identity) only for finitely many x. Under a local gauge transformation the basic fields introduced so far transform as follows :

$$\begin{aligned}
&\phi(x) \to \phi^{\underset{\sim}{h}}(x) = U^{\phi}(h_x) \; \phi(x) \\
&\psi^{1}(x) \to \psi^{1\underset{\sim}{h}}(x) = \overline{U^{\psi}}(h_x) \; \psi^{1}(x) \\
&\psi^{2}(x) \to \psi^{2\underset{\sim}{h}}(x) = U^{\psi}(h_x) \; \psi^{2}(x) \\
&w_{x,b} \to w^{\underset{\sim}{h}}_{x,b} = w_{x,b} \; h_x^{-1}
\end{aligned} \tag{2.23}$$

From the last equation in (2.23) we get

$$g_{xy} \to g^{\underset{\sim}{h}}_{xy} = h_x \; g_{xy} \; h_y^{-1} \tag{2.24}$$

The following lemma is easy, but important. It is therefore stated explicitly, but the proof is left as an exercise to the reader.

Lemma 2.1 :

(1) The expectations $< - >_o^\Phi$, $< - >_o^\psi$, $< - >_o^G$, and hence $< - >_o^M$ and $< - >_o$, are invariant under local gauge transformations.

(2) The expectation $< - >_o^\psi$ is invariant under arbitrary transformations of V_Λ^ψ - see (2.8) - of determinant 1. In particular, it is invariant under the transformation : $\psi^k(x) \to v_{j(k,x)}\ \psi^k(x)$ - see (2.6), (2.7) - for all x in an arbitrary subset of Λ .

We now consider a subset $\Lambda \subset \mathbb{Z}^\nu_{\frac{1}{2}}$ invariant under a reflection r at some given hyperplane perpendicular to the j-direction, $j \in \{0,\ldots,\nu-1\}$, which lies in between two lattice planes containing sites; e.g. r may be reflection at $\{x^o = 0\}$.

Let $\Lambda_\pm$ denote the part of Λ lying "above", resp. "below" the given hyperplane. Clearly $\Lambda_- = r\Lambda_+$. If Λ is a rectangle with opposite faces identified (i.e. Λ is wrapped on a torus, corresponding to periodic boundary conditions) then the reflections r are defined relative to a pair of hyperplanes decomposing Λ into two subsets, Λ_+ and Λ_- , of equal size.

Given a reflection r , we introduce an anti-linear map (anti-morphism $\theta : \mathfrak{A}_{\Lambda_\pm} \to \mathfrak{A}_{\Lambda_\mp}$, defined by

$$\text{(a)} \qquad \theta[F(\Phi(x))] = \overline{F(\Phi(rx))} \tag{2.25}$$

(with $\overline{F}$ the complex conjugate of F) ,

$$\text{(b)} \qquad \theta[z\ \psi^k_{\alpha a}(x)] = \overline{z}(v_j\ \psi^{3-k})_{\alpha a}\ (rx) \quad , \tag{2.26}$$

where z is an arbitrary complex number and j is the direction perpendicular to the hyperplane of reflection; [17] .

(c) $$\theta[A(w_{x,b})] = \overline{A(w_{rx,rb})} \quad , \tag{2.27}$$

with $rb = < rx,ry >$, for $b = < x,y >$,

Moreover,

$$\theta[B(g_{xy})] = \overline{B(g_{rx\ ry})} \quad ,$$

The map θ can be extended in a unique way to an anti-linear map from $\overline{\mathfrak{A}}_\Lambda$ to $\overline{\mathfrak{A}}_\Lambda$ satisfying

$$\theta[F \cdot G] = \theta[G]\ \theta[F] \ , \tag{2.28}$$

for arbitrary F,G in $\overline{\mathfrak{A}}_\Lambda$.

For completeness we recall the following result of [17] which plays an important role in the sequel.

<u>Lemma 2.2</u> :

For arbitrary $F_1,\ldots,F_n$ in $\overline{\mathfrak{A}}_{\Lambda_+}$,

$$< \prod_{j=1}^{n} \{F_j\theta[F_j]\} >_o \ \geqq 0 \ ,$$

in particular,

$$< F\theta[F] >_o \ \geqq 0 \tag{2.29}$$

for $F \in \overline{\mathfrak{A}}_{\Lambda_+}$.

<u>Proof</u> : It is shown in Section II. 3 of [17] that

$$\prod_{j=1}^{n} \{F_j\theta[F_j]\} = (\prod_{j=1}^{n} F_j)\ (\theta[\prod_{j=1}^{n} F_j]) \ , \text{ i.e.}$$

$$P = \{ \sum_{i=1}^{n} F_i \theta[F_i] : F_i \in \mathfrak{A}_{\Lambda_+} , \text{ for all } i \} \tag{2.30}$$

is a multiplicative cone. (This follows from the commutation - , resp. anti - commutation properties of the F_j's and equ. (2.28)). Hence we must only prove (2.29). It is not hard to see that it suffices to prove (2.29) for F the following monomial in ψ^1, ψ^2 :

$$F = f(\underset{\sim}{w}_+, \underset{\sim}{\phi}_+) \prod_{\substack{\alpha, a \\ x \in \Lambda_+}} \psi^1_{\alpha,a}(x) \wedge \psi^2_{\alpha,a}(x) , \tag{2.31}$$

with $f \in \mathfrak{A}^{\phi}_{\Lambda_+} \otimes \overline{\mathfrak{A}}^{G}_{\Lambda_+}$. We use (2.25) - (2.28) to compute $\theta[F]$. Then we apply Lemma 2.1 , (2) to evaluate $< F\theta[F] >_o$. This gives

$$< F\theta[F]) >_o = | < f >_o^{\phi,G} |^2 \geq 0$$

Q.E.D.

Remark :

Let F and G be in $\mathfrak{A}_{\Lambda_+}$. Since $< - >_o$ is linear, $< F\theta[G] >_o$ is linear in the first and anti-linear in the second argument. Moreover $< F\theta[F]_o > \geq 0$, by (2.29). Thus $< \cdot \ \theta[\cdot] >_o$ is a positive semi-definite inner product on $\overline{\mathfrak{A}}_{\Lambda_+}$, and we have the Schwarz inequality

$$| < F\theta[G] >_o | \leq < F\theta[F]>_o^{\frac{1}{2}} < G\theta[G] >_o^{\frac{1}{2}} \tag{2.32}$$

Since $\mathfrak{A}^M_{\Lambda_+}$ can be identified with the subalgebra $1 \otimes \mathfrak{A}^M_{\Lambda_+}$ of $\overline{\mathfrak{A}}_{\Lambda_+}$, inequality (2.32) gives

$$| < F\theta[G] >_o^M | \leq [< F\theta[F] >_o^M]^{\frac{1}{2}} [< G\theta[G] >_o]^{\frac{1}{2}} , \tag{2.33}$$

for arbitrary F and G in $\mathfrak{A}^M_{\Lambda_+}$.

So far this set up us completely analogous to the one developped in [18,21] for classical and quantum lattice systems in statistical mechanics. In order to describe interacting lattice gauge theories in a general way we could thus follow [18,21] . A brief outline is given in Subsection 2.2. The material given there may eventually be important for the study of the renormalization group ("block spin transformation"), applied to lattice gauge theories. The reader not interested in "abstract nonsense" should directly proceed to 2.3.

2.2. Interaction potentials and interacting expectations

Following [18,45] we may introduce a "cone of reflection-negative potentials" in one of the standard Banach spaces [46,47] of potentials used in statistical mechanics. A large space of "actions" for interacting lattice gauge theories may be needed for doing renormalization group (block spin) transformations and permits us to study long range interactions leading to interesting phenomena such as phase transitions, [18,21]. (We note that, by the general results of [47], there do exist lattice gauge theories with first order phase transitions). We briefly recall the definition of potentials : A potential U is a map from bounded subsets, X , of the lattice $\mathbb{Z}^{\nu}_{\frac{1}{2}}$ to $\bar{\mathfrak{A}}_{\mathbb{Z}^{\nu}_{\frac{1}{2}}}$ satisfying

(1) $U : X \to U_X \in \mathfrak{A}^e_X$

(2) U is translation invariant and reflection covariant (i.e. $\theta U_X = U_{rX}$) ; see [46,18] .

(3) Each U_X is a polynomial in $\{\psi^k(x)\}_{x \in X}$ with coefficients (anti-symmetric tensors) in $\mathfrak{A}^G_X \otimes \mathfrak{A}^{\Phi}_X$. A norm $\|U_X\|$ can be defined e.g. as the sum of the supremum norms of these coefficients. One requires e.g.

$$\sum_{X \ni 0} \|U_X\| < \infty .$$

(4) U is invariant under local gauge transformations.

(5) U is reflection-negative, see [18]. This condition guarantees that the resulting lattice gauge theory satisfies Osterwalder-Schrader (= reflection) positivity, or, in other words, has a selfadjoint, generalized transfer matrix.

(6) U has a formal continuum limit (as the lattice spacing tends to 0) which is compatible with Euclidean invariance; in particular U is isotropic (i.e. satisfies "Nelson's symmetry").

Given a potential U satisfying (1) - (6) , an <u>action</u> A_Λ for a lattice gauge theory in the region Λ is introduced by

$$A_\Lambda = \sum_{X \subset \Lambda} U_X \quad ,$$

and an interacting expectation $< - >_{A_\Lambda}$ by

$$< - >_{A_\Lambda} = < e^{-A_\Lambda} >^{-1} < - e^{-A_\Lambda} >_o \qquad (2.34)$$

(It is shown in Proposition 2.7 below that $< e^{-A_\Lambda} >_o > 0$) .

The definition (2.34) of interacting expectations can be generalized by introducing (reflection positive [18]) boundary conditions. One can then apply Dobrushin-Lanford-Ruelle equations [47] to characterize general interacting expectations; see also [47,18] .

The resulting lattice gauge theory is <u>invariant under local gauge transformations</u>, satisfies <u>reflection positivity</u> (for reflection symmetric Λ) , has at least one limit as $\Lambda \uparrow \mathbb{Z}^\nu_{\frac{1}{2}}$ which is gauge - and <u>translation invariant</u> and reflection positive (use a sequence of hypercubes with periodic b.c.) and satisfies <u>Nelson's symmetry</u>.

All results discussed throughout Section 2 except Proposition 2.7, (2), are valid in the general context described here. The reader familiar with [18,45] may verify this as an exercise. Here we do not give further details, but we emphasize that the concepts described in this section will appear quite natural in the light of Sections 2.3 and 2.4.

2.3 Matter interacting with an external gauge field

In this section we prove the diamagnetic inequality : Theorem A of Section 1.2.

Given a bond $< x,y >$ we define a Dirac matrix γ_{xy} by

$$\gamma_{xy} = \begin{cases} \gamma_j & \text{if } x-y = e_j \\ -\gamma_j & \text{if } x-y = -e_j \ , \end{cases} \tag{2.35}$$

where e_j is the basis vector of $\mathbb{Z}^\nu$ in the positive j-direction, $j = 0,\dots,\nu-1$.

We set

$$\begin{aligned} &\psi^1(x) \cdot U^\psi(g_{xy})\ \gamma_{xy}\ \psi^2(y) \\ &= \sum_{\alpha,\beta} U^\psi(g_{xy})_{ab}\ (\gamma_{xy})_{\alpha\beta}\ \psi^1_{\alpha a}(x) \wedge \psi^2_{\beta b}(y) \end{aligned} \tag{2.36}$$

This combination is clearly gauge invariant, i.e.

$$\begin{aligned} &\psi^{1\underset{\sim}{h}}(x) \cdot U^\psi(g^{\underset{\sim}{h}}_{xy})\ \gamma_{xy}\ \psi^{2\underset{\sim}{h}}(y) \\ &= \psi^1(x) \cdot U^\psi(g_{xy})\ \gamma_{xy}\ \psi^2(y) \end{aligned} \tag{2.37}$$

It can be rewritten more symmetrically as

$$\overline{U}^{\psi}(w_{x,<x,y>})\ \psi^1(x)\cdot U^{\psi}(w_{y,<y,x>})\ \gamma_{xy}\ \psi^2(y) \tag{2.38}$$

Remark : There is a wellknown problem with lattice Fermi fields [48,41] : They describe too many degrees of freedom and therefore fail to reproduce the standard perturbation theory in the continuum limit. There are various ways to cure this disease (at the price of giving up chiral invariance even when the mass is 0), involving replacing γ_{xy} by some matrix Γ_{xy} such that

(1) $\gamma_{xy}\ \Gamma^*_{xy}\ \gamma_{xy} = \Gamma_{xy}$

(2) $[\Gamma_{xy}, U^{\psi}[G]] = 0$;

(1) ensures Osterwalder-Schrader positivity (see [17] and Section 2.4) and (2) gauge invariance. Γ_{xy} can be a linear combination of 1, γ_{xy} , $i\gamma_5$, [17] ; (for a different proposal see [41,49]) . In the sequel all will remain unchanged if γ_{xy} is replaced by some Γ_{xy} obeying (1) and (2) !

Furthermore, we define

$$\begin{aligned}\Phi(x)\cdot U^{\Phi}(g_{xy})\ \Phi(y) &\equiv (\Phi(x),\ U^{\Phi}(g_{xy})\ \Phi(y))\\ &= U^{\Phi}(w_{x,<x,y>})\ \Phi(x)\cdot U^{\Phi}(w_{y,<y,x>})\ \Phi(y)\end{aligned} \tag{2.39}$$

where $(\cdot\ ,\ \cdot)$ is the scalar product on V^{Φ} . Again, this is clearly gauge invariant.

An action is now defined in terms of the building blocks (2.37) - (2.39):

$$\begin{aligned}A^M_\Lambda(\underset{\sim}{\Phi},\ \underset{\sim}{\psi},\ \underset{\sim}{g}) = &-\sum_{<x,y>\subset\Lambda}\{\gamma_{xy}\psi^1(x)\cdot U^{\psi}(g_{xy})\ \psi^2(y)\\ &\qquad + \Phi(x)\cdot U^{\Phi}(g_{xy})\ \Phi(y)\}\\ &+\sum_{x\in\Lambda} V(\psi^1(x),\ \psi^2(x),\ \Phi(x))\end{aligned} \tag{2.40}$$

where V is some G-invariant, scalar function of $\psi^1(x)$, $\psi^2(x)$ and $\phi(x)$, i.e. $V \in \mathfrak{A}^{M,e}_{\{x\}}$ and

$$V(\psi^{1\underset{\sim}{h}}(x),\ \psi^{2\underset{\sim}{h}}(x),\ \phi^{\underset{\sim}{h}}(x)) = V(\psi^1(x),\ \psi^2(x),\ \phi(x))$$

(independent of $\underset{\sim}{g}$) . For example,

$$\begin{aligned} V(\psi^1(x),\ \psi^2(x),\ \phi(x)) &= P[(\phi(x),\phi(x))] \\ &+ \psi^1(x)\ \tau\psi^2(x)\cdot\phi(x) + M\psi^1(x)\ \psi^2(x)\ , \end{aligned} \tag{2.41}$$

where $\tau = \{\tau^j\}$ are $N = \dim V^{\phi}$ matrices of size $M \times M$, with $M = \dim V^{\psi^k}$, satisfying

$$U^{\psi}(h)^*\ \tau^j U^{\psi}(h) = \sum_{\ell} U^{\phi}(h)^j_{\ell}\ \tau^{\ell}\ , \tag{2.42}$$

and $\psi^1(x)\ \tau^j\psi^2(x) = \sum_{\substack{\alpha,a \\ \alpha',a'}} \psi^1_{\alpha a}(x) \wedge \tau^j_{\alpha a,\alpha' a'}\ \ \psi^2_{\alpha' a'}(x)$,

P is a polynomial bounded from below, and M is the bare mass of the Fermions; (More generally, M can be a mass matrix acting on flavour indices. It may also contain terms proportional to $i\gamma_5$ [17] without being incompatible with conditions (1) - (6) of Section 2.2 !). We define

We define

$$Z_{\Lambda}(\underset{\sim}{g}) = < e^{-A^M_{\Lambda}(\underset{\sim}{\phi},\underset{\sim}{\psi};\underset{\sim}{g})} >^M_o\ . \tag{2.43}$$

The first main result we propose to prove is Theorem A of Section 1.2.

Theorem 2.3 :

Choose a bounded, rectangular region $\Lambda \subset \mathbb{Z}^{\nu}_{\frac{1}{2}}$ with opposite faces identified (torus) and sides of even length, and impose periodic boundary conditions on A_Λ . Then

$$|Z_\Lambda(\underset{\sim}{g})| \leqq Z_\Lambda(\underset{\sim}{1}) .$$

Remark : If the "vacuum energy density", ϵ_Λ , is defined by

$$\epsilon_\Lambda(\underset{\sim}{g}) = - \frac{1}{|\Lambda|} \log |Z_\Lambda(\underset{\sim}{g})|$$

the inequality of Theorem 2.3 says

$$\epsilon_\Lambda(\underset{\sim}{g}) \geqq \epsilon_\Lambda(\underset{\sim}{1}) , \tag{2.44}$$

which is Theorem A.

Proof : We choose a pair of hyperplanes, π , lying in between lattice planes and cutting the rectangle Λ into two pieces, Λ_+ and Λ_- , of equal size. (Since the sides of Λ have even length, many such π's exist. In fact, there are $|\Lambda|/_{2}\nu$ different choices of π , where $|\Lambda|$ is the number of sites in Λ). We label the sites bordering π by $1,\dots,M, 1',\dots,M'$, where $|M| = |M'|$ is an even integer, with $|M| = \#$ of sites in $\{1,\dots,M\}$; $\{1,\dots,M\} \subset \Lambda_+$ and $\{1',\dots,M'\} \subset \Lambda_-$.

The basic idea of our proof is to find an upper bound for $Z_\Lambda(\underset{\sim}{g})$ in terms of Z_Λ's in which all the gauge fields, $\{g_{\ell\ell'}\}_{\ell=1,\dots,M}$, crossing π are replaced by 1 (the unit element in G).

In accordance with the above decomposition of Λ we write the action $A^M_\Lambda(\underset{\sim}{g})$ as a sum of three terms, $A^M_{\Lambda_+}(\underset{\sim}{g}_+)$, $A^M_{\Lambda_-}(\underset{\sim}{g}_-)$ and $A^M_{\Lambda_+,\Lambda_-}(\underset{\sim}{g}_\pi)$. Here $\underset{\sim}{g}_\pm = \{g_{xy}\}_{<x,y> \subset \Lambda_\pm}$, $\underset{\sim}{g}_\pi = \{g_{\ell\ell'}\}_{\ell=1,\dots,M}$; the terms $A^M_{\Lambda_\pm}(\underset{\sim}{g}_\pm)$ are given by (2.40), with Λ replaced by $\Lambda_\pm$, (and $\underset{\sim}{\phi}_\pm$, $\underset{\sim}{\psi}_\pm$ suppressed in our notation).

In order to define $A^M_{\Lambda_+,\Lambda_-}(\underset{\sim}{g}_\pi)$ we write each g_{xy} as $w^{-1}_{x,<x,y>} \cdot w_{y,<y,x>}$, e.g. $w_{\ell,<\ell,\ell'>} = g^{-1}_{\ell\ell'}$, $w_{\ell',<\ell',\ell>} = 1$, etc.

We then set

$$A^M_{\Lambda_\pm}(\underset{\sim}{w}_\pm) \equiv A^M_{\Lambda_\pm}(\underset{\sim}{g}_\pm) \quad \text{, and} \tag{2.45}$$

$$\begin{aligned} A^M_{\Lambda_+,\Lambda_-}(\underset{\sim}{g}_\pi) &\equiv A^M_{\Lambda_+,\Lambda_-}(\underset{\sim}{w}^\pi_+ , \underset{\sim}{w}^\pi_-) \\ = - \sum_{\ell=1,\dots,M} &\{v_{\ell\ell'}\, \overline{U}^\psi(w_{\ell,<\ell,\ell'>})\, \psi^1(\ell) \cdot U^\psi(w_{\ell',<\ell',\ell>})\, \psi^2(\ell') \\ &+ v_{\ell'\ell}\, \overline{U}^\psi(w_{\ell',<\ell',\ell>})\, \psi^1(\ell') \cdot U^\psi(w_{\ell,<\ell,\ell'>})\, \psi^2(\ell)\} \\ - \sum_{\ell=1,\dots,M} &\{U^\phi(w_{\ell,<\ell,\ell'>})\, \phi(\ell) \cdot U^\phi(w_{\ell'<\ell',\ell>})\, \phi(\ell') \\ &+ (\ell \leftrightarrow \ell')\} \end{aligned} \tag{2.46}$$

Recalling now the definition (2.25) - (2.28) of θ, (2.36) and (2.39) and regarding the w's as external variables, one verifies that

$$e^{-A^M_{\Lambda_-}(\underset{\sim}{w}_-)} = \theta[e^{-A^M_{\Lambda_+}(\underset{\sim}{w}_{r-})}] , \tag{2.47}$$

and using, in addition, (2.46) one sees that

$$A^M_{\Lambda_+,\Lambda_-}(\underset{\sim}{w}^\pi_+, \underset{\sim}{w}^\pi_-) = - \sum F_i(\underset{\sim}{w}^\pi_+)\; \theta[F_i(\underset{\sim}{w}^\pi_{r-})] , \tag{2.48}$$

with $F_i(\underset{\sim}{w}^\pi_+)$ and $F_i(\underset{\sim}{w}^\pi_{r-})$ in $\mathfrak{A}^M_{\Lambda_+}$, and $\underset{\sim}{w}^\pi_{r-}$ defined in the obvious way. Equation (2.48) is the basic ingredient.

Next, we note that $A^M_{\Lambda_\pm}$ and $A^M_{\Lambda_+,\Lambda_-}$ are in $\mathfrak{A}^e_\Lambda$ (i.e. even in ψ^1, ψ^2) so that they all commute with each other. Thus

$$Z_\Lambda(\underset{\sim}{g}) = \langle e^{-A^M_{\Lambda_+}(\underset{\sim}{w}_+)} \, e^{-A^M_{\Lambda_-}(\underset{\sim}{w}_-)} \, e^{-A^M_{\Lambda_+,\Lambda_-}(\underset{\sim}{w}^\pi_+, \underset{\sim}{w}^\pi_-)} \rangle^M_o$$

$$= \langle e^{-A^M_{\Lambda_+}(\underset{\sim}{w}_+)} \, \theta[e^{-A^M_{\Lambda_+}(\underset{\sim}{w}_{r-})}] \, e^{\sum_i F_i(\underset{\sim}{w}^\pi_+)\, \theta[F_i(\underset{\sim}{w}^\pi_{r-})]} \rangle^M_o$$

If we expand the third exponential on the r.s. we see that $Z_\Lambda(\underset{\sim}{g})$ is of the form

$$\sum_\alpha \langle \prod_i \{D^\alpha_i(\underset{\sim}{w}_+)\;\theta[D^\alpha_i(\underset{\sim}{w}_{r-})]\} \rangle^M_o \quad ,$$

for some $D^\alpha_i(\underset{\sim}{w}_+)$, $D^\alpha_i(\underset{\sim}{w}_{r-})$ in $\mathfrak{A}^M_{\Lambda_+}$.

Now we apply Lemma 2.2, resp. (2.29) and (2.33) to conclude that

$$\langle \prod_i D^\alpha_i(\underset{\sim}{w}_+)\;\theta[D^\alpha_i(\underset{\sim}{w}_{r-})] \rangle^M_o =$$

$$= \langle \{\prod_i D^\alpha_i(\underset{\sim}{w}_+)\}\;\theta[\prod_i D^\alpha_i(\underset{\sim}{w}_{r-})] \rangle^M_o$$

$$\leq \{ \langle \prod_i D^\alpha_i(\underset{\sim}{w}_+)\;\theta[D^\alpha_i(\underset{\sim}{w}_+)] \rangle^M_o \}^{\frac{1}{2}}$$

$$\times \{ \langle \prod_i D^\alpha_i(\underset{\sim}{w}_{r-})\;\theta[D^\alpha_i(\underset{\sim}{w}_{r-})] \rangle^M_o \}^{\frac{1}{2}} \quad ,$$

for all α . Applying now the Schwarz inequality with respect to the sum $\sum_\alpha$ and resumming under the square roots we obtain

$$|Z_\Lambda(\underset{\sim}{g})| \leq \{ \langle e^{-A^M_{\Lambda_+}(\underset{\sim}{w}_+)} \, \theta[e^{-A^M_{\Lambda_+}(\underset{\sim}{w}_+)}] \, e^{\sum_i F_i(\underset{\sim}{w}^\pi_+)\;\theta[F_i(\underset{\sim}{w}^\pi_+)]} \rangle^M_o \}^{\frac{1}{2}}$$

$$\times \{ \langle e^{-A^M_{\Lambda_+}(\underset{\sim}{w}_{r-})} \, \theta[e^{-A^M_{\Lambda_+}(\underset{\sim}{w}_{r-})}] \, e^{\sum_i F_i(\underset{\sim}{w}^\pi_{r-})\,\theta[F_i(\underset{\sim}{w}^\pi_{r-})]} \rangle^M_o \}^{\frac{1}{2}}$$

$$= \{ \langle e^{-A^M_{\Lambda_+}(\underset{\sim}{w}_+)} \, e^{-A^M_{\Lambda_-}(\underset{\sim}{w}_{r+})} \, e^{-A_{\Lambda_+,\Lambda_-}(\underset{\sim}{w}^\pi_+, \underset{\sim}{w}^\pi_{r+})} \rangle^M_o \}^{\frac{1}{2}}$$

$$\times \{ \langle e^{-A^M_{\Lambda_+}(\underset{\sim}{w}_{r-})} \, e^{-A^M_{\Lambda_-}(\underset{\sim}{w}_-)} \, e^{-A^M_{\Lambda_+,\Lambda_-}(\underset{\sim}{w}^\pi_{r-}, \underset{\sim}{w}^\pi_-)} \rangle^M_o \}^{\frac{1}{2}} \qquad (2.49)$$

Next,

$$A^M_{\Lambda_+,\Lambda_-}(\underset{\sim}{w}^\pi_{+},\underset{\sim}{w}^\pi_{r+})$$

$$= - \sum_{\ell=1,\ldots,M} \{ \gamma_{\ell\ell'}\, \overline{U}^\psi(w_{\ell,<\ell,\ell'>})\, \psi^1(\ell)\cdot U^\psi(w_{\ell,<\ell,\ell'>})\, \psi^2(\ell')$$

$$+ \gamma_{\ell'\ell}\, \overline{U}^\psi(w_{\ell,<\ell,\ell'>})\, \psi^1(\ell')\cdot U^\psi(w_{\ell,<\ell,\ell'>})\, \psi^2(\ell) \}$$

$$- \sum_{\ell=1,\ldots,M} \{ U^\Phi(w_{\ell,<\ell,\ell'>})\, \Phi(\ell)\cdot U^\Phi(w_{\ell,<\ell,\ell'>})\, \Phi(\ell') + \ldots \}$$

$$= A^M_{\Lambda_+,\Lambda_-}(\underset{\sim}{1}\ ,\ \underset{\sim}{1})\ , \qquad (2.50)$$

since, by (2.36) and (2.38),

$$\gamma_{\ell\ell'}\, \overline{U}^\psi(w_{\ell,<\ell,\ell'>})\, \psi^1(\ell)\cdot U^\psi(w_{\ell,<\ell,\ell'>})\, \psi^2(\ell')$$

$$= \gamma_{\ell\ell'}\, \psi^1(\ell)\cdot U^\psi(w_{\ell,<\ell,\ell'>})^*\, U^\psi(w_{\ell,<\ell,\ell'>})\, \psi^2(\ell')$$

$$= \gamma_{\ell\ell'}\, \psi^1(\ell)\cdot\psi^2(\ell')\ ,\ \text{etc}\ldots$$

By the same reasoning,

$$A^M_{\Lambda_+,\Lambda_-}(\underset{\sim}{w}^\pi_{r-},\ \underset{\sim}{w}^\pi_{-}) = A^M_{\Lambda_+,\Lambda_-}(\underset{\sim}{1}\ ,\ \underset{\sim}{1}) \qquad (2.51)$$

so by (2.49) - (2.51) ,

$$|Z_\Lambda(\underset{\sim}{g})| = |Z_\Lambda(\underset{\sim}{g}_+,\underset{\sim}{g}_-,\ \underset{\sim}{g}_\pi)| \le Z_\Lambda(\underset{\sim}{g}_+,\underset{\sim}{g}_{r+},\underset{\sim}{1})^{\frac{1}{2}}\ Z_\Lambda(\underset{\sim}{g}_{r-},\underset{\sim}{g}_-,\underset{\sim}{1})^{\frac{1}{2}} \qquad (2.52)$$

Note that on the r.s. of (2.52) all gauge fields on bonds crossing π have been replaced by 1 ! This is the basic inequality. We now iterate it : In both factors on the r.s. of (2.52) we choose a new pair of hyperplanes $\pi' \neq \pi$ and apply (2.52) again. As a result, all gauge fields on bonds crossing

π and π' are replaced by 1 . To replace all gauge fields by 1 we make all $|\Lambda|/2^{\nu}$ possible choices of pairs of hyperplanes (perpendicular to all ν directions of the lattice). This yields

$$|Z_{\Lambda}(\underset{\sim}{g})| \leq \{Z_{\Lambda}(\underset{\sim}{1})^{2^{\nu}/|\Lambda|}\}^{|\Lambda|/2^{\nu}} = Z_{\Lambda}(\underset{\sim}{1}) \tag{2.53}$$

Q.E.D.

Remarks : The theorem just proven is reminiscent of Theorem 2.3 of [18b]. The kind of book-keeping necessary to arrive at (2.53) can be replaced by an inductive argument; (see e.g. [18a]) .

Theorem 2.3 extends to the more general framework considered in Subsection 2.2. Also note that we did not make use of gauge invariance in the proof so that the inequality can be extended to certain actions with gauge dependent terms.

We believe that Theorem 2.3 is true for a more general class of boundary conditions and more general lattices. Indeed, for the abelian Higgs models we prove Theorem 2.3 under very general assumptions in Section 4.

Theorem 2.3 has two noteworthy corollaries.

Let
$$z = \lim_{\Lambda \uparrow \mathbb{Z}^{\nu}_{\frac{1}{2}}} [Z_{\Lambda}(\underset{\sim}{1})]^{1/|\Lambda|} .$$

Standard arguments of statistical mechanics show that this limit exists and is independent of b.c. Mimicking proofs of related inequalities in continuum field theories [50] based on infinitely many applications of the Schwarz inequalities (2.49), (2.52) (and using the DLR equations to show that boundary conditions have a negligible effect, see e.g. [45]) we obtain

Corollary 2.4 :

For arbitrary, bounded regions Λ and "arbitrary" b.c. at $\partial\Lambda$

$$|Z_\Lambda(\underset{\sim}{g})| \leq z^{|\Lambda|} e^{O(|\partial\Lambda|)}$$

Next, let $\delta A_x(\psi^k(x),\phi(x))$ be a perturbation of the action A_Λ^M localized at site x, i.e. depending only on $\{\psi^1(x),\psi^2(x),\phi(x)\}$. Let

$$\delta A_\Lambda = \sum_{x\in\Lambda} \delta A_x(\psi^k(x),\phi(x)) \,, \quad \text{and}$$

$$\delta A_\Lambda^x = \sum_{y\in\Lambda} \delta A_x(\psi^k(y),\phi(y)) \,. \quad \text{Define}$$

$$Z_\Lambda(\underset{\sim}{g};\delta A_\Lambda) = \langle e^{-A_\Lambda^M(\underset{\sim}{\psi},\underset{\sim}{\phi};\underset{\sim}{g}) - \delta A_\Lambda(\underset{\sim}{\psi},\underset{\sim}{\phi})}\rangle_o^M$$

Corollary 2.5 :

$$|Z_\Lambda(\underset{\sim}{g};\delta A_\Lambda)| \leq \prod_{x\in\Lambda} Z_\Lambda(1;\delta A_\Lambda^x)^{1/|\Lambda|} \,.$$

Proof : This follows again by applying the Schwarz inequalities (2.49), (2.52), corresponding to all possible choices of pairs of hyperplanes, as in Theorem 2.3, and keeping track of all terms produced in this way : See the proof of Theorem 2.2 of [18a] .

Q.E.D.

Next, we briefly discuss gauge invariance.

Definition :

The subalgebra of $\mathfrak{A}_\Lambda$ invariant under local gauge transformations is denoted $\mathfrak{A}_\Lambda^{inv.}$, i.e. $F(\underset{\sim}{\phi},\underset{\sim}{\psi};\underset{\sim}{g}) \in \mathfrak{A}_\Lambda^{inv.}$ if $F \in \mathfrak{A}_\Lambda$ and

$$F(\underset{\sim}{\phi}^{\underset{\sim}{h}}, \underset{\sim}{\psi}^{\underset{\sim}{h}}; \underset{\sim}{g}^{\underset{\sim}{h}}) = F(\underset{\sim}{\phi},\underset{\sim}{\psi};\underset{\sim}{g}) \,, \tag{2.54}$$

for arbitrary, local gauge transformations $\underset{\sim}{h}$; see (2.22) - 2.24) .

For $F \in \overline{\mathfrak{N}}_\Lambda$, let

$$Z_\Lambda(\underset{\sim}{w};F(\underset{\sim}{w})) = < F(\underset{\sim}{\bar\psi},\underset{\sim}{\psi};\underset{\sim}{w})\, e^{-A_\Lambda^M(\underset{\sim}{\bar\psi},\underset{\sim}{\psi};\underset{\sim}{g})} >_o^M \quad .$$

If $F \in \mathfrak{N}_\Lambda$ we write this as $Z_\Lambda(\underset{\sim}{g};F(\underset{\sim}{g}))$.

Theorem 2.6 : (Gauge-invariance)

For $F \in \mathfrak{N}_\Lambda^{inv.}$

$$Z_\Lambda(\underset{\sim}{g}^h;F(\underset{\sim}{g}^h)) = Z_\Lambda(\underset{\sim}{g};F(\underset{\sim}{g})) \, ,$$

for arbitrary local gauge transformations.

The proof is a direct consequence of Lemma 2.1.

Remarks :

Theorem 2.6 is a strong restriction on the possible form of the functional $Z_\Lambda(\underset{\sim}{g};F(\underset{\sim}{g}))$: It depends only on products of g_{xy}'s along closed loops. Furthermore, this theorem permits us to choose any gauge that is convenient to calculate (or estimate) Green's functions of gauge invariant observables. This is very helpful in the construction of continuum $Higgs_2$; (see also Section 5). By expanding both sides of Theorem 2.6 in a parameter (e.g. the loop parameter $\hbar$, or a coupling constant) one obtains Ward identities for the lattice theories.

Finally, we remark that $Z_\Lambda(\underset{\sim}{g})$ is real : Let $\underset{\sim}{g}^* = \{g^*_{xy}\}_{<x,y> \subset \Lambda}$, with $g^*_{xy} \equiv g_{yx}$. It is not hard to show, using arguments similar to Lemma 2.1 (e.g. invariance of $< - >_o^\psi$ under $\psi^k \to \psi^{3-k}$) that

$$\overline{Z_\Lambda(\underset{\sim}{g})} = Z_\Lambda(\underset{\sim}{g}^*) = Z_\Lambda(\underset{\sim}{g}) \quad . \tag{2.55}$$

2.4 Fully interacting lattice gauge theories.

Following Wilson's proposal [9] one introduces an action for the gauge field as follows; see also [10,17] : Choose a representation U^{YM} of G . Four nearest neighbor sites $xyzu$ in a plane, forming a unit square, are called a plaquette, abbreviated by P

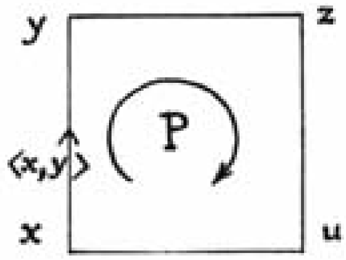


Fig. 2

We define

$$A_P^{YM} = -\frac{1}{2g_o^2} \text{ Re Tr}[U^{YM}(g_{xy}\, g_{yz}\, g_{zu}\, g_{ux})] \quad , \tag{2.56}$$

where g_o is the bare Yang-Mills coupling constant. (From the point of view of general lattice theories one could omit taking the real part, but then a consistent, universal orientation of the plaquettes must be chosen). This action can be written as

$$2g_o^2\, A_P^{YM} = -\text{ Re Tr } C^P D^P = -\tfrac{1}{2}\sum_{ij} C_{ij}^P\, D_{ji}^P - \tfrac{1}{2}\sum_{ij} \overline{C}_{ij}^P\, \overline{D}_{ji}^P \;, \tag{257}$$

where $C^P = U^{YM}(w_{y,<y,x>}\, g_{yz}\, w_{z,<z,u>}^{-1})$,

and $D^P = U^{YM}(w_{u,<u,z>}\, g_{ux}\, w_{x,<x,y>}^{-1})$.

A straighforward computation shows that

$$\theta(C_{ij}^P) = D_{ji}^P$$

and therefore

$$2g_o^2\, A_P^{YM} = -\tfrac{1}{2}\sum_{ij} C_{ij}^P\, \theta[C_{ij}^P] - \tfrac{1}{2}\sum_{ij} \overline{C}_{ij}^P\, \theta[\overline{C}_{ij}^P] \tag{2.58}$$

Furthermore, if P is strictly above π

$$\theta[A_P^{YM}] = A_{rP}^{YM} \quad , \tag{2.59}$$

where rP is the reflection of P (note that the orientation of the plaquettes is unimportant when we take real parts, but (2.59) remains true in general).

We now introduce the action for the gauge (Yang-Mills) field :

$$A_\Lambda^{YM} = \sum_{P\subset\Lambda} A_P^{YM} \quad . \tag{2.60}$$

The total action of a lattice gauge theory in the region Λ is then given by

$$A_\Lambda = A_\Lambda^{YM}(\underset{\sim}{g}) + A_\Lambda^M(\underset{\sim}{\bar\psi},\underset{\sim}{\psi};\underset{\sim}{g}) \tag{2.61}$$

The interacting expectation of this theory is defined by

$$< - >_{A_\Lambda} = < e^{-A_\Lambda} >_o^{-1} \; < - \; e^{-A_\Lambda} >_o \tag{2.62}$$

For $F = F(\underset{\sim}{\bar\psi},\underset{\sim}{\psi};\underset{\sim}{w}) \in \mathfrak{A}_\Lambda$, this can also be written as

$$< F >_{A_\Lambda} = [< Z_\Lambda(\underset{\sim}{g}) \; e^{-A_\Lambda^{YM}(\underset{\sim}{g})} >_o^G]^{-1} \; < Z_\Lambda(\underset{\sim}{w};F(\underset{\sim}{w})) \; e^{-A_\Lambda^{YM}(\underset{\sim}{g})} >_o^G$$

In order for these definitions to make sense we must check that $< e^{-A_\Lambda} >_o$ is strictly positive. This is asserted in

<u>Proposition 2.7</u> :

(i) Let Λ be a rectangular region with sides of length 2^{m_j} , $j = 0,\ldots,\nu-1$, and impose free or periodic boundary conditions. Then

$$< e^{-A_\Lambda} >_o \geqq 1$$

(2) For an arbitrary rectangle Λ with sides of <u>even</u> length and periodic boundary conditions,

$$< e^{-A_\Lambda} >_o \geq 1 \quad .$$

Although quite simple to prove, this proposition is a basic result for studying lattice gauge theories in the weak coupling region (where high temperature expansions, see [17] , do not converge). We prove it after the following

<u>Theorem 2.8</u>. (Osterwalder-Schrader positivity)

Choose a hyperplane π (resp. a pair of hyperplanes) and a bounded region (resp. rectangle) Λ reflection symmetric with respect to π , i.e. $\Lambda = \Lambda_+ \cup \Lambda_-$ with $r\Lambda_+ = \Lambda_-$. Let θ be given by (2.25) - (2.28) , and let $F \in \mathfrak{A}_{\Lambda_+}$. Then, for free (resp. periodic) boundary conditions,

$$< F\theta[F] >_{A_\Lambda} \geq 0 \ .$$

<u>Proof</u> : By (2.62) and Proposition 2.7 it suffices to show

$$< F\theta[F]\, e^{-A_\Lambda} >_o \geq 0 \qquad (2.63)$$

By (2.47) and (2.59)

$$\theta[e^{-A_{\Lambda_+}}] = e^{-A_{\Lambda_-}}$$

Set $A^{YM}_{\Lambda_+,\Lambda_-} = \sum_{P \cap \pi \neq \emptyset} A^{YM}_P$. Then (2.58) implies

$$A^{YM}_{\Lambda_+,\Lambda_-} = -\frac{1}{2g^2} \sum_{P \cap \pi \neq \emptyset} \tfrac{1}{2} \Big[\sum_{ij} C^P_{ij}\ \theta[C^P_{ij}] + \sum_{ij} \bar{C}^P_{ij}\ \theta[\bar{C}^P_{ij}] \Big] \ ,$$

with $C^P_{ij} \in \bar{\mathfrak{A}}^G_{\Lambda_+}$. This and (2.48) prove that

$$A_{\Lambda_+,\Lambda_-} = A^{YM}_{\Lambda_+,\Lambda_-} + A^{M}_{\Lambda_+,\Lambda_-}(\underset{\sim}{w}^{\pi}_{+} , \underset{\sim}{w}^{\pi}) = - \sum G_i \theta[G_i]$$

with $G_i \in \overline{\mathfrak{A}}_{\Lambda_+}$, for all i.

Moreover, A_{Λ_+}, A_{Λ_-} and A_{Λ_+,Λ_-} all commute with each other, since they are even in ψ^1, ψ^2. Therefore

$$< F\theta[F]\, e^{-A_\Lambda} >_o = < F\theta[F]\, e^{-A_{\Lambda_+}}\, \theta[e^{-A_{\Lambda_+}}]\, e^{\sum_i G_i \theta[G_i]} >_o \qquad (2.64)$$

By expanding the exponential we see that the r.s. is of the form

$$\sum_\alpha < \Pi\{D^\alpha_i\ \theta[D^\alpha_i]\} >_o$$

which is non-negative by Lemma 2.2. (This proof is almost identical to one given in Section 2.3 of [17], except that we do not have to choose a special gauge).

Q.E.D.

<u>Proof of Proposition 2.7</u>:

By (2.64), with $F = 1$,

$$< e^{-A_\Lambda} >_o = \sum_{m_i = 0,1,2,\ldots} \Pi \frac{1}{m_i!} < e^{-A_{\Lambda_+}}\, \theta[e^{-A_{\Lambda_+}}]\, \Pi\, G_i^{m_i} \theta[G_i]^{m_i} >_o$$
$$\geq < e^{-A_{\Lambda_+}}\, \theta[e^{-A_{\Lambda_+}}] >_o \quad ,$$

since the terms for which $m_i > 0$, for some i, are all non-negative by Lemma 2.2. Hence

$$< e^{-A_\Lambda} >_o \ \geq\ < e^{-A_{\Lambda_+}} >_o^2 \ . \qquad (2.65)$$

Now we choose a hyperplane π' bisecting Λ_+ : $\Lambda_+ = \Lambda^1_+ \cup \Lambda^1_-$, with $r\Lambda^1_+ = \Lambda^1_-$, and we apply (2.65) again. This yields

$$< e^{-A_\Lambda} >_o \geq < e^{-A_{\Lambda^1_+}} >_o^4 \text{ , etc...}$$

Since each side of Λ has length 2^{m_j}, $m_j = 1,2,\ldots,$ we arrive after a large number, n, of applications of (2.65) at a one-site set Λ^n_+. Clearly

$$< e^{-A_{\Lambda^n_+}} > = < 1 >_o = 1 .$$

This completes the proof of (1); (this proof extends to the general situation described in Section 2.2).

<u>Proof of (2)</u> : Let $\Lambda = [-L_o + \frac{1}{2}, L_o - \frac{1}{2}] \times \cdots \times [-L_{\nu-1} + \frac{1}{2}, L_{\nu-1} - \frac{1}{2}]$.

Note that A_Λ only couples nearest neighbors. This and (2.63) permit us to define a positive semi-definite transfer matrix $T = T_{L_1,\ldots,L_\nu}$ (the square of a hermitean matrix) such that

$$< e^{-A_\Lambda} >_o = \mathrm{Tr}(T^{L_o}) .$$

It follows from the definition of the trace and the concavity of x^α, for $0 < \alpha < 1$, that

$$\mathrm{Tr}(T^{L_o}) \geq \mathrm{Tr}(T^{2^{m_o}})^{L_o/2^{m_o}}$$

if m_o is so large that $2^{m_o} \geq L_o$.

Details of very similar arguments (applied to classical statistical mechanics) may be found in Section 4 of [51]. Applying the same inequality in the other ν-1 directions ("Nelson's symmetry") we find

$$< e^{-A_\Lambda} >_o \geq < e^{-A_{\tilde\Lambda}} >^{|\Lambda|/|\tilde\Lambda|} \geq 1 ,$$

where $\tilde\Lambda$ is a rectangle with sides of length $2^{m_j+1} \geq 2L_j$, $j = 0,\ldots,\nu-1$.

Q.E.D.

The vacuum energy density, ϵ , is defined as

$$\epsilon_\Lambda = - \frac{1}{|\Lambda|} \log < e^{-A_\Lambda} >_o \equiv - \alpha_\Lambda \tag{2.66}$$

Let $(m) \subset \mathbb{Z}^\nu_{\frac{1}{2}}$ denote a cube with sides of length m .

Corollary 2.9 :

For rectangular regions and periodic boundary condtions we have

(1) $\epsilon_\Lambda \leqq 2^\nu \epsilon_{(4)} - (2^\nu - 1)\, \epsilon_{(2)} \leqq 1$.

(2) If $\{\Lambda_n\}$ is an arbitrary increasing sequence of rectangles with sides of even length then

$$\epsilon_{\Lambda_n} \text{ is increasing in } \Lambda_n .$$

(3) $\lim\limits_{\Lambda \uparrow \mathbb{Z}^\nu_{\frac{1}{2}}} \epsilon_\Lambda \equiv \epsilon_\infty$ exists.

Proof : This is Lemma 4.6 and Corollary 4.7 of [51].

(The proofs of (1) and (2) follow by refining the arguments used in the proof of Proposition 2.7, (2); and (3) follows from (1) and (2)).

The final results of Section 2 are the chessboard estimate and infrared bounds.

Given two sites, x and y , we define

$$\sigma_j = (y^j - x^j) \bmod. 2.$$

Let r_j denote reflection at the hyperplane $\{x^j = 0\}$ (lying in between two lattice planes of $\mathbb{Z}^\nu_{\frac{1}{2}}$) , and let θ_j be given by (2.25) - (2.28) , $j = 0,\ldots,\nu-1$. Let τ_{xy} be the translation from $\left(\prod_{j=o}^{\nu-1} r_j^{\sigma_j}\right)x$ to y .

Given a function F_x of $\{\bar{\psi}(x),\psi(x);w_{x,b}, b \in \mathcal{B}_x\}$, i.e. $F_x \in \bar{\mathfrak{A}}_{\{x\}}$, and given a site y we define

$$F_{(xy)} = \tau_{xy}(\{\prod_{j=o}^{\nu-1} \theta_j^{\sigma_j}\}[F_x])$$

The following portraits this definition for $\nu = 2$:

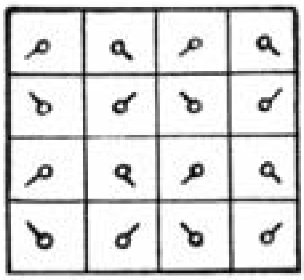


Fig. 3

<u>Theorem 2.10</u> : (Chessboard estimate)

Let Λ be a rectangle with sides of even length. Then, for periodic boundary conditions,

$$\left|< \prod_{x\in\Lambda} F_x >_{A_\Lambda}\right| \leq \prod_{x\in\Lambda} < \prod_{y\in\Lambda} F_{(xy)} >_{A_\Lambda}^{1/|\Lambda|}$$

<u>Proof</u> : Given Theorem 2.8, this follows directly from Theorem 2.2 of [18a] (as explained there in a somewhat different context). The reader can construct a proof by using the Schwarz inequality with respect to $< \cdot\,\theta[\cdot\,] >_{A_\Lambda}$ many times as in the proof of Theorem 2.3 and Corollaries 2.4 and 2.5.

Q.E.D.

<u>Remark</u> : Theorem 2.10 is a basic tool for the proof of Theorems C - E of Section 1.2; see III.

We end this section with a brief sketch of <u>infrared bounds</u> for the non-abelian lattice gauge theories. For this purpose we define a distorted action for the Yang-Mills field

$$A_P^{YM}(h_P) = -\frac{1}{2g_o^2}\,\mathrm{Tr}[U^{YM}(w_{x,<x,y>}^{-1}\, h_{xy}\, w_{y,<y,x>}\, w_{y,<y,z>}^{-1}\, h_{yz}$$

$$\times\, w_{z,<z,y>}\, w_{z,<,u>}^{-1}\, h_{zu}\, w_{u,<u,x>}^{-1}\, h_{ux}\, w_{x,<x,u>})]$$

with $P = (x,y,z,u)$ as in Fig. 1 and (2.56), and $h_P = (h_{xy}, h_{yz}, h_{zu}, h_{ux})$ four (arbitrary) elements of G. We then set

$$A_\Lambda(\underset{\sim}{h}) = \sum_{P\subset\Lambda} A_P^{YM}(h_P) + A_\Lambda^M(\underset{\sim}{\phi},\underset{\sim}{\bar\phi};\underset{\sim}{g}) .$$

<u>Theorem 2.11</u> : (Infrared bounds)

$$< e^{-A_\Lambda(\underset{\sim}{h})} >_o \;\leq\; < e^{-A_\Lambda} >_o$$

<u>Proof</u> : As in (2.57) - (2.58) one shows that, for π a plane bisecting P (see after (2.57)),

$$2g_o^2\, A_P^{YM}(h_P) = -\sum C_{ij}^P(h_{yz})\; U_{j\ell}^{YM}(h_{zu})\; \theta[C_{m\ell}^P(h_{ux})]\;\; U_{mi}^{YM}(h_{xy}) ,$$

with the obvious definition of $C_{ij}^P(h)$. With this equation at hand, the proof is completed by repeating the arguments used in the proof of Theorem 2.8 and noting that

$$\sum \theta[U_{j\ell}^{YM}(h_{zu})\; \theta[C_{m\ell}^P(h_{ux})]\, U_{mi}^{YM}(h_{xy})]\quad U_{jk}^{YM}(h_{zu})\; \theta[C_{nk}^P(h_{ux})]\, U_{ni}^{YM}(h_{xy})$$

$$= \sum C_{m\ell}^P(h_{ux})\; \theta[C_{m\ell}^P(h_{ux})] ,$$

because $U^{YM}(h)^*\, U^{YM}(h) = 1$. See also proof of Lemma 2.2. Details are very similar to the ones given in the proof of Theorems 2.3 and 4.7 of ref.[18b].

Q.E.D.

<u>Remarks</u> : (1) "Infrared" upper bounds on certain expectations (two point functions) are obtained from Theorem 2.11 by setting $h_{xy} = 1 + \varepsilon\tau_{xy} + \frac{\varepsilon^2}{2}\tau_{xy}^2 + O(\varepsilon^3)$, where

τ_{xy} is an element of the Lie algebra of G, expanding the ℓ.s. of Theorem 2.11 in ϵ, dividing by ϵ^2 and taking the limit $\epsilon \searrow 0$; see [21]. In the non-abelian case these estimates only take a simple form in the formal continuum limit. This is not so in the abelian case for which these estimates are further discussed in Section 4.

(2) Further infrared bounds can be obtained by changing the parametrization of the gauge field or inserting h's at other places, e.g.
$A^M(\underset{\sim}{\phi},\underset{\sim}{\psi};\underset{\sim}{g}) \to A^M(\underset{\sim}{\phi},\underset{\sim}{\psi};\ w^{-1}hw)$.

Theorems 2.10 and 2.11 show that the choice of parametrization of the gauge field (here the w-"parameters") can be quite crucial, at least technically. We believe that this, in fact, might be one of the key problems to be resolved in quantizing Yang-Mills fields in the continuum.

3. Estimates on Green's Functions and Diamagnetic Inequality Revisited

In this section we review, within the context of lattice theories, some important inequalities due to Schrader et al. [6 ,19] and [7]. They are important in our construction of the continuum abelian Higgs model in two dimensions; see II. Therefore, and because our proofs, inspired by [24a], are very short and simple, we feel it worth reporting them here. The main results of this section are

- "Kato-inequalities" [6, 7] for the Euclidean propagator of the free, scalar lattice field in a fixed,external gauge field, i.e. bounds on the Green's function of the covariant finite-difference Laplacean

- the (Schrader-R. Seiler) special diamagnetic inequality for a free scalar field in an external gauge field, with general boundary conditions (and on general lattices).

3.1. The basic method

In this section the action is given by

$$A^M_\Lambda(\underset{\sim}{\phi};\underset{\sim}{g}) = -\tfrac{1}{2} \sum_{\langle x,y\rangle \subset \Lambda} (\phi(x),\, U^{\phi}(g_{xy})\ \phi(y)) + (\nu + \frac{m_o^2}{2}) \sum_{x\in\Lambda} (\phi(x),\phi(x)) \tag{3.1}$$

It is quadratic in $\underset{\sim}{\phi}$. Thus the two point function (Euclidean propagator)

$$\lim_{\Lambda\uparrow \mathbb{Z}^\nu_{\frac{1}{2}}} [\langle e^{-A^M_\Lambda(\underset{\sim}{\phi};\underset{\sim}{g})}\rangle^M_o]^{-1} \langle \phi_\alpha(x)\ \phi_\beta(y)\ e^{-A^M_\Lambda(\underset{\sim}{\phi};\underset{\sim}{g})}\rangle^M_o$$

is given by
$$(-\Delta_{\underset{\sim}{g}} + m_o^2)^{-1}_{\alpha\beta}(x,y)\ , \tag{3.2}$$

where $\Delta_{\underset{\sim}{g}}$ is the finite difference, covariant Laplacean which we now define in terms of its "integral" kernel :

$$
\underset{\sim}{A}_{g}(x,y) = \begin{cases} 1 \otimes (U^{\Phi}(g_{xy}))_C \text{ , if } x \text{ and } y \text{ are nearest neighbors} \\ -2\nu\, 1 \otimes (1)_C \text{ , if } x = y \\ 0 \text{ , otherwise} \end{cases} \tag{3.3}
$$

Clearly, the first factors on the r.s. of (3.3) are irrelevant in this section, and we ignore them henceforth. Defining $A_{\underset{\sim}{g}}$ by

$$
A_{\underset{\sim}{g}}(x,y) = \begin{cases} \frac{1}{2\nu+m_o^2} U^{\Phi}(g_{xy}) \text{ , if } x \text{ and } y \text{ are nearest neighbors} \\ 0 \text{ , otherwise} \end{cases} \tag{3.4}
$$

we have

$$
(-\Delta_g + m_o^2)^{-1} = (2\nu + m_o^2)^{-1} (1 - A_{\underset{\sim}{g}})^{-1} \tag{3.5}
$$

In order to derive estimates on the Green's function (3.2) we expand the r.h.s. of (3.5) in a Neumann series. (It is clear from (3.4) that, for $m_o > 0$, this Neumann series converges absolutely, for <u>all</u> $\nu \geq 1$. If $m_o = 0$ one gets convergence only in $\nu \geq 3$ dimensions, whereas in $\nu = 1,2$ dimensions there are the well known infrared divergences).

Using (3.4) we observe that each term in the Neumann series for $(1 - A_{\underset{\sim}{g}})^{-1}_{\alpha\beta}(x,y)$ can be labelled by a path, ω , starting at x and ending at y ; see also [24a]. Given a path ω , we introduce a path parameter $s \in \{0,1,2,\ldots N(\omega)\}$ with $\omega_i \equiv \omega(0) = x$ and $\omega_f \equiv \omega(N(\omega)) = y$. Then

$$
\begin{aligned}
&(2\nu + m_o^2)^{-1} (1 - A_{\underset{\sim}{g}})^{-1}_{\alpha\beta}(x,y) \\
&= \sum_{\substack{\omega \\ \omega_i = x \\ \omega_f = y}} [\overrightarrow{\prod}_{s=0,1,\ldots,N(\omega)-1} (2\nu + m_o^2)^{-1} U^{\Phi}(g_{\omega(s)\omega(s+1)})]_{\alpha\beta} (2\nu + m_o^2)^{-1}
\end{aligned} \tag{3.6}
$$

Hence, for $z \in V^{\Phi}$,

$$|\Sigma\, \bar{z}_{\alpha}(-\Delta_{\underset{\sim}{g}} + m_o^2)^{-1}_{\alpha\beta}(x,y) z_{\beta}|$$

$$\leq \sum_{\substack{\omega \\ \omega_i = x \\ \omega_f = y}} |(z, \prod^{\rightarrow}_{s=0,1,\ldots,N(\omega)-1} (2\nu + m_o^2)^{-1} U^{\Phi}(g_{\omega(s)\omega(s+1)})z)|(2\nu + m_o^2)^{-1}$$

$$\leq (z,z) \sum_{\substack{\omega \\ \omega_i = x \\ \omega_f = y}} \prod_{s=0}^{N(\omega)} \frac{1}{2\nu + m_o^2} \quad , \quad \text{since} \quad \|U^{\Phi}(g)\| = 1 \quad ,$$

$$= \sum_{\substack{\omega \\ \omega_i = x \\ \omega_f = y}} (z, \prod^{\rightarrow}_{s=0,1,\ldots,N(\omega)-1} (2\nu + m_o^2)^{-1}\; U^{\Phi}(1)z)\; (2\nu + m_o^2)^{-1}$$

$$= \Sigma\, \bar{z}_{\alpha}(-\Delta_{\underset{\sim}{1}} + m_o^2)^{-1}_{\alpha\beta}(x,y)\; z_{\beta} \;, \quad \text{i.e.}$$

$$(z, (-\Delta_{g} + m_o^2)^{-1}(x,y)z) \leq (z,(-\Delta + m_o^2)^{-1}(x,y)z) \;, \tag{3.7}$$

where $\Delta \equiv \Delta_{\underset{\sim}{1}}$, for all $z \in V^{\Phi}$ and all x,y in the lattice. This is Theorem B, (1) of the introduction.

Next, we study the kernel of $e^{t\Delta_{g}}$. Clearly

$$e^{t\Delta_{g}} = e^{-2\nu t} \sum_{n=o}^{\infty} \frac{t^n}{n!} B^n_{\underset{\sim}{g}} \quad , \tag{3.8}$$

with $B_{g} \equiv (2\nu + m_o^2)A_{g}$ (the off diagonal part of Δ_{g}). Clearly the series on the r.h.s. of (3.8) converges absolutely, for all t .

For each n the kernel of $B^n_{\underset{\sim}{g}}$, $(B^n_{\underset{\sim}{g}})_{\alpha\beta}(x,y)$, can again be written as a sum over paths, ω , starting at x and ending at y . (Depending on the distance between x and y , the sum is empty for small n , and the corresponding kernel is 0). The previous reasoning implies

$$(z,(e^{t\Delta_{\underset{\sim}{g}}})(x,y)z) \leq (z,(e^{t\Delta})(x,y)z) \;, \tag{3.9}$$

for all $z \in V^{\Phi}$ and all x,y in the lattice.

One is often interested in writing $e^{t\Delta_{\underset{\sim}{g}}}$ in terms of a Feynman-Kac formula and, moreover, exhibiting the dependence on the lattice spacing.

The lattice spacing is given by an arbitrary, positive number a , and the covariant finite difference Laplacean, $\Delta^a_{\underset{\sim}{g}}$, on a lattice $\mathcal{L}_a$ $(= a\,\mathbb{Z}^{\nu}_{\Phi})$ with lattice spacing a is related to $\Delta_{\underset{\sim}{g}}$ by

$$\Delta^a_{\underset{\sim}{g}}(x,y) = a^{-2}\Delta_{\underset{\sim}{g}}(a^{-1}x,a^{-1}y) \;.$$

From this, the definition of $B_{\underset{\sim}{g}}$, see (3.4) , and (3.8) we obtain

$$(e^{t\Delta^a_{\underset{\sim}{g}}})(x,y) = e^{-2\nu a^{-2}t} \sum_{\substack{\omega \\ \omega_i = x \\ \omega_f = y}} \frac{1}{N(\omega)!} \prod_{s=0}^{N(\omega)-1} a^{-2}tU^{\Phi}(g_{\omega(s)\omega(s+1)}) \tag{3.6'}$$

This Feynman-Kac formula clearly proves (3.9). It is, however, rather inconvenient for taking the limit $a \to 0$. Therefore we rewrite the r.s. of (3.6') in another form : When $\underset{\sim}{g} = \underset{\sim}{1}$, (3.6') shows that $(e^{t\Delta^a})(x,y)$ is non-negative. Hence it is the transition function of a Markov process. Let $\Xi_a = \mathcal{L}^{\times \mathbb{R}_+}$ be the space of paths $\xi(t) \in \mathcal{L}_a$, $t \geq 0$, and let $Pr^t_{xy}(d\xi)_a$ denote the usual, Poisson-type path space measure of this process. Then we obtain from the r.s. of (3.6')

$$(e^{t\Delta^a_{\underset{\sim}{g}}})(x,y) = \int_{\Xi_a} Pr^t_{xy}(d\xi)_a \lim_{\delta\to 0} \prod_{\tau=0}^{(t/\delta)-1} U^{\Phi}(g_{\omega(\delta\tau)\omega(\delta(\tau+1))}) \;, \tag{3.10}$$

with the convention that $g_{xy} = 1$ if $x = y$. (The limit under the integral on the r.s. of (3.10) exists almost every where).

A standard argument shows that $Pr^t_{xy}(d\xi)_a$ can be viewed as a measure on $\Xi_o = (\mathbb{R}^\nu)^{\times \mathbb{R}_+}$, and one can show that

$$\lim_{a\to o} Pr^t_{xy}(d\xi)_a \equiv Pr^t_{xy}(d\xi)$$

exists as a weak limit on $C(\Xi_o)$, the space of continuous functions on Ξ_o. Thus we obtain from (3.10) the following formal expression for $a = 0$:

$$(e^{t\Delta_{\underset{\sim}{A}}})_{\alpha\beta}(x,y) = \int_{\Xi_o} Pr^t_{xy}(d\xi) P[e^{i\int_o^t A_\mu(\xi(t'))d\xi^\mu(t')}]_{\alpha\beta} \tag{3.10'}$$

where P indicates "path ordering" -see (3.10) - which can be omitted in the abelian case, and A_μ is the vector potential, an element of the Lie algebra of $U^\Phi(G)$. Clearly (3.10') yields a formal proof of (3.9) in the continuum limit, $a = 0$. In the abelian case (3.10') can be given a rigorous meaning, see e.g. [52], so that it proves (3.9) for $a = 0$. (Alternately one can show that, in (3.9), both sides have a limit, as $a \to 0$, if one sets $U^\Phi(g_{xy}) = e^{ia\, A_{<x,y>}(\frac{x+y}{2})}$, and $A_\mu(\cdot)$ is uniformly bounded and C^∞ on $\mathbb{R}^\nu$. These results are of some importance in the proof of convergence of the lattice approximation for the abelian Higgs model in two space-time dimensions. See II).

If we integrate $(e^{t\Delta_{\underset{\sim}{g}}})_{\alpha\beta}(x,y)$ with an arbitrary positive, finite measure $d\rho(t)$ supported on $[0,\infty)$ we obtain further inequalities analogous to (3.9). In particular, for $d\rho(t) = e^{-m_o^2 t}dt,\ t \geq 0$, we recover (3.7) from (3.9).

3.2 Some generalizations

Consider a Euclidean propagator

$$G^E_{\alpha\beta}(x,y;\underset{\sim}{g}) = \int_0^\infty (-\Delta_{\underset{\sim}{g}} + a)^{-1}_{\alpha\beta}(x,y)\, d\rho(a);\; d\rho \geq 0 \tag{3.11}$$

(which comes from some long range action, quadratic in $\underset{\sim}{\phi}$, that still yields Osterwalder-Schrader positivity; see [18b]) . As an example we mention

$$(-\Delta_{\underset{\sim}{g}} + m_o^2)^{-\alpha} = \frac{1}{\pi}\int_0^\infty a^{-\alpha} \sin(\pi\alpha)(-\Delta_{\underset{\sim}{g}} + m_o^2 + a)^{-1}\, da \;, \tag{3.12}$$

with $0 < \alpha \leq 1$. Obviously, (3.11), (3.12) and (3.7) immediately show that

$$(z, G^E_{\alpha\beta}(x,y;\underset{\sim}{g})z) \leq (z, G^E_{\alpha\beta}(x,y;\underset{\sim}{1})z) \;, \tag{3.13}$$

and by (3.7) and (3.12) this remains true for $\alpha > 1$ in (3.12). Furthermore using

$$(-\Delta_{\underset{\sim}{g}} + m_o^2)^{\alpha} = (2\nu + m_o^2)^{\alpha} \sum_{n=0}^{\infty} (-1)^n \binom{\alpha}{n} A^n_{\underset{\sim}{g}} \;,\; 0 < \alpha \leq 1 \;, \tag{3.14}$$

with $A_{\underset{\sim}{g}}$ given by (3.4), we see by expanding the exponential in a double power series, using (3.14) and (3.8), that

$$(z,[e^{-t(-\Delta_{\underset{\sim}{g}}+m_o^2)^{\alpha}}](x,y)z) \leq (z,[e^{-t(-\Delta+m_o^2)^{\alpha}}](x,y)z) \;,$$

for all $z \in V^{\phi}$, all x,y in the lattice, $0 < \alpha \leq 1$ and $m_o \geq 0$. (This inequality may be useful to analyze relativistic, spinless particles in an external vector potential; the case where $\alpha = \frac{1}{2}$) .

3.3 The diamagnetic inequality of R. Schrader and R. Seiler

For the action introduced in (3.1), $Z_{\Lambda}(\underset{\sim}{g})$ (defined in (2.43)) is given by

$$Z_\Lambda(\underset{\sim}{g}) = \{\det(-\Delta^\beta_{\underset{\sim}{g}} + m_o^2)^{-1}\}^{\frac{1}{2}} = e^{\frac{1}{2}\,\mathrm{Tr}\,\ln(-\Delta^\beta_{\underset{\sim}{g}} + m_o^2)^{-1}} \tag{3.16}$$

where β denotes a boundary condition at $\partial\Lambda$ for which inequality (3.7) remains true. As an exercise, the reader may check that (3.7) is valid for the classical lattice boundary conditions, (Dirichlet,...). Thus

$$\begin{aligned} Z_\Lambda(\underset{\sim}{g})/Z_\Lambda(\underset{\sim}{1}) = \exp[\lim_{T\to\infty}\int_o^T dt \sum_{x\in\Lambda}\sum_\alpha \{(-\Delta^\beta_{\underset{\sim}{g}} + m_o^2 + t)^{-1}_{\alpha\alpha}(x,x) \\ - (-\Delta^\beta + m_o^2 + t)^{-1}_{\alpha\alpha}(x,x)\}] \\ \leqq 1\,, \end{aligned}$$

by (3.7). Hence

$$Z_\Lambda(\underset{\sim}{g}) \leqq Z_\Lambda(\underset{\sim}{1})\,,\ \text{for all b.c. } \beta \tag{3.17}$$

for which (3.7) is valid. This is the argument of Schrader-Seiler [19]. Clearly

$$\det(G^E(\cdot,\cdot;\underset{\sim}{g}))^{\frac{1}{2}} \leqq \det(G^E(\cdot,\cdot;\underset{\sim}{1}))^{\frac{1}{2}}\ , \tag{3.18}$$

for any propagator G^E satisfying (3.13); (same proof). This remark and (3.13) permits one to regularize the propagator $(-\Delta_{\underset{\sim}{g}} + m_o^2)^{-1}$ in such a way that the continuum limit of $Z_\Lambda(\underset{\sim}{g})$ exists <u>without</u> destroying the diamagnetic inequality! In contrast to the proof of Theorem 2.3, the arguments leading to (3.17), (3.18) can be applied for a large class of boundary conditions and lattices.

Note that, for free massive Fermions in an external gauge field with periodic b.c., Theorem 2.3 and identities analogous to the ones used here prove that the Fermionic Green's functions cannot obey an inequality of the form (3.13)!

Finally we wish to present an alternate, somewhat more instructive proof of (3.17), yielding a stronger result. By (3.16),

$$\frac{\det((2\nu + m_o^2)\mathbb{1})^{1/2}}{\det(-\Delta_{g}^{\beta} + m_o^2)^{1/2}} = \det(\mathbb{1} - A_{g})^{-1/2} = \text{const. } Z_\Lambda(g) \quad ,$$

(and we impose e.g. Dirichlet boundary conditions at $\partial\Lambda$). Now, use the loop expansion

$$\begin{aligned}\det(\mathbb{1} - A_{g})^{-1/2} &= \exp[-1/2 \operatorname{Tr} \ln(1-A_{g})] \\ &= \exp[1/2 \sum_{n=0}^{\infty} (1/n) \operatorname{Tr}(A_{g}^{n})] \quad ,\end{aligned} \tag{3.19}$$

which converges for $m_0^2 > 0$. Using (3.4) we see that $\sum_{n=0}^{\infty}(1/n)\operatorname{Tr}(A_{g}^{n})$ is a sum of characters of products of group elements, g_{xy}, along closed loops, with positive coefficients. Thus the maximum is taken when $g = \mathbb{1}$ which proves (3.17). In the abelian case, the above fact implies that the photon-photon interaction will be attractive, [35]!

For abelian Higgs models without Dirac fields very general diamagnetic inequalities are proven in Section 4.

The results of this and the next section will be applied, in an essential way, in our construction of the continuum abelian Higgs model in two space-time dimensions which is presented in paper II.

Part 2

4. Abelian Higgs Models : Strong Diamagnetic Inequalities and Infrared Bounds

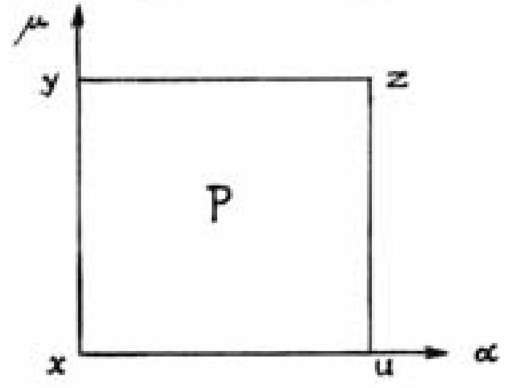
In Part 2 we investigate the abelian Higgs models, resp. scalar QED, on the lattice emphasizing those results which will survive taking the continuum limit (whenever it exists) and, in fact, are important in our construction of the continuum Higgs model in two space-time dimensions. For this reason we shall display the dependence of all quantities on the lattice spacing explicitly. In the abelian case it is convenient to represent elements in the gauge group $G = U(1)$ as exponentials of elements in the Lie algebra :

$$g_{xy} = e^{ieaA_{xy}}, \; A_{xy} \in \mathbb{R} \quad , \tag{4.1}$$

where e is the electric charge and $a > 0$ is the lattice spacing. Let $< x,y >$ be a nearest neighbor bond in the direction $\mu \in \{0,1,\ldots,\nu\text{-}1\}$ with $x^\mu < y^\mu$ and $x^\alpha = y^\alpha$, for $\alpha \neq \mu$. We also use the notation

$$A_\mu(x) \equiv A_{< x,y >} \equiv A_{xy} \quad . \tag{4.2}$$

Given a plaquette $P = \{x,y,z,u\}$, in the (μ,α) plane, (i.e. $< x,y >$ points in the positive μ - and $< y,z >$ in the positive α-dirention),



we define

$$B_P = (\mathrm{curl}\ A)(P) = \frac{1}{a}(A_{xy} + A_{yz} + A_{zu} + A_{ux})$$
$$= \frac{1}{a}\left[A_\mu(x) - A_\mu(u)\right] + \frac{1}{a}\left[A_\alpha(y) - A_\alpha(x)\right] \quad . \tag{4.3}$$

A local gauge transformation is stil given by a map $\underset{\sim}{h} : x \longrightarrow h_x \in U(1)$ of compact support which we now write as $h_x = e^{i\chi_x}$, $\chi_x \in \mathbb{R}$. Then

$$h_x g_{xy} h_y^{-1} = \exp\{iea(A_{xy} + \frac{1}{ea}[\chi_x - \chi_y])\}$$
$$= \exp\{iea(A_\mu(x) - \frac{1}{e}(\partial_\mu\chi)(x))\} \quad , \tag{4.4}$$

with $< x,y >$ pointing in the positive μ -direction and

$$(\partial_\mu\chi)(x) = \frac{1}{a}[\chi_y - \chi_x] \quad .$$

Thus gauge transformations can be defined within the Lie algebra :

$$A_\mu(x) \longrightarrow A_\mu^\chi(x) = A_\mu(x) - \frac{1}{e}(\partial_\mu\chi)(x) \quad . \tag{4.5}$$

Clearly B_P is <u>gauge-invariant</u>.

In the abelian case the Higgs field is a complex scalar field $\underset{\sim}{\Phi} : x \longrightarrow \Phi(x) \in \mathbb{C}$. (There are no Dirac fields throughout Part II).

The usual action for the Higgs field in an electromagnetic field described by $\underset{\sim}{A} : < x,y > \longrightarrow A_{xy} \in \mathbb{R}$ is given by

$$A^M_\Lambda(\underset{\sim}{\Phi};\underset{\sim}{A}) \equiv A^M_{\Lambda,a}(\underset{\sim}{\Phi};\underset{\sim}{A})$$
$$= \frac{1}{2}\sum_{<x,y>\subset\Lambda} a^\nu \left|\frac{1}{a}(\Phi(x) - e^{ieaA_{xy}}\Phi(y))\right|^2 + \sum_{x\in\Lambda} a^\nu V(|\Phi(x)|)$$
$$= -\frac{1}{2}(\underset{\sim}{\Phi}, \underset{\sim}{\Delta_A}\underset{\sim}{\Phi}) + \sum_{x\in\Lambda} a^\nu V(|\Phi(x)|) \tag{4.6}$$

where $\Delta_{\underset{\sim}{A}}$ is the covariant Laplacean introduced in Section 3 with Dirichlet or periodic boundary conditions at $\partial\Lambda$, V is some function of $|\phi|$ bounded from below. The measure $d\rho$ introduced in (2.5), Part I, is chosen to be a Gaussian measure

$$d\rho(\phi(x)) = e^{-\frac{m^2}{2} a^\nu |\phi(x)|^2} d\left(\frac{\phi(x)+\overline{\phi(x)}}{2}\right) d\left(\frac{\phi(x)-\overline{\phi(x)}}{2i}\right) \qquad . \tag{4.7}$$

Some other actions are considered in III. In contradistinction to the non-abelian case one has many natural, gauge-invariant options for defining the action of the gauge field, due to the gauge invariance of B_P . Here some prominent ones :

$$\text{(W)}\ A^{YM}_{\Lambda,a} = -\sum_{P\subset\Lambda} a^{\nu-4}\cos(a^2 B_P) \qquad ; \tag{4.8}$$

this corresponds precisely to definitions (2.56), (2.60).

This action is periodic in A_{xy} with period $\frac{2\pi}{a}$, i.e. can be restricted to $[-\frac{\pi}{a}, \frac{\pi}{a}]$. The expectation for a pure gauge field is given by

$$(Z^{YM}_{\Lambda,a})^{-1}\, e^{-A^{YM}_{\Lambda,a}} \prod_{<x,y>\subset\Lambda} dA_{xy} \qquad , \tag{4.9}$$

where dA_{xy} is the Lebesgue measure on $[-\frac{\pi}{a}, \frac{\pi}{a}]$ and $Z^{YM}_{\Lambda,a}$ is the obvious normalization factor. This example was proposed by Wilson [9]. It must be assumed that the lectric charge, e , is an integer, so that $e^{ieaA_{xy}}$ has period $\frac{2\pi}{ea}$, hence $\frac{2\pi}{a}$.

$$\begin{aligned} \text{(P)}\ & F_a(B_P) = \sum_{n_P\in\mathbb{Z}} e^{-\frac{a^{\nu-4}}{2}(a^2 B_P + 2\pi\, n_P)^2} \quad , \\ & e^{-A^{YM}_{\Lambda,a}} = \prod_{P\subset\Lambda} F_a(B_P) \qquad . \end{aligned} \tag{4.10}$$

Clearly $A^{YM}_{P,a}$ is periodic in A_{xy} with period $\frac{2\pi}{a}$. The function F_a is the kernel of $\exp \frac{1}{2} \Delta$ with periodic boundary conditions at $\pm \frac{\pi}{a}$. (The action defined in (4.10) is useful to discuss the relations between the "Euclidean" and the Hamiltonian formulation of lattice gauge theories). The remaining definitions and constraints are as in (W) . This example was proposed by Polyakov [26] and has been used in [27,28] and III. The relation between (W) and (P) is identical to the one between the classical rotator model and its Villain approximation, [29] .

(G) Finally one can also choose a quadratic action

$$A^{YM}_{\Lambda,a} = \sum_{P\subset\Lambda} a^{\nu} B_P^2 + \frac{m_A^2}{2} \sum_{<x,y>\subset\Lambda} A_{xy}^2 , \quad m_A \geq 0 \quad . \tag{4.11}$$

Of course, the mass term on the r.s. of (4.11) breaks gauge invariance when $m_A > 0$, but this does not cause problems in an abelian model, because $\underset{\sim}{A}$ couples to a conserved current. In various situations we shall be able to pass to the limit $m_A = 0$, recovering gauge invariance. See papers II and III. In particular, Theorems D and E of the introduction are results for the case where $m_A = 0$! Note that, for $m_A = 0$, (4.10) and (4.11) become equivalent.

The measure for the pure gauge field is then defined as the Gaussian

$$(Z^{YM}_{\Lambda,a})^{-1} e^{-A^{YM}_{\Lambda,a}} \prod_{<x,y>\subset\Lambda} dA_{x,y} , \quad A_{xy} \in \mathbb{R} \quad ,$$

which is well defined, for $m_A > 0$. It has a unique limit (in the sense of convergence of moments or generating functionals), as $\Lambda \uparrow a\mathbb{Z}^{\nu}_{1/2}$. This limiting Gaussian measure is denoted $d\mu_{C_{m_A}}(\underset{\sim}{A})$. It has mean 0 and covariance C_{m_A} determined by

$$C^{-1}_{m_A} = \text{curl}^* \quad \text{curl} \ + m_A^2 \qquad . \tag{4.12}$$

Example (G) is the lattice approximation to be used to construct the Higgs models in the continnum limit ; see Sections 5,6 and paper II. Clearly, any value of the electric charge, e , is possible in example (G).

4.1 <u>Strong Diamagnetic Inequalities</u>.

In the proofs of the following results it is convenient to choose polar coordinates for the Higgs field $\underset{\sim}{\phi}$:

$$\phi(x) = r_x \, e^{i\theta_x} \ , \ \theta_x \in [0,2\pi) \ , \ r_x \in \mathbb{R}^+ \quad .$$

Then

$$\frac{a^\nu}{2}\left|\frac{1}{a}(\phi(x)-e^{ieaA_{xy}}\phi(y))\right|^2$$
$$= -a^{\nu-2} r_x r_y \cos(-\theta_x + eaA_{xy} + \theta_y) + \frac{a^{\nu-2}}{2}[r_x^2 + r_y^2] \quad . \tag{4.13}$$

We propose to consider a general, non-translation invariant action

$$A^{M}_{\Lambda}(\underset{\sim}{c};\underset{\sim}{\phi};\underset{\sim}{A}) = - \sum_{\langle x,y\rangle \subset \Lambda} c_{xy} \ r_x \ r_y \ \cos(-\theta_x + eaA_{xy} + \theta_y) \tag{4.14}$$

where $\{c_{xy}\}$ are arbitrary non-negative numbers. At each site $x \in \Lambda$ we are given a finite measure $d\rho_x(r_x)$ on $\mathbb{R}^+$ with the property that

$$\int e^{\alpha r^2} \, d\rho_x(r) < \infty \qquad ,$$

for all $\alpha \geqq 0$ and all $x \in \Lambda$. (This condition is somewhat stronger than necessary).

Let $\partial\Lambda$ be the set of all sites in $\Lambda^c = a\,\mathbb{Z}^\nu_{1/2} \backslash \Lambda$ with a nearest neighbor in Λ , and $\bar{\Lambda} = \Lambda \cup \partial\Lambda$.

Given an arbitrary set $\tilde{\Lambda} \subseteq a\,\mathbb{Z}^{\nu}_{1/2}$, we set $r_{\tilde{\Lambda}} = \{r_x : x \in \tilde{\Lambda}\}$ and $\theta_{\tilde{\Lambda}} = \{\theta_x : x \in \tilde{\Lambda}\}$. Also $\underset{\sim}{A} = \{A_{xy} : \langle x,y \rangle \subset a\,\mathbb{Z}^{\nu}_{1/2}\}$.

Given a function $G_{\tilde{\Lambda}}(r_{\tilde{\Lambda}},\theta_{\tilde{\Lambda}},\underset{\sim}{A})$ we let $\tilde{G}_{\tilde{\Lambda}}(r_{\tilde{\Lambda}},n_{\tilde{\Lambda}},\underset{\sim}{A})$ denote the partial Fourier transform of $G_{\tilde{\Lambda}}$ in the variables $\theta_{\tilde{\Lambda}}$; $n_{\tilde{\Lambda}}$ are the variables conjugate to $\theta_{\tilde{\Lambda}}$. Furthermore, $\hat{G}_{\tilde{\Lambda}}(r_{\tilde{\Lambda}},n_{\tilde{\Lambda}},\underset{\sim}{a})$ is the Fourier transform of $G_{\tilde{\Lambda}}$ in $\theta_{\tilde{\Lambda}}$ <u>and</u> $\underset{\sim}{A}$, with $\underset{\sim}{a}$ the variables conjugate to $\underset{\sim}{A}$.

If $G_{\tilde{\Lambda}} = G_{\partial\Lambda}(r_{\partial\Lambda},\theta_{\partial\Lambda},\underset{\sim}{A})$ only depends on $r_{\partial\Lambda}$, $\theta_{\partial\Lambda}$ we say that $G_{\partial\Lambda}$ is a <u>boundary condition</u>. We define

$$Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{A}) = \int \prod_{x\in\Lambda} d\rho_x(r_x)\frac{d\theta_x}{2\pi}\, e^{-A^M_{\Lambda}(\underset{\sim}{c};\underset{\sim}{\Phi};\underset{\sim}{A})}\, G_{\tilde{\Lambda}}(r_{\tilde{\Lambda}},\theta_{\tilde{\Lambda}},\underset{\sim}{A}) \qquad . \qquad (4.15)$$

We add some <u>examples</u> for the choice of $d\rho_x$, c_{xy} , G :

$$d\rho_x(r) = e^{a^{\nu}h_x r^2}\, e^{-a^{\nu}[\lambda r^4 + \frac{m^2}{2} r^2 + \frac{\nu}{a^2} r^2]}\, r dr \qquad ,$$

with dr the Lebesque measure on $\mathbb{R}^+$, $\underset{\sim}{h} : x \longrightarrow h_x$ a real-valued function on Λ , $\lambda > 0$.

$c_{xy} = a^{\nu-2}[1+f_{xy}]$, with $f : \langle x,y \rangle \longrightarrow f_{xy}$ a non-negative function on the bonds of Λ .

e.g.

$$G_{\tilde{\Lambda}} = \prod_{x_j \in X \subseteq \Lambda} r_{x_j}^{m_j}\, e^{\pm i m_j \theta_{x_j}}\, G_{\partial\Lambda}$$

, where $G_{\partial\Lambda}$ is a function imposing lattice Dirichlet - or Neumann - or periodic b.c. at $\partial\Lambda$. For these choices, $Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{A})$ represents <u>unnormalized Green's functions of a Higgs field in an external vector potential</u> (resp., for $X = \emptyset$, the generating functional of gauge-invariant, unnormalized Green's functions), and the usual partition function of this theory is obtained by setting $\underset{\sim}{h} = \underset{\sim}{f} = 0$ and $X = \emptyset$.

Theorem 4.1 :

(1) If $|\tilde{G}_{\bar{\Lambda}}(r_{\bar{\Lambda}},n_{\bar{\Lambda}},\underset{\sim}{A})| \leqq \tilde{G}_{\bar{\Lambda}}(r_{\bar{\Lambda}},n_{\bar{\Lambda}},\underset{\sim}{0})$ then $|Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{A})| \leqq Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{0})$ (strong diamagnetic inequality).

(2) If $\hat{G}_{\Lambda}$ is non-negative then $Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{A})$ is of positive type in $\underset{\sim}{A}$; in particular, (1) holds.

Remark : It is easy to check that, in the above examples, the hypotheses of Theorem 4.1, (1) and (2) are satisfied.

Proof : We apply a "duality transformation" : Let $F_{xy}(r_x r_y;n_{xy})$ be the Fourier transform of $\exp[c_{xy}r_x r_y \cos(\theta_{xy})]$ in the variable $\theta_{xy} = \theta_y - \theta_x$. By power series expansion of the exponential one verifies the well known fact that

$$F_{xy}(r_x r_y;n_{xy}) \geqq 0 \quad . \tag{4.16}$$

The Fourier transform of $\exp[c_{xy}r_x r_y \cos(\theta_{xy} + eaA_{xy})]$ in $\theta_{xy} = \theta_y - \theta_x$ is clearly given by

$$F_{xy}(r_x r_y;n_{xy})\, e^{ieaA_{xy}n_{xy}} \quad . \tag{4.17}$$

If $< x,y >$ points in the positive μ-direction we write $n^{\mu}(x)$ for n_{xy} . Let $\vec{e}_{\mu}$ be the unit vector in the positive μ-direction. Then $\vec{n}(x) = \sum_{\mu=0}^{\nu-1} n^{\mu}(x)\vec{e}_{\mu}$ defines a vector field on the lattice. We define

$$(\mathrm{div}\,\vec{n})(x) = \sum_{\mu=0}^{\nu-1} n^{\mu}(x)-n^{\mu}(x-\vec{e}_{\mu}) \quad . \tag{4.18}$$

Given a site x , the $2\nu + 1$ functions $\exp[c_{xy}r_x r_y \cos(\theta_y + eaA_{xy} - \theta_x)$, with y a nearest neighbor of x , and $G_{\bar{\Lambda}}(r_{\bar{\Lambda}},\theta_{\bar{\Lambda}},\underset{\sim}{A})$ depend on θ_x . Representing them by their Fourier series we obtain a factor $\exp[-i\theta_x\{(\mathrm{div}\,\vec{n})(x) - m_x\}]$ and we can do the θ_x-integral explicitly :

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta_x \, e^{-i\theta_x\{(\mathrm{div}\,\vec{n})(x) - m_x\}} = \delta_{(\mathrm{div}\,\vec{n})(x),m_x} \quad .$$

This shows that

$$\begin{aligned} Z_\Lambda(\underset{\sim}{c},G;\underset{\sim}{A}) = \int \prod_{x\in\Lambda} d\rho_x(r_x) \sum_{\vec{n},m_\Lambda} \prod_{<x,y>\subset\Lambda} F_{xy}(r_x r_y;n_{xy}) \\ \times e^{iean_{xy}A_{xy}} \,\widetilde{G}(r_{\bar\Lambda},m_{\bar\Lambda};\underset{\sim}{A}) \prod_{x\in\Lambda} \delta_{(\mathrm{div}\,\vec{n})(x),m_x} \quad . \end{aligned} \tag{4.19}$$

Therefore, using (4.16) and the hypothesis of Theorem 4.1, (1) we find

$$\begin{aligned} |Z_\Lambda(\underset{\sim}{c},G;\underset{\sim}{A})| &\leqq \int \prod_{x\in\Lambda} d\rho_x(r_x) \sum_{\vec{n},m_\Lambda} \prod_{<x,y>\subset\Lambda} F_{xy}(r_x r_y;n_{xy}) \\ &\quad \times |e^{iean_{xy}A_{xy}}|\,|\widetilde{G}(r_{\bar\Lambda},m_{\bar\Lambda};\underset{\sim}{A})| \prod_{x\in\Lambda} \delta_{(\mathrm{div}\,\vec{n})(x),m_x} \\ &\leqq \int \prod_{x\in\Lambda} d\rho_x(r_x) \sum_{\vec{n},m_\Lambda} \prod_{<x,y>\subset\Lambda} F_{xy}(r_x r_y;n_{xy}) \\ &\quad \times \widetilde{G}(r_{\bar\Lambda},m_{\bar\Lambda};\underset{\sim}{0}) \prod_{x\in\Lambda} \delta_{(\mathrm{div}\,\vec{n})(x),m_x} \\ &= \int \prod_{x\in\Lambda} d\rho\, r_x)\frac{d\theta_x}{2\pi}\, e^{-A^M_\Lambda(\underset{\sim}{c};\underset{\sim}{\theta};\underset{\sim}{0})} G_{\bar\Lambda}(r_{\bar\Lambda},\theta_{\bar\Lambda};\underset{\sim}{0}) \\ &= Z_\Lambda(\underset{\sim}{c},G;\underset{\sim}{0}) \end{aligned}$$

which completes the proof of (1).

To prove (2) we note that, by (4.16), $F_{xy}(r_x r_y;n_{xy})e^{iean_{xy}A_{xy}}$ is of positive type in A_{xy}. Since, by hypothesis of (2), $\hat{G}_{\bar\Lambda}(r_{\bar\Lambda},m_{\bar\Lambda},\underset{\sim}{a})$ is non-negative, $\widetilde{G}_{\bar\Lambda}(r_{\bar\Lambda},m_{\bar\Lambda},\underset{\sim}{A})$ is of positive type in $\underset{\sim}{A}$, for all $r_{\bar\Lambda},m_{\bar\Lambda}$. Since a product of functions of positive type is again of positive type, (4.19) shows that $Z_\Lambda(\underset{\sim}{c},G;\underset{\sim}{A})$ is of positive type in $\underset{\sim}{A}$. Finally, each function $f(\underset{\sim}{A})$ of positive type in $\underset{\sim}{A}$ satisfies $|f(\underset{\sim}{A})| \leqq f(\underset{\sim}{0})$, so that (2) implies (1).

Q.E.D.

Remarks :

(1) In the proof we have only used that $\exp(c_{xy} r_x r_y \cos(\theta_{xy}))$ is a function of positive type in θ_{xy} . Therefore Theorem 4.1 remains true if we modify the action $A^M_\Lambda(\underset{\sim}{c};\underset{\sim}{\Phi},\underset{\sim}{A})$ in any way that satisfies this constraint. In particular, we may replace each factor $\exp(c_{xy} r_x r_y \cos(\theta_y + eaA_{xy} - \theta_x))$ in $e^{-A^M_\Lambda}$ by

$$\sum_{k_{xy} \in \mathbb{Z}} \exp\left[-\frac{c_{xy} r_x r_y}{2} (\theta_y + eaA_{xy} - \theta_x + 2\pi k_{xy})^2\right] \quad , \tag{4.20}$$

thus obtaining Theorem 4.1 for the Higgs models in the Polyakov - Villain approximation which is considered in III.

(2) Theorem 4.1 and its proof can obviously be extended to a large class of lattices.

(3) We feel that the method of proof of Theorem 4.1 can be extended to non-abelian Higgs models without fermions, but this is not investigated here.

Let $d\mu_C(\underset{\sim}{A})$ be a normalized Gaussian measure for the vector potential $\underset{\sim}{A}$ with mean 0 and covariance C . We say that a sequence $\{C_n\}_{n=1}^{\infty}$ of covariances is increasing iff $0 \leqq C_n \leqq C_{n+1}$, in the sense of quadratic forms, for all $n = 1,2,3,\ldots$.

Corollary 4.2 :

Under the hypotheses of Theorem 4.1, (2)

$$I(e,C) \equiv \int Z_\Lambda(\underset{\sim}{c};G;\underset{\sim}{A})\, d\mu_C(\underset{\sim}{A}) \quad ,$$

where e is the electric charge, is monotone decreasing in C . In particular $I(e,C)$ is monotone decreasing in $|e|$, and for $C = C_{m_A}$ + gauge terms, with C_{m_A} given by (4.12) and a gauge-invariant G_Λ , $I(e,C_{m_A})$ is monotone increasing in the mass m_A of the vector potential.

<u>Remark</u> : This result also holds in Wilson's and Polyakov's lattice theories ; i.e. if $A^{YM}_{\Lambda,a}$ is the action for the <u>abelian</u> gauge field introduced in (W), resp. (P) and $e = 1$ in A^M_Λ then

$$[< e^{-g_o^{-2} A^{YM}_{\Lambda,a}} >^G_o]^{-1} < Z_\Lambda(\underset{\sim}{c},G;\underset{\sim}{A}) e^{-g_o^{-2} A^{YM}_{\Lambda,a}} >^G_o$$

is monotone decreasing in $|g_o|$. This is an example of a general class of monotonicity results which follow from Theorem 4.1 , (2) and correlation inequalities of [35,53] .

<u>Proof</u> :

Since $Z(\underset{\sim}{c},G;\underset{\sim}{A})$ is of positive type in $\underset{\sim}{A}$ (Theorem 4.1, (2)) it has the representation

$$Z_\Lambda(\underset{\sim}{c},G;\underset{\sim}{A}) = \int d\alpha_\Lambda(\underset{\sim}{m}) \prod_{<x,y>} e^{i m_{xy} A_{xy}} \quad , \tag{4.21}$$

where $\underset{\sim}{m}$ is real-valued, and $d\alpha_\Lambda$ is a positive measure. But

$$\int \prod_{<x,y>} e^{i m_{xy} A_{xy}} \, d\mu_C(\underset{\sim}{A}) = \exp[-\tfrac{1}{2}(\underset{\sim}{m},C\underset{\sim}{m})]$$

is monotone decreasing in C . This proves the first part of Corollary 4.2 . To prove the second part we set $\underset{\sim}{A}' = e\underset{\sim}{A}$ which makes $A^M_\Lambda(\underset{\sim}{c};\underset{\sim}{\phi};\underset{\sim}{A}')$ independent of e , (<u>provided normal ordering is independent of</u> e , <u>which is only possible if</u> $a > 0$!) See (4.14).

We note that $\underset{\sim}{A}'$ has mean 0 and covariance e^2C . Hence

$$I(e,C) = I(1,e^2C) \quad . \tag{4.22}$$

Clearly e^2C is monotone increasing in $|e|$.

Finally, we know from Theorem 2.6 (see also Scetion 5.1) that, for G_{Λ} gauge-invariant, $Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{A})$ is gauge-invariant. Therefore gauge terms in the covariance C can be chosen at convenience, i.e. $I(e,C)$ is independent of gauge terms. To prove the last part we may therefore choose

$$C^{-1}_{m_A} = \delta_{\mu\nu}(-\Delta + m_A^2) \qquad ,$$

so that C_{m_A} is decreasing in m_A .

Q.E.D.

Next, we recall a few well known properties of functions of positive type : Let $F(x)$, $W(x)$ and $H(x)$ be functions of positive type on $\mathbb{R}^N$. Then $W(x)\,H(x)$ is of positive type, too, and

$$\begin{aligned}
&\left|\int d^N x\; F(x+a)\; W(x)\; H(x)\right| \\
&= \left|\int d^N p\; e^{ipa}\; \hat{F}(p)(\hat{W} * \hat{H})(p)\right| \\
&\leq \int d^N p\; \hat{F}(p)\; (\hat{W} * \hat{H})(p) \\
&= \int d^N x\; F(x)\; W(x)\; H(x) \qquad . \qquad (4.23)
\end{aligned}$$

Let

$$F(\underset{\sim}{A}) = \prod_{P \subset \chi} e^{-\frac{1}{2} a^{\nu} B_P^2} \equiv \exp[-\tfrac{1}{2}(\underset{\sim}{A},Q_o\;\underset{\sim}{A})] \qquad ,$$

where

$$\chi \supseteq \Lambda\,,\; \underset{\sim}{A} = \{A_{xy}\}_{<x,y>\subset \chi} \qquad ,$$

and Q_o is a positive semi-definite quadratic form.

Let

$$W(\underset{\sim}{A}) = \exp[-\tfrac{1}{2}\,(\underset{\sim}{A},Q\;\underset{\sim}{A})] \qquad ,$$

where Q is a positive semi-definite quadratic form ; W is a Gaussian which fixes a gauge. The "unitary gauge" for a massive vector field corresponds to

$$(\underset{\sim}{A}, Q\underset{\sim}{A}) = \sum_{<x,y>\subset\tilde{\Lambda}} a^{\nu} m_A^2 \, A_{xy}^2 \quad .$$

Another example is the "Feynman gauge" :

$$Q = \delta_{\mu\nu}(-\Delta + m_A^2) - Q_o \geqq 0 \quad .$$

Clearly F and W are of positive type in $\underset{\sim}{A}$, a well known property of Gaussians with mean 0 , and

$$H(\underset{\sim}{A}) = Z_{\Lambda}(\underset{\sim}{c}, G; \underset{\sim}{A})$$

is of positive type, by Theorem 4.1, (2), for $\hat{G}_{\Lambda} \geqq 0$. So we may apply inequality (4.23).

Given a bond $< x,y >$, there are $2(\nu-1)$ plaquettes, $P_1, P_1', \ldots, P_{\nu-1}, P_{\nu-1}'$ with $< x,y >$ as a common bond and such that P_i and P_i' are in the same plane, $i=1,\ldots,\nu-1$. For a function $\underset{\sim}{h} : P \longrightarrow h_P \in \mathbb{R}$ on plaquettes in χ , we define

$$(\delta h)_{xy} = h_{P_1} - h_{P_1'} + \ldots + h_{P_{\nu-1}} - h_{P_{\nu-1}'} \quad . \tag{4.24}$$

Applying (4.23) we obtain

$$\begin{aligned} &\left|\int F(\underset{\sim}{A}+\delta\underset{\sim}{h})\; W(\underset{\sim}{A})\; H(\underset{\sim}{A}) \prod_{<x,y>\subset\chi} dA_{xy}\right| \\ &\quad \leqq \int F(\underset{\sim}{A})\; W(\underset{\sim}{A})\; H(\underset{\sim}{A}) \prod_{<x,y>\subset\chi} dA_{xy} \quad . \end{aligned} \tag{4.25}$$

But

$$\begin{aligned} F(\underset{\sim}{A} \pm \delta\underset{\sim}{h}) &= e^{\pm \sum_{P\subset\chi} a^{\nu} B_P h_P}\, e^{-\frac{1}{2}\sum_{P\subset\chi} a^{\nu} h_P^2}\, F(\underset{\sim}{A}) \\ &\equiv e^{\pm B(\underset{\sim}{h})}\, e^{-1/2\|\underset{\sim}{h}\|_2^2}\, F(\underset{\sim}{A}) \quad . \end{aligned} \tag{4.26}$$

Since B is gauge-invariant, (4.25) and (4.26) provide an a priori upper bound on the interacting expectation of $\exp \pm B(\underset{\sim}{h})$ which is independent of the gauge chosen, i.e. holds in any gauge, and of the coupling constants of the theory ; (see also Section 5).

The same method supplies an upper bound on the interacting expectation of

$$\exp[\pm \sum_{<x,y>} a^\nu A_{xy} g_{xy}]$$

(which of course <u>does</u> depend on the gauge chosen) :

$$\begin{aligned} &\left|\int (F.W)(\underset{\sim}{A} + \underset{\sim}{g})\, H(\underset{\sim}{A}) \prod_{<x,y>\subset\Lambda} dA_{xy}\right| \\ &\leqq \int (F.W)(\underset{\sim}{A})\, H(\underset{\sim}{A}) \prod_{<x,y>\subset\Lambda} dA_{xy} \quad . \end{aligned} \tag{4.27}$$

Upon normalization the bound obtained from (4.27) is still independent of the coupling constants. We will not use it.

Suppose now that Q is translation-invariant (up to a boundary condition) and strictly positive. Then, for a suitable choice of C and of Z_Λ ,

$$d\mu_C(\underset{\sim}{A}) = \lim_{\Lambda \nearrow \mathbb{Z}^\nu} Z_\Lambda^{-1} \left(\prod_{P\subset\Lambda} e^{-\frac{a^\nu}{2} B_P^2}\right) e^{-\frac{1}{2}(\underset{\sim}{A},Q,\underset{\sim}{A})} \prod_{<x,y>\subset\Lambda} dA_{xy} \quad \text{exists,}$$

(in the sense that the characteristic functionals and moments converge). We now get from (4.25) and (4.26).

<u>Theorem 4.3</u> :

$$\begin{aligned} &\int e^{\pm B(\underset{\sim}{h})} Z_\Lambda(\underset{\sim}{c},G;\underset{\sim}{A})\, d\mu_C(\underset{\sim}{A}) \\ &\leqq e^{1/2\|h\|_2^2} \int Z_\Lambda(\underset{\sim}{c},G;\underset{\sim}{A})\, d\mu_C(\underset{\sim}{A}) \quad . \end{aligned}$$

Remarks :

(1) Let $r^2(g) = \sum_{x \in \Lambda} a^\nu r_x^2 g_x$ with g a real-valued function on Λ. In the definition (4.14) of A^M we set $c_{xy} = a^{\nu-2}$, and we choose the distribution

$$d\rho_x(r) = e^{-a^\nu r_x^2 g_x} d\rho(r) \qquad ,$$

(with $d\rho$ independent of x). Furthermore $G_{\bar{\Lambda}} = G_{\partial\Lambda}$ is a boundary condition with $\hat{G}_{\partial\Lambda} \geqq 0$. Finally, $\langle \text{---} \rangle_{A_\Lambda}$ denotes the normalized, interacting expectation determined by $d\mu_C(A)$ and the above choices for $A_\Lambda^M(c;\phi;A)$ and $d\rho_x = d\rho(g = 0)$. Then Theorem 4.3 contains as a special case

$$\langle e^{\pm B(h)} e^{r^2(g)} \rangle_{A_\Lambda} \leqq e^{1/2\|h\|_2^2} \langle e^{r^2(g)} \rangle_{A_\Lambda} \qquad . \tag{4.28}$$

In Section 6 we shall show that, for $g \geqq 0$,

$$\langle e^{r^2(g)} \rangle_{A_\Lambda} \leqq \langle e^{r^2(g)} \rangle_{A_\Lambda(e=0)} \tag{4.29}$$

i.e. the expectation of $\exp r^2(g)$ is bounded above by the one for 0 electric charge, provided Wick ordering can be chosen to be independent of e, (i.e. $a > 0$).

(2) Clearly, Theorem 4.3 and (4.28) - (4.29) represent a stronger version of Theorem 2.11, restricted to the abelian case. (Of course, the proof of the latter applies to the present case -see Section 5 - but yields weaker bounds).

(3) Obviously the methods used to prove Theorem 4.3 can be applied to the abelian gauge theories obtained from the actions (W) and (P), since those are also of the form $-\log F(A)$, with $F(A)$ of positive type. This observation yields a priori upper bounds for the expectations of a class of functions of B. Since we shall not use them, we leave it to the reader to write them down explicitly.

5. Gauge invariance, Osterwalder-Schrader positivity and gauge-variant Green's functions

The set up in this section is the same as in Section 4 : We restrict our discussion to abelian models, although some of the general statements in 5.1 and 5.3 extend to non-abelian models. The action $A_\lambda^M(\underset{\sim}{c};\underset{\sim}{\Phi};\underset{\sim}{A})$ is defined as in (4.14), and the gauge field action is one of the actions introduced in (4.8)-(4.11).

5.1. Gauge-invariance

Gauge transformations are defined by (4.4), (4.5), i.e.

$$A_\mu(x) \longrightarrow A_\mu^\chi(x) = A_\mu(x) - \frac{1}{e}(\partial_\mu\chi)(x) \qquad . \tag{5.1}$$

Let $\{h_\varkappa\}$ be a family of bounded, strongly decreasing functions on the lattice with the property that

$$h_\varkappa \longrightarrow \delta_o \quad , \quad \text{as} \quad \varkappa \longrightarrow \infty \qquad . \tag{5.2}$$

For purposes that will become clear in II (convergence of the lattice approximation as $a \searrow 0$) we define a gauge field $\underset{\sim}{A}_{\mu,\varkappa}$ with ultraviolet cutoff $\varkappa$ by

$$A_{\mu,\varkappa}(x) = (h_\varkappa * A_\mu)(x) \tag{5.3}$$

with the aim of showing that gauge-invariance and Osterwalder-Schrader positivity can be discussed in the presence of certain ultraviolet cutoffs ; (see also Section 6) . Furthermore we emphasize that <u>all</u> results of Section 4 are still valid when $\underset{\sim}{A}$ is replaced by $\underset{\sim}{A}_\varkappa$! (The verification is trivial).

Definitions (5.1) and (5.3) give

$$\begin{aligned} \underset{\sim}{A}^\chi_{\mu,\varkappa} &= \underset{\sim}{A}_{\mu,\varkappa} - \frac{1}{e}\,\partial_\mu(h_\varkappa * \underset{\sim}{\chi}) \\ &\equiv \underset{\sim}{A}_{\mu,\varkappa} - \frac{1}{e}\,\partial_\mu\underset{\sim}{\chi}_\varkappa \\ &= (\underset{\sim}{A}_\mu - \frac{1}{e}\,\partial_\mu\underset{\sim}{\chi})_\varkappa \qquad . \end{aligned} \tag{5.4}$$

By Lemma 2.1 and Theorem 2.6 we have

$$Z_{\Lambda}(\underset{\sim}{c}, G; \underset{\sim x}{A}) = Z_{\Lambda}(\underset{\sim}{c}, G; \underset{\sim x}{A} - \frac{1}{e} \partial \underset{\sim x}{\chi}) \tag{5.5}$$

(gauge invariance)

when G_{Λ} is gauge-invariant. More generally,

$$Z_{\Lambda}(\underset{\sim}{c}, G; \underset{\sim x}{A}) = Z_{\Lambda}(\underset{\sim}{c}, G^{\chi}; \underset{\sim x}{A} - \frac{1}{e} \partial \underset{\sim x}{\chi}) \quad , \tag{5.6}$$

with G^{χ} the gauge-transformed of G .

We note that if $G_{\Lambda} = G_{\partial\Lambda}$ is strictly localized near $\partial\Lambda$, for all Λ , and χ is a local gauge transformation then $G^{\chi} = G$, as $\Lambda \uparrow a\mathbb{Z}^{\nu}_{1/2}$. As a result one shows that, <u>unless</u> gauge fixing terms are added to the action, the expectation of a lattice gauge theory is automatically gauge-invariant in the thermodynamic limit. See also Guerra et al. [11] and Section 2.2. Next, we discuss the consequences of (5.5) - (5.6) for fully interacting abelian lattice gauge theories in case (G) ; see (4.11). Let $d\mu_C(\underset{\sim}{A})$ be a Gaussian measure with mean 0 and covariance C and let $d\nu(\chi)$ be an arbitrary probability measure for a random field χ_x , $x \in a\mathbb{Z}^{\nu}_{1/2}$, with values in $\mathbb{R}$. We define

$$\hat{\underset{\sim}{A}} = \underset{\sim}{A} - \frac{1}{e} \partial \underset{\sim}{\chi} \quad ,$$

and

$$d\mu(\hat{\underset{\sim}{A}}) = d\mu_C(\underset{\sim}{A}) d\nu(\chi) \quad . \tag{5.7}$$

Let B be the curl of $\underset{\sim}{A}$, defined in (4.3) which is gauge-invariant, i.e. $B = \hat{B}$. Then we obtain from (5.4) and (5.5)

$$\int d\mu_C(\underset{\sim}{A}) e^{B(\underset{\sim}{h})} Z_\Lambda(\underset{\sim}{c}, G; \underset{\sim}{A}_\varkappa)$$

$$= \int d\mu_C(\underset{\sim}{A}) d\nu(\chi)\ e^{B(\underset{\sim}{h})} Z_\Lambda(\underset{\sim}{c}, G; \underset{\sim}{A}_\varkappa)$$

$$= \int d\mu_C(\underset{\sim}{A}) d\nu(\chi) e^{\hat{B}(\underset{\sim}{h})} Z_\Lambda(\underset{\sim}{c}, G; \underset{\sim}{A}_\varkappa - \frac{1}{e} \partial \underset{\sim}{\chi}_\varkappa)$$

$$= \int d\mu(\hat{\underset{\sim}{A}}) e^{\hat{B}(\underset{\sim}{h})} Z_\Lambda(c, G; \hat{\underset{\sim}{A}}_\varkappa) \quad . \tag{5.8}$$

These identities extend to the expectations determined by the actions (W) and (P) and (mutatis mutandis, $\varkappa = \infty$) to the non-abelian case. The Gaussian expectations for $\underset{\sim}{A}$ are the ones of primary importance for II. See [17] for a discussion of the Faddeev-Popov procedure in non-abelian lattice theories.

We now consider the case when $d\nu$ is a Gaussian measure, $d\nu_F$, of mean 0 and covariance $F \geqq 0$. Then $d\mu(\hat{\underset{\sim}{A}}) = d\mu_{\hat{C}}(\hat{\underset{\sim}{A}})$, with

$$\hat{C} = C + \partial\ F\ \partial^* \quad , \tag{5.9}$$

and

$$\int d\mu_C(\underset{\sim}{A}) e^{\underset{\sim}{A}(\underset{\sim}{g})} Z_\Lambda(\underset{\sim}{c}, G; \underset{\sim}{A})$$

$$= \int e^{-(1/2e^2)(\partial^* \underset{\sim}{g}, F\ \partial^*\ \underset{\sim}{g})}$$

$$\times \int d\mu_C(\underset{\sim}{A}) d\nu_F(\chi) e^{(\underset{\sim}{A} - e^{-1} \partial \underset{\sim}{\chi})(\underset{\sim}{g})} Z_\Lambda(\underset{\sim}{c}, G; \underset{\sim}{A}_\varkappa - \frac{1}{e} \partial \underset{\sim}{\chi}_\varkappa)$$

$$= e^{-(1/2e^2)(g, \partial F \partial^* g)} \int d\mu_{\hat{C}}(\hat{\underset{\sim}{A}}) e^{\hat{\underset{\sim}{A}}(\underset{\sim}{g})} Z_\Lambda(\underset{\sim}{c}, G; \hat{\underset{\sim}{A}}_\varkappa) \quad . \tag{5.10}$$

For $\underset{\sim}{g} = \delta\underset{\sim}{h}$, see (4.24) , $\hat{\underset{\sim}{A}}(\delta\underset{\sim}{h}) = \hat{B}(\underset{\sim}{h}) = B(\underset{\sim}{h})$, and $(g, \partial F \partial^* g) = 0$.

Let C_u be the "unitary" covariance given by

$$C_u^{-1} = (\mathrm{curl})^*\ \mathrm{curl} + m_A^2\ ,\ m_A > 0 \quad , \tag{5.11}$$

see (G) . Let F be some quadratic form on $\ell^2(a\mathbb{Z}^\nu_{1/2})$ such that

$$C_F = C_u + \partial F \partial^* \quad \text{is non-negative .} \tag{5.12}$$

By writing F as $F_1 - F_2$, with $F_1 \geqq 0$ and $F_2 \geqq 0$, and applying (5.10) twice, for F_1 in one and for F_2 in the other direction - with suitable choices for C and $\hat{C}$ - we find

Theorem 5.1 : If $G_{\bar{\Lambda}}$ is gauge-invariant

$$\int d\mu_{C_n}(\underset{\sim}{A})e^{\underset{\sim}{A}(\underset{\sim}{g})} Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{A}_{\varkappa})$$
$$= \int e^{-(1/2e^2)(g,\partial F\partial^* g)} \int d\mu_{C_F}(\underset{\sim}{A})e^{\underset{\sim}{A}(\underset{\sim}{g})} Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{A}_{\varkappa}) \quad , \tag{5.13}$$

in particular

$$\int d\mu_{C_u}(\underset{\sim}{A})e^{B(\underset{\sim}{h})} Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{A}_{\varkappa}) = \int d\mu_{C_F}(\underset{\sim}{A})e^{B(\underset{\sim}{h})} Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{A}_{\varkappa}) \quad . \tag{5.14}$$

Remarks :

(1) (5.13) shows that, in a gauge which deviates from the "unitary" gauge by a change in the covariance of $\underset{\sim}{A}$, the "ghost degrees of freedom" of $\underset{\sim}{A}$ have Gaussian distribution and decouple.

(2) From (5.14) we find, using an interpolating covariance, $C_s = sC_u + (1-s)C_F$, for which (5.14) remains clearly true, by differentiation in s

$$(C_u - C_F)_{\mu\nu}(x,y)\frac{\delta}{\delta A_\mu(x)}\frac{\delta}{\delta A_\nu(y)} \; Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{A}_{\varkappa}) = 0 \quad .$$

This follows also from the infinitesimal form of equation (5.5) :

$$\partial_\mu \frac{\delta}{\delta A_\mu(x)} Z_{\Lambda}(\underset{\sim}{c},G;\underset{\sim}{A}_{\varkappa}) = 0 \quad . \tag{5.15}$$

This identity is a summed up version of the Ward identities. (It is not hard to see that the usual Ward identities, in the form valid on the lattice, can be recovered from (5.15) to all orders in e^2) .

(3) Identity (5.8) and Theorem 5.1 permit us to choose a gauge for $\underset{\sim}{A}$ adapted to the problem under consideration. In the construction of the limit $a \downarrow 0$, in two dimensions, and the existence proof of the continuum Higgs model in two space-time dimensions, see II, we choose the gauge with covariance C given, in momentum space, by

$$C_{\mu\nu}(k) = (\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2+m_A^2})(k^2 + m_A^2)^{-1} \quad . \tag{5.16}$$

The Green's functions of all gauge-invariant fields computed in the gauge given by (5.16) are <u>identical</u> to the ones in the "unitary" gauge. This, of course, is shown by proving the convergence of the lattice approximation in the gauge determined by (5.16) (corresponding to a particular choice of F in (5.12)) and applying Theorem 5.1. The covariance (5.16) has good power counting properties and is therefore convenient to prove that the limit $\varkappa \longrightarrow \infty$ (removal of ultraviolet cutoff) exists in the continuum limit. It also permits us to discuss the limit $m_A \downarrow 0$ (using Corollary 4.2, for $a > 0$) which is painful when using the "unitary" covariance.

5.2 <u>Osterwalder-Schrader positivity</u>

In accordance with our parametrization of the general gauge field $\underset{\sim}{g}$ by an auxiliary field $\underset{\sim}{w}$ in Section 2 of Part 1 we introduce a field $\underset{\sim}{\alpha}$:

$$\underset{\sim}{\alpha} = \{\alpha_{x,b}\}_{x \in a\mathbb{Z}^\nu_{1/2},\, b \in \mathcal{B}_x} \quad , \text{ (see Section 2, c) ,}$$

(2.14)-(2.15)). This is a real-valued random field in terms of which the gauge A_{xy} is given by

$$A_{xy} = \frac{1}{a}\,(\alpha_{y,<y,x>} - \alpha_{x,<x,y>}) \quad . \tag{5.17}$$

The gauge covariance properties of $\underset{\sim}{\alpha}$ are given by

$$\alpha_{x,<x,y>} \longrightarrow \alpha^{\chi}_{x,<x,y>} = \alpha_{x,<x,y>} - \frac{1}{e}\chi_x \quad . \tag{5.18}$$

Note that (5.17)-(5.18) are consistent with (4.4)-(4.5). Let π be a hyperplane (or a pair, in case of periodic boundary conditions) lying in between lattice planes. E.g. $\pi = \{x : x^o = 0\}$. Let r denote reflection at π , and

$$\left.\begin{array}{ll} \theta_* \alpha_{x,<x,y>} & = \alpha_{rx,<rx,ry>} \\ \theta[F(\underset{\sim}{\alpha})] & = \overline{F(\theta_* \underset{\sim}{\alpha})} \end{array}\right\} \tag{5.19}$$

<u>Lemma 5.2</u> :

Let $F(A_{xy})$ be a function of positive type in A_{xy} . Then

(1) $\qquad \theta[F(A_{xy})] = \overline{F(A_{rxry})} = F(-A_{rxry})$, and

(2) $\qquad$ for $y = rx$,

$$\begin{aligned} F(A_{xy}) &= F(\alpha_{y,<x,y>} - \alpha_{x,<x,y>}) \\ &= \int d\varphi(p) e^{-ip\alpha_{x,<x,y>}} e^{ip\theta_* \alpha_{x,<x,y>}} \\ &= \int d\varphi(p) e^{-ip\alpha_{x,<x,y>}} \theta[e^{-ip\alpha_{x,<x,y>}}] \quad , \end{aligned}$$

for some non-negative measure $d\varphi(p)$.

<u>Proof</u> : Since F is of positive type, it is of the form

$$F(x) = \int d\varphi(p) e^{ipx} \quad , \tag{5.20}$$

for some measure $d\varphi \geqq 0$. Hence

$$\begin{aligned} \theta[F(A_{xy})] &= \int d\varphi(p) e^{-ipA_{rxry}} = \overline{\int d\varphi(p) e^{ipA_{rxry}}} \\ &= F(-A_{rxry}) = \overline{F(A_{rxry})} \quad ; \end{aligned}$$

(2) follows from (5.19) and (5.20) .

Q.E.D.

Let Λ be a bounded subset of $a\mathbb{Z}^{\nu}_{1/2}$ symmetric with respect to π, i.e. $\Lambda = \Lambda_+ \cup \Lambda_- = \Lambda_+ \cup r\Lambda_+$. Let $\bar{G}_{\Lambda_{(+)}}$ be the algebra of all bounded, continuous functions of $\underset{\sim}{\phi}_{(+)} = \{\phi(x)\}_{x \in \Lambda_{(+)}}$ and $\underset{\sim}{\alpha}_{(+)} = \{\alpha_{x,b}\}_{x \in \Lambda_{(+)}, b \in \mathcal{B}_x}$. Let $\vec{e}_o$ be the lattice vector perpendicular to π. Let the ultraviolet cutoff functions h_κ be of the form

$$h_\kappa(x - x.\vec{e}_o)\delta_{o,x.\vec{e}_o}) \quad \text{for all } \kappa \quad , \tag{5.21}$$

i.e. there is no cutoff in the direction perpendicular to π. Suppose the a priori measure $d\rho_x(r)$, $r_x = |\phi(x)|$, is independent of x and assume that the boundary condition $G_{\partial\Lambda}$ is reflection positive [18b,45] in the sense that

$$< F\, \theta[F] G_{\partial\Lambda} >_o \geqq 0 \quad , \text{ for all } F \in \bar{G}_{\Lambda_+} \quad , \tag{5.22}$$

where $< - >_o$ is the uncorrelated expectation defined as in (2.21). Set

$$A^{\kappa}_{\Lambda} = A^{YM}_{\Lambda, a} + A^{M}_{\Lambda}(\underset{\sim}{c};\underset{\sim}{\phi};\underset{\sim}{A}_\kappa) \quad ,$$

with $A^{YM}_{\Lambda, a}$ defined as in (W) or (P) (see (4.8), (4.10), resp.) and A_Λ as in (4.14) with

$$\underset{\sim}{c} = \{c_{xy}\} = a^{\nu-2} \times \text{ constant} \quad .$$

Let

$$< F >_{A^{\kappa}_{\Lambda}, G_{\partial\Lambda}} = < e^{-A^{\kappa}_{\Lambda}} G_{\partial\Lambda} >_o^{-1} < F\, e^{-A^{\kappa}_{\Lambda}} G_{\partial\Lambda} >_o \quad , \tag{5.23}$$

for arbitrary $F \in \bar{G}_\Lambda$.

In case (G) (see (4.11)) we define

$$< - >_{A^{\varkappa}_{\Lambda}, G_{\partial\Lambda}} = \Big[\int d\mu_{C_u}(\underset{\sim}{A}) \; Z_{\Lambda}(\underset{\sim}{c}, G_{\partial\Lambda}; \underset{\sim}{A}_{\varkappa}) \Big]^{-1}$$

$$\times \int d\mu_{C_u}(\underset{\sim}{A}) \; o < o \; e^{-A^{M}_{\Lambda}(\underset{\sim}{c}; \underset{\sim}{\phi}; \underset{\sim}{A}_{\varkappa})} \; G_{\partial\Lambda} >^{M}_{o} \quad . \tag{5.24}$$

From Lemma 2.2 and Lemma 5.2 we now obtain, by the arguments already used in the proof of Theorem 2.8 ,

<u>Theorem 5.3</u> : (Osterwalder-Schrader positivity).

In all cases, (W), (P) and (G), and under the hypotheses stated above,

$$< F \; \theta[F] >_{A^{\varkappa}_{\Lambda}, G_{\partial\Lambda}} \geqq 0 \quad , \quad \text{for all} \quad F \in \bar{G}_{\Lambda_+} \quad .$$

<u>Remark</u> : We emphasize that property (5.21) of the ultraviolet cutoffs $h_{\varkappa}$ is essential. (If it were violated Lemma 5.2 would not be applicable, and Theorem 5.3 would be false!).

Let $G^{\text{inv.}}_{\Lambda_{(+)}}$ be the <u>gauge-invariant</u> subalgebra of $\bar{G}_{\Lambda_{(+)}}$. Combining Theorem 5.3 with (5.8), resp. Theorem 5.1 we obtain

<u>Corollary 5.4</u> :

Under the same hypotheses as in Theorem 5.3, the inequality

$$< F \; \theta[F] >_{A^{\varkappa}_{\Lambda}, G_{\partial\Lambda}} \geqq 0 \quad , \quad \text{for all} \quad F \in G^{\text{inv.}}_{\Lambda_+} \quad ,$$

holds in <u>every gauge</u>, for all $\varkappa \leqq \infty$, for all $a > 0$.

Remarks :

(1) If there are no Fermi fields in a theory and the action has only nearest neighbor couplings (of the general form studied in [18a]), in particular for the models studied here, Theorem 5.3 and Corollary 5.4 are also true for the case where π is a lattice plane. (A proof can be found in [18a], in a different context). In this case one can define time 0-fields and a "Schrödinger representation". (Moreover, the lattice theory has the "Markov property", even in the thermodynamic limit).

(2) The standard boundary conditions (periodic, half-Dirichlet,...) correspond to a $G_{\partial\Lambda}$ which is RP, in the sense of inequality (5.22).

(3) Corollary 5.4 is a basic tool for proving Osterwalder-Schrader positivity of the gauge-invariant Euclidean Green's functions of the continuum Higgs model in two space-time dimensions (in the limits $\varkappa \longrightarrow \infty$, $\Lambda \nearrow \mathbb{R}^2$). See II.

5.3 Gauge-dependent Green's functions

There seems to be some amount of confusion about gauge-variant Green's functions in lattice gauge theories and about the rôle of the "global symmetry group" associated with a gauge group of the second kind which might justify the following comments.

Three facts, not too surprising in the light of the previous sections, are noteworthy ; (we do not claim to be original here) :

(1) A gauge group of the second kind is not (or only accidentally) associated with a physical, global symmetry group : If one adopts the point of view that all the physics of a gauge theory can be extracted, in principle, from its gauge-invariant

Green's functions, this fact is obvious. More significantly, in a gauge theory with matter fields (e.g. a Georgi-Glashow model) particles do, in general, not form G -multiplets (G is the gauge group). This remains true in theories with instantons : Such theories have in general particle multiplets which are not classified by the representations of G , (the instanton corrections to the mass spectrum in a model like the Georgi-Glashow model are small), and the instantons "restore the symmetry " only in the sense that an unphysical (gauge-dependent) order parameter vanishes which was formerly believed to be non-zero.
These statements can be tested rigorously, in principle , for lattice gauge theories. We shall further discuss them in III, at least in the abelian case.

(2) If, in a lattice gauge theory, no gauge is fixed (or, equivalently, one integrates over the group of all local gauge transformations) then, of course, the expectation of a gauge-dependent observable (in the thermodynamic limit, and for arbitrary b.c.) is equal to its average over the group of all local gauge transformations, in particular the usual gauge-variant Green's functions vanish ; see also Guerra et al. [11].

On the other hand we know from Section 5.1 (this remain true in the non-abelian theories, [17]) that one can fix many different gauges, by adding terms to the action of the gauge field (recall the measure $d\nu(\chi)$). Once a gauge is fixed and the integration over the group of local gauge transformations has thereby been reduced, gauge-variant Green's functions do in general not vanish, at all!

For example, let the action of the gauge field be defined as in (W) or (P) and choose a suitable gauge-fixing measure $d\nu(\chi)$ - see Section 5.1 . Let $\langle - \rangle_{\mu,\nu,G_{\partial\Lambda}}$ denote the corresponding interacting expectation with boundary condition $G_{\partial\Lambda}$. Then, for suitable subsets X , Y of Λ with $|X| = |Y|$,

$$< \prod_{x \in X \subseteq \Lambda} \phi^*(x) \prod_{y \in Y \subseteq \Lambda} \phi(y) >_{\mu,\nu,G_{\partial\Lambda}} \neq 0 \quad .$$

This is quite easy to check e.g. in the axial gauge obtained by the following gauge transformation, $\underset{\sim}{\chi}$:

$$\chi_y - \chi_x = eA_{xy} \quad , \quad \text{with} \quad y = x + \vec{e}_o$$

$$d\nu(\chi) = \prod_{x \in \Lambda} \delta_o(\chi_x) d\chi_x \quad .$$

Then one can use a cluster expansion to show that gauge-variant Green's functions of the type defined above are non-vanishing, for suitable coupling constants.

It is of interest to consider non gauge-invariant boundary conditions $G_{\partial\Lambda}$. For example, $G_{\partial\Lambda}$ may specify the field configuration $\underset{\sim}{\phi}$ on $\partial\Lambda$ as follows :

$$\phi(x) = \phi_o \quad , \quad x \in \partial\Lambda \quad , \tag{5.25}$$

where ϕ_o is a positive number.

After a gauge has been fixed it is a <u>meaningful</u> problem to analyse the unphysical (obviously gauge-dependent) order parameter $< \phi(o) >_{\mu,\nu,G_{\partial\Lambda}}$. In fact, it is most interesting to analyze whether $\lim_{\Lambda \uparrow a\mathbb{Z}^\nu_{1/2}} < \phi(o) >_{\mu,\nu,G_{\partial\Lambda}}$, with $G_{\partial\Lambda}$ as in (5.25), vanishes or not, (depending on the gauge chosen by ν).

Suppose, the a priori distribution $d\rho(r)$ for the Higgs field is strongly peaked at $r = \phi_o > 0$, the boundary condition is as in (5.25) and the gauge field is abelian. In dimension $\nu = 2$

$$\lim_{\Lambda \uparrow a\mathbb{Z}^\nu_{1/2}} < \phi(o) >_{\mu,\nu,G_{\partial\Lambda}} = 0 \quad , \tag{5.26}$$

in any gauge [53], but, for three or more dimensions, we conjecture that

$$\lim_{\Lambda \uparrow a\mathbb{Z}^{\nu}_{1/2}} < \phi(o) >_{\mu,\nu,G_{\partial\Lambda}} \neq 0 \quad , \quad \simeq \phi_o \quad , \tag{5.27}$$

<u>for a suitable gauge</u> ν , (with indefinite metric). This conjecture is related to the fact that the three or more dimensional continuum abelian Higgs models have <u>no</u> instantons, whereas the two dimensional model has ; (the Nielsen-Olesen vortices).

Using a cluster expansion (see [17],III) one can prove, on the other hand, that when $d\rho$ is strongly peaked at $r = 0$

$$\lim_{\Lambda \uparrow a\mathbb{Z}^{\nu}_{1/2}} < \phi(o) >_{\mu,\nu,G_{\partial\Lambda}} = 0 \quad , \tag{5.28}$$

for "all" b.c. $G_{\partial\Lambda}$ and in an arbitrary gauge. Hence, a proof of (5.27) might be suggestive of the existence of a Higgs $\longrightarrow$ scalar Q.E.D. phase transition (at least in dimension 4 or more).

(3) It should be pointed out that there is no contradiction between (5.26) and the fact that, in dimension 2 or more, the photon is <u>massive</u> and there is a standard Higgs mechanism in some region of the coupling constant space, [17]. Moreover, the fact that fractionally charged quarks are <u>confined</u> (see Theorem E, Section 1.2) does <u>not at all</u> imply that the photon is massless or that the usual Higgs mechanism breaks down! See also III and [28] .

6. Some new correlation inequalities for abelian Higgs models.

In this section we prove new correlation inequalities of the Ginibre type [33] which we apply to prove bounds, monotonicity properties and the existence of the thermodynamic limit for gauge-variant and - invariant Green's functions. With a view on our applications to the continuum Higgs model in two space-time dimensions presented in paper II we consider the action (G) for the vector potential and, as in Section 5, we impose an arbitrary ultraviolet cutoff on the gauge field in the matter action, defined in (4.6), (4.14).

Our inequalities are a synthesis of the ones of Ginibre [33] and their extensions proven in [34,35].

For the action (W) Guerra et al. [11] have pointed out that the Ginibre inequalities apply and yield the existence of the thermodynamic limit. For the action (P) correlation inequalities and applications to e.g. proving a confinement bound for the U(1) theory are due to [53], based on [35].

In this paper we explain the basic method and prove the most important inequalities. Extensions and applications appear in papers II and III.

6.1 Definitions and the main inequality

We consider an arbitrary finite lattice Λ , e.g. a bounded subset of $a\mathbb{Z}^{\nu}_{1/2}$. We use polar coordinates for the Higgs field $\underset{\sim}{\phi}$:

$$\phi(x) = r_x e^{i\theta_x}, \theta_x \in [-2\pi, 2\pi), r_x \in \mathbb{R}_+ \quad , \tag{6.1}$$

for all $x \in \Lambda$. Note that the angles θ_x vary over twice the circle with distribution given by the Lebesgue measure on $[-2\pi,2\pi)$; the distribution of r_x is given by a finite measure $d\tilde{\rho}_x(r_x)$, as in Section 4.1. We set $\underset{\sim}{r} = \{r_x\}_{x \in \Lambda}$, $\underset{\sim}{\theta} = \{\theta_x\}_{x \in \Lambda}$.

The matter action is still defined as in (4.14), i.e.

$$A^M_\Lambda(\underset{\sim}{c};\underset{\sim}{\phi};\underset{\sim}{\bar{A}}) = - \sum_{<x,y>\subset\Lambda} c_{xy} r_x r_y \cos(-\theta_x + \bar{A}_{xy} + \theta_y) \quad , \tag{6.2}$$

where $<x,y>$ is an arbitrary pair of neighboring sites in Λ , $c_{xy} \geqq 0$, $\bar{A}_{xy} = ea\, A_{xy,\varkappa}$, with $\varkappa$ an arbitrary ultraviolet cutoff ; see Section 5 , (5.3).

Furthermore, $d\mu_C(\underset{\sim}{A})$ is an arbitrary Gaussian measure with mean 0 and covariance $C \geqq 0$.

The total expectation is given by

$$< F > \equiv < F >_{A^M_\Lambda, G_{\partial\Lambda}} \quad , \; F \in \bar{G}_\Lambda \tag{6.3}$$

where $< - >_{A^M_\Lambda, G_{\partial\Lambda}}$ is defined in (5.24).

For convenience we choose

$$G_{\partial\Lambda} = 1 \quad , \tag{6.4}$$

i.e. we impose Dirichlet boundary conditions on $\underset{\sim}{\phi}$, but more general b.c. can be accomodated in Theorem 6.1, below ; e.g. periodic b.c. when Λ is a rectangular region in $a\mathbb{Z}^\nu_{1/2}$.

Let $\underset{\sim}{x}$ be a collection of real variables. We define $C_{\underset{\sim}{x}}$ to be the multiplicative, positive cone of all polynomials in $\underset{\sim}{x}$ with positive coefficients. Let $n = \{n_x\}_{x\in\Lambda}, m, \ldots$ be functions on Λ with values in the integers, and $f = \{f_{xy}\}_{<x,y>\subset\Lambda}$, $g, \ldots$ real-valued functions on pairs of neighboring sites in Λ . We define

$$n.\theta + f.\bar{A} = \sum_x n_x \theta_x + \sum_{<x,y>\subset\Lambda} f_{xy} \bar{A}_{xy} \quad .$$

For F and G in $\bar{G}_\Lambda$ we set

$$< F;G > = < FG > - < F > < G > \quad .$$

Our main inequality is

Theorem 6.1 :

For all P and Q in $C_{\underset{\sim}{r}}$ and n,m,f and g as above,

$$< P(\underset{\sim}{r})\cos(n.\theta + f.\bar{A}) \; ; \; Q(\underset{\sim}{r}) \cos(m.\theta + g.\bar{A}) > \geqq 0 \quad .$$

Remark :

The trigonometric identity

$$\prod_{j=1}^{k} \cos \alpha_j = (\tfrac{1}{2})^k \sum_{\{e_j\}} \cos(\sum_{j=1}^{k} e_j\alpha_j) \quad , \tag{6.5}$$

with $e_j = \pm 1$, $j = 1,2,\ldots,k$, yields obvious generalizations of the inequality in Theorem 6.1.

6.2 Proof of Theorem 6.1

Following [33] we introduce identically distributed duplicate variables $\underset{\sim}{\bar{A}},\underset{\sim}{\bar{A}}',\underset{\sim}{r},\underset{\sim}{r}',\underset{\sim}{\theta}$ and $\underset{\sim}{\theta}'$ with expectations $< - >$ and $< - >' = < - >$. The product expectation $<-> \otimes <->'$ is henceforth also denoted $<->$. We then define new random variables $\underset{\sim}{\alpha},\underset{\sim}{\beta},\underset{\sim}{\rho},\underset{\sim}{\lambda},\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\delta}$ as follows :

$$\begin{aligned} A_{xy} &= \alpha_{xy} - \beta_{xy} \; , \; (\bar{A}_{xy} = \bar{\alpha}_{xy} - \bar{\beta}_{xy}) \\ A'_{xy} &= \alpha_{xy} + \beta_{xy} \; , \; (\bar{A}'_{xy} = \bar{\alpha}_{xy} + \bar{\beta}_{xy}) \quad , \end{aligned} \tag{6.6}$$

and the linear maps $\alpha \longrightarrow \bar{\alpha}$, $\beta \longrightarrow \bar{\beta}$ are identical to the one taking A to $\bar{A}$ and A' to $\bar{A}'$. The transformations (6.6) are the composition of an orthogonal transformation and a dilation by a factor $1/\sqrt{2}$, so that

$$d\mu_C(\underset{\sim}{A})\, d\mu_C(\underset{\sim}{A}') = d\mu_{1/2C}(\underset{\sim}{\alpha})\, d\mu_{1/2C}(\underset{\sim}{\beta}) \quad . \tag{6.7}$$

Furthermore

$$r_x = \rho_x + \lambda_x \quad , \quad r_x' = \rho_x - \lambda_x \quad , \tag{6.8}$$

so that

$$\rho_x = \frac{1}{2}(r_x + r_x') \geqq 0 \quad . \tag{6.9}$$

Finally

$$\theta_x = \epsilon_x - \delta_x \quad , \quad \theta_x' = \epsilon_x + \delta_x \quad . \tag{6.10}$$

All functions ("observables") in $\bar{G}_\Lambda$, $\bar{G}_\Lambda'$ are periodic in θ_x , resp. θ_x' with period 2π . Therefore

$$\begin{aligned} &\frac{1}{2}\, \chi_{[-2\pi,2\pi)}(\theta_x)\chi_{[-2\pi,2\pi)}(\theta_x')d\theta_x d\theta_x' \\ &\qquad = \chi_{[-\pi,\pi)}(\epsilon_x)\chi_{[-\pi,\pi)}(\delta_x)d\epsilon_x d\delta_x \quad , \end{aligned} \tag{6.11}$$

as conditional expectations on $\bar{G}_\Lambda \otimes \bar{G}_\Lambda'$ [33] (i.e. when restricted to functions of θ_x, θ_x' with period 2π).

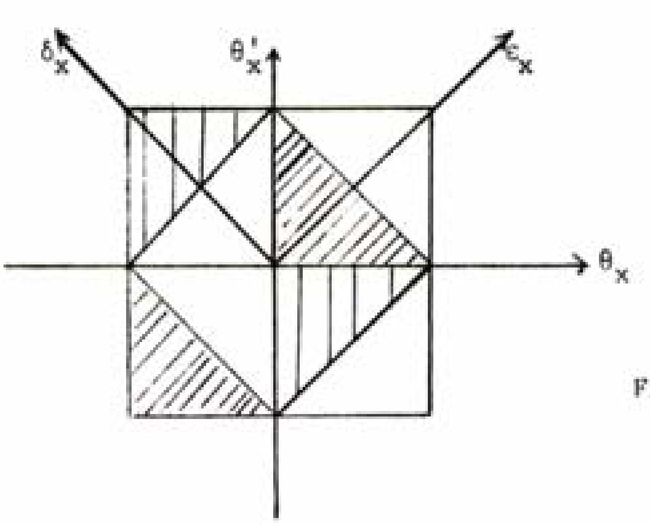


Fig. 4

We also set

$$\chi_{xy} = -\epsilon_x + \bar{\alpha}_{xy} + \epsilon_y \ , \ \Psi_{xy} = -\delta_x + \overline{\beta_{xy}} + \delta_y \quad .$$

Next, we summarize some basic identities.

$$r_x r_y = a_{xy} + b_{xy} \quad , \text{ where}$$

$$a_{xy} = \rho_x \rho_y + \lambda_x \lambda_y \ , \ b_{xy} = \rho_x \lambda_y + \rho_y \lambda_x \quad , \tag{6.12}$$

and

$$r'_x \ r'_y = a_{xy} - b_{xy} \quad . \tag{6.13}$$

Using (6.12), (6.13), (6.8), (6.10) and the identity

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

we find

$$r_x r_y \cos(-\theta_x + \bar{A}_{xy} + \theta_y) + r'_x r'_y \cos(-\theta'_x + \bar{A}'_{xy} + \theta'_y)$$

$$= 2a_{xy} \cos \chi_{xy} \cos \Psi_{xy} + 2b_{xy} \sin \chi_{xy} \sin \Psi_{xy} \quad . \tag{6.14}$$

Hence

$$\exp A^M_\Lambda(\underset{\sim}{c};\underset{\sim}{\Phi};\underset{\sim}{\bar{A}}) \exp A^M_\Lambda(\underset{\sim}{c};\underset{\sim}{\Phi}',\underset{\sim}{\bar{A}}')$$

$$= \prod_{\langle x,y \rangle \subset \Lambda} \exp 2c_{xy}\left[a_{xy} \cos \chi_{xy} \cos \Psi_{xy} + b_{xy} \sin \chi_{xy} \sin \Psi_{xy}\right] \ . \tag{6.15}$$

For later use we note

$$r_x r_y \cos(-\theta_x + \bar{A}_{xy} + \theta_y) - r'_x r'_y \cos(-\theta'_x + \bar{A}'_{xy} + \theta'_y)$$

$$= 2b_{xy} \cos \chi_{xy} \cos \Psi_{xy} + 2a_{xy} \sin \chi_{xy} \sin \Psi_{xy} \quad . \tag{6.16}$$

Finally we have the general identities

$$P(\underset{\sim}{r})\cos(n.\theta + f.\bar{A})$$

$$= \tilde{P}(\underset{\sim}{\rho},\underset{\sim}{\lambda})\big[\cos(n.\varepsilon + f.\bar{\alpha})\cos(n.\delta + f.\bar{\beta}) \tag{6.17}$$

$$+ \sin(n.\varepsilon + f.\bar{\alpha})\sin(n.\delta + f.\bar{\beta})\big] \quad ,$$

with $\tilde{P} \in C_{\underset{\sim}{\rho},\underset{\sim}{\lambda}}$, and

$$P(\underset{\sim}{r})\cos(n.\theta + f.\bar{A}) - P(\underset{\sim}{r}')\cos(n.\theta' + f.\bar{A}')$$

$$= P_o(\underset{\sim}{\rho},\underset{\sim}{\lambda})\cos(n.\varepsilon + f.\bar{\alpha})\cos(n.\delta + f.\bar{\beta}) \tag{6.18}$$

$$+ P_e(\underset{\sim}{\rho},\underset{\sim}{\lambda})\sin(n.\varepsilon + f.\bar{\alpha})\sin(n.\delta + f.\bar{\beta}) \quad ,$$

where $P_o \in C_{\underset{\sim}{\rho},\underset{\sim}{\lambda}}$ is odd in $\underset{\sim}{\lambda}$, and $P_e \in C_{\underset{\sim}{\rho},\underset{\sim}{\lambda}}$ is even in $\underset{\sim}{\lambda}$; (6.16) is an explicit special case of (6.18). Hence, using (6.7), (6.11) and (6.15) to rewrite $< — > \otimes < — >'$ and then applying (6.17), (6.18) and identity (6.5) we obtain

$$< P(\underset{\sim}{r})\cos(n.\theta + f.\bar{A})\ ;\ Q(\underset{\sim}{r})\cos(n.\theta + g.\bar{A}) >$$

$$= < P(\underset{\sim}{r})\cos(n.\theta + f.\bar{A})\big[Q(\underset{\sim}{r})\cos(m.\theta + g.\bar{A})$$

$$- Q(\underset{\sim}{r}')\cos(m.\theta' - g.\bar{A}')\big] >$$

$$= Z^{-2}\sum_{\underset{\sim}{n}',\underset{\sim}{f}',\underset{\sim}{g}'}\int\prod_{x\in\Lambda}\big[\ d\tilde{\rho}_x(r_x)\,d\tilde{\rho}_x(r'_x)\big]\ d\mu_{1/2C}(\underset{\sim}{\alpha})\prod_{x\in\Lambda}\Big[\int_{-\pi}^{\pi}d\epsilon_x\Big]$$

$$\times\ d\mu_{1/2C}(\underset{\sim}{\beta})\prod_{x\in\Lambda}\Big[\int_{-\pi}^{\pi}d\delta_x\Big]\prod_{<x,y>\subset\Lambda}\Big[e^{2c_{xy}a_{xy}\cos\chi_{xy}\cos\Psi_{xy}}$$

$$\times\ e^{2c_{xy}b_{xy}\sin\chi_{xy}\sin\Psi_{xy}}\Big]\,P_{\underset{\sim}{n}',\underset{\sim}{f}',\underset{\sim}{\sigma}'}(\underset{\sim}{\rho},\underset{\sim}{\lambda})$$

$$\times\Big[\sum_j\cos(n'_j.\varepsilon + f'_j.\bar{\alpha} - \sigma'_j\,\tfrac{\pi}{2})\Big]\Big[\sum_j\cos(n'_j.\delta + f'_j.\bar{\beta} - \sigma'_j\,\tfrac{\pi}{2})\Big] \quad , \tag{6.19}$$

where we have written $\sin\alpha$ as $\cos(\alpha - \frac{\pi}{2})$, $Z > 0$ is the partition function, $P_{\underset{\sim}{n}',\underset{\sim}{f}',\underset{\sim}{\sigma}'} \in C_{\underset{\sim}{\rho},\underset{\sim}{\lambda}}$ for all $\underset{\sim}{n}',\underset{\sim}{f}',\underset{\sim}{\sigma}'$ (here we use that $C_{\underset{\sim}{\rho},\underset{\sim}{\lambda}}$ is multiplicative), and $\sigma_j' \in \{0,1\}$, for all j.

Now we expand the exponentials. If we apply identity (6.5) to each term in this expansion and recall that $c_{xy} \geqq 0$, for all $< x,y >$, we see, using the multiplicativity of $C_{\underset{\sim}{\rho},\underset{\sim}{\lambda}}$ again, that there results a sum of terms of the form

$$P_{\underset{\sim}{n}'',\underset{\sim}{f}'',\underset{\sim}{\sigma}''}(\underset{\sim}{\rho},\underset{\sim}{\lambda})\Big[\sum_j \cos(n_j''.\epsilon + f_j''.\bar{\alpha} - \sigma_j'' \tfrac{\pi}{2})\Big] \times \Big[\sum_j \cos(n_j''.\delta + f_j''.\bar{\beta} - \sigma_j'' \tfrac{\pi}{2})\Big] \quad , \tag{6.20}$$

with $P_{\underset{\sim}{n}'',\underset{\sim}{f}'',\underset{\sim}{\sigma}''} \in C_{\underset{\sim}{\rho},\underset{\sim}{\lambda}}$, for all $\underset{\sim}{n}'',\underset{\sim}{f}''$ and $\underset{\sim}{\sigma}''$. Next we integrate (6.20) over $\underset{\sim}{\alpha},\underset{\sim}{\epsilon}$ and $\underset{\sim}{\beta},\underset{\sim}{\delta}$ with the measure written out in (6.19). Clearly, this integration yields

$$\hat{P} = K_{\underset{\sim}{n}'',\underset{\sim}{f}'',\underset{\sim}{\sigma}''}\; P_{\underset{\sim}{n}'',\underset{\sim}{f}'',\underset{\sim}{\sigma}''} \quad ,$$

with $K_{\underset{\sim}{n}'',\underset{\sim}{f}'',\underset{\sim}{\sigma}''} \geqq 0$, i.e.

$$\hat{P} \in C_{\underset{\sim}{\rho},\underset{\sim}{\lambda}} \quad . \tag{6.21}$$

In order to determine the sign of the integral

$$\int \prod_{x \in \Lambda} d\tilde{\rho}_x(r_x)\, d\tilde{\rho}_x(r_x')\, \hat{P}(\underset{\sim}{\rho},\underset{\sim}{\lambda}) \quad ,$$

we note that the measure $d\mu_x(\rho_x,\lambda_x) \equiv d\tilde{\rho}_x(r_x)\, d\tilde{\rho}_x(r_x')$ is invariant under the substitution

$$r_x \longrightarrow r_x' \,,\; r_x' \longrightarrow r_x \quad , \text{ i.e.}$$

$$\rho_x \longrightarrow \rho_x \,,\; \lambda_x \longrightarrow -\lambda_x \quad , \text{ (see (6.8))} ,$$

for all $x \in \Lambda$. Therefore

$$\int_{x} \prod_{\in \Lambda} d\mu_x(\rho_x,\lambda_x)\hat{P}\,(\underset{\sim}{\rho},\underset{\sim}{\lambda}) \\ = \int \prod_{x \in \Lambda} d\mu_x(\rho_x,\lambda_x)\hat{P}_e(\underset{\sim}{\rho},\underset{\sim}{\lambda}) \quad , \tag{6.22}$$

where $\hat{P}_e$ is the part of $\hat{P}$ that is <u>even</u> in λ_x , for all $x \in \Lambda$. Clearly $\hat{P}_e \in C_{\underset{\sim}{\rho},\underset{\sim}{\lambda}}$. Hence

$$\hat{P}_e(\rho,\lambda) \geqq 0 \quad , \tag{6.23}$$

since $\rho_x \geqq 0$, for all $x \in \Lambda$, (see (6.9)) .

But (6.19)-(6.23) complete the proof of Theorem 6.1 .

Q.E.D.

6.3 <u>Further inequalities</u>

The method presented in the last section and variations can clearly be used to prove many more correlation inequalities ; see also [33,54]. Here we report some important ones (without attempting an exhaustive list). Let

$$r^2(g) = \sum_{x \in \Lambda} r_x^2 g_x \ , \ A(f) = \sum_{<x,y> \subset \Lambda} A_{xy} f_{xy} \quad .$$

<u>Theorem 6.2 </u>:

For $g \geqq 0$, f real and h an arbitrary complex-valued function,

(1) $\quad < e^{r^2(g)} e^{iA(f)} ; |A(h)|^2 > \leqq 0$

(2) $\quad < e^{r^2(g)} e^{iA(f)} ; r_x r_y \cos(-\theta_x + \bar{A}_{xy} + \theta_y) > \geqq 0$

(3) $\quad < e^{-r^2(g)} e^{A(f)} ; |A(h)|^2 > \geqq 0$

(4) $\quad < e^{-r^2(g)} e^{A(f)} ; r_x r_y \cos(-\theta_x + \bar{A}_{xy} + \theta_y) > \leqq 0 \quad .$

<u>Proof</u> : Inequalities (1) and (2) are immediate consequences of Theorem 6.1 : First, $e^{r^2(g)}$ is a limit of polynomials in $C_{\underset{\sim}{r}}$. Second, one can replace $e^{iA(f)}$ by $\cos(A(f))$, since the expectation $\langle - \rangle$, $|A(h)|^2$ and $r_x r_y \cos(-\theta_x + \bar{A}_{xy} + \theta_y)$ are invariant under the substitution $\underset{\sim}{A} \longrightarrow -\underset{\sim}{A}$, $\underset{\sim}{\theta} \longrightarrow -\underset{\sim}{\theta}$. Finally, for real h ,

$$A(h)^2 = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon^2}(1-\cos \varepsilon A(h)) \quad , \tag{6.24}$$

and for $h = h_1 + ih_2$,

$$|A(h)|^2 = A(h_1)^2 + A(h_2)^2 \quad . \tag{6.25}$$

We also recall identity (6.16) which one applies in the proof of (2).

The proofs of (3) and (4) are constructed by repeating the arguments given in the proof of Theorem 6.1, but exchanging the roles of $\underset{\sim}{r}$ and $\underset{\sim}{r}'$ $(r_x = \rho_x - \lambda_x$, $r'_x = \rho_x + \lambda_x)$, $\underset{\sim}{\theta}$ and $\underset{\sim}{\theta}'$, $\underset{\sim}{\bar{A}}$ and $\underset{\sim}{\bar{A}}'$ and noticing, furthermore, that

(a) $\qquad e^{-r^2(g)} = e^{-\rho^2} e^{-\lambda^2(g)} e^{(\rho.\lambda)(g)}$, where $e^{(\rho.\lambda)(g)}$ is a limit of polynomials in $C_{\underset{\sim}{\rho},\underset{\sim}{\lambda}}$,

and

(b) $\qquad$ the measure $d\mu_x^g$, defined by

$$d\mu_x^g(\rho_x,\lambda_x) = e^{-\rho_x^2 g_x} e^{-\lambda_x^2 g_x} d\widetilde{\rho}_x(r_x) d\widetilde{\rho}_x(r'_x)$$

is invariant under $\rho_x \longrightarrow \rho_x$, $\lambda_x \longrightarrow -\lambda_x$.

Q.E.D.

Let $\langle - \rangle_\Lambda = \langle - \rangle \equiv \langle - \rangle_{A^M_\Lambda , G_{\partial\Lambda}}$ when the boundary condition $G_{\partial\Lambda}$ is given by (6.4), (i.e. Dirichlet b.c. for $\underset{\sim}{\Phi}$, and Λ is a bounded subset of $a\mathbb{Z}^\nu_{1/2}$) .

The following corollary is an easy consequence of (6.2)-(6.4) and Theorem 6.2 ; see [55,35] and II.

<u>Corollary 6.3</u> :

(1) $< e^{r^2(g)} e^{iA(f)} >$ is <u>decreasing</u> in the covariance C of $\underset{\sim}{A}$, $< e^{-r^2(g)} e^{A(f)} >$ is <u>increasing</u> in C .

(2) $< e^{r^2(g)} e^{iA(f)} >_{\Lambda}$ is <u>increasing</u> in Λ , $< e^{-r^2(g)} e^{A(f)} >_{\Lambda}$ is <u>decreasing</u> in Λ .

<u>Remarks</u> :

(1) follows from Theorem 6.2, (1) resp. (3), as explained in [35] (Proof of Corollary 3.2) ; (2) follows from Theorem 6.2, (2) resp. (4) ; see [35] (Section 4).

As special cases of Corollary 6.3, (1) we have

$$\begin{aligned} &< e^{r^2(g)} > (e) \leqq < e^{r^2(g)} > (e = 0) \\ &< e^{-r^2(g)} > (e) \geqq < e^{-r^2(g)} > (e = 0) \quad , \end{aligned} \tag{6.26}$$

where e is the electric charge, $g \geqq 0$. The proof is obtained from Corollary 6.3 by the substitution of Section 4.1, (4.21)-(4.22).

Among our applications of Corollary 6.3 in papers II and III are :

(1) Construction of the thermodynamic limit, including the construction of Euclidean invariant, gauge-invariant Green's functions satisfying Osterwalder-Schrader positivity for the continuum Higgs model in two space-time dimensions in the thermodynamic limit. (This is derived from Corollary 6.3 by the method of generating functionals. The inequalities of Corollary 6.3 are not affected

when $r^2 = |\phi|^2$ is Wick ordered)!

(2) Construction of the limit $m_A \downarrow o$, including a derivation of the correct O-mass (continuum limit) Feynman rules (possibly of interest in non-abelian two dimensional Yang-Mills theories, too).

(3) Inequalities between correlations in the θ-vacua (see Theorem D, Section 1.2), $0 < \theta < 2\pi$, and correlations in the standard $\theta = 0$ vacuum.

The material of Sections 3-6 plays a rather basic role in our further analyses of the abelian Higgs models, in particular in our construction of the continuum Higgs model in two space-time dimensions.

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