Some Frontiers in Constructive Quantum Field Theory and Equilibrium Statistical Mechanics

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Some Frontiers in Constructive Quantum Field Theory and Equilibrium Statistical Mechanics[1]

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Abstract:
We present and discuss a list of important, mostly open problems in constructive quantum field theory and equilibrium statistical mechanics the solution of which requires (in rare cases : required) new ideas going beyond high - and low - temperature expansions guided by standard (super-renormalizable and infrared finite) perturbation theory about the critical points of some action or Hamilton function, beyond Peierls-type arguments and their variants and beyond spin wave theory and its rigorous counterparts. This list of problems includes higher order phase transitions, critical phenomena, long range forces, gauge theories, quantum solitons, etc.

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I.. Introduction:

A list of important problems and table of contents.

I.1 Personal problems and acknowlegements.

"Die Phantasie wird nur von dem erregt, was man noch nicht oder nicht mehr besitzt;..." (R. Musil, Der Mann ohne Eigenschaften).

A free translation of this quote might read as follows: Our imagination gets only excited (inspired) by what we do not possess yet, or not possess anymore.

When I recently learned this quote I felt it would be the right motto for these notes to two lectures I presented at $M \cap \phi$ in Rome. First reason: In these notes I try to speak about some problems in theoretical and mathematical physics whose solutions we do not possess, yet. At best we have some vague ideas of how to approach them or some preliminary results. My hope is that stating those problems in a precise way will stimulate our imagination and eventually lead to significant progress. Second reason: I found those ten days in Rome very exciting not only because of the interesting news about $M \cap \phi$ I learned, but at least as much because Rome is a place that inspires our imagination by showing us witnesses of some wealth we do not possess anymore : an overwhelming variety of past culture and civilization sunken into history; (and it excites our imagination by its wide variety of future possiblities).

Visitors of Rome face a serious problem. Unless they have a vast amount of time available they have to make a choice:
1) They might just enjoy themselves, relax and have Frascati, Espresso and good meals.
2) They might concentrate on seeing only some of the antique, or the Renaissance or the modern sites.
3) They might rush through all or most of Rome and then try to look at this or that in more detail.

When preparing my lectures and writing these notes I was facing a similar problem: Should I relax and just write a few pages of trivialities, should I concentrate on one specific problem and try to discuss it carefully, or should I rush through many cf the problems that excite me and look only at a few in some more detail?

These notes are probably a bad compromise of alternatives 2) and 3). It might well be that they show nothing more than the author's ignorance, somewhat contrary to his intention and presumably the one of all those people from whom he has profited in innumerable discussions (or through correspondence): E. H. Lieb, O. McBryan, Y. M. Park, E. Seiler, B. Simon, T. Spencer and others. They should have written these notes. Apart from those people I wish to thank the organizers of $M \cap \phi$ for their great work and for giving me the opportunity to present ideas that are in part, to say the least, doubtful.

I.2 <u>The main theme and table of contents</u>.

In these notes we are concerned with problems in constructive quantum field theory and equilibrium statistical mechanics a complete solution of which requires to go beyond - standard (super-renormalizable and infrared finite) perturbation theory about finitely many isolated (constant) degenerate minimas of some classical action or Hamilton function and its convergent versions: High and low temperature expansions, Peierls-type contour arguments, etc.; - super-renormalizable perturbation expansions or approximations (e.g. spin wave theory) about infinitely many, non-isolated (but constant) degenerate minimas of some classical action or Hamilton function and its rigorous versions: Spin wave analysis and Infrared (Gaussian) domination, the Goldstone theorem (and scattering theory for zero mass particles or excitations).

Among such problems there are

A. Rigorous treatment of non-super-renormalizable ultraviolet divergences, field strength - and charge renormalization.

B. Gauge theories in general, (meaning of gauge invariance in the presence of instantons, infrared divergences, confinement, lattice approximation, etc.); super-renormalizable gauge theories, such as the abelian Higgs model in two space-time dimensions (which has instantons) or QED in three dimensions (existence, physical positivity, phase transitions, etc.).

C. The theory of (topological) charges and super-selection sectors; quantum solitons.

D. Higher order phase transitions, critical phenomena and infrared divergences, the theory of critical points, interactions of very long range.

E. Scattering of charged particles interacting with the radiation field.

None of the problems A.-E. has so far been understood-not to mention solved-in a mathematically rigorous way. (The great importance of these problems for theoretical physics need not be explained here).

To make it clear at the beginning: I have <u>nothing interesting</u> to say <u>about A</u>. Although mathematical physicists (Schrader[1] and Glimm-Jaffe[2], see also [3])have tried to formulate this problem in a precise way and developed some preliminary ideas, one is far from knowing what the main difficulties are and one could view it as a scandal that we still do not have any concrete ideas about how the predictions of the renormalization group (e.g. asymptotic freedom and its converse;one may also think of supersymmetry) can be made into precise hints to the constructivists or, more ambitiously, into provable results.

I shall not say much of interest about <u>problem B</u>. either. (A preliminary outline of a program towards <u>constructing</u> continuum gauge quantum field theories and some <u>rigorous</u> results for simple models in two space-time dimensions were first given in [4]). What I could say about B. may well not be of much interest and, furthermore, it would require much more space. It is limited to some partly rigorous [4,5,6] and partly semi-rigorous [7] results on two dimensional gauge theories and some comments on lattice theories and on the meaning of gauge invarinace in theories with instantons [28]. However, I do want to recommend the following references to the reader's attention: [8,9,10,11,12] and [13,14,15,16,27,28]. In Section III a few results are sketched. In particular, we find phase transitions and a breakdown of the Higgs mechanism in approximate models of gauge theories with instantons; (for the $\theta = \pi$ vacuum): A new result that might be interesting for particle physics.

For reasons of page limitation I cannot describe the recent rigorous work concerning C. (quantum solitons) either; see [17,18,19]. But I want to emphasize that in these references a point of view has been developed which I feel is the correct one and will survive (e.g. because of its mathematical precision, which has not yet been widely appreciated, though). A rather general theory of Poincaré covariant superselection sectors with non-trivial (e.g. topological) charges is now available [20,17,7,21], and for a large class of two dimensional models with non-trivial superselection (soliton) sectors a quantum field theory of solitons has been developed [17], and it has been proven that, to leading order, the mass of the quantum soliton is given by the rest energy of the classical soliton [19]. The question of whether an expansion in $\hbar$ of all interesting quantum soliton effects about classical soliton solutions is asymptotic at $\hbar = 0$ can now be posed in a precise fashion and is presently studied; see also [22,23,24,25,7].

A discussion of D. (higher order phase transitions, critical phenomena,...) is the main part of these notes. As to the methods available for proving rigorous results in the field of critical phenomena one is still almost entirely limited to using correlation inequalities, infrared domination (and reflection positivity) - see Sections II, III - and some special inequalities (e.g. for Coulomb systems) - or else rely on exactly solvable models [26] about which I have nothing to say. Such methods are insufficient and may not lend themselves to much hard analysis. What is missing is a constructive version of the renormalization group (or other methods for setting up expansions about zero mass situations) applicable to physically interesting models and amenable to rigorous mathematics. An exception is the very recent work of Glimm and Jaffe [27] concerning the U(1) lattice gauge theory in four dimensions which may turn out to be interesting for statistical mechanics, too. In Section II we give a new derivation of their approximation and in Section III we present some results complementary to theirs. Our methods also apply to the abelian Higgs model on the lattice [28].

As to problem E: The reader is advised to consult the contribution of D. Buchholz to these proceedings and refs. [29,30,31]. Buchholz' results [29] and earlier proposals and results of the author [30] may supply a suitable axiomatic framework for understanding the scattering of charged particles and photons. This framework has been tested and partially confirmed in a simple model of non-relativistic electrons interacting with massless, scalar photons which has infrared divergences typical of QED, [31].

II. Models, mathematical structures, inequalities.

II.1) Lattice spin systems and - gauge theories.

Let $\mathbb{Z}^\nu$ be the simple, cubic lattice in ν dimensions. At each site $i \in \mathbb{Z}^\nu$ there is a random variable (classical spin) $\vec{S}_i \in \mathbb{R}^N$ distributed according to a (generally, but not always finite) measure $d\lambda(\vec{S})$ on $\mathbb{R}^N$. With a bounded cube $\Lambda \subset \mathbb{Z}^\nu$ we associate a Hamilton function

$$H(\{\vec{S}\}_\Lambda) = -\sum_{i,j\in\Lambda} J(i-j)\vec{S}_i \cdot \vec{S}_j + \vec{h}\cdot\left(\sum_{i\in\Lambda} \vec{S}_i\right) . \qquad \text{(II.1)}$$

We usually impose periodic (Λ viewed as a torus) or free ($\vec{S}_i = 0$, for all $i \notin \Lambda$) boundary conditions. The couplings $J(m)$ are assumed to be non-negative for $m \neq 0$ (ferromagnetic), of exponential decrease in $|m|$, reflection positive [32] (which is equivalent to the existence of a selfadjoint transfer matrix [32]) and isotropic (w.r. to interchanging lattice axes). Finally $\vec{h}$ is a fixed external field which we assume, from now on, to point in the 1-direction : $\vec{h} = h\cdot e_1$.

We let $\langle - \rangle_\Lambda(\beta, h)$ denote the Gibbs equilibrium expectation of the system so defined. We set $\langle - \rangle_\Lambda(\beta) = \langle - \rangle_\Lambda(\beta, 0)$. Here β is the inverse temperature.

For measures $d\lambda$ of compact support a standard compactness argument gives existence of at least one limiting Gibbs expectation, $\langle - \rangle(\beta, h)$, as $\Lambda \uparrow \mathbb{Z}^\nu$, and periodic boundary conditions (or correlation inequalities) guarantee translation invariance. The susceptibility χ is defined by

$$\chi(\beta, h) = \sum_j \langle \vec{S}_0 \cdot \vec{S}_j \rangle(\beta, h) , \qquad \text{(II.2)}$$

and the inverse correlation length (mass) by

$$m(\beta, h) = -\lim_{n\to\infty} \frac{1}{n} \log\langle \vec{S}_0 \cdot \vec{S}_{ne_\alpha} \rangle(\beta, h) , \qquad \text{(II.3)}$$

where e_α is the unit vector in the positive α-direction. Note that $m(\beta, h) > 0$ implies $\chi(\beta, h) < \infty$.

We now consider some examples:

II.1) N-vector models : $N = 1,2,3,\ldots,$

$$d\lambda(\vec{S}) = \delta(|\vec{S}| - 1)d^N S .$$

For $N = 1$ this is the Ising model, for $N = 2$ the rotator and for $N = 3$ the classical Heisenberg model. The rotator model can be rewritten in terms of angle variables:

$$\vec{S}_i = (\cos\theta_i, \sin\theta_i), \ \theta_i \in [0, 2\pi]$$

$$d\lambda(\vec{S}) \longmapsto \frac{d\theta}{2\pi} ,$$

$$H(\{\theta\}_\Lambda) = -\sum_{i,j\in\Lambda} J(i-j)[\cos(\theta_i - \theta_j) - 1] + h \sum_{i\in\Lambda} \cos\theta_i \tag{II.4}$$

For the Ising and the rotator model it is known that the equilibrium expectation in the limit $\Lambda = \mathbb{Z}^\nu$ is unique for $h \neq 0$ [33,34] and for $h = 0$ in the absence of spontaneous magnetization (i. e. $\langle S_i^1\rangle(\beta, 0_\pm) = 0$), [33,35].

II.2) Dual Villain - (or V-) model:

This is the model with $N = 1$, $h = 0$, $d\lambda(S) = (\sum_{m\in\mathbb{Z}} \delta(S-m))dS$ and free boundary conditions. If $J(m)$ is the kernel of the finite difference Laplacean on $\mathbb{Z}^\nu$ (nearest neighbor coupling) and $\nu = 2$ (in which case the moments $\langle\pi_{i\in A} S_i\rangle(\beta)$ do presumably not exist for small β !) this model is an approximation to the dual of the nearest neighbor rotator model at $h = 0$, provided one sets

$$\beta_V \propto \beta_{rot}^{-1} . \tag{II.5}$$

This follows from replacing $\exp \beta[\cos\theta - 1]$ by

$$\sum_{n\in\mathbb{Z}} \exp[-\frac{\beta}{2}(\theta + 2\pi n)^2] \tag{II.6}$$

in the definition of the partition function and the Gibbs expectation of the rotator model, then taking the Fourier transform with respect to $\{\theta_i\}$ and making a change of variables; see e.g. [36] and refs. given there. Next, we introduce abelian lattice gauge theories [14,15,16,27]. For this purpose we consider "p-form valued random fields on $\mathbb{Z}^\nu$". Such a p-form valued r.f. ω is of the form

$$\omega_i = \sum_{\alpha_1<\cdots<\alpha_p} \Omega_i^{\alpha_1\cdots\alpha_p} e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_p} , \tag{II.7}$$

where, for each i and given $\alpha_1,\ldots,\alpha_p$ $\Omega_i^{\alpha_1\cdots\alpha_p}$ is a real random variable which is totally antisymmetric in $\alpha_1,\ldots,\alpha_p$. For such p-forms one can define the usual duality map $*$ so that $*\omega$ is a $(\nu-p)$-form when ω is a p-form. Furthermore, we define

$$d\omega_i = \sum_{\alpha,\alpha_1<\cdots<\alpha_p} \partial^\alpha\Omega_i^{\alpha_1\cdots\alpha_p} e_\alpha\wedge e_{\alpha_1}\wedge\cdots\wedge e_{\alpha_p} , \tag{II.8}$$

with

$$\partial^\alpha\Omega_i^{\#} = \Omega_{i+e_\alpha}^{\#} - \Omega_i^{\#} \tag{II.9}$$

II.3) The U(1) lattice gauge theory:

At each site $i\in\mathbb{Z}^\nu$ we are given a 1-form

$$\theta_i = \sum_\alpha \theta_i^\alpha e_\alpha \; , \quad 0 \leq \theta_i^\alpha \leq 2\pi \; . \qquad \text{(II.10)}$$

We write $d\,\theta_i$ as

$$\phi_i = \sum_{\alpha_1<\alpha_2} \phi_i^{\alpha_1\alpha_2} e_{\alpha_1} \wedge e_{\alpha_2}$$

with

$$\phi_i^{\alpha_1\alpha_2} = \partial^{\alpha_1}\theta_i^{\alpha_2} - \partial^{\alpha_2}\theta_i^{\alpha_1} \; . \qquad \text{(II.11)}$$

The single spin distribution $d\lambda$ of the $U(1)$ lattice gauge theory is given by $\prod_{\alpha=1}^{\nu} (2\pi)^{-1}\, d\theta^\alpha$, and the Hamilton function (which should be called <u>action</u> in this context) by

$$H(\{\theta^\alpha\}_\Lambda) = -\sum_{i\in\Lambda} \sum_{\alpha_1<\alpha_2} [\cos(\phi_i^{\alpha_1\alpha_2}) - 1] \qquad \text{(II.12)}$$

These definitions also determine the partition function and the Gibbs equilibrium expectation of this model which is non-trivial only for $\nu \geq 3$. Similar expressions define the <u>abelian Higgs model</u> on the lattice which describes an additional pair of random fields (ρ_i, χ_i), $\rho_i \in \mathbb{R}^+$, $\chi_i \in [0, 2\pi]$, [37].

Suppose we now replace, in the definition of the partition function and the Gibbs expectations, the factors

$$\exp \beta[\cos(\phi^{\alpha_1\alpha_2}) - 1] \quad \text{by}$$

$$\sum_{n_i^{\alpha_1\alpha_2}\in\mathbb{Z}} \exp[-\tfrac{\beta}{2}(\phi_i^{\alpha_1\alpha_2} + 2\pi\, n_i^{\alpha_1\alpha_2})^2] \qquad \text{(II.13)}$$

and take the Fourier transform with respect to $\{\theta_i^\alpha\}_{i\in\Lambda}$.

Then we obtain, after some straight-forward calculations, rewriting a 2-form as the * of a $(\nu-2)$-form and using Poincaré's lemma

$$d\omega = 0 \implies \omega = d\gamma \qquad \text{(II.14)}$$

(<u>always</u> valid on the lattice) : [27,28]

II.4) For <u>$\nu = 3$</u> the nearest neighbor Villain model with $\beta_V \propto \beta_{U(1)}^{-1}$;

II.5) For <u>$\nu = 4$</u> a model (which we call the vector Villain - or VV-model, [28]) for a 1-form lattice random field a_i^α with single spin distribution

$$\left.\begin{aligned} d\lambda(a_i) &= \prod_{\alpha=1}^{4} d\bar{\lambda}(a_i^\alpha) , \\ d\bar{\lambda}(a) &= \Big(\sum_{m\in\mathbb{Z}} \delta(a-m)\Big)da , \end{aligned}\right\} \qquad \text{(II.15)}$$

Hamilton function (or action)

$$H(\{a^\alpha\}_\Lambda) = \frac{1}{2}\sum_{i\in\Lambda} (da_i)^2 \qquad \text{(II.16)}$$

and with
$$\beta_{VV} \propto \beta_{U(1)}^{-1} . \qquad \text{(II.17)}$$

In both cases we impose free boundary conditions at the boundary $\partial\Lambda$ of Λ. Similar gauge-invariant approximations can be made for the abelian Higgs model [28]. Finally we briefly discuss an isomorphism of the V- and the VV-model onto Coulomb-type models:

Given a V-model with couplings $J(m)$, let $\hat{J}(m)$ be - the (convolution) inverse of $J(m)$ and let $d\lambda(\frac{q}{2\pi})$ be as in defintion II.2) of the V-model (with S replaced by $^q/_{2\pi}$), but replace the Hamilton function H by

$$\hat{H}(\{q\}_\Lambda) = \frac{1}{2}\sum_{i,j\in\Lambda} \hat{J}(i-j)q_iq_j \qquad \text{(II.18)}$$

The model so obtained is called the $\hat{V}$-model.

If $\nu = 2$ and J is the kernel of the finite difference Laplacean $\hat{J}$ is the two dimensional lattice Coulomb potential (which is only conditionally positive definite; see II.2) and we replace $\prod_{i\in\Lambda} d\lambda(q_i)$ by $\delta(0, \sum_{i\in\Lambda} q_i)\prod_{i\in\Lambda} d\lambda(q_i)$, where $\delta(m, n)$ is the Kronecker δ. (II.19)

Finally we introduce a $\hat{VV}$-model: $d\lambda(\frac{q_i}{2\pi})$ is as in (II.15) ($^{q_i}/2\pi$ replacing a_i). We choose as an a priori measure on $\mathbb{R}^{4|\Lambda|}$ the measure

$$\prod_{i\in\Lambda} d\lambda(q_i)\delta(0, {*d*}\, q_i) \qquad \text{(II.20)}$$

and as Hamilton function (action)

$$H(\{q\}_\Lambda) = \frac{1}{2}\sum_{i,j\in\Lambda} W_\Lambda(i-j)q_iq_j , \qquad \text{(II.21)}$$

where W_Λ is the Green's function of the finite difference Laplacean on $\mathbb{Z}^4$ (the lattice Coulomb potential) with free boundary conditions at $\partial\Lambda$, q_i is a conserved vector charge. Using generalizations of the simple identities

$$\int e^{-\frac{1}{2}ax^2} e^{ibx}dx = \sqrt{\frac{2\pi}{a}}\, e^{-\frac{b^2}{2a}} , \qquad \text{(II.22)}$$

and

$$\sum_{n\in\mathbb{Z}} e^{2\pi inx} = \sum_{m\in\mathbb{Z}} \delta(x-m) \qquad \text{(II.23)}$$

we obtain

Theorem II.1:

1)[36] The V-model at inverse temperature β_V is isomorphic to the $\hat{V}$-model defined in (II.18) (II.19) at $\beta_{\hat{V}} = \beta_V^{-1}$; in particular the nearest neighbor V-model is mapped onto the $\hat{V}$-model with the Coulomb potential as couplings.

2)[27,28] The VV-model at inverse temperature β_{VV} is isomorphic to the $\hat{VV}$-model defined in (II.20) (II.21) at $\beta_{\hat{VV}} = \beta_{VV}^{-1}$.

In both cases, the partition function of the V-, resp. VV-model is the product of the partition function of the $\hat{V}$-, resp. $\hat{VV}$-model and a spin wave partition function $\det(-\sqrt{2\pi}\, J_\Lambda)^{-1}$. The $\hat{VV}$-model was first discussed in [67].

Remark: In Theorem II.1, 2) we recover (in a novel way) the Glimm-Jaffe approximation to the U(1)-model, [27]. See [27,28] for details.

II.2 Classical gases

Let x be a position vector in a configuration space $C = \mathbb{R}^\nu$ or $\mathbb{Z}^\nu$, and let q be a generalized charge, a vector in some topological vector space Q .

The potential between a particle with charge q at position x and one with charge q' at position x' is given by a function $V(q,x;\, q',x')$ on $(Q \times C)^2$ of positive type, satisfying translation invariance and

$$V(q,x;\, q',x') = -V(-q,x;\, q',x') = -V(q,x;\, -q',x') \qquad \text{(II.24)}$$

and $V(q,x;\, q,x) \leq v < \infty$; see also [38].

The potential for n particles with parameters $W_i = (q_i,x_i)$, $i=1,\ldots,n$ is given by

$$U((W)_n) = \sum_{1\leq i<j\leq n} V(W_i;\, W_j) \; . \qquad \text{(II.25)}$$

We let $d\lambda$ be a finite, positive measure on Q with $d\lambda(q) = d\lambda(-q)$. (II.26)

Set $d\lambda(q)_n = \prod_{i=1}^{n} d\lambda(q_i)$, $d(x)_n = \prod_{i=1}^{n} d^\nu x_i$.

The grand canonical partition function for the system in a bounded region $\Lambda \subset C$ is given by

$$\Xi_\Lambda(\beta,z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{Q^n} d\lambda(q)_n \int_{\Lambda^n} d(x)_n \, e^{-\beta U((W)_n)}$$

where z is the activity (and $\int_{\Lambda^n} d(x)_n$ denotes the sum over all sites in $\Lambda^n = \Lambda^{\times n}$, when $C = \mathbb{Z}^\nu$). Pressure $p_\Lambda(\beta,z)$ and correlation functions $\rho_\Lambda(\beta,z;\, W_1,\ldots,W_n)$ are defined in the usual way; see [39].

Examples:

II.6) Coulomb-type potentials

$Q = \mathbb{R}$ and $d\lambda(q) \overset{e.g.}{=} (\delta(q-1) + \delta(q+1))dq$, $V(q,x;\ q',x') = q\cdot q'\ W(x-x')$, where W is a potential on C of positive type and arbitrarily long range. The following cases are of special interest:

6.a) $C = \mathbb{Z}^\nu$, $\nu \geq 2$, W the Green's function of the finite difference Laplacean (i.e. the lattice Coulomb potential). When $\nu = 2$ W is not of positive type, but it is conditionally of positive type, i.e.

$$\sum \bar{c}_i\ c_j\ W(x_i - x_j) \geq 0\ , \tag{II.28}$$

for arbitrary complex numbers $c_1, c_2, \ldots$ with $\sum_i c_i = 0$. This will suffice for a study of the neutral Coulomb gas; (see also (II.19)). As $z \to \infty$ this model converges to the nearest neighbor V-model at $\beta_V = \beta^{-1}_{\text{Coulomb}}$, [38].

6.b) $C = \mathbb{R}^\nu$, $\nu \geq 2$, W a regularized version of the Coulomb potential; ("ultra-violet cutoff").

6.c) $C = \mathbb{R}^2$, W the two dimensional Coulomb potential; see [40].

In cases 6.b), $\nu = 2$, and 6.c) the same comment as in 6.a), i.e. (II.28), applies.

II.7) Dipole potentials

$Q = \mathbb{R}^\nu$, $d\lambda(q) \overset{e.g.}{=} \delta(|q|-1)d^\nu q$, $V(q,x;\ q',x') = (q\cdot\nabla_x)(q'\cdot\nabla_{x'})W(x-x')$, with W a potential of positive type on $\mathbb{R}^\nu$ (or $\mathbb{Z}^\nu$), e.g. a (regularized) Coulomb potential such that $V(q,x;\ q,x) < \infty$.

II.3 Functional integrals

Let $\mathcal{H}_V$ be the Hilbert space of real functions $f,g,h,\ldots$ on $Q \times C$ with scalar product

$$\begin{aligned} \langle f,g\rangle_{\beta V} = \int & d^\nu x\ d^\nu x'\ d\lambda(q)d\lambda(q')\overline{f(q,x)} \\ & \times \beta V(q,x;\ q',x')g(q',x') \end{aligned} \tag{II.29}$$

Let ϕ be the Gaussian process with mean 0 and covariance βV, indexed by $\mathcal{H}_V$, and let $\langle - \rangle_{\beta V}$ denote the corresponding Gaussian expectation (given by a Gaussian measure $d\mu_{\beta V}$ with mean 0, covariance βV, defined on a suitable measure space; see [41]).

One defines Wick ordering with respect to $\langle - \rangle_{\beta V}$ by

$$:e^{i\phi(f)}: \ = e^{i\phi(f)}\langle e^{i\phi(f)}\rangle^{-1}_{\beta V} \tag{II.30}$$

We set

$$C_\Lambda \equiv -S_\Lambda \equiv \int_Q d\lambda(q) \int_\Lambda d^\nu x\ :\cos\phi:\ (q,x) \tag{II.31}$$

The following gives a connection between the quantities Ξ_Λ and ρ_Λ introduced in II.2 and (Gaussian) functional integrals.

Theorem II.2, [42,40,38]:

$$\Xi_\Lambda(\beta,z) = \langle \exp z\, C_\Lambda \rangle_{\beta V} \;, \quad \text{and}$$

$$\begin{aligned}\rho_\Lambda(\beta,z;\, W_1,\ldots,W_n) &= \Xi_\Lambda(\beta,z)^{-1}\, z^n \\ &\quad \times \langle \prod_{i=1}^{n} :e^{i\phi(W_i)}:\; e^{zC_\Lambda} \rangle_{\beta V} \\ &\equiv z^n \langle \prod_{i=1}^{n} :e^{i\phi(W_i)}: \rangle_\Lambda(\beta,z) \;. \end{aligned} \tag{II.32}$$

Next we discuss a general inequality extending Ginibre's inequality for the rotator [43] and some recent inequalities of Park and the author [44,38].

Let ϕ and ϕ' be two isomorphic random fields indexed by a real Hilbert space $\mathcal{H}$ and distributed according to some measure $d\mu$. Let $\langle - \rangle_\mu$ denote expectation w.r. to $d\mu$. Let X be some measure space and $d\rho$ a finite measure on X. Let $x \longmapsto \ell_x$ be a measurable mapping $: X \longrightarrow \mathcal{H}_1 \subseteq \mathcal{H}$. Define

$$C(\rho) = \int_X d\rho(x) \cos \phi(\ell_x) \;,$$

$$\langle - \rangle(\mu,\rho) \equiv \langle e^{C(\rho)} \rangle_\mu^{-1} \langle - \, e^{C(\rho)} \rangle_\mu \;, \text{ and}$$

$$\langle A;\, B\rangle(\mu,\rho) \equiv \langle AB\rangle(\mu,\rho) - \langle A\rangle(\mu,\rho)\cdot\langle B\rangle(\mu,\rho) \;.$$

Theorem II.3, [38]:

Let S be the class of random variables of the form $\prod_i [\cos \phi(f_i) \pm \cos \phi'(f_i)]$, $f_i \in \mathcal{H}_1$. Suppose that

$$d\mu(\phi)d\mu(\phi') \restriction S = d\nu\!\left(\frac{\phi+\phi'}{\sqrt{2}}\right) d\nu\!\left(\frac{\phi-\phi'}{\sqrt{2}}\right) \restriction S \;,$$

for some finite measure $d\nu$.

Then, for arbitrary f and g in $\mathcal{H}_1$,

1) $\langle \cos \phi(f)\rangle(\mu,\rho) \geq 0$, (provided μ is of positive type)

2) $\langle \cos \phi(f);\, \cos \phi(g)\rangle(\mu,\rho) \geq 0$

Remark: 2) implies e.g. that $\langle \cos \phi(f)\rangle(\mu,\rho)$ is monotone increasing in ρ.

(II.33)

Application 1: Ginibre's inequalities [43] for the rotator, the $U(1)$ lattice gauge theory and the abelian Higgs model; (the latter two cases have been noticed in [45,37]. They are straightforward consequences of [43], resp. Theorem II.3).

Application 2: The inequalities of Park and the author [38] for the correlation functions of classical gases, e.g., for all f, g in $\mathcal{H}_V$,

$$\left.\begin{array}{l} \langle :e^{i\phi(f)}:\rangle_\Lambda(\beta,z) \geqq 0 \\ \langle :e^{i\phi(f)}: ; :\cos\phi(g):\rangle_\Lambda(\beta,z) \geqq 0 \\ \langle :e^{i\phi(f)}:\rangle_\Lambda(\beta,z) \uparrow \text{ in } \Lambda \text{ and } z \end{array}\right\} \quad (II.34)$$

(They follow from (II.30) and Thm. II.3:1), 2), (II.33), resp., by noting that $:e^{i\phi(f)}: = :\cos\phi(f): + i\, :\sin\phi(f):$ and that $\langle - \rangle_\Lambda(\beta,z)$ is invariant under $\phi \longmapsto -\phi$).

II.4 Functional integrals and quantum field theory

Functional integrals are also used to construct relativistic quantum field models such as the $\phi^4_{2,3}$ - [46,47,48], $(\vec{\phi}\cdot\vec{\phi})^2_{2,3}$ - [49,50,51] or the sine-Gordon model [4,40,5,44], etc.

In the context of Euclidean field theory q labels different fields in the theory, x is a Euclidean space-time point; $\phi(q,x)$ is a generalized stochastic process with expectation $\langle - \rangle$ given by some probability measure on a space of generalized functions. The moments $\langle \prod_{i=1}^{n} \phi(q_i,x_i)\rangle$ are tentatively interpreted as the Euclidean Green (or Schwinger) functions of a relativistic quantum field theory satisfying all the Wightman axioms (except possibly uniqueness of the vacuum). Sufficient conditions for this interpretation to be correct have been given in a basic paper of Osterwalder and Schrader [52]. For detailed, rigorous information on the Euclidean description of relativistic quantum field theory and functional integrals see also [47,48] and refs. given there, and [31,53]. A formal version of the conditions of Osterwalder and Schrader [52,31] is as follows: Let $S(\phi) \equiv \int d^\nu x \; S(\phi(\cdot,x))$ be the classical Euclidean action of some field theory. Formally, the expectation $\langle - \rangle := \langle - \rangle_S$ is given by the Euclidean Gell'Mann-Low formula

$$"\langle - \rangle_S = \left[\int e^{-\hbar^{-1}:S_{ren.}(\hbar^{1/2}\phi):} \Pi\mathcal{D}\phi(q,x)\right]^{-1} \times \int - \, e^{-\hbar^{-1}:S_{ren.}(\hbar^{1/2}\phi):} \Pi\mathcal{D}\phi(q,x)" , \quad (II.35)$$

where $\hbar$ is (proportional to) Planck's constant, the double colons denote some Wick order (depending on the curvature of the classical action at some of its absolute minimas), and $S_{ren.}$ is the renormalized action (possibly including ∞ (for $\hbar > 0$) counterterms).

Quasi-Theorem II.4:

Suppose (II.35) can be given a rigorous meaning, and

1) $:S_{ren.}(\hbar^{1/2}\phi):$ is Euclidean invariant;

2) if θ represents Euclidean time reflection on random variables

(with expectation $\langle - \rangle_S$) then $\theta : \overline{S_{\text{ren.}}(\hbar^{1/2}\phi(\vec{x},t))} : = : S_{\text{ren.}}(\hbar^{1/2}\phi(\vec{x},-t)):$ with $(\vec{x},t) \equiv x$.

3) $\langle e^{\phi(f)} \rangle_S$ exists for a suitable class of test functions. Then the moments of $\langle - \rangle_S$ exist and are the Schwinger functions of a unique relativistic quantum field theory.

For
$$S(\phi) = S_o(\phi) \equiv \frac{1}{2} [(\nabla\phi)^2 + m^2\phi^2] , \qquad \text{(II.36)}$$

$\langle - \rangle_S$ ("given" by (II.35)) is the Gaussian expectation with mean 0 and covariance $(-\Delta+m^2)^{-1}$, the kernel of which is the Yukawa-($m > 0$), resp. the Coulomb potential ($m=0$). (II.37)

Its moments are the Schwinger functions of the free field. For $m = 0$ and

$$S(\phi) = S_o(\phi) + \lambda \cos \phi \qquad \text{(II.38)}$$

we obtain the sine-Gordon theory. If we compare (II.35) - (II.38) with (II.30)-(II.32) and with model II.6.c) we obtain

Theorem II.5:

1) For $\hbar < 4\pi$ ("no ultraviolet divergences") the sine-Gordon theory is isomorphic to the two dimensional, two component Coulomb gas II.6.c) [40], and

2) The inequalitites of Theorem II.3 apply to the Schwinger functions of the fields $:e^{i\phi}:$ of the sine-Gordon theory, [44].

Rigorous connections between the standard ϕ^4_ν - and $(\vec{\phi}\cdot\vec{\phi})^2_\nu$-models $(\vec{\phi} = (\phi_1,\phi_2))$ and the classical N-vector models, example II.1), due to [48,50,54], are by now well known.

The case $\nu = 2$,
$$\left.\begin{aligned} S_o(\phi) &= \sum_{i,j\in\mathbb{Z}^2} J(i-j)\phi_i\phi_j \\ S(\phi) &= S_o(\phi) + \lambda \sum_i \cos(\phi_i - \theta), \ \theta \in [0,2\pi], \end{aligned}\right\} \qquad \text{(II.39)}$$

with $J^{-1} = q^2 V_C + W$, where V_C is the two dim. lattice Coulomb potential and W is a positive type potential of very short range, gives an approximate description of the Euclidean vortex (magnetic) field in the abelian Higgs model in two space-time dimensions if q is chosen to be the ratio of the electric charge of a massless fermion Dirac field and the Higgs scalar [7]. The interaction term $\lambda \sum_i \cos(\phi_i - \theta)$ comes from the instantons of this model: the Nielsen-Olesen vortices[55]. For $q = 0$ the angle θ parametrizes the θ-vacua first described in [11,12]. Using Theorem II.2 one sees that our approximation is obviously a version of the dilute gas approximation of Polyakov [9] and others [10,13]. In this form it is proposed and discussed in [7]. The lattice Higgs model in the Villain approximation leads to a similar effective field theory, but there are some important differences.

III Critical points, critical phenomena, long range forces

III.1 Critical point in the N-vector and $|\vec{\phi}|^4$-models

First we consider the N-vector models defined in Section II.1, for $N = 1, 2, 3$, i.e. the Ising, the rotator and the classical Heisenberg model. They are known to satisfy the Lee-Yang theorem of [56,50]. From this we get

Theorem III.1:

1) For $\mathrm{Re}\, h \neq 0$ the correlation functions $\langle \prod_{i \in A} S_i^{\alpha_i} \rangle (\beta,h)$ are analytic in h and real analytic in β in an h-dependent neighborhood of $(0,\infty)$. Periodic and free boundary conditions coincide in the thermodynamic limit.

2) For real $h \neq 0$, $m(\beta,h) > 0$, and $m(\beta,h) = O(h)$ when $m(\beta,0) = 0$. For real $h \neq 0$ the exponential decay rate of all truncated correlations is uniformly bounded away from 0.

This theorem has been derived in [49] as a consequence of the Lee-Yang theorem. It holds for the $(\vec{\phi}\cdot\vec{\phi})^2_{2,3}$-models $(\vec{\phi}=(\phi^1,\dots,\phi^N)$, $N = 1, 2, 3)$, too [49]. Part 2) extends earlier results of [57,58].

Open problem: Do Lee-Yang theorem and Theorem III.1 remain true for $N > 3$?

Theorem III.2:

1) For $N = 1$, $\nu \geq 2$, and for $N = 2, 3$, $\nu \geq 3$, there exists $\bar{\beta}_c < \infty$ such that, for $\beta > \bar{\beta}_c$, $\lim_{h\to 0} \langle S^1 \rangle (\beta,h) \neq 0$ (spontaneous magnetization).

2) For $h = 0$, $N = 1, 2, 3, 4$ there exists $\underline{\beta}_c \leq \bar{\beta}_c$ (with $\bar{\beta}_c < \infty$ for $N = 1$, $\nu \geq 2$ and $N = 2, 3, 4$, $\nu \geq 3$) such that $\lim_{\beta\uparrow\underline{\beta}_c} m(\beta) = 0$, $\lim_{\beta\uparrow\underline{\beta}_c} \chi(\beta) = \infty$.

For $\nu \geq 3$ the expectation $\langle - \rangle(\underline{\beta}_c) \equiv \lim_{\beta\uparrow\underline{\beta}_c} \langle - \rangle(\beta)$ is clustering. I.e. there exists a critcal model with mass = 0, infinite susceptibility, but no long range order.

3) If $\langle \vec{S}_0 \cdot \vec{S}_j \rangle (\underline{\beta}_c) \approx |j|^{-(d-2+\eta)}$, as $|j| \to \infty$, then $0 \leq \eta \leq 2$.

Remarks and comments: 1) For $N = 1, 2$ $m^{-1}(\beta)$ and $\chi(\beta)$ are monotone increasing in β (for $h=0$): A consequence of Theorem II.3, resp. [43]. 2) For $N \geq 2$, part 1) follows from infrared domination [59]; for more details concerning this and related results for these and a class of quantum models see [60,61] and Lieb's contribution to these proceedings. Part 2) is based on infrared bounds [59] and the "Lebowitz inequalities" $\langle S_i^\alpha ; S_j^\alpha ; S_\ell^\alpha ; S_m^\alpha \rangle (\beta) \leq 0$, and

$$\langle S_i^\alpha \, S_j^\alpha ; S_\ell^\gamma \, S_m^\gamma \rangle (\beta) \leq 0$$

proven in [62,63]. See [64,65,21] for $N = 1$, and [28] for $N = 2, 3, 4$. Finally 3) follows from infrared domination [59] and 2); see [59, 21].

3) Clearly $\overline{\beta}_C = \infty$ for $N \geqq 2$ and $\nu = 2$.

4) Theorem III.2 extends to $(\vec{\phi}\cdot\vec{\phi})^2_{2,3}$(N=1,2,3) for which the following additional results hold: Absence of two particle bound states [66,63] and existence of N degenerate elementary particles [21] (for almost every physical mass > 0) at $h = 0$, in the one phase region.

Open problems:

1) Show (or disprove that $\underline{\beta}_C = \overline{\beta}_C$ for $\nu \geqq 3$, (or $\nu = 2$ and $N \geqq 3$, [67])

and that $\lim\limits_{\beta \downarrow \overline{\beta}_C} \langle S^1_i \rangle (\beta, 0+) = 0$; (for $N = 1$ this would imply $m(\beta, 0+) \downarrow 0$, as

$\beta \downarrow \overline{\beta}_C$, [31]).

2) Prove (or disprove) the existence of a Euclidean invariant scaling limit (and hence of an associated relativistic quantum field theory, e.g. [32,53]) for <-> ($\underline{\beta}_C$). For the $\nu = 2$ Ising model this problem has been partially solved in [69] by rather direct, very difficult calculations. A proper, general and rigorous understanding of the scaling limit is however still missing; see e.g. [53].

3) Does the scaling limit in $\phi^4_{2,3}$ [23] teach us something about non-super renormalizable ultraviolet divergences and triviality or non-triviality of ϕ^4_4 [1,2,53] ?

Theorem III.3:

Let $N = 2, 3, \ldots$ and $\nu = 2$.

1) [70] For arbitrary $\varepsilon > 0$ there exists $\beta_o(\varepsilon) < \infty$ such that for all $\beta > \beta_o(\varepsilon)$

$$\langle \vec{S}_o \cdot \vec{S}_j \rangle (\beta) \leq \text{const.} \cdot |j|^{-[(2\pi+\varepsilon)\beta]^{-1}}$$

2) [51] $m(\beta) \leq \text{const.}\ e^{-\text{const.}\,\beta N^{-1}}$, for $\beta >> 1$.

Remark: This result has been extended to all truncated correlations in [63]. A new proof of 1) has been found in [71]. Part 2) also holds for the field theory case [72], and it seems to us that the methods of [71] presumably give 1) for the field theory case, too.

Conjecture III.4, [67,36,73]:

For $N \geqq 3$, $\nu = 2$ $\underline{\beta}_C$ is infinite, for $N = 2$, $\nu = 2$ $\underline{\beta}_C$ is finite, more specifically, the bound of Theorem III.3.1) is saturated for $N = 2$, and the factor "N^{-1}" in the exponent on the r.h.s. of 2) can be replaced by $(N-2)^{-1}$, provided β is very large; see also [68].

Obvious open problem: Prove Conjecture III.4.

A proof would be an impressive and promising beginning in our understanding of higher order phase transitions and critical phenomena.

III.2 Gases with long range forces

Theorem III.5, [38]:

In the notations and under the hypotheses of Section II.2

$$\lim_{\Lambda \uparrow C} p_\Lambda(\beta, z) \equiv p(\beta, z) \quad \text{and, for all } n, \tag{III.1}$$

$$\lim_{\Lambda \uparrow C} \rho_\Lambda(\beta, z; W_1, \dots, W_n) \equiv \rho(\beta, z; W_1, \dots, W_n) \tag{III.2}$$

exist and are independent of $\{\Lambda\}$. Moreover, the correlation functions $\rho_{(\Lambda)}(\beta, z; \cdots)$ are monotone increasing in z, the Fourier transform of the "effective potential function" $\langle\hat{\phi}(q, k)\hat{\phi}(q,-k)\rangle\ (\beta, z)$ is monotone decreasing in z and bounded above by $\beta\hat{V}(q,k; q,-k)$, (its value for $z = 0$!). (III.3)

This Theorem is a direct consequence of Theorems II.2 and II.3. It is proven in [38], where it has also been shown to be true for the corresponding quantum gases with "Boltzmann statistics". To our knowledge this is the first existence theorem for thermodynamic and correlation functions valid for potentials of arbitrarily long range and for all positive β, z. Part (III.1) extends to certain gases with statistics and to potentials that include hard cores ; (III.3) is an infrared bound to be compared with the one of [59]. Under various additional assumptions it implies clustering of the correlation functions in the thermodynamic limit [38].

Corollary III.6:

The thermodynamic limit of the pressure (resp. vacuum energy density) and all correlation (resp. Schwinger) functions of the following models exists and is shape independent:

1) The two dimensional Coulomb gas-example II.6.c)-above collapse temperature [40], equivalently (see Theorem II.5), the $\nu = 2$ sine-Gordon theory for $\mathcal{K} < 4\pi$; see [44]. The "bosonized" [74,5] $\nu = 2$ Yukawa - and a model for $\nu = 2$ QED of massive fermions and massive photons [4]; see [6].

2) The classical gases of examples II.6.a) - c) and II.7), the V- and VV-models (examples II.2), II.6.a), II.5)), and the isomorphic $\hat{V}$- and $V\hat{V}$-models; (see Theorem II.1).

3) The rotator [43], the U(1) lattice gauge theory [45,37] and a class of abelian Higgs models on the lattice [37], resp. their Villain approximation [28].

Remarks:

1) Cor. III.6 is a direct consequence of Thms. II.2, II.3 [38,28]. It would be of considerable interest to prove Thm. II.3 resp. Cor. III.6.1) for the Yukawa model in the Matthews-Salam-Seiler [75] representation.

to the U(1) (resp. the abelian Higgs [28]) lattice gauge theory complementary to the (deeper!) one of Glimm and Jaffe [27].

Next we state a beautiful result due to Brydges [76].

Theorem III.7:

For the lattice Coulomb gas, example II.6.a) in ν dimensions exponential Debye screening is valid in a region of high enough temperature and activity approximately given by the scaling properties of the corresponding continuum Coulomb gases.

Remarks: 1) This result is a lot more difficult to prove than a corresponding result that affirms exponential Debye screening for the $\hat{V}$- and $\hat{V}\hat{V}$-models which are isomorphic to special types of Coulomb gases (Thm. II.1). Brydges' methods [76] are based on the difficult "expansion in phase boundaries" due to Glimm, Jaffe and Spencer [77] (which can be applied to this problem thanks to Thms. II.2, II.5 [40], whereas in the case of $\hat{V}$-resp. $\hat{V}\hat{V}$-models standard Peierls arguments (convertable into high temperature expansions) suffice.

2) Brydges' methods apply to a larger class of lattice gases than the one he considers: If $W^{-1}(m)$ is of exponential decrease in $|m|$ one always gets screening in some range of (high) temperatures and activities. This may show that screening (in particular Debye screening) may not really depend too much on special properties of the Coulomb potential (such as Newton's theorem [78]).

3) Applied to the $\nu = 2$ lattice Coulomb gas (example II.6.a)) Brydges' results give Debye screening only for high temperatures. The obvious open problem is thus: Is there a $\beta_C < \infty$ such that for $\beta \geqq \beta_C$ Debye screening disappears, e.g. in the sense that the susceptibility (defined in terms of the effective potential function) is infinite? The following inequalities are relevant to this problem.

Theorem III.8 [38,28]:

Let $\langle\hat{S}(k)\hat{S}(-k)\rangle_V(\beta)$ be the two point function of the nearest neighbor V-model and $\langle\hat{\phi}(k)\hat{\phi}(-k)\rangle\ (\beta, z)$ the effective potential function of the lattice Coulomb gas, in ν-dim. momentum space. Then

1)
$$\langle\hat{S}(k)\hat{S}(-k)\rangle_V(\beta^{-1}) = \langle\hat{\phi}(k)\hat{\phi}(-k)\rangle\ (\beta, z = \infty)$$
$$\leqq \langle\hat{\phi}(k)\hat{\phi}(-k)\rangle\ (\beta, z) \leqq O(\beta k^{-2}).$$

2) In any V-model $\langle\hat{S}(k)\hat{S}(-k)\rangle_V(\beta^{-1})$ is monotone increasing in β.

3) in the VV-model $\langle|da(k)|^2\rangle_{VV}(\beta^{-1})$ is monotone increasing in β and $\leq O(\beta)$.

Remark: The V- and the VV-model satisfy "inverse" Lebowitz inequalities [38].
This theorem follows from Theorems II.2, II.3, III.5.

Some consequences of it are:

If the susceptibilities $\chi_V(\beta)$ and $\chi_{VV}(\beta)(= [k^{-2}<|da(k)|^2>_{VV}(\beta)]_{k=0})$ of the V-resp. VV-model are infinite for some $\beta = \beta_c$ they are infinite for all $\beta \leq \beta_c$, and screening in the $\hat{V}$- resp. $\hat{VV}$-model disappears. Furthermore

$$<\hat{\phi}(0)\hat{\phi}(0)>(\beta,z) \geq \chi_V(\beta^{-1}), \forall (\beta,z),$$

so that the ν dim. lattice Coulomb gas II.6.a) has a higher order phase transition (break down of Debye screening) if the V-(resp. $\hat{V}$-) model have one.

Furthermore, the connection between the $\nu = 2$ V-model and the $\nu = 2$ rotator suggests [36,73] that the $\nu = 2$ rotator has a higher order phase transition, provided the $\nu = 2$ V-model has one. Motivated by the results of Section III, the renormalization group [36] and [67,27] we make the following

<u>Conjecture III.9</u>, [36,67,73,27]:

1) The $\nu = 2$ rotator, V-model and lattice Coulomb gas II.6.a) have a critical interval (β_c, ∞), $\beta_c < \infty$ on which $m(\beta) = 0$ and the susceptibility is infinite; (see also Thm. III.2.2)).

2) The $\nu = 3$ U(1) lattice gauge theory [27], the $\nu = 3$ V-model and, presumably, the $\nu = 3$ lattice Coulomb gas have some form of screening for all $\beta < \infty$. However, the $\nu = 3$ VV-model has a phase transition [28].

3) The $\nu = 4$ U(1) lattice gauge theory and the $\nu = 4$ VV-model [27] have a phase transition of the form described in 1).

<u>Remarks</u>: 1) From Theorem III.8 (see also Thm. III.2.2)) we know that it suffices to exhibit one β_c such that $m(\beta_c) = 0$, $(\chi(\beta_c) = \infty)$. 2) Results of Glimm and Jaffe [27,79] will probably soon provide a proof of 3) and possibly of 1), too. See also [67].

An important <u>open problem</u> is to rigorously investigate the scaling properties of these models for $\beta \geq \beta_c$ (once $\beta_c < \infty$ is established) and their continuum limits.

Finally we come to some <u>comments on the $\nu = 2$ abelian Higgs model</u> in the approximation of Section II.4, (II.39). (We refer the reader to II.4 for the definition of the approximate, effective field theory for the vorticity, the angle θ and the charge ratio q). In the approximation (II.39) to the $\nu = 2$ continuum abelian Higgs model the following results are rigorous [7,28,6]:

1) For $q = 0$ and $\theta = 0$ the expectation $<\text{-}>_S$ has cluster properties (follows from (III.3) and [80]), i.e. the vacuum is unique [52], for all $\hbar$ and λ. For small $\hbar$ and all λ or small λ and all $\hbar$ they are <u>exponential</u>.

2) For $q = 0$ and $\theta = \pi$ we find, for small $\hbar$ and large λ, a first order phase transition in the vorticity and two different vacua (corresp. to $\pi+0$, $\pi-0$) with opposite <u>spontaneous vorticity</u>; (this follows from a Peierls argument [81,18,61]).

We note that 1) also holds for an abelian Higgs model on the lattice [16], resp. its

Villain approximation [28] (which we call H-V model).

The standard lattice Higgs, resp. H-V model only gives an analogue of the $\theta = 0$ vacuum. However a modified lattice model [28] gives the θ-vacua; it has a first order phase transition at $\theta = \pi$, and the existence of two different Gibbs expectations (with opposite spontaneous "vorticity") can be proven rigorously in some range of coupling constants [28]. In particular, the usual Higgs mechanism can be proven to occur for arbitrary coupling constants only for the $\theta = 0$ H-V model. At the critical point of the $\theta = \pi$ theory the Higgs mechanism breaks down. A heuristic approximation to a non-abelian Higgs model with instantons in four space-time dimensions (similar in spirit to the approximation (II.39) of Section II.4 with $q = 0$ - one may think of an SU(2) Higgs model [8] without fermions) suggests that a first order phase transition accompanied by spontaneous instanton density and the break down of the Higgs mechanism at the critical point may be typical features of the $\theta = \pi$ theory; see [28]. However, for the three - or more dimensional (abelian) H-V models there is only one vacuum (equilibrium state), and the Higgs mechanism occurs for arbitrary values of the coupling constants [28].

3) For the two dimensional, abelian Higgs model coupled to massless fermions in the approximation II.4, (II.39), i.e. for $q > 0$, and for $0 < \hbar q^2 << 1$ and large enough λ a slight variation of Brydges' results [76] (see Remark 2 following Theorem III.7) gives exponential screening which can be interpreted as dynamical mass generation for the fermions [13,7]. If Conjecture III.9.1) is true this dynamical mass generation disappears at large value of $\hbar q^2$ (which plays the role of β in the $\nu = 2$ lattice Coulomb gas II.6.a));the collapse phenomenon encountered in model II.6.c) [40] and the renormalization group [36] predict $\hbar q^2 \gtrsim 4\pi$ as the critical value).

4) For arbitrary $q > 0$ chiral invariance is broken, and fermionic charges which are not an integer multiple of the charge of the Higgs scalar are confined. See [13,7].

Remark: The resulting picture for the $\nu = 2$ continuum abelian Higgs model has similarities to the one found for $\nu = 2$ massive QED in [82,5].

Open problems:

1) Construct the $\nu = 2$ continuum abelian Higgs model rigorously and show the approximation II.4, (II.39) discussed above gives qualitatively correct results.

2) Which version of gauge-invariant lattice approximation to this model has a continuum limit equal to it?

3) Does the $\nu = 2$ Higgs model teach us something about more interesting gauge theories with instantons?

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