GRAVITATIONAL, DILATONIC AND AXIONIC RADIATIVE DAMPING OF COSMIC STRINGS

Alessandra BUONANNO and Thibault DAMOUR

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Février 1998

IHES/P/98/06

Submitted for publication to Physical Review D; gr-qc 9801105

Gravitational, dilatonic and axionic radiative damping of cosmic strings,
A. Buonanno and T. Damour.

Corrigendum:

the tension renormalization result, Eq. (3.24), should read

$$\mu(\delta) = \mu_{\mathrm{R}} + (4\,\alpha^2\,G\,\mu^2 - 4\,G\,\lambda^2)\,\log\left(\frac{\Delta_{\mathrm{R}}}{\delta}\right)\,,$$

see gr-qc/9801105 v2, and hep-th/9803025.

Gravitational, dilatonic and axionic radiative damping of cosmic strings

Alessandra Buonanno[1] and Thibault Damour[1,2]

[1] *Institut des Hautes Etudes Scientifiques, 91440 Bures-sur-Yvette, France*

[2] *DARC, CNRS-Observatoire de Paris, 92195 Meudon, France*

Abstract

We study the radiation reaction on cosmic strings due to the emission of dilatonic, gravitational and axionic waves. After discussing the renormalization of the string tension induced by time-symmetric self-interactions (thereby correcting errors in the literature concerning the effect of the dilaton field), we concentrate on the finite radiation damping force associated with the half-retarded minus half-advanced "reactive" fields. We revisit a recent proposal of using a "local back reaction approximation" for the reactive fields. We find, contrary to previous claims, that this proposal leads to *antidamping* in the case of the axionic field, and to *zero* (integrated) *damping* in the case of the gravitational field. One gets normal *positive damping* only in the case of the dilatonic field. We propose to use a suitably modified version of the local dilatonic radiation reaction as a substitute for the exact (non local) gravitational radiation reaction. The incorporation of such a local approximation to gravitational radiation reaction should allow one to complete, in a computationally non intensive way, string network simulations and to give better estimates of the amount and spectrum of gravitational radiation emitted by a cosmologically evolving network of massive strings.

I. INTRODUCTION

Cosmic strings are predicted, within a wide class of elementary particle models, to form at phase transitions in the early universe [1], [2]. The creation of a network of cosmic strings can have important astrophysical consequence, notably for the formation of structure in the universe [3], [4]. A network of cosmic strings might also be a copious source of the various fields or quanta to which they are coupled. Oscillating loops of cosmic string can generate observationally significant stochastic backgrounds of: gravitational waves [5], massless Goldstone bosons [6], light axions [7], [8], or light dilatons [9]. The amount of radiation emitted by cosmic strings depend: (i) on the nature of the considered field; (ii) on the coupling parameter of this field to the string; (iii) on the dynamics of individual strings; and (iv) on the distribution function and cosmological evolution of the string network. It is important to note that the latter network distribution function in turn depends on the radiation properties of strings. Indeed, numerical simulations suggest that the characteristic size of the loops chopped off long strings at the epoch t will be on order of the smallest structures on the long strings, which is itself arguably determined by radiative back reaction [10], [11]. For instance, if one considers grand unified theory (GUT) scale strings, with tension $\mu \sim \Lambda_{\rm GUT}^2$, gravitational radiation (possibly together with dilaton radiation which has a comparable magnitude [9]) will be the dominant radiative mechanism, and will be characterized by the coupling parameter $G\mu \sim (\Lambda_{\rm GUT}/m_{\rm Planck})^2 \sim 10^{-6}$. It is then natural to expect that the same dimensionless parameter $G\mu$ will control the radiative decay of the small scale structure (crinkles and kinks) on the horizon-sized strings, thereby determining also the characteristic size relative to the horizon of the small loops produced by the intersections of long strings: $\ell_{\rm loops} \equiv \widehat{\alpha}\, ct$, with $\widehat{\alpha} \sim \Gamma_{\rm kink}\, G\mu$, $\Gamma_{\rm kink}$ being some dimensionless measure of the network-averaged radiation efficiency of kinky strings [10], [11], [12], [13]. If one considers "global" strings, i.e. strings formed when a global symmetry is broken at a mass scale f_a, emission of the Goldstone boson associated to this symmetry breaking will be the dominant radiation damping mechanism and will be characterized by the dimensionless parameter $f_a^2/\mu_{\rm effective} \sim (\log(L/\delta))^{-1} \sim 10^{-2}$, where the effective tension $\mu_{\rm effective}$ is renormalized by a large logarithm (see, e.g., [2]).

Present numerical simulations of string networks do not take into account the effect of radiative damping on the actual string motion. The above mentioned argument concluding in the case of GUT strings to the link $\widehat{\alpha} \sim \Gamma_{\text{kink}} G\mu$ between the loop size and radiative effects has been justified by Quashnock and Spergel [11] who studied the gravitational back reaction of a sample of cosmic string loops. However, their "exact", non local approach to gravitational back reaction is numerically so demanding that there is little prospect to implementing it in full string network simulations. This lack of consideration of the dynamical effects of radiative damping is a major deficiency of string network simulations which leaves unanswered crucial questions such as: Is the string distribution function attracted to a solution which "scales" with the horizon size down to the smallest structures? and What is the precise amount and spectrum of the gravitational (or axionic, in the case of global strings) radiation emitted by the combined distribution of small loops and long strings?

Recently, Battye and Shellard [14], [15] proposed a new, computationally much less intensive, approach to the radiative back reaction of (global) strings. They proposed a "local back reaction approximation" based on an analogy with the well-known Abraham-Lorentz-Dirac result for a self-interacting electron. Their approach assumes that the dominant contribution to the back reaction force density at a certain string point comes from string segments in the immediate vicinity of that point. They have endeavored to justify their approach by combining analytical results (concerning approximate expressions of the local, axionic radiative damping force) and numerical simulations (comparison between the effect of their local back reaction and a direct field-theory evolution of some global string solutions).

In this paper, we revisit the problem of the back reaction of cosmic strings associated to the emission of gravitational, dilatonic and axionic fields, with particular emphasis on the "local back reaction approximation" of Battye and Shellard. Our results will be somewhat at variance with previous ones in the literature. First, we find that the results of Ref. [16] for the (renormalizable) *infinite* contributions to the string tension due to its coupling to the dilaton and axion fields are incorrect. Second, we find that the *finite* contributions to the axionic self-field given in Refs. [14], [15] are in error. We give below the correct results, and we find that the resulting reaction force leads to *antidamping* rather than damping as claimed in Refs. [14], [15]. We also investigate below the local approximations to gravitational and dilatonic self-forces and find *zero damping* in the gravitational case, and a normal, *positive damping* for the dilatonic case. These paradoxical results (which are explained in detail below) show the severe limitations of the "local back reaction approximation" of Refs. [14], [15]. However, we argue below that the meaningful (positive-damping) dilatonic local back reaction force can be used, after some modification, as a convenient effective *substitute* for the exact (non local) gravitational back reaction force.

In the next section, we present our formalism for treating self-interactions of strings. We describe in Section III our results for the renormalizable, divergent self-action terms, and, in Section IV, our results for the finite contributions to the "local" reaction force. In Section V we indicate how the local dilatonic damping force could be used in full-scale network simulations to simulate the dynamical effects of gravitational radiation. Section VI contains our conclusions. Some technical details are relegated to the Appendix.

As signs will play a crucial role below, let us emphasize that we use the "mostly positive" signature $(-,+,+,+)$ for the space-time metric $g_{\mu\nu}$ $(\mu,\nu = 0,1,2,3)$, and the corresponding $(-,+)$ signature for the worldsheet metric γ_{ab} $(a,b = 0,1$ being worldsheet indices).

II. COSMIC STRINGS INTERACTING WITH GRAVITATIONAL, DILATONIC AND AXIONIC FIELDS

We consider a closed string $z^\mu(\sigma^a)$ (with $\sigma^0 = \tau$, $\sigma^1 = \sigma$, $0 \leq \sigma < L$) interacting with gravitational $g_{\mu\nu}(x^\lambda) \equiv \eta_{\mu\nu} + h_{\mu\nu}(x^\lambda)$, dilatonic $\varphi(x)$, and axionic (Kalb-Ramond) $B_{\mu\nu}(x)$ fields. The action for the string coupled to $g_{\mu\nu}$, φ and $B_{\mu\nu}$ reads

$$S_s = -\int \mu(\varphi)\, dA - \frac{\lambda}{2}\int B_{\mu\nu}\, dz^\mu \wedge dz^\nu\,. \tag{2.1}$$

Here $dA = \sqrt{\gamma}\, d^2\sigma$ (with $\gamma \equiv -\det\gamma_{ab}$; $\gamma_{ab} \equiv g_{\mu\nu}(z)\,\partial_a z^\mu\,\partial_b z^\nu$ denoting the metric induced on the worldsheet) is the string area element and the dilaton dependence of the string tension μ can be taken to be exponential

$$\mu(\varphi) = \mu\, e^{2\alpha\varphi}\,. \tag{2.2}$$

At the linearized approximation where we shall work the form (2.2) is equivalent to a linear coupling $\mu(\varphi) \simeq \mu(1 + 2\alpha\varphi)$. The dimensionless parameter α measures the strength of the coupling of φ to cosmic strings (our notation agrees with the tensor-scalar notation of Ref. [17]), while the coupling strength of the axion field is measured by the parameter λ with dimension $(\text{mass})^2$. Due to our "gravitational normalization" of the kinetic term of $B_{\mu\nu}$, the link between λ and the mass scale f_a used in Refs. [14], [15] is $2G\,\lambda^2 = \pi f_a^2$.

The action for the fields is

$$S_f = \frac{1}{16\pi G} \int d^4x \sqrt{g} \left[\mathcal{R} - 2 \nabla^\mu \varphi \nabla_\mu \varphi - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} \right] , \tag{2.3}$$

where $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$, $g \equiv -\det(g_{\mu\nu})$, and where we use the curvature conventions $\mathcal{R}^\mu{}_{\nu\rho\sigma} = \partial_\rho \Gamma^\mu_{\nu\sigma} - \ldots$, $\mathcal{R}_{\mu\nu} = \mathcal{R}^\rho{}_{\mu\rho\nu}$. With this notation, a tree-level coupled fundamental string (of string theory) has $\mu(\varphi) = \mu\, e^{2\varphi}$ (so that $\alpha = 1$), $\lambda = \mu$ and an extra factor $e^{-4\varphi}$ (in 4 dimensions) multiplying the kinetic term of the B field.

More explicitly, the string action (2.1) can be written (using the Polyakov form) as

$$S_s = -\frac{\mu}{2} \int d^2\sigma\, e^{2\alpha\varphi} \sqrt{\widehat{\gamma}}\, \widehat{\gamma}^{ab}\, \partial_a z^\mu\, \partial_b z^\nu\, g_{\mu\nu} - \frac{\lambda}{2} \int d^2\sigma \epsilon^{ab}\, \partial_a z^\mu\, \partial_b z^\nu\, B_{\mu\nu} , \tag{2.4}$$

where the worldsheet metric $\widehat{\gamma}_{ab}$ must be independently varied and where $\epsilon^{01} = -1$, $\epsilon^{10} = 1$. The equation of motion of $\widehat{\gamma}_{ab}$ is the constraint that it be conformal to the induced metric $\gamma_{ab} = g_{\mu\nu}(z)\, \partial_a z^\mu\, \partial_b z^\nu$. In the following, we shall often use the conformal gauge $\sqrt{\widehat{\gamma}}\, \widehat{\gamma}^{ab} = \sqrt{\gamma}\, \gamma^{ab} = \eta^{ab}$ (where $\eta^{00} = -1$, $\eta^{11} = +1$), i.e. we shall choose the (τ, σ) parametrization of the worldsheet so that

$$\dot{z}^\mu\, \dot{z}^\nu\, g_{\mu\nu} + z'^\mu\, z'^\nu\, g_{\mu\nu} = 0 , \qquad \dot{z}^\mu\, z'^\nu\, g_{\mu\nu} = 0 . \tag{2.5}$$

Here $\dot{z} \equiv \partial_0 z \equiv \partial z / \partial \tau$ and $z' \equiv \partial_1 z \equiv \partial z / \partial \sigma$. Note also the expression, in this gauge, of the worldsheet volume density

$$\sqrt{\gamma} = g_{\mu\nu}\, z'^\mu\, z'^\nu = -g_{\mu\nu}\, \dot{z}^\mu\, \dot{z}^\nu . \tag{2.6}$$

Let us note that the string contribution to the energy-momentum tensor,

$$T^{\mu\nu} = \frac{2}{\sqrt{g}} \frac{\delta S_m}{\delta g_{\mu\nu}} , \tag{2.7}$$

reads

$$T^{\mu\nu} = \frac{\mu}{\sqrt{g}} \int d^2\sigma\, e^{2\alpha\varphi}\, U^{\mu\nu}\, \delta^4(x - z(\sigma)) , \tag{2.8}$$

where $\int d^4x\, \delta^4(x) = 1$ and

$$U^{\mu\nu} \equiv -\sqrt{\gamma}\, \gamma^{ab}\, \partial_a z^\mu\, \partial_b z^\nu = \dot{z}^\mu\, \dot{z}^\nu - z'^\mu\, z'^\nu \qquad \text{(in conformal gauge)} \tag{2.9}$$

is the "vertex operator" for the interaction of the string with the gravitational field $g_{\mu\nu}$. The corresponding vertex operator for the interaction with the dilaton φ is simply the trace $U \equiv g_{\mu\nu}\, U^{\mu\nu}$, while the one corresponding to the axion $B_{\mu\nu}$ is

$$V^{\mu\nu} \equiv -\epsilon^{ab}\,\partial_a\, z^\mu\,\partial_b\, z^\nu = \dot{z}^\mu\, z'^\nu - \dot{z}^\nu\, z'^\mu\,. \tag{2.10}$$

The exact equation of motion of the string can be written (in any worldsheet gauge) as

$$-\,\mu\,\partial_a(\sqrt{\gamma}\,\gamma^{ab}\,\partial_b z^\mu) = \mathcal{F}^\mu\,, \tag{2.11}$$

where the force density $\mathcal{F}^\mu$ is given by

$$\mathcal{F}^\mu = -\mu\,\Gamma^\mu_{\alpha\beta}\,U^{\alpha\beta} - 2\mu\,\alpha\,(\partial_\alpha\varphi)\,U^{\alpha\mu} + \mu\,\alpha\,U\,g^{\mu\alpha}\,(\partial_\alpha\varphi) + \frac{1}{2}\,\lambda\,V^{\beta\gamma}\,g^{\mu\alpha}\,H_{\alpha\beta\gamma}\,. \tag{2.12}$$

Note that, in the conformal gauge, the equations of motion of the string are simply

$$\ddot{z}^\mu - z^{\mu\prime\prime} = \mathcal{F}^\mu\,, \tag{2.13}$$

but that one must remember the $g_{\mu\nu}$-dependence of the constraints (2.5). In the linearized approximation (i.e. at first order in the couplings to the fields) the force density reads

$$\mathcal{F}_\mu \equiv \eta_{\mu\nu}\,\mathcal{F}^\nu = \mathcal{F}^\varphi_\mu + \mathcal{F}^h_\mu + \mathcal{F}^B_\mu\,, \tag{2.14}$$

where

$$\mathcal{F}^\varphi_\mu = \alpha\,\mu\,U\,\partial_\mu\varphi - 2\alpha\,\mu\,g_{\mu\alpha}\,U^{\alpha\beta}\,\partial_\beta\varphi\,, \tag{2.15}$$

$$\mathcal{F}^h_\mu = \frac{\mu}{2}\,U^{\alpha\beta}\,\partial_\mu h_{\alpha\beta} - \mu\,U^{\alpha\beta}\,\partial_\alpha h_{\beta\mu}\,, \tag{2.16}$$

$$\mathcal{F}^B_\mu = \frac{\lambda}{2}\,V^{\alpha\beta}\,\partial_\mu B_{\alpha\beta} + \lambda\,V^{\alpha\beta}\,\partial_\alpha B_{\beta\mu}\,. \tag{2.17}$$

When fixing the gauge freedom of the gravitational and axionic fields in the usual way ($g^{\alpha\beta}\,\Gamma^\mu_{\alpha\beta} = 0$; $\nabla^\nu\,B_{\mu\nu} = 0$), the field equations derived from S_f read, at linearized order,

$$\Box\varphi(x) = -4\pi\int d^2\sigma\,\Sigma^\varphi\,\delta^4(x - z(\sigma))\,, \tag{2.18}$$

$$\Box h_{\mu\nu}(x) = -4\pi\int d^2\sigma\,\Sigma^h_{\mu\nu}\,\delta^4(x - z(\sigma))\,, \tag{2.19}$$

$$\Box B_{\mu\nu}(x) = -4\pi\int d^2\sigma\,\Sigma^B_{\mu\nu}\,\delta^4(x - z(\sigma))\,, \tag{2.20}$$

where the corresponding source terms are defined as

$$\Sigma^\varphi = \alpha G\,\mu\,U\,, \qquad \Sigma^h_{\mu\nu} = 4\,G\,\mu\,\widetilde{U}_{\mu\nu}\,, \qquad \Sigma^B_{\mu\nu} = 4\,G\,\lambda\,V_{\mu\nu}\,. \tag{2.21}$$

Here, $\widetilde{U}_{\mu\nu} \equiv U_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} U$, and, in the present approximation, the vertex operators entering these source terms are simply $U_{\mu\nu} = \dot{z}_\mu \dot{z}_\nu - z'_\mu z'_\nu$, $U = \eta^{\mu\nu} U_{\mu\nu}$, $V_{\mu\nu} = \dot{z}_\mu z'_\nu - \dot{z}_\nu z'_\mu$, where we freely use the flat metric $\eta_{\mu\nu}$ to move indices.

The field equations (2.18)–(2.20) are classically solved by introducing the four dimensional retarded Green function

$$G_{\rm ret}(x-y) = \frac{1}{2\pi}\,\theta(x^0-y^0)\,\delta((x-y)^2)\,; \tag{2.22}$$

$$\Box G_{\rm ret}(x-y) = -\delta^{(4)}(x-y)\,. \tag{2.23}$$

This "retarded" Green function incorporates the physical boundary condition of the non-existence of preexisting radiation converging from infinity toward the string source. The (unphysical) time-reverse of $G_{\rm ret}$ is the "advanced" Green function

$$G_{\rm adv}(x-y) = \frac{1}{2\pi}\,\theta(-(x^0-y^0))\,\delta((x-y)^2)\,, \tag{2.24}$$

$$= G_{\rm ret}\,(y-x)\,. \tag{2.25}$$

Let us consider, as a general model for Eqs. (2.18)–(2.20), the generic field equation

$$\Box A(x) = -4\pi \int d^2\sigma\, \Sigma(\sigma)\, \delta^4(x - z(\sigma))\,. \tag{2.26}$$

Its classical solution reads

$$A_{\rm ret}(x) = +4\pi \int d\sigma'\, d\tau'\, \Sigma(\sigma',\tau')\, G_{\rm ret}(x - z(\sigma',\tau'))\,. \tag{2.27}$$

Applying the formula $\delta\,(F(\tau')) = \sum_{\tau_0} \delta\,(\tau'-\tau_0)/|\partial\, F(\tau_0)/\partial\,\tau_0|$, where the sum runs over all the solutions τ_0 of $F(\tau') = 0$, one can effectuate the integral over τ' in Eq. (2.27) with the result

$$A_{\rm ret}(x) = \int d\sigma' \left(\frac{\Sigma(\sigma',\tau')}{|\Omega\cdot\dot{z}|}\right)\Big|_{\tau'=\tau_{\rm ret}}\,. \tag{2.28}$$

Here, we have defined $\Omega^\mu\,(x,\tau') \equiv x^\mu - z^\mu\,(\tau')$, and $\tau_{\rm ret}\,(x^\mu)$ as being the retarded solution (i.e. such that $x^0 - z^0\,(\tau_{\rm ret}(x)) > 0$) of $\eta_{\mu\nu}\,\Omega^\mu(\tau')\,\Omega^\nu(\tau') = 0$. In the following, we use also the quantity $\partial_\mu\, A_{\rm ret}$ which, after using the formula

$$\partial_\mu\, \delta(F(x,\tau')) = \partial_\mu\, F\;\; \delta'(F(x,\tau')) = \frac{\partial_\mu\, F}{(\partial\, F/\partial\,\tau')}\,\frac{\partial\,\delta(F)}{\partial\,\tau'} \tag{2.29}$$

and integrating by parts, can be written as

$$\partial_\mu A_{\rm ret}(x) = \int d\sigma' \left[\frac{1}{|\Omega\cdot\dot z|} \frac{d}{d\tau'} \left(\frac{\Omega_\mu\, \Sigma(\sigma',\tau')}{\Omega\cdot\dot z} \right) \right]_{|_{\tau'=\tau_{\rm ret}}} . \tag{2.30}$$

The corresponding results for the advanced fields are

$$A_{\rm adv}(x) = \int d\sigma' \left(\frac{\Sigma(\sigma',\tau')}{|\Omega\cdot\dot z|} \right)_{|_{\tau'=\tau_{\rm adv}}} , \tag{2.31}$$

$$\partial_\mu\, A_{\rm adv}(x) = \int d\sigma' \left[\frac{1}{|\Omega\cdot\dot z|} \frac{d}{d\tau'} \left(\frac{\Omega_\mu\, \Sigma(\sigma',\tau')}{\Omega\cdot\dot z} \right) \right]_{|_{\tau'=\tau_{\rm adv}}} , \tag{2.32}$$

where $\tau_{\rm adv}(x^\mu)$ is the advanced solution of $\eta_{\mu\nu}\, \Omega^\mu(\tau')\, \Omega^\nu(\tau') = 0$. Note that the scalar product $\Omega\cdot\dot z$ is *negative* for $\tau' = \tau_{\rm ret}$ and *positive* for $\tau' = \tau_{\rm adv}$.

III. RENORMALIZATION OF THE STRING TENSION FROM DIVERGENT SELF-INTERACTIONS

We consider the problem of a cosmic string interacting with its own (linearized) gravitational, dilatonic and axionic fields. The equations of motion of such a string contain a force density $\mathcal{F}^\mu$ obtained by inserting the retarded fields $\varphi^{\rm ret}(x)$, $h^{\rm ret}_{\mu\nu}(x)$, $B^{\rm ret}_{\mu\nu}(x)$ in Eqs. (2.15)–(2.17) and evaluating the result at a point z^μ on the string. As in the case of a self-interacting point particle, the force $\mathcal{F}^\mu(x=z)$ is infinite because of the divergent contribution generated when the source point $z^\mu(\tau',\sigma')$ coincides with the field point $x^\mu = z^\mu(\tau,\sigma)$. It was emphasized long ago by Dirac [18], in the case of an electron moving in its own electromagnetic field, that this problem can be cured by renormalizing the mass, thereby absorbing the divergent part of the self-force. More precisely, Dirac introduced a cut-off radius δ around the electron and found a corresponding (ultraviolet divergent) self-force $\mathcal{F}^\mu(\delta) = -(e^2/2\,\delta)\, \ddot z^\mu + \mathcal{F}^\mu_R$ where $\mathcal{F}^\mu_R$ is a finite (renormalized) contribution. If the mass of the electron plus its δ-surrounding depends on δ according to

$$m(\delta) = m_R - \frac{e^2}{2\,\delta}, \tag{3.1}$$

where m_R denotes a finite, "renormalized" mass, the ultraviolet divergent equations of motion $m(\delta)\, \ddot z^\mu = \mathcal{F}^\mu(\delta)$ give the finite result $m_R\, \ddot z^\mu = \mathcal{F}^\mu_R$. Note that the δ-dependence of $m(\delta)$ (for a fixed m_R) is compatible with the idea that $m(\delta)$ represents the total mass-energy of the particle plus that of the electromagnetic field contained within the radius $\delta : m(\delta_2) - m(\delta_1) = + \int_{\delta_1}^{\delta_2} d^3x\, (8\pi)^{-1}\, (e/r^2)^2$. Dirac also found that the remaining finite self-force was given by (using a proper-time normalization of $\tau : \dot{z}^2 = \eta_{\mu\nu}\, dz^\mu/d\tau\, dz^\nu/d\tau = -1$)

$$\mathcal{F}_R^\mu = \mathcal{F}_{\text{reac}}^\mu \equiv \frac{1}{2}\,(F_{\text{ret}}^{\mu\nu} - F_{\text{adv}}^{\mu\nu})\,\dot{z}_\nu = \frac{2}{3}\,e^2(\dddot{z}^\mu + (\dot{z}\cdot\dddot{z})\,\dot{z}^\mu)\,. \tag{3.2}$$

The analogous problem for self-interacting cosmic strings has been studied by Dabholkar and Quashnock [19] for the coupling to the axion field (see also [15]), and by Copeland, Haws and Hindmarsh [16] for the couplings to gravitational, dilatonic and axionic fields. We found, however, that the results of Ref. [16] are in error. For completeness, let us indicate our results for the renormalization caused by the three fields.

The divergent contributions to the self-interaction force come from the fact that the various field derivatives $\partial_\alpha\,\varphi(x)$, $\partial_\alpha\, h_{\beta\gamma}(x)$, $\partial_\alpha\, B_{\beta\gamma}(x)$ blow up when the field point x sits on the worldsheet: $x^\mu \to z^\mu(\tau,\sigma)$. To give a meaning to Eq. (2.30) when $x \to z(\tau,\sigma)$ we introduce an ultraviolet cutoff δ in the σ'-integration, i.e. we replace the integral over a full period of σ', $\int_{\sigma_0}^{\sigma_0+L} d\sigma'$, on the right-hand side of Eq. (2.30) by $\int_{\sigma_0}^{\sigma-\delta} d\sigma' + \int_{\sigma+\delta}^{\sigma_0+L} d\sigma'$. We then need the expansions in powers of $\sigma'-\sigma$ and $\tau'-\tau$ of all the quantities entering Eq. (2.30):

$$\Omega_\mu(\tau',\sigma') \simeq -(\sigma'-\sigma)\,z'_\mu - (\tau'-\tau)\,\dot{z}_\mu - \frac{1}{2}(\sigma'-\sigma)^2\,z''_\mu - \frac{1}{2}(\tau'-\tau)^2\,\ddot{z}_\mu$$
$$-(\sigma'-\sigma)\,(\tau'-\tau)\,\dot{z}'_\mu\,, \tag{3.3}$$

$$(\Omega\cdot z)(\tau',\sigma') \simeq -(\tau'-\tau)\,\dot{z}^2 - \frac{3}{2}(\tau'-\tau)^2\,(\dot{z}\cdot\ddot{z}) + \frac{1}{2}(\sigma'-\sigma)^2\,(z''\cdot\dot{z})$$
$$+(\sigma'-\sigma)\,(\tau'-\tau)\,(\ddot{z}\cdot z')\,, \tag{3.4}$$

$$\dot{z}_\mu(\tau',\sigma') \simeq \dot{z}_\mu + (\tau'-\tau)\,\ddot{z}_\mu + (\sigma'-\sigma)\,\dot{z}'_\mu\,. \tag{3.5}$$

Inserting these expansions in Eq. (2.30) we get

$$\partial_\mu A_{\text{ret}}(z) = \frac{1}{(\dot{z}^2)^2}\log\left(\frac{1}{\delta}\right)\Big[-\Sigma\,\ddot{z}_\mu + \Sigma\,z''_\mu + 4\Sigma\,z'_\mu\left(\frac{z''\cdot z'}{\dot{z}^2}\right) + 4\Sigma\,\dot{z}_\mu\left(\frac{\ddot{z}\cdot\dot{z}}{\dot{z}^2}\right)$$
$$+2\Sigma'\,z'_\mu - 2\dot{\Sigma}\,\dot{z}_\mu\Big] \quad + \quad \text{finite terms}\,. \tag{3.6}$$

The rather complicated-looking terms proportional to $(z'\cdot z'')/\dot{z}^2$ and $(\dot{z}\cdot\ddot{z})/\dot{z}^2$ in Eq. (3.6) are, actually, "connection" terms linked to the fact that the source Σ is a worldsheet density (conformal weight 2) rather than a worldsheet scalar (conformal weight 0). Let us associate to each source Σ a corresponding worldsheet scalar defined by

$$S \equiv \frac{1}{\sqrt{\gamma}}\,\Sigma\,. \tag{3.7}$$

Here $\sqrt{\gamma} = (-\det\gamma_{ab})^{1/2}$ is the area-density $dA/d^2\sigma$, which reads, in conformal gauge: $\sqrt{\gamma} = z'^2 = -\dot{z}^2$. Then Eq. (3.6) simplifies to

$$\partial_\mu\,A_{\rm ret}(z) = \frac{1}{\sqrt{\gamma}}\,\log\left(\frac{1}{\delta}\right)\,\left[-S\,\ddot{z}_\mu + S\,z''_\mu - 2\,\dot{S}\,\dot{z}_\mu + 2\,S' z'_\mu\right] \quad + \quad \text{finite terms.} \tag{3.8}$$

Let us also note, for completeness, the corresponding results for $A_{\rm ret}(z)$:

$$A_{\rm ret}(z) = \frac{1}{\sqrt{\gamma}}\,\log\left(\frac{1}{\delta}\right)\,[2\,\Sigma] + \text{finite} = \log\left(\frac{1}{\delta}\right)\,[2\,S] + \text{finite}\,. \tag{3.9}$$

As a check on the above results one can verify that the divergent parts satisfy

$$\frac{\partial}{\partial\tau}\,A_{\rm ret}(z) = \dot{z}^\mu\,\partial_\mu\,A_{\rm ret}(z)\,, \tag{3.10}$$

$$\frac{\partial}{\partial\sigma}\,A_{\rm ret}(z) = z^{\mu\prime}\,\partial_\mu\,A_{\rm ret}(z)\,. \tag{3.11}$$

To check these links one must use the following consequence of the conformal gauge constraints $0 \equiv T_{ab} \equiv g_{\mu\nu}(z)\,\partial_a\,z^\mu\,\partial_b\,z^\nu - \frac{1}{2}\,\eta_{ab}\,\eta^{cd}\,g_{\mu\nu}(z)\,\partial_c\,z^\mu\,\partial_d\,z^\nu$:

$$0 = \sqrt{\gamma}\,\nabla_b\,T^b_a = g_{\mu\nu}(z)\,\partial_a\,z^\mu\,\eta^{cd}\,(\partial_{cd}\,z^\nu + \Gamma^\nu_{\alpha\beta}\,\partial_c\,z^\alpha\,\partial_d\,z^\beta)\,. \tag{3.12}$$

In a first-order contribution such as Eqs. (2.15)–(2.17) one can neglect $\mathcal{O}(h_{\mu\nu})$ terms in Eq. (3.12) which yields simply

$$\dot{z}^\mu(\ddot{z}_\mu - z''_\mu) = 0 = z'^\mu(\ddot{z}_\mu - z''_\mu).$$

Because of the logarithmic divergence entering Eq. (3.6) we need to introduce, besides the ultraviolet cutoff scale δ (which can be thought of as the width of the cosmic string), an arbitrary, finite, renormalization length scale Δ_R. Then, we can *define* precisely the "infinite part" (IP) of $\partial_\mu\,A_{\rm ret}(z)$, i.e. the part which blows up when $\delta \to 0$, by replacing in Eq. (3.6) the logarithm by $\log(\Delta_R/\delta)$, and by discarding any other finite contribution. To apply this definition to the three fields φ, $h_{\mu\nu}$, and $B_{\mu\nu}$, we need to use the corresponding sources, Eq. (2.21). Using the easily verified identities satisfied by the vertex operators,

$$U_{\mu\nu}\,\widetilde{U}^{\mu\nu} \equiv U_{\mu\nu}\,U^{\mu\nu} - \frac{1}{2}U^2 = 0\,, \quad U_{\mu\sigma}\,\widetilde{U}^{\nu\sigma} = 0\,, \quad U_{\mu\nu}\,\dot{\widetilde{U}}^{\mu\nu} = 0\,, \tag{3.13}$$

$$V_{\mu\nu}\,V^{\mu\nu} = -2(\dot{z}^2)^2\,, \quad \dot{V}_{\mu\nu}\,\dot{V}^{\mu\nu} = -2\ddot{z}^2\,\dot{z}^2 + 2\dot{z}^2\,\dot{z}'^2 - 4(\ddot{z}\cdot\dot{z})^2 - 4(\dot{z}'\cdot\dot{z})^2\,, \tag{3.14}$$

we obtain

$$\mathrm{IP}(\partial_\mu\varphi) = -2\alpha\,G\,\mu\,(\ddot{z}_\mu - z''_\mu)\,\frac{1}{\dot{z}^2}\,\log\left(\frac{\Delta_R}{\delta}\right)\,, \tag{3.15}$$

$$\mathrm{IP}(U^{\alpha\beta}\,\partial_\alpha h_{\beta\mu}) = -8G\,\mu\,(\ddot{z}_\mu - z''_\mu)\,\log\left(\frac{\Delta_R}{\delta}\right)\,, \tag{3.16}$$

$$\mathrm{IP}(U^{\alpha\beta}\,\partial_\mu h_{\alpha\beta}) = 0\,, \tag{3.17}$$

$$\mathrm{IP}(V^{\alpha\beta}\,\partial_\mu B_{\alpha\beta}) = 8G\,\lambda\,(\ddot{z}_\mu - z''_\mu)\,\log\left(\frac{\Delta_R}{\delta}\right)\,, \tag{3.18}$$

$$\mathrm{IP}(V^{\alpha\beta}\,\partial_\alpha B_{\beta\mu}) = -8G\,\lambda\,(\ddot{z}_\mu - z''_\mu)\,\log\left(\frac{\Delta_R}{\delta}\right)\,. \tag{3.19}$$

Finally, the infinite part of the various force densities (2.15)–(2.17) read

$$\mathrm{IP}(\mathcal{F}^\varphi_\mu) = -4\alpha^2\,G\,\mu^2\,(\ddot{z}_\mu - z''_\mu)\,\log\left(\frac{\Delta_R}{\delta}\right)\,, \tag{3.20}$$

$$\mathrm{IP}(\mathcal{F}^h_\mu) = 8G\,\mu^2\,(\ddot{z}_\mu - z''_\mu)\,\log\left(\frac{\Delta_R}{\delta}\right)\,, \tag{3.21}$$

$$\mathrm{IP}(\mathcal{F}^B_\mu) = -4G\,\lambda^2\,(\ddot{z}_\mu - z''_\mu)\,\log\left(\frac{\Delta_R}{\delta}\right)\,. \tag{3.22}$$

As in point-particle electrodynamics, the original ultraviolet-divergent equations of motion (where $\mu(\delta)$ denotes the "bare", regularized but not renormalized, string tension)

$$\mu(\delta)\,(\ddot{z}^\mu - z^{\mu\prime\prime}) = \mathcal{F}^\mu(\delta) \equiv \mathrm{IP}\,(\mathcal{F}^\mu(\delta)) + \mathrm{FP}\,(\mathcal{F}^\mu(\delta)) \tag{3.23}$$

are renormalizable. Analogously to Eq. (3.1), one must assume that the bare tension depends on δ according to

$$\mu(\delta) = \mu_R + \left[-4\,\alpha^2\,G\,\mu^2 - 4\,G\,\lambda^2 + 8\,G\,\mu^2\right]\,\log\left(\frac{\Delta_R}{\delta}\right)\,. \tag{3.24}$$

The equations of motion then contain only finite (i.e. δ-independent) quantities:

$$\mu_R\,(\ddot{z}^\mu - z^{\mu\prime\prime}) = \mathcal{F}^\mu_R\,. \tag{3.25}$$

The renormalized force density appearing on the right-hand side of Eq. (3.25) is defined as the "finite part" (FP) of the bare force density

$$\mathcal{F}^\mu_R \equiv \mathrm{FP}\,(\mathcal{F}^\mu) \equiv \lim_{\delta\to 0}\,[\mathcal{F}^\mu(\delta) - \mathrm{IP}\,(\mathcal{F}^\mu(\delta))]\,. \tag{3.26}$$

As in the electrodynamics case, the signs (and the actual values) of the tension renormalizations due to the φ and $B_{\mu\nu}$ fields are that expected from $\mu(\delta_2)-\mu(\delta_1) = + \int_{\delta_1}^{\delta_2} d^3x \, (T_\varphi^{00}+T_B^{00})$, with

$$T_\varphi^{\mu\nu} = \frac{1}{4\pi G} \left[\partial^\mu \varphi \, \partial^\nu \varphi - \frac{1}{2} \eta^{\mu\nu} (\partial\varphi)^2 \right] , \tag{3.27}$$

$$T_B^{\mu\nu} = \frac{1}{32\pi G} \left[H^{\mu\rho\sigma} \, H^\nu_{\ \rho\sigma} - \frac{1}{6} \eta^{\mu\nu} \, H^2 \right] . \tag{3.28}$$

Our purely dynamical calculation agrees with the results of Dabholkar et al. [20], [21] who discussed a field-energy approach to tension renormalization. The opposite sign (and the value) of the gravitational contribution in Eq. (3.24) also agrees with their calculation of the (non positive definite) gravitational energy pseudotensor. As emphasized by in [20], [21] the latter opposite sign allows for a compensation between gravity and the two other bosonic fields. In particular, the couplings of the bosonic fields of fundamental strings at string tree level, namely $\alpha = 1$ and $\lambda = \mu$ leads to a nonrenormalization of the string tension. Ref. [16] comments also on such a nonrenormalization, though they get it only accidentally right as their force-renormalization calculation differs from Eq. (3.24) in having a vanishing coefficient for the φ-contribution and a doubled one for the B-one. For what concerns the B-contribution to the μ-renormalization, we note that the result (3.24) confirms the similar dynamical calculations of Refs. [19], [14], [15].

One important distinguishing feature between the usual point-particle renormalization (3.1) and the string one (3.24) is the appearance of an arbitrary, renormalization length scale Δ_R. Note that, by definition, the bare (regularized) quantities $\mu(\delta)$, $\mathcal{F}^\mu(\delta)$ do not depend on the choice of Δ_R. By contrast, the renormalized quantities μ_R, $\mathcal{F}_R^\mu$ depend on the choice of Δ_R, but in such a way that the content of the renormalized equations of motion is left invariant at first order in the field couplings. Indeed,

$$\mu_R(\Delta'_R) = \mu_R(\Delta_R) + C \, \log \left(\frac{\Delta'_R}{\Delta_R} \right) , \tag{3.29}$$

$$\mathcal{F}_R^\mu(\Delta'_R) = \mathcal{F}_R^\mu(\Delta_R) + C(\ddot{z}^\mu - z^{\mu\prime\prime}) \, \log \left(\frac{\Delta'_R}{\Delta_R} \right) , \tag{3.30}$$

with $C = 4\,\alpha^2\,G\,\mu^2 + 4\,G\,\lambda^2 - 8\,G\,\mu^2 + \mathcal{O}(G^2)$. As we work only to first order in the field couplings, note that the quantity μ appearing in C can, formally, be considered as being a renormalized value, rather than the bare one, thereby leading to the renormalization group equation $\partial\,\mu_R/\partial \, \log \, \Delta_R = C(\mu_R)$. [The nonrenormalizability of the gravitational interaction makes it delicate to extend this argument to higher orders in G. By contrast, if we consider only a canonically normalized axionic field, with coupling $\sqrt{G}\,\lambda = \sqrt{\pi/2}\,f_a$, C does not depend on μ and the first-order renormalization result is exact.]

IV. RENORMALIZED FORCE DENSITY AND THE LOCAL BACK-REACTION APPROXIMATION

A. Reactive part of the self-force

At the linearized approximation, at which we work here, the renormalized force density $\mathcal{F}^\mu_R$, Eq. (3.26), is a linear (non local) functional of the field derivatives. Following Dirac [18] it is useful to decompose any field $A_{\rm ret}(x)$ in two parts:

$$A_{\rm ret}(x) = A_{\rm sym}(x) + A_{\rm reac}(x)\,, \tag{4.1}$$

$$A_{\rm sym}(x) \equiv \frac{1}{2}\,(A_{\rm ret}(x) + A_{\rm adv}(x))\,, \tag{4.2}$$

$$A_{\rm reac}(x) \equiv \frac{1}{2}\,A_{\rm rad}(x) \equiv \frac{1}{2}\,(A_{\rm ret}(x) - A_{\rm adv}(x))\,. \tag{4.3}$$

Note the definition of two fields, $A_{\rm reac}$ and $A_{\rm rad}$, differing by a factor 2, associated to the difference $A_{\rm ret} - A_{\rm adv}$. Both fields play a special role in the discussion below. They are both finite, as well as their derivatives, when considered at a point $x = z$ of the source. Therefore the contribution to the self-force corresponding to $A_{\rm reac}$ is finite and does not need to be renormalized. [Hence, we shall dispense in the following with the label R when considering $\mathcal{F}^\mu_R(A_{\rm reac})$.]

Let us first prove why, very generally, in the decomposition of the force corresponding to Eq. (4.1),

$$\mathcal{F}^\mu_R(A_{\rm ret}) = \mathcal{F}^\mu_R(A_{\rm sym}) + \mathcal{F}^\mu(A_{\rm reac}) \equiv \mathcal{F}^\mu_{R\,\rm sym} + \mathcal{F}^\mu_{\rm reac}\,, \tag{4.4}$$

the term $\mathcal{F}^\mu_{\rm reac}$ can be considered as defining the full radiation reaction force, responsible for draining out of the mechanical system on which it acts (the string in our case) the energy lost to infinity in the form of waves of the A field. Indeed, for any field (in the linear approximation) we can define a field energy-momentum tensor $T^{\mu\nu}_f(A)$ which is quadratic in (the derivatives of) A. The total energy tensor $T^{\mu\nu} = T^{\mu\nu}_s + T^{\mu\nu}_f$ (where $T^{\mu\nu}_s$ denotes the energy tensor of source) is conserved: $0 = \partial_\nu T^{\mu\nu}$. This leads to the equations of the source: $\partial_\nu T_s^{\mu\nu} = F^\mu(A)$ where $F^\mu(A) \equiv -\partial_\nu T_f^{\mu\nu}(A)$ represents the spacetime (rather than worldsheet) version of the force density acting on the source. [We work here with the bare force density.] Let us consider, as a formal simplification, the case where the coupling between the source and the field A is (adiabatically) turned off in the far past and the far future. Then the energy-momentum lost by the source during the entire interaction with the field, $P^\mu_{s\,\text{lost}} = -\int d^3x\, [T_s^{\mu 0}(+\infty) - T_s^{\mu 0}(-\infty)]$, can be written as

$$P^\mu_{s\,\text{lost}} = -\int d^4x\, \partial_\nu T_s^{\mu\nu} = -\int d^4x\, F^\mu(A) = P^\mu_{f\,\text{gained}}, \tag{4.5}$$

where $P^\mu_{f\,\text{gained}} = +\int d^3x\, [T_f^{\mu 0}(+\infty) - T_f^{\mu 0}(-\infty)]$ is the energy-momentum gained by the field. When applying this result to the usual interaction force $F^\mu(A_{\text{ret}}) = -\partial_\nu T_f^{\mu\nu}(A_{\text{ret}})$ one has zero energy in A_{ret} in the far past, so that $P^\mu_{s\,\text{lost}} = \int d^3x\, T_f^{\mu\nu}(A_{\text{ret}}(t=+\infty))$. The field energy momentum tensor $T_f^{\mu\nu}(A)$ is quadratic in the field and can always be written as the diagonal value of a symmetric quadratic form $T_f^{\mu\nu}(A) = Q^{\mu\nu}(A,A)$. It is easy to see that the generic structure $F^\mu(A) \equiv -\partial_\nu T_f^{\mu\nu}(A) = \mathcal{S}_A \cdot \partial A$, where $\mathcal{S}_A$ is a source term for the field A, and where the dot product denotes some contraction of indices, is generalized, when considering $Q^{\mu\nu}$ to: $-\partial_\nu Q^{\mu\nu}(A_1, A_2) = \frac{1}{2}[\mathcal{S}_{A_1} \cdot \partial A_2 + \mathcal{S}_{A_2} \cdot \partial A_1]$. We can apply this to the case where $A_1 = A_{\text{ret}}$ and $A_2 = A_{\text{rad}} = A_{\text{ret}} - A_{\text{adv}}$ (for which $\mathcal{S}_{A_2} = 0$) with the result:

$$\frac{1}{2}\mathcal{S} \cdot \partial A_{\text{rad}} = \mathcal{S} \cdot \partial A_{\text{reac}} = F^\mu(A_{\text{reac}}) = -\partial_\nu Q^{\mu\nu}(A_{\text{ret}}, A_{\text{rad}}), \tag{4.6}$$

where $\mathcal{S}$ is the usual source, and $F^\mu(A_{\text{reac}})$ the result of replacing A_{ret} by $A_{\text{reac}} = \frac{1}{2} A_{\text{rad}}$ in the usual force density. Integrating the latter formula over spacetime gives

$$-\int d^4x\, F^\mu(A_{\text{reac}}) = \int d^3x\, [Q^{\mu 0}(A_{\text{ret}}, A_{\text{rad}})|_{t=+\infty} - Q^{\mu 0}(A_{\text{ret}}, A_{\text{rad}})|_{t=-\infty}]. \tag{4.7}$$

Again, one has zero energy from the far past contribution (because $A_{\text{ret}}(-\infty) = 0$), while the far future contribution is simply, thanks to $A_{\text{adv}}(+\infty) = 0$, $Q^{\mu 0}(A_{\text{ret}}, A_{\text{ret}}) = T_f^{\mu 0}(A_{\text{ret}})$ so that

$$-\int d^4x\, F^\mu(A_{\text{reac}}) = \int d^3x\, T_f^{\mu 0}(A_{\text{ret}}(+\infty)) = -\int d^4x\, F^\mu(A_{\text{ret}}) = P^\mu_{s\,\text{lost}}. \tag{4.8}$$

This proves, for any field treated in the linear approximation, that the contribution to the self-force due to A_{reac} contains, when integrated over time, the full effect of radiation damping, ensuring conservation with the energy-momentum lost to radiation. The contribution $F^\mu(A_{\text{reac}})$ can be called the "reactive" part of the self-force $F^\mu(A_{\text{ret}})$.

Summarizing the results at this point, the renormalized self-interaction force (returning now to the worldsheet distributed force density) can be written as

$$\mathcal{F}^\mu_R = \mathrm{FP}_{\Delta_R}\mathcal{F}^\mu = \mathrm{FP}_{\Delta_R}\mathcal{F}^\mu_{\rm sym}(\delta) + \mathcal{F}^\mu_{\rm reac}\,, \tag{4.9}$$

where FP_{Δ_R} denotes Hadamard's Finite Part ("Partie Finie") operation [22] (i.e., in our case, the result of subtracting a term $\propto \log(\delta/\Delta_R)$ from the ultraviolet-cutoff integral $\mathcal{F}^\mu(\delta) = \int_{\sigma_0}^{\sigma-\delta} d\sigma'[\ldots] + \int_{\sigma+\delta}^{\sigma_0+L} d\sigma'[\ldots]$). Note that only the *symmetric* contribution, obtained by replacing $A_{\rm ret}$ by $A_{\rm sym} = \frac{1}{2}(A_{\rm ret} + A_{\rm adv})$ in the force density, needs to be renormalized, and thereby acquires a dependence on the arbitrary scale Δ_R. This symmetric contribution does not contribute, after integration over time, to the overall damping of the source. The finite (Δ_R-independent), reactive contribution $\mathcal{F}^\mu_{\rm reac} \equiv \mathcal{F}^\mu(A_{\rm reac})$ embodies (on the average) the full effect of radiation damping.

The advantage of the above decomposition is to isolate, very cleanly, the radiation damping force from the other non cumulative, self-interactions. Its disadvantage is to write the nonlocal, but *causal* self-force FP $\mathcal{F}^\mu(A_{\rm ret})$ as a sum of two *acausal* (meaning future-dependent) contributions. In principle one can work directly with the full, causal $\mathcal{F}^\mu_R$ (as done, e.g., in Ref. [11]), but this is computationally very intensive. [A simplification, used by the latter authors, is that the self-force $\mathcal{F}^\mu(\delta)$ becomes, as is clear from Eq. (3.23), finite as $\delta \to 0$ when evaluated on free-string trajectories, satisfying $\ddot{z}^\mu - z^{\mu\prime\prime} = 0$.] We shall follow Refs. [14], [15] in working only with the (finite) reactive force $\mathcal{F}^\mu_{\rm reac}$ and in trying to define a simple approximation for it.

B. Local back-reaction terms in dimensional regularization

The reaction force $\mathcal{F}^\mu_{\rm reac}$ is linear in $\partial_\mu A_{\rm reac}(z)$, which is itself given by the following integral

$$\partial_\mu A_{\rm reac}(z) = \int_0^L d\sigma'\, B^z_\mu(\sigma')\,, \tag{4.10}$$

with

$$B^z_\mu(\sigma') = \frac{1}{2}\left\{\left[\frac{1}{|\Omega\cdot\dot{z}|}\frac{d}{d\tau'}\left(\frac{\Omega_\mu\,\Sigma(\sigma',\tau')}{\Omega\cdot\dot{z}}\right)\right]\bigg|_{\tau'=\tau_{\rm ret}} - \left[\frac{1}{|\Omega\cdot\dot{z}|}\frac{d}{d\tau'}\left(\frac{\Omega_\mu\,\Sigma(\sigma',\tau')}{\Omega\cdot\dot{z}}\right)\right]\bigg|_{\tau'=\tau_{\rm adv}}\right\}. \tag{4.11}$$

The integrand $B^z_\mu(\sigma')$ is the finite difference between two terms that blow up when $\sigma' \to \sigma$ (σ being such that $z = z(\sigma,\tau)$). When σ' is well away from σ (say, for long, horizon-sized strings) $B^z_\mu(\sigma')$ is expected to decrease roughly as the inverse spatial distance $|\Omega \cdot \dot{z}|$, i.e. roughly as $|\sigma'-\sigma|^{-1}$. In other words, a very rough representation of the typical behaviour of $B_\mu(\sigma)$ is $B(\sigma') \sim (2(\sigma'-\sigma))^{-1}\,[f(\tau-(\sigma'-\sigma)) - f(\tau+(\sigma'-\sigma)]$, where the "effective source function" $f(\tau)$ is expected to oscillate as τ varies. If we think in terms of one Fourier mode, say $f(\tau) = f_\omega\, e^{-i\omega\tau}$, these considerations suggest that the field derivative ∂A is roughly given by an integral of the form

$$\partial A = \int d\sigma'\, B(\sigma') \sim i f_\omega\, e^{-i\omega\tau} \int_{-\infty}^{+\infty} d\sigma'\, \frac{\sin\omega(\sigma'-\sigma)}{(\sigma'-\sigma)}\,. \tag{4.12}$$

The latter integral is equal to π, so that one can finally replace the oscillatory and decreasing integrand $B(\sigma')$ by an effective δ-function, $B_{\rm eff}(\sigma') = B(0)\,\Delta\,\delta(\sigma'-\sigma)$, with (in our example) $B(0) = -\dot{f}(0) = i\omega f_\omega$ and $\Delta = \pi/\omega$, or, in other words, $\partial A = \int d\sigma'\, B(\sigma')$ is replaced by $\Delta\, B(0)$. The analogous proposal of replacing the complicated, non local integral (4.10) giving $\partial_\mu A_{\rm reac}$ simply by the local expression

$$[\partial_\mu A_{\rm reac}]^{\rm local} = \Delta\, B^z_\mu(0)\,, \tag{4.13}$$

where Δ is some length scale linked to the wavelength of the main Fourier component of the radiation, was made by Battye and Shellard [14], [15] (see also [19]). In effect, this proposal is equivalent to replacing the σ'-extended source $\Sigma(\tau',\sigma')$ by the σ'-local effective source $\Delta\,\Sigma(\tau',\sigma)\,\delta(\sigma'-\sigma)$. One of the main aims of the present paper is to study critically the consequences of this proposal.

Though this "local back reaction approximation" drastically simplifies the evaluation of the reaction force $\mathcal{F}^\mu_{\rm reac}$, there remains the non trivial analytical task of computing the $\sigma' \to 0$ limit of the difference between the two complicated (and divergent) terms making up $B^z_\mu(\sigma')$. We found very helpful in this respect to use *dimensional regularization*, i.e. to use, instead of the normal (singular) four dimensional Green's functions (2.22), (2.24), their analytic continuation to a spacetime of (formal) dimension $n = 4-\epsilon$. This technique is well known to be quite useful in quantum field theory, but it (or, at least, a variant of it) has also been shown long ago to be technically very convenient in the classical theory of point particles [23], [24], [25], [26].

Riesz [23] has shown that the retarded and advanced Green's functions in dimension $n \equiv 4 - \epsilon$ read

$$G^{(n)}_{\substack{\rm ret\\ \rm adv}}(x-y) = \frac{1}{H_n(2)} \left(-(x-y)^2\right)^{\frac{2-n}{2}} \theta(-(x-y)^2)\, \theta(\pm(x^0-y^0))\,, \tag{4.14}$$

with $H_n(2) = 2\,\pi^{\frac{n-2}{2}}\,\Gamma\left(\frac{4-n}{2}\right)$ and

$$\Box\, G^{(n)}(x-y) = -\delta^n(x-y)\,. \tag{4.15}$$

Note that, when $\epsilon = 4 - n \to 0$, the coefficient appearing in Eq. (4.14) becomes

$$\frac{1}{H_n(2)} = \frac{\epsilon}{4\pi}\,(1 + \mathcal{O}(\epsilon))\,. \tag{4.16}$$

To save writing, we shall neglect in the following the factor $1 + \mathcal{O}(\epsilon)$ in Eq. (4.16) which plays no role in the terms we consider. Then, we write the retarded solution of our model field equation (2.1) in dimension $n = 4 - \epsilon$ as

$$A_{\rm ret}(x) = \epsilon \int d\sigma' \int_{-\infty}^{\tau_{\rm ret}} d\tau'\, \Sigma\, (\Omega^2)^{\frac{2-n}{2}}\, \theta(\Omega^2)\,, \tag{4.17}$$

where $\Omega^2 \equiv -(x - z(\tau', \sigma'))^2$. (Note the inclusion of a minus sign so that $\Omega^2 > 0$ within the light cone). Again neglecting a factor $1 + \mathcal{O}(\epsilon)$, the field derivative reads

$$\partial_\mu A_{\rm ret}(x) = 2\epsilon \int d\sigma' \int_{-\infty}^{\tau_{\rm ret}} d\tau'\, \Sigma\, \Omega_\mu\, (\Omega^2)^{-n/2} \theta(\Omega^2)\,. \tag{4.18}$$

Using some efficient tools of dimensional regularization (which are explained in Appendix A) we get our main technical results: the explicit expressions of the reactive field, and its derivatives, in the local back reaction approximation

$$[A_{\rm reac}(z)]^{\rm local} = \frac{\Delta}{\dot z^2}\left[\dot\Sigma - \Sigma\left(\frac{\dot z \cdot \ddot z}{\dot z^2}\right)\right]\,, \tag{4.19}$$

$$\begin{aligned}[\partial_\mu A_{\rm reac}(z)]^{\rm local} = \frac{\Delta}{(\dot z^2)^2}\Bigg[&\frac{1}{3}\Sigma\,\dddot z_\mu + \dot\Sigma\,\ddot z_\mu + \ddot\Sigma\,\dot z_\mu - 4\dot\Sigma\,\dot z_\mu\left(\frac{\dot z\cdot\ddot z}{\dot z^2}\right) - 2\Sigma\ddot z_\mu\left(\frac{\dot z\cdot\ddot z}{\dot z^2}\right)\\ &-\Sigma\,\dot z_\mu\left(\frac{\ddot z\cdot\ddot z}{\dot z^2}\right) - \frac{4}{3}\,\Sigma\,\dot z_\mu\left(\frac{\dot z\cdot\dddot z}{\dot z^2}\right) + 6\Sigma\,\dot z_\mu\left(\frac{\dot z\cdot\ddot z}{\dot z^2}\right)^2\Bigg]\,. \end{aligned} \tag{4.20}$$

Some simplifications occur if we introduce, instead of the worldsheet density Σ, the corresponding worldsheet scalar $S \equiv \Sigma/\sqrt{\gamma}$. We find

$$[A_{\rm reac}(z)]^{\rm local} = \Delta\left[-\dot S - S\left(\frac{\dot z\cdot\ddot z}{\dot z^2}\right)\right]\,, \tag{4.21}$$

$$\begin{aligned}[\partial_\mu\, A_{\rm reac}(z)]^{\rm local} = \frac{\Delta}{\dot z^2}\Bigg\{&S\left[-\frac{1}{3}\,\dddot z_\mu - \frac{2}{3}\,\dot z_\mu\left(\frac{\dot z\cdot\dddot z}{\dot z^2}\right) + 2\dot z_\mu\left(\frac{\dot z\cdot\ddot z}{\dot z^2}\right)^2 - \dot z_\mu\left(\frac{\ddot z\cdot\ddot z}{\dot z^2}\right)\right]\\ &-\ddot z_\mu\,\dot S - \dot z_\mu\,\ddot S\Bigg\}\,. \end{aligned} \tag{4.22}$$

Note that Eqs. (4.19), (4.20) and Eqs. (4.21), (4.22) satisfy the compatibility condition $\dot{z}^\mu \partial_\mu A = \dot{A}$, but because of the lack of worldsheet covariance (broken by the introduction of Δ) the analog condition for z' is not verified.

C. Dilaton radiation reaction

Let us first apply our results to the case of the dilaton field φ, which has not been previously studied in the literature. The corresponding (worldsheet scalar) source is then simply

$$S_\varphi = \frac{1}{z'^2} \Sigma_\varphi = \frac{1}{z'^2} \alpha\, G\mu\, U = -2\,\alpha\, G\mu\,. \tag{4.23}$$

S_φ being a constant, the preceding formulas simplify very much:

$$[\varphi_{\rm reac}(z)]^{\rm local} = 2\alpha\, G\,\mu\,\Delta \left(\frac{\dot{z}\cdot\ddot{z}}{\dot{z}^2}\right), \tag{4.24}$$

$$[\partial_\mu \varphi_{\rm reac}(z)]^{\rm local} = 2\alpha\, G\,\mu\, \frac{\Delta}{\dot{z}^2}\left[\frac{1}{3}\dddot{z}_\mu + \frac{2}{3}\dot{z}_\mu \left(\frac{\dot{z}\cdot\dddot{z}}{\dot{z}^2}\right) - 2\dot{z}_\mu \left(\frac{\dot{z}\cdot\ddot{z}}{\dot{z}^2}\right)^2 + \dot{z}_\mu \left(\frac{\ddot{z}\cdot\ddot{z}}{\dot{z}^2}\right)\right]. \tag{4.25}$$

Inserting these results in the dilaton self-force (2.15), we get

$$\mathcal{F}^\varphi_\mu = \frac{4}{3}\alpha^2\, G\,\mu^2\,\Delta\left[\dddot{z}_\mu - \dot{z}_\mu\left(\frac{\dot{z}\cdot\dddot{z}}{\dot{z}^2}\right) + z'_\mu \left(\frac{z'\cdot\dddot{z}}{\dot{z}^2}\right)\right]. \tag{4.26}$$

Consistently with our choice of conformal gauge (which, in the case of the dilaton coupling, is the same as in flat space, see Eq. (3.12)), we see that the reaction force (4.26) is orthogonal to the two worldsheet tangent vectors, $\dot{z}^\mu$ and z'^μ:

$$\dot{z}^\mu\, \mathcal{F}^\varphi_\mu \equiv 0 \equiv z'^\mu\, \mathcal{F}^\varphi_\mu\,. \tag{4.27}$$

Let us now show that the putative, local approximation to the dilaton reaction force, Eq. (4.26) conveys some of the correct physical characteristics expected from a radiation damping force. In particular, let us check that the overall sign of Eq. (4.26) is the correct one. First, we remark that we can work iteratively and therefore consider that the reaction force (4.26), and its integrated effects, can be evaluated on a free string trajectory. In other words, when evaluating the total four momentum lost by the string under the action of $\mathcal{F}_\mu = \mathcal{F}^\varphi_\mu + \mathcal{F}^h_\mu + \mathcal{F}^B_\mu$,

$$P^{\rm lost}_{s\,\mu} = -\int d\sigma\, d\tau\, \mathcal{F}_\mu\,, \tag{4.28}$$

we can insert a free string trajectory on the right-hand side of (4.28)[1]. This being the case, we can now further restrict the worldsheet gauge by choosing a *temporal* conformal gauge, i.e. such that $t = z^0(\tau,\sigma) = \tau$. Geometrically, this means that the $\tau =$ const. sections of the worldsheet coincide with $x^0 =$ const. space-time coordinate planes. [The choice $z^0 = \tau$ is consistent for free string trajectories because for them $\ddot{z}^\mu - z^{\mu\prime\prime} = 0$.] In this gauge, we have

$$\dot{z}^0 = 1\,, \qquad -\dot{z}^2 = 1 - \mathbf{v}^2 = \mathbf{z}'^2\,, \qquad \dot{z}\cdot\dddot{z} = \mathbf{v}\cdot\ddot{\mathbf{v}} \qquad z'\cdot\dddot{z} = \mathbf{z}'\cdot\ddot{\mathbf{v}}\,, \tag{4.29}$$

where we have introduced the 3-velocity $\mathbf{v} \equiv \dot{\mathbf{z}}$. The zero component of Eq. (4.28) then reads

$$\mathcal{F}^0_\varphi = +\frac{4}{3}\,\alpha^2\,G\mu^2\,\Delta\,\frac{\mathbf{v}\cdot\ddot{\mathbf{v}}}{1-\mathbf{v}^2}\,. \tag{4.30}$$

Assuming that the scale Δ is constant, we can integrate by parts and write for the total energy lost by the string

$$E^{\rm lost}_\varphi = \frac{4}{3}\alpha^2\,G\,\mu^2\,\Delta\int d\sigma\,d\tau\,\left[\frac{\dot{\mathbf{v}}^2}{1-\mathbf{v}^2} + 2\,\frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{(1-\mathbf{v}^2)^2}\right]\,. \tag{4.31}$$

The integrand of Eq. (4.31) is positive definite, ensuring that the reaction force (4.26) has the correct sign for representing a radiation damping force.

We can further check that the total 4-momentum lost by the string is, as it should, time-like. First, let us note that the relation

$$U^{\rho\mu}\,\partial_\rho\varphi = -\partial_a(\sqrt{\gamma}\,\gamma^{ab}\,\partial_b z^\mu\,\varphi) + \varphi\,\partial_a(\sqrt{\gamma}\,\gamma^{ab}\,\partial_b z^\mu)\,, \tag{4.32}$$

shows that, as far as its integrated effects are concerned, the dilaton reaction force (2.15) is equivalent to

$$\mathcal{F}^{\varphi\,\rm equiv.}_\mu = \alpha\,\mu\,U\,\partial_\mu\,\varphi_{\rm reac} = \frac{1}{G}\,\Sigma_\varphi\,\partial_\mu\,\varphi_{\rm reac}\,. \tag{4.33}$$

Inserting Eq. (4.20), or better, Eq. (4.22) into Eq. (4.33) yields, after integration by parts, a total 4-momentum loss

---

[1]Strictly speaking the integral in Eq. (4.28) is infinite because free string trajectories are periodic. The meaning of Eq. (4.28), and similar integrals below is to give, after division by the total coordinate time span τ, the time-averaged energy-momentum loss.

$$P_\mu^{\mathrm{lost}\,\varphi} = \frac{4}{3}\,\alpha^2\, G\mu^2\,\Delta \int d\tau\, d\sigma\, \pi_\mu\,, \tag{4.34}$$

with integrand

$$\pi_\mu = \dot z_\mu \left[2\left(\frac{\dot z\cdot \ddot z}{\dot z^2}\right)^2 - \left(\frac{\ddot z\cdot \ddot z}{\dot z^2}\right)\right] + 2\ddot z_\mu \left(\frac{\dot z\cdot \ddot z}{\dot z^2}\right)\,. \tag{4.35}$$

The square of π_μ reads

$$\pi_\mu\,\pi^\mu = \frac{1}{(\dot z^2)^4}\left[12\dot z^2\,(\dot z\cdot\ddot z)^4 + (\dot z^2)^3\,(\ddot z^2)^2 - 4\ddot z^2\,(\dot z^2)^2\,(\dot z\cdot \ddot z)^2\right]\,. \tag{4.36}$$

This is negative definite in the temporal gauge $t = \tau$, showing that $|\mathbf{P}_\varphi^{\mathrm{lost}}| < E_\varphi^{\mathrm{lost}}$, as physically expected.

D. Gravitational and axionic radiation reaction

We are going to see that the generalization of the dilaton results to the case of the gravitational and axionic fields is nontrivial, and leads to physically nonsensical results. Let us first generalize Eq. (4.33). The relations

$$U^{\alpha\beta}\,\partial_\alpha h_{\beta\mu} = -\partial_a(\sqrt{\gamma}\,\gamma^{ab}\,h_{\alpha\mu}\,\partial_b z^\alpha) + h_{\alpha\mu}\,\partial_a(\sqrt{\gamma}\,\gamma^{ab}\,\partial_b z^\alpha)\,, \tag{4.37}$$

$$V^{\lambda\nu}\,\partial_\lambda B_{\nu\mu} = \partial_b(\epsilon^{ab}\,\partial_a z^\nu\, B_{\nu\mu}) - \partial_b(\epsilon^{ab}\,\partial_a z^\nu)\,B_{\nu\mu}\,, \tag{4.38}$$

show that, as far as their integrated effects are concerned, the gravitational and axionic reaction forces (2.16), (2.17) are equivalent, respectively, to:

$$\mathcal{F}_\mu^{h\,\mathrm{equiv.}} = \frac{1}{2}\,\mu\, U^{\alpha\beta}\,\partial_\mu\, h_{\alpha\beta}^{\mathrm{reac}}\,, \tag{4.39}$$

$$\mathcal{F}_\mu^{B\,\mathrm{equiv.}} = \frac{1}{2}\,\lambda\, V^{\alpha\beta}\,\partial_\mu\, B_{\alpha\beta}^{\mathrm{reac}}\,. \tag{4.40}$$

It is important to note that, as in the dilaton case Eq. (4.33), these equivalent reaction forces are simple bilinear forms in the vertex operators and the derivatives of the fields. They can both be written as

$$\mathcal{F}_\mu^{\mathrm{equiv.}} = \frac{1}{8G}\,\Sigma\cdot\partial_\mu\, A_{\mathrm{reac}} \tag{4.41}$$

where, as in Eq. (2.26), Σ denotes the source of the field $A = h_{\alpha\beta}$ or $B_{\alpha\beta}$, and where the dot denotes a certain symmetric bilinear form acting on symmetric or antisymmetric tensors. With the normalization of Eq. (4.41) these bilinear forms are, respectively,

$$U_{\alpha\beta} = U_{\beta\alpha} : \qquad U \cdot U \equiv U_{\alpha\beta} U^{\alpha\beta} - \frac{1}{2} U^2 \equiv U_{\alpha\beta} \widetilde{U}^{\alpha\beta} , \tag{4.42}$$

$$V_{\alpha\beta} = -V_{\beta\alpha} : \qquad V \cdot V \equiv V_{\alpha\beta} V^{\alpha\beta} . \tag{4.43}$$

One can recognize here the quadratic forms defined by the residues of the gauge-fixed propagators of the h and B fields. Note that if we wish to rewrite the scalar reaction force (4.33) in the same format (4.41) as the tensor ones we have to define the dot product for scalar sources as

$$\Sigma_\varphi \cdot \Sigma_\varphi \equiv 8 \, \Sigma_\varphi^2 . \tag{4.44}$$

Using this notation, and the results above on the reaction fields, it is possible to compute in a rather streamlined way the total 4-momentum lost under the action of the local reaction force:

$$P_\mu^{\rm lost} = -\frac{1}{8G} \int \int d\sigma \, d\tau \, \Sigma \cdot [\partial_\mu A_{\rm reac}]^{\rm local} . \tag{4.45}$$

The calculation is simple if one uses the form (4.22). Let us note that the worldsheet-scalar sources ($S = \Sigma / \sqrt{\gamma}$) for the three fields we consider (φ, h and B) satisfy

$$S \cdot S = \text{const.} , \tag{4.46}$$

$$S \cdot \dot{S} = 0 . \tag{4.47}$$

Indeed, if we introduce the scalarized vertex operators (with conformal dimension zero) $\widehat{U} \equiv U / \sqrt{\gamma}$, $\widehat{U}_{\alpha\beta} \equiv U_{\alpha\beta} / \sqrt{\gamma}$ and $\widehat{V}_{\alpha\beta} \equiv V_{\alpha\beta} / \sqrt{\gamma}$, it is easily seen that

$$\begin{array}{ll} \varphi : & \widehat{U} \cdot \widehat{U} \equiv 8 \, \widehat{U}^2 = +32 , \\ h : & \widehat{U}_{\alpha\beta} \cdot \widehat{U}^{\alpha\beta} \equiv \widehat{U}_{\alpha\beta} \widehat{U}^{\alpha\beta} - \frac{1}{2} \widehat{U}^2 = 0 , \\ B : & \widehat{V}_{\alpha\beta} \cdot \widehat{V}^{\alpha\beta} = -2 . \end{array} \tag{4.48}$$

The relations (4.46), (4.47) simplify very much the evaluation of $P_\mu^{\rm lost}$. In particular, the constancy of $S \cdot S$ allows one to integrate by parts on $\dddot{z}_\mu$, etc ... without having to differentiate the $S \cdot S$ factors. By some simple manipulations, using also the consequence

$$\dot{S} \cdot \dot{S} + S \cdot \ddot{S} = 0 , \tag{4.49}$$

of Eq. (4.47), we get

$$P_\mu^{\rm lost} = \frac{1}{8G} \int \int d\sigma \, d\tau \int \dot{z}_\mu \left\{ (\dot{S} \cdot \dot{S}) + (S \cdot S) \left[\frac{1}{3} \dot{u}^2 + \frac{1}{12} \dot{\phi}^2 \right] + \frac{1}{3} \ddot{z}_\mu (S \cdot S) \, \dot{\phi} \right\} . \tag{4.50}$$

Here we introduced a special notation for the conformal factor (Liouville field),

$$ds^2 = e^\phi(-d\tau^2 + d\sigma^2)\,, \qquad e^\phi = \sqrt{\gamma} = z'^2 = -\dot{z}^2\,, \tag{4.51}$$

and we defined the *unit* time like vector $u^\mu = e^{-\phi/2}\,\dot{z}^\mu$, and its first derivative

$$\dot{u}_\mu = \frac{d}{d\tau}\,(e^{-\phi/2}\,\dot{z}_\mu)\,, \qquad \dot{u}^2 = -\frac{\ddot{z}^2}{\dot{z}^2} + \frac{1}{4}\,\dot{\phi}^2 > 0\,. \tag{4.52}$$

Let us now prove the remarkable result that the contribution proportional to $\dot{S}\cdot\dot{S}$ in Eq. (4.50) vanishes for all three fields when evaluated (as we are iteratively allowed to do) on a free string trajectory:

$$\int\!\!\int d\sigma\, d\tau\, \dot{z}_\mu\,(\dot{S}\cdot\dot{S}) = 0\,. \tag{4.53}$$

Indeed, for the scalar case $\widehat{U} = -2$ and $\dot{\widehat{U}} = 0$, while for the other fields a straightforward calculation gives

$$\dot{\widehat{U}}_{\alpha\beta}\,\dot{\widehat{U}}^{\alpha\beta} - \frac{1}{2}\,\dot{\widehat{U}}^2 = -\Box_\eta\,\phi\,, \tag{4.54}$$

$$\dot{\widehat{V}}_{\alpha\beta}\,\dot{\widehat{V}}^{\alpha\beta} = +\Box_\eta\,\phi\,, \tag{4.55}$$

when taking into account the vanishing of terms proportional to the worldsheet derivatives of $\Box_\eta\, z^\mu = -\ddot{z}^\mu + z^{\mu\prime\prime}$. [These results have a nice geometrical interpretation linked to the Gauss-Codazzi relations.] Integrating by parts, we see that the contribution (4.53) is proportional to $\int\!\int d\sigma\, d\tau\,(\Box_\eta\,\dot{z}_\mu)\,\phi$ which vanishes, again because of the free string equations of motion.

Finally, remembering the constancy of $S\cdot S$, we get the very simple result

$$P_\mu^{\rm lost} = \frac{1}{3}\,\Delta\,\frac{S\cdot S}{8G}\int\!\!\int d\sigma\, d\tau\,\pi_\mu\,, \tag{4.56}$$

where the integrand

$$\pi_\mu = \dot{z}_\mu\left(\dot{u}^2 + \frac{1}{4}\,\dot{\phi}^2\right) + \ddot{z}_\mu\,\dot{\phi} \tag{4.57}$$

is easily seen to coincide with the one which appeared above, Eq. (4.35), in our direct calculation of the dilaton reaction. Let us recall that the present calculation applies uniformly to all three fields if we define the dot product between dilatonic vertex operators with an extra factor 8, see Eq. (4.44).

The conclusion is that the local approximation to back reaction for the three fields φ, h and B leads to energy-momentum losses which are proportional to the same quantity $\int\int d^2\sigma\, \pi_\mu$ with coefficients respectively given by (using Eqs. (4.48) above)

$$\frac{\Delta}{3}\,\frac{S_\varphi \cdot S_\varphi}{8G} = \frac{\Delta}{3}\,(\alpha G\mu)^2\,(\widehat{U})^2 = +\frac{4}{3}\,\Delta\, G\,\alpha^2\,\mu^2\,, \tag{4.58}$$

$$\frac{\Delta}{3}\,\frac{S_h \cdot S_h}{8G} = \frac{\Delta}{3}\,\frac{(4G\mu)^2}{8G}\,\widehat{U}_{\alpha\beta}\,\widehat{\widehat{U}}^{\alpha\beta} = 0\,, \tag{4.59}$$

$$\frac{\Delta}{3}\,\frac{S_B \cdot S_B}{8G} = \frac{\Delta}{3}\,\frac{(4G\lambda)^2}{8G}\,\widehat{V}_{\alpha\beta}\,\widehat{V}^{\alpha\beta} = -\frac{4}{3}\,\Delta\, G\,\lambda^2\,. \tag{4.60}$$

The result (4.58) coincides with Eq. (4.34) above (for which we have verified that the overall sign is correct). We therefore conclude that the "local reaction approximation" (4.13) yields: (i) a *vanishing*, net energy-momentum loss for the gravitational field, and (ii) the *wrong sign* (antidamping) for the axionic field. The latter result disagrees with Refs. [14], [15] (see the Appendix) which claimed to obtain positive damping. It is for clarifying this important sign question that we have presented above a streamlined calculation showing that the overall sign can simply be read from the contraction of the vertex operators of the fields. Indeed, finally the physical energy-loss sign is simply determined by the easily checked (and *signature independent*) signs in Eqs. (4.48).

E. Gauge invariance and mass-shell-only positivity

Why is the "local back reaction approximation" giving wrong answers in the cases of gravitational and axionic fields but a physically acceptable one in the case of the dilatonic field? The basic reason for this difference between $h_{\mu\nu}$ and $B_{\mu\nu}$ on one side, and φ on the other is the *gauge invariance* of the former. Indeed, a gauge symmetry (here $h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu\,\xi_\nu + \partial_\nu\,\xi_\mu$, $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\,A_\nu - \partial_\nu\,A_\mu$) means that some of the components of $h_{\mu\nu}$ and $B_{\mu\nu}$ are not real physical excitations. This is associated with the fact that some of the components of $h_{\mu\nu}$ and $B_{\mu\nu}$ (namely h_{0i} and B_{0i}) have kinetic terms with the *wrong sign*, i.e. that they (formally) carry negative energy. Therefore, approximating radiation damping is very delicate for gauge fields. A slight violation of gauge invariance by the approximation procedure can lead to antidamping (the literature of gravitational radiation damping is full of such errors, see e.g. [30]). A more precise way of seeing why the local back reaction approximation is dangerous in this respect is the following.

We have proven above that an exact expression for the 4-momentum of the source lost to radiation is given (for φ, $h_{\mu\nu}$ and $B_{\mu\nu}$, and more generally for any linearly coupled field) by an expression of the form

$$P_\mu^{\rm lost} = -\,k \int d^4x \, J(x) \cdot \partial_\mu \, A_{\rm reac}(x) \, , \tag{4.61}$$

where $J(x)$ is the source of $A(x)$

$$\Box \, A(x) = -J(x) \, , \tag{4.62}$$

and where k is a *positive* coefficient which depends on the normalization of the kinetic terms of $A(x)$ [$4\pi k = 1/8G$ when using the above normalizations, the extra factor 4π compensating for our present way of writing the field equation (4.62).] The spacetime source $J(x)$ is linked to our previous string distributed sources by $J(x) = 4\pi \int d^2\sigma \, \Sigma \, \delta^4(x-z)$. The dot product in Eq. (4.61) is the symmetric bilinear form defined in Eqs. (4.42), (4.43), (4.44) above for the three cases h, B and φ. Introducing Fourier transforms, with the conventions,

$$J(p) = \int d^4x \, e^{-ipx} \, J(x) \, , \tag{4.63}$$

$$G_{\rm reac}(x) = \frac{1}{2} \left[G_{\rm ret}(x) - G_{\rm adv}(x) \right] = \int \frac{d^4p}{(2\pi)^4} \, G_{\rm reac}(p) \, e^{+ipx} \, , \tag{4.64}$$

the energy loss (4.61) reads

$$P_\mu^{\rm lost} = -\,k \int \frac{d^4p}{(2\pi)^4} \, ip_\mu \, G_{\rm reac}(p) \, J(-p) \cdot J(p) \, . \tag{4.65}$$

To see the positivity properties of $P_\mu^{\rm lost}$ we need to insert the explicit expression of the Fourier transform of $G_{\rm reac}$.

The Fourier decomposition of the retarded and advanced Green functions ($\Box \, G = -\delta^4$) read

$$G_{\substack{\rm ret \\ \rm adv}}(x) = \int \frac{d^4p}{(2\pi)^4} \, \frac{e^{ipx}}{\mathbf{p}^2 - (p^0 \pm i\eta)^2} = \int \frac{d^4p}{(2\pi)^4} \, \frac{e^{ipx}}{p^2 \mp i \, \eta \, p^0} \, , \tag{4.66}$$

where η is any positive infinitesimal. Using the formula

$$\frac{1}{x-a\pm i\eta}=P\,\frac{1}{x-a}\mp i\pi\delta(x-a)\,, \tag{4.67}$$

where P denotes the principal part, one finds

$$G_{\rm reac}(p)=\frac{1}{2}\,[G_{\rm ret}(p)-G_{\rm adv}(p)]=i\pi\;{\rm sign}\,(p^0)\,\delta(p^2)\,. \tag{4.68}$$

Inserting (4.68) into (4.65) one gets

$$\begin{aligned} P_\mu^{\rm lost} &= +\,k\pi\int\frac{d^4p}{(2\pi)^4}\;{\rm sign}\,(p^0)\,p_\mu\,\delta(p^2)\,J(-p)\cdot J(p) \\ &= +\,k\int_{V_+}\widetilde{dp}\,p_\mu\,J^*(p)\cdot J(p)\,, \end{aligned} \tag{4.69}$$

where V_+ denotes the positive mass shell $p^0=+\sqrt{\mathbf{p}^2}$ and $\widetilde{dp}=(2\pi)^{-3}\,d^3\mathbf{p}/2p^0$ the natural integration measure on V_+. Here, we have used the reality of the source: $J^*(x)=J(x)\Rightarrow J^*(p)=J(-p)$.

As in the case of Eq. (4.28) and its kin, the meaning of Eq. (4.69) is formal when evaluated on a (periodic) free string trajectory. However, it is, as usual, easy to convert Eq. (4.69) in a result for the average rate of 4-momentum loss by using Fermi's golden rule:

$$[\delta(p^0-n\omega)]^2=\frac{1}{2\pi}\,\delta(p^0-n\omega)\int d\tau\,. \tag{4.70}$$

One then recovers known results for the average energy radiation from periodic string motions [2], [9].

The integrand in the last result has the good sign (i.e. defines a vector within the future directed light cone) if the dot product $J^*(p)\cdot J(p)>0$. This is clearly the case for a scalar source, but for the gauge fields $h_{\mu\nu}$ and $B_{\mu\nu}$ one has integrands

$$J^*_{h\mu\nu}(p)\,J_h^{\mu\nu}(p)-\frac{1}{2}\,|J^\lambda_{h\lambda}(p)|^2\,, \tag{4.71}$$

and

$$J^*_{B\mu\nu}(p)\,J_B^{\mu\nu}(p) \tag{4.72}$$

which are not explicitly positive because of the wrong sign of the mixed components J_{0i}. As is well known this potential problem is cured by one consequence of gauge invariance, namely some conservation conditions which must be satisfied by the source. In our case the gravitational source $J^h_{\mu\nu}(x)\propto\widetilde{T}_{\mu\nu}(x)$ must satisfy $\partial^\nu\,\widetilde{J}^h_{\mu\nu}=0$, while the axionic source must satisfy $\partial^\nu J^B_{\mu\nu}(x) = 0$. In the Fourier domain this gives $p^\nu \tilde{J}^h_{\mu\nu}(p) = 0$ or $p^\nu J^B_{\mu\nu}(p) = 0$. These transversality constraints are just enough to ensure that the integrands (4.71), (4.72) are positive *when evaluated on the mass shell* V_+. What happens in the "local back reaction approximation" is that one replaces the Green function $G_{\rm reac}(x)$ by a distributional kernel $G_{\rm loc}(x)$ with support (in x space) localized at $x = 0$. Its Fourier transform $G_{\rm loc}(p)$ is no longer localized on the light cone $p^2 = 0$, and therefore the delicate compensations ensuring the positivity of the integrands (4.71), (4.72) do not work anymore.

V. IMPROVED DILATONIC REACTION AS SUBSTITUTE TO GRAVITATIONAL REACTION

As the main motivation of the present study is to find a physically reasonable, and numerically acceptable, approximation to gravitational radiation damping, the results of the previous Section would seem to suggest that the local back reaction approach fails to provide such an approximation. However, we wish to propose a more positive interpretation. Indeed, both the direct verification of Section IV C, and the argument (in Fourier space) of Section IV E shows that the local back reaction approximation can make sense when applied to scalar fields. On the other hand, Damour and Vilenkin [9] in a recent study of dilaton emission by cosmic strings have found that, in spite of their genuine physical differences, gravitational radiation and dilatonic radiation from strings are globally rather similar. For the samples of cuspy or kinky loops explored in Ref. [9], the global energy losses into these fields turned out to be roughly proportional to each other. Even when considering in more detail the physically important problem of the amount of radiation from cusps, it was found that (despite an expected difference linked to the spin 2 transversality projection) both radiations were again roughly similar.

Let us also recall that this similarity, or better brotherhood, between gravitational and dilatonic couplings is technically apparent in the similarity of their vertex operators (which are both subsumed in the form $\zeta_{\mu\nu}\, \partial^a z^\mu\, \partial_a z^\nu$ with a generic symmetric polarization tensor $\zeta_{\mu\nu}$) and is a very important element of superstring theory. This leads us to propose to use, after a suitable normalization, the physically acceptable local dilatonic back reaction force as a *substitute* for the gravitational radiation one. In other words, we propose to use as "approximation" to gravitational radiation damping a local reaction force of the form (in conformal gauge)

$$\mathcal{F}_\mu = \frac{4}{3} G\mu^2 \Delta \left[\dddot{z}_\mu - \dot{z}_\mu \left(\frac{\dot{z} \cdot \dddot{z}}{\dot{z}^2} \right) + z'_\mu \left(\frac{z' \cdot \dddot{z}}{\dot{z}^2} \right) \right] . \tag{5.1}$$

We note also that, though there are more differences between axionic and gravitational radiations than between the dilatonic and gravitational ones, they are still roughly similar in many ways (as witnessed again by the brotherhood of their vertex operators $\zeta_{\mu\nu}\,\partial^a z^\mu \,\partial_a z^\nu$ with now a generic asymmetric polarization tensor) so that one can hope to be able also to represent in an acceptable manner axionic radiation damping by a force of the type (5.1) with the replacement $G\mu^2 \to G\lambda^2$ and another, suitable choice of Δ. [Actually, due to their sign error, this last proposal agrees with the practical proposition made in Refs. [14], [15].]

As mentioned above, the algebraic structure of Eq. (5.1) makes it compatible with the standard (flat space) conformal gauge conditions. It remains, however, to precise the choice of Δ in Eq. (5.1). Up to now we have implicitly assumed that Δ was constant. There are, however, several reasons for suggesting a non constant Δ. The first reason concerns energy-momentum losses associated with cusps. To see things better, let us use a temporal gauge $t = \tau$ and concentrate on the energy loss implied by Eq. (5.1). One finds simply

$$E^{\rm lost} = -\frac{4}{3} G\mu^2 \int\!\!\int d\sigma \, d\tau \, \Delta \, \frac{\mathbf{v} \cdot \ddot{\mathbf{v}}}{1 - \mathbf{v}^2} , \tag{5.2}$$

where $\mathbf{v}(\sigma,\tau) \equiv \dot{\mathbf{z}}(\sigma,\tau)$. At a cusp $\mathbf{v}^2(\sigma,\tau) = 1$. As $\mathbf{v}^2(\sigma,\tau) \leq 1$ everywhere, near a cusp one will have $\mathbf{v}^2(\sigma,\tau) = 1 - (a\sigma^2 + b\sigma\tau + c\tau^2) + \mathcal{O}((\sigma+\tau)^3)$ where the parenthesis is a positive definite quadratic form. This shows that, if Δ is constant, the integral $E^{\rm lost} \sim \int\!\!\int d\sigma \, d\tau \, (1 - \mathbf{v}^2)^{-1}$ is logarithmically divergent (as we explicitly verified on specific string solutions). As the real energy loss to gravitational or dilatonic radiation from (momentary) cusps is finite, this shows that Eq. (5.1) overestimates the importance of back reaction due to cusps. In other words, if one tries to complete the equations of motion of a string by adding the force (5.1) with $\Delta = $ const, this reaction force will prevent the appearance of real cusps. As the calculations of Ref. [11], using the "exact" nonlocal gravitational radiation, find that cusps are weakened but survive, it is clear that one must somehow soften the "local" force (5.1) if we wish to represent adequately the physics of cusps. At this point it is useful to note that the proposal (5.1) lacks worldsheet covariance, which means that Δ has introduced a local coordinate length or time scale on the worldsheet, rather than an invariant interval. As the ratio between coordinate lengths and times and proper intervals is locally given by the square root of the conformal factor $e^\phi = z'^2 = -\dot{z}^2$ ($= 1 - \mathbf{v}^2$ in temporal gauge), it is natural to think that a better measure of the coordinate interval Δ to use in Eq. (5.1) might vary along the worldsheet because it incorporates some power of e^ϕ. This might (if this power is positive) prevent the logarithmic divergence of the integral (5.2). At this stage, a purely phenomenological proposal is to take Δ in Eq. (5.1) of the form

$$\Delta(\sigma, \tau) = f\,(-\dot{z}^2)^\eta\, 2\lambda\,, \tag{5.3}$$

where f is a dimensionless factor, η is a positive power, and λ the wavelength of the radiatively dominant mode emitted by the string. We introduced a factor two for convenience because, in the case of loops for which the fundamental mode is dominant, the wavelength is $L/2$ where L is the invariant length of the loop. On the other hand, if we consider a loop carrying mainly high-frequency excitations, or an infinite string, it is clear that Δ should not be related to the total length L, but to a length linked to the scale of the principal modes propagating on the string.

A first check of the physical consistency of the proposal (5.3) consists in verifying that, despite the nonconstancy of Δ, the integrated energy loss (5.2) will be positive for all possible loop tranjectories. Integrating by parts Eq. (5.2) one finds

$$E^{\rm lost} = \frac{4}{3}\, G\,\mu^2\, f(2\lambda) \int\!\!\int d\sigma\, d\tau\, (1-\mathbf{v}^2)^\eta \left[\frac{\dot{\mathbf{v}}\cdot\dot{\mathbf{v}}}{1-\mathbf{v}^2} + 2(1-\eta)\,\frac{(\mathbf{v}\cdot\dot{\mathbf{v}})^2}{(1-\mathbf{v}^2)^2}\right]. \tag{5.4}$$

This is manifestly positive (and finite) as long as $0 < \eta < 1$. Assuming this to be the case, the question is then: Are there values of f and η (after having decided on a precise definition of λ) such that the corresponding damping force (5.1) gives a reasonably accurate description of the "exact" effects of energy loss to gravitational radiation? We did not try to answer this question in full generality. For simplicity, we fixed the power η to the value $\eta = \frac{1}{2}$ (which seems intuitively preferred as it evokes a Lorentz contraction factor arising because we look at string elements "moving" with relativistic speeds). Then we compared the energy loss due to (5.1) to the energy radiated in gravitational waves as (computed (using Eq. (4.69)) in the literature. As a sample of loop trajectories we consider Burden loops [28]

$$\mathbf{z}(\tau, \sigma) = \frac{1}{2}\left[\mathbf{a}(u) + \mathbf{b}(v)\right], \tag{5.5}$$

$$\mathbf{a} = \frac{L}{2\pi}\left[\frac{1}{m}\cos(m\,u)\,\vec{e}_3 + \frac{1}{m}\sin(m\,u)\,\vec{e}_1\right], \tag{5.6}$$

$$\mathbf{b} = \frac{L}{2\pi}\left[\frac{1}{n}\cos(n\,v)\,\vec{e}_3 - \frac{1}{n}\sin(n\,v)\,\vec{e}_1'\right], \tag{5.7}$$

where

$$u = \frac{2\pi}{L}(\tau - \sigma), \qquad v = \frac{2\pi}{L}(\tau + \sigma), \qquad \vec{e}_1^{\,\prime} = \cos\psi\,\vec{e}_1 + \sin\psi\,\vec{e}_2\,. \tag{5.8}$$

This family of solutions depends on the overall scale L, which is the total invariant length of the loop ($M = \mu L$), on two integers m and n, and on the angle ψ. Our parameter ψ coincides with the angle ψ in [28], denoted φ in [2]. The actual oscillation period of the loop is $T = L/(2mn)$ which leads us to choosing $2\lambda = 2T = L/mn$ in Eq. (5.3). With this choice we computed the energy loss (5.2). The calculation is simplified by noting, on the one hand, that, for this family of loops, $\mathbf{v}\cdot\ddot{\mathbf{v}} = -\left(\frac{2\pi}{L}\right)^2\left(\frac{m^2+n^2}{2}\right)\mathbf{v}^2$, and on the other hand that the worldsheet integral in (5.2) can be rewritten in terms of an average over linear combinations of the two angles $2\pi m(\tau-\sigma)/L$ and $2\pi n(\tau+\sigma)/L$. This yields simply for the average rate of energy loss

$$\Gamma_{m,n} \equiv \frac{\dot{E}^{\rm lost}}{G\mu^2} = \frac{4}{3}\,f\,\frac{m^2+n^2}{2mn}\,\gamma\,, \tag{5.9}$$

where

$$\gamma = 4\int_0^\pi dx \int_0^\pi dy \left[-\sqrt{1-\mathbf{v}^2} + \frac{1}{\sqrt{1-\mathbf{v}^2}}\right], \tag{5.10}$$

with

$$\mathbf{v}^2 = \frac{1}{2}\left[1 - \frac{1}{2}(1+\cos\psi)\,\cos x - \frac{1}{2}(1-\cos\psi)\,\cos y\right]. \tag{5.11}$$

We plot in Fig. 1 $\Gamma_{m,n}$ as a function of the angle ψ, for the nominal value $f = 1$ and for the two cases $(m,n) = (1,1)$, $(m,n) = (1,3)$. [As said above there is a simple scaling law for the dependence on m and n.] If one compares this Figure with the figures published in [28], [2] (Fig. 7.6, p. 205 there) one sees that they give a roughly adequate numerical representation of energy losses to gravitational radiation if

$$f \simeq 0.8\,. \tag{5.12}$$

The fact that our present "best fit" value of the factor f leads to values of Δ which are numerically comparable to L (when $(m,n) = (1,1)$) rather than to a smaller fraction of L should not be considered as physically incompatible with the idea of using a local approximation to back reaction. Indeed, on the other hand, the rough justification of the local

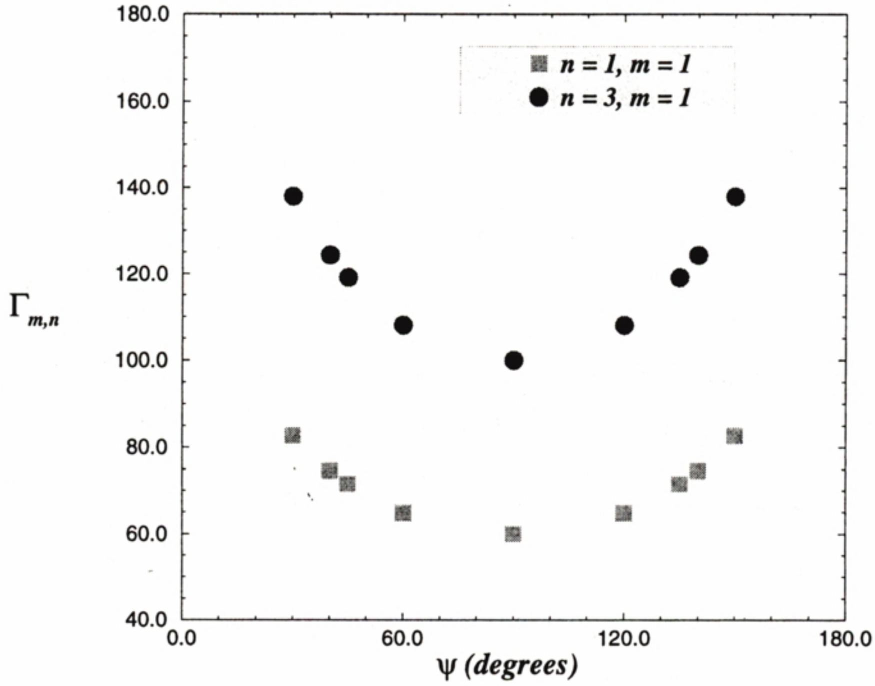

FIG. 1. *Dimensionless energy loss rate Burden loops with* $(m, n) = (1, 1)$ *and* $(1, 3)$.

approximation given in Section IV suggested $\Delta \sim \pi/\omega \sim \lambda/2$, i.e. something like $L/4$, and, on the other hand, numerical computations show that the energy lost to dilaton waves (with coupling $\alpha = 1$) is *smaller* than that lost to gravitational waves by a factor of order 3 or so (part of which is simply due to the fact that there are two independent tensor modes against one scalar mode). Therefore, as we use Δ only as an *effective* parameter to model gravitational damping it is normal to end up with an increased value of Δ/L.

Clearly, more work would be needed to confirm that the modified local dilaton reaction (5.1) can be used as a phenomenological representation of gravitational reaction. Our main purpose here was to clarify the crucial sign problems associated to gauge fields, and to give a first bit of evidence indicating that Eq. (5.1) deserves seriously to be considered as an interesting candidate for mimicking, in a computationally non intensive way, the back reaction of gravitational radiation. We are aware that several important issues will need to be further studied before being able to use Eq. (5.1) in a network simulation. Some numerically adequate definition of λ will have to be provided beyond a case by case definition, which in the case of long loops decorated by a regular array of kinks, as in Ref. [11], would be something like $2\lambda \sim L/N$ where N is the total number of kinks. We note in this respect that a Burden loop with $m = 1$ and $n \gg 1$ provides a simple model of a long, circular loop decorated by a travelling pattern of small transverse oscillations. However, the local approximation (5.1) cannot expected to be accurate in this case, because the radiation from purely left-moving or right-moving modes is known to be suppressed [2]. This suppression is not expected to hold in the more physical generic case where the transverse oscillations move both ways. The accuracy of the local approximation (5.1) should therefore be tested only in such more generic cases.

The explicit expression (5.1) must be rewritten in the temporal, but not necessarily conformal, worldsheet gauges used in numerical simulations, and the higher time derivatives in $\mathcal{F}_\mu$ must be eliminated by using (as is standard in electrodynamics [29] and gravitodynamics [27]) the lowest-order equations of motion. [These last two issues have already been treated in Refs. [14], [15].] Finally, we did not try to explore whether $\eta = 1/2$ is the phenomenologically preferred value. To study this point one should carefully compare the effects of (5.1) on the weakening of cusps and kinks with the results based on the exact, nonlocal reaction force [11]. [The facts that the curves in Fig. 1 are flatter than the corresponding figures in [28], [2] suggest that a smaller value of η might give a better fit.]

VI. CONCLUSIONS

In this paper we studied the problem of radiation reaction on cosmic strings caused by the emission of gravitational, dilatonic and axionic fields.

We corrected errors present in the literature regarding the contribution of the dilaton and axion field to the renormalization of the string tension and we analyzed in detail the proposal of Battye and Shellard [14], [15] to treat the back reaction of cosmic strings in a *local* approximation, based on an analogy with the Abraham-Lorentz-Dirac treatment of self-interacting point-charges. For this purpose we found very convenient to use *dimensional regularization*, a well known technique in quantum field theory. Our results can be easily summarized:

- The local back reaction approximation gives antidamping for the axionic field and a vanishing net energy-momentum loss for the gravitational one. We argued that these physically unacceptable results ultimately come from a violation of gauge invariance (and/or mass shell conditions) in the local back reaction method.

- Up till now the radiation reaction for the dilaton field had not been studied in the literature and we found that in this case the local reaction force has the correct sign for describing a radiation damping force. Hence, taking into account the similarity between gravitational and dilatonic couplings [9], we propose to use, after a suitable normalization of the length scale Δ, Eq. (5.3) (which includes a "redshift" factor important near the cusps), the local dilatonic back reaction, Eq. (5.1), to mimic the gravitational radiation one.

It will be interesting to see what are the consequences of considering the effective reaction force, Eq. (5.1), in full-scale network simulations (done for several different values of $G\mu$) of gravitational radiation. Until such simulations (keeping track of the damping of small scale structure on long strings) are performed, one will not be able to give any precise prediction for the amount and spectrum of stochastic gravitational waves that the forthcoming LIGO/VIRGO network of interferometric detectors, possibly completed by cryogenic bar detectors, might observe.

ACKNOWLEDGEMENTS

We thank Bruce Allen, Richard Battye, Francois Bouchet and Alex Vilenkin for useful exchanges of ideas.

APPENDIX A:

In this Appendix we will give some details on the derivation of Eqs. (4.19), (4.20) using dimensional continuation.

A nice feature of analytic continuation is that it allows one to work "as if" many singular terms were regular. For instance, the factors $(\Omega^2)^{(2-n)/2}$ and $(\Omega^2)^{-n/2}$ that appear in Eqs. (4.17), (4.18) blow up on the light cone ($\Omega^2 = 0$) when $n = 4$. However, if we take the real part of $\epsilon = 4 - n$ large enough (even so large as corresponding to negative values for $\mathrm{Re}(n)$), these Ω-dependent factors become finite, and actually *vanishing*, on the light cone. This remark allows one to deal efficiently with the Ω-dependent factors appearing in Eqs. (4.17), (4.18). We are here interested in the contributions to $A_{\rm ret}(z)$ and $\partial_\mu A_{\rm ret}(z)$ coming from a small neighbourhood $z' = z(\tau', \sigma')$ of $z = z(\tau, \sigma)$ on the worldsheet. Let us, for simplicity, denote $\omega \equiv \Omega^2$. We first remark that when $(\tau', \sigma') \to (\tau, \sigma)$, $\omega = -(z(\tau,\sigma) - z(\tau',\sigma'))^2$ admits an expansion in powers of $\tau' - \tau$ and $\sigma' - \sigma$ of the form

$$\omega = \omega_2 + \omega_3 + \omega_4 + \dots\,, \tag{A1}$$

with $\omega_2 = -\dot{z}^2\,[(\tau'-\tau)^2 - (\sigma'-\sigma)^2]$, and

$$\omega_3 = \mathcal{O}\left((\tau'-\tau)^3 + (\tau'-\tau)^2\,(\sigma'-\sigma) + (\tau'-\tau)\,(\sigma'-\sigma)^2 + (\sigma'-\sigma)^3\right),\ \text{etc}\dots \tag{A2}$$

Then we can formally expand the Ω-dependent factors of Eqs. (4.17), (4.18) in powers of $\tau' - \tau$ and $\sigma' - \sigma$ as follows

$$T\,[\omega^\alpha\,\theta(\omega)] = \left[\omega_2^\alpha + \alpha\,\omega_2^{\alpha-1}(\omega_3+\omega_4+\dots) + \frac{\alpha(\alpha-1)}{2}\,\omega_2^{\alpha-2}(\omega_3+\dots)^2 + \dots\right] \times$$
$$\left[\theta(\omega_2) + \delta(\omega_2)\,(\omega_3+\omega_4+\dots) + \delta'(\omega_2)\,(\omega_3+\dots)^2 + \dots\right].$$

Here and below, the symbol T will be used to denote a (formal) Taylor expansion of any quantity following it. This expansion is valid (at any finite order) when $\mathrm{Re}(\alpha)$ is large enough, and is therefore valid (by analytic continuation) in our case where $\alpha = (2-n)/2$ or $-n/2$. A technically very useful aspect of the above expansion is that all the terms containing $\delta(\omega_2)$ or its derivatives give *vanishing* contributions (because $\omega_2^{\alpha-k}\,\delta^{(\ell)}(\omega_2)$ vanishes if $\mathrm{Re}(\alpha)$ is large enough, so that, by analytic continuation, $\omega_2^{\alpha-k}\,\delta^{(\ell)}(\omega_2) = 0$ for all values of α). The net effect is that the contribution coming from a small string segment $-\frac{\Delta}{2} < (\sigma'-\sigma) < \frac{\Delta}{2}$ around σ (with Δ being much smaller that the local radius of curvature of the worldsheet) can be simply (and correctly) written as the following expansion:

$$\begin{aligned}
[A_{\rm ret}(z)]^\Delta &\equiv \epsilon \int_{-\Delta/2}^{\Delta/2} d\sigma' \int_{-\infty}^{\tau_1} d\tau'\,\Sigma\,\omega^{\frac{2-n}{2}}\,\theta(\omega) \\
&= \epsilon \int_{-\Delta/2}^{\Delta/2} d\sigma' \int_{-\infty}^{\tau_1} d\tau'\,T\left(\Sigma\,\omega^{\frac{2-n}{2}}\right)\,\theta(\omega_2) \\
&= \epsilon \int_{-\Delta/2}^{\Delta/2} d\sigma' \int_{-\infty}^{\tau-|\Delta\sigma|} d\tau'\,T\left(\Sigma\,\omega^{\frac{2-n}{2}}\right).
\end{aligned}$$

Here, we have introduced an arbitrary upper limit τ_1, submitted only to the constraint $\tau_{\rm ret} < \tau_1 < \tau_{\rm adv}$ (for instance τ_1 could be τ), and which replaces the missing theta function

$\theta(z^0 - z'^0)$ by selecting the retarded portion of the other theta function $\theta(\omega)$. As above, the symbol T denotes a formal Taylor expansion. The expansion $T(\Sigma\,\omega^\alpha)$ is simply obtained by multiplying the expansion (A1) of ω with that of $\Sigma(\tau', \sigma')$, namely

$$T\left[\Sigma(\tau', \sigma')\right] = \Sigma(\tau, \sigma) + (\tau' - \tau)\,\dot{\Sigma} + (\sigma' - \sigma)\,\Sigma' + \dots \tag{A3}$$

Similarly we have

$$[\partial_\mu A_{\rm ret}(z)]^\Delta = 2\epsilon \int_{-\Delta/2}^{\Delta/2} d\sigma' \int_{-\infty}^{\tau - |\Delta\sigma|} d\tau'\, T\,(\Sigma\,\Omega_\mu\,\omega^{-n/2})\,, \tag{A4}$$

as well as corresponding expressions for the advanced fields

$$[A_{\rm adv}(z)]^\Delta = \epsilon \int_{-\Delta/2}^{\Delta/2} d\sigma' \int_{\tau + |\Delta\sigma|}^{+\infty} d\tau'\, T\,\left(\Sigma\,\omega^{\frac{2-n}{2}}\right)\,, \tag{A5}$$

$$[\partial_\mu A_{\rm adv}(z)]^\Delta = 2\epsilon \int_{-\Delta/2}^{\Delta/2} d\sigma' \int_{\tau + |\Delta\sigma|}^{+\infty} d\tau'\, T\,\left(\Sigma\,\Omega_\mu\,\omega^{-n/2}\right)\,. \tag{A6}$$

As a check, we first computed the ultraviolet divergent contributions to $A_{\rm ret}(z)$ and $\partial_\mu A_{\rm ret}(z)$. We find

$$A_{\rm ret}(z) = -\frac{1}{\dot{z}^2}\left(\frac{2}{\epsilon}\right) 2\,\Sigma\,, \tag{A7}$$

$$\partial_\mu A_{\rm ret}(z) = \frac{1}{(\dot{z}^2)^2}\left(\frac{2}{\epsilon}\right)\left[-\Sigma\,\ddot{z}_\mu + \Sigma\,z''_\mu + 4\Sigma\,z'_\mu\left(\frac{z'\cdot z''}{\dot{z}^2}\right) + 4\Sigma\,\dot{z}_\mu\left(\frac{\dot{z}\cdot\ddot{z}}{\dot{z}^2}\right)\right.$$
$$\left. + \,2\Sigma'\,z'_\mu - 2\dot{\Sigma}\,\dot{z}_\mu\right]\,. \tag{A8}$$

As it should, Eq. (A8) yields exactly the same divergences as we found in Sec. III by introducing a cut-off δ in the σ' integration in four dimensions. More precisely, Eq. (A8) coincides with Eq. (3.6) if we change $2/\epsilon \to \log 1/\delta$. Let us note that, in the present approach, the renormalization scale Δ_R would enter by being introduced as a dimension-preserving factor in the dimensionful coupling constants, like Newton's constant G, say $G^{(n)} = G^{(n=4)}\,\Delta_R^\alpha$.

Our main interest is to compute the "local approximations" to the reaction field

$$A_{\rm reac}(x) = \frac{1}{2}\left(A_{\rm ret}(x) - A_{\rm adv}(x)\right)\,, \tag{A9}$$

and its derivatives. Dimensional continuation gives an efficient tool for computing these. Indeed, combining the previous expansions we can write

$$A_{\text{reac}}(x) = -\epsilon \int_{-\Delta/2}^{\Delta/2} d\sigma' \int_{\tau+|\Delta\sigma|}^{+\infty} d\tau' \, \theta(\Omega_0^2) \, T_{(\tau'-\tau)\,\text{odd}} \left(\Sigma \, (\Omega^2)^{\frac{2-n}{2}} \right) , \tag{A10}$$

$$\partial_\mu A_{\text{reac}}(x) = -2\epsilon \int_{-\Delta/2}^{\Delta/2} d\sigma' \int_{\tau+|\Delta\sigma|}^{+\infty} d\tau' \, \theta(\Omega_0^2) \, T_{(\tau'-\tau)\,\text{odd}} \left(\Sigma \, \Omega_\mu \, (\Omega^2)^{-\frac{n}{2}} \right) , \tag{A11}$$

where $T_{(\tau'-\tau)\,\text{odd}}$ denotes the part of the Taylor expansion which is odd in $\tau' - \tau$. Moreover, as we know in advance (and easily check) that the σ'-integrands in Eqs. (A10) and (A11) are regular at $\sigma' = 0$, we can very simply write the result of the local approximation (4.13) (with a corresponding definition for $A_{\text{reac}}^{\text{local}}(z)$) by replacing $\sigma' = \sigma$ in the integrands of Eqs. (A10), (A11)

$$[A_{\text{reac}}(z)]^{\text{local}} = -\epsilon\Delta \int_{\tau}^{+\infty} d\tau' \, T^{\sigma'=\sigma}_{(\tau'-\tau)\,\text{odd}} \left[\Sigma (\Omega^2)^{\frac{2-n}{2}} \right] , \tag{A12}$$

$$[\partial_\mu \, A_{\text{reac}}(z)]^{\text{local}} = -2\epsilon\Delta \int_{\tau}^{+\infty} d\tau' \, T^{\sigma'=\sigma}_{(\tau'-\tau)\,\text{odd}} \left[\Sigma \, \Omega_\mu \, (\Omega^2)^{-\frac{n}{2}} \right] . \tag{A13}$$

Here, $T^{\sigma'=\sigma}_{(\tau'-\tau)\,\text{odd}}$ denotes the operation of replacing σ' by σ and keeping only the odd terms in the remaining Taylor expansion in $\tau' - \tau$. This simplifies very much the computation of the reactive terms (making it only a slight generalization of the well known point-particle results, as given for a general source in, e.g. [26]). Indeed, inserting the following expansions

$$\Omega_\mu(\tau', \sigma) \simeq -(\tau' - \tau) \, \dot{z}_\mu - \frac{1}{2} (\tau' - \tau)^2 \, \ddot{z}_\mu - \frac{1}{6} (\tau' - \tau)^3 \, \dddot{z}_\mu \, , \tag{A14}$$

$$\Sigma(\tau', \sigma) \simeq \Sigma(\tau, \sigma) + (\tau' - \tau) \, \dot{\Sigma} + + \frac{1}{2} (\tau' - \tau)^2 \, \ddot{\Sigma} \, , \tag{A15}$$

$$\begin{aligned} \Omega^2(\tau', \sigma) \simeq -\dot{z}^2 \, (\tau' - \tau)^2 \Big[1 + (\tau' - \tau) \left(\frac{\dot{z} \cdot \ddot{z}}{\dot{z}^2} \right) + \frac{1}{4} \, (\tau' - \tau)^2 \left(\frac{\ddot{z} \cdot \ddot{z}}{\dot{z}^2} \right) + \\ \frac{1}{3} \, (\tau' - \tau)^2 \left(\frac{\dddot{z} \cdot \dot{z}}{\dot{z}^2} \right) \Big] , \end{aligned} \tag{A16}$$

in Eqs. (A12), (A13) we get our main results

$$[A_{\text{reac}}(z)]^{\text{local}} = \frac{\Delta}{\dot{z}^2} \left[\dot{\Sigma} - \Sigma \left(\frac{\dot{z} \cdot \ddot{z}}{\dot{z}^2} \right) \right] , \tag{A17}$$

$$\begin{aligned} [\partial_\mu A_{\text{reac}}(z)]^{\text{local}} = \frac{\Delta}{(\dot{z}^2)^2} \Big[\frac{1}{3} \Sigma \, \dddot{z}_\mu + \dot{\Sigma} \, \ddot{z}_\mu + \ddot{\Sigma} \, \dot{z}_\mu - 4 \dot{\Sigma} \, \dot{z}_\mu \left(\frac{\dot{z} \cdot \ddot{z}}{\dot{z}^2} \right) - 2 \Sigma \ddot{z}_\mu \left(\frac{\dot{z} \cdot \ddot{z}}{\dot{z}^2} \right) \\ - \Sigma \, \dot{z}_\mu \left(\frac{\ddot{z} \cdot \ddot{z}}{\dot{z}^2} \right) - \frac{4}{3} \, \Sigma \, \dot{z}_\mu \left(\frac{\dot{z} \cdot \dddot{z}}{\dot{z}^2} \right) + 6 \Sigma \, \dot{z}_\mu \left(\frac{\dot{z} \cdot \ddot{z}}{\dot{z}^2} \right)^2 \Big] . \end{aligned} \tag{A18}$$

These results were also obtained (as a check) from Eqs. (A10), (A11) without using in advance the simplification of putting $\sigma' = \sigma$ in the integrand.

We have also performed a direct check on these final expressions by comparing them to the well known point-particle case [24], [25], [26]. Indeed, we have seen above that $A^{\rm local}_{\rm reac}$ and $\partial_\mu A^{\rm local}_{\rm reac}$ could be thought of as being generated by the effective source $\Sigma^{\rm eff.}(\tau', \sigma') = \delta(\sigma' - \sigma)\, \Delta\, \Sigma(\tau', \sigma)$, i.e. a source along the world-line $\mathcal{L}_\sigma$, defined by $\sigma' = \sigma$. For any given value of σ, by transforming the coordinate time τ' into the proper time $s = \int e^{\phi/2}\, d\tau'$ along $\mathcal{L}_\sigma$ and by renormalizing in a suitable way the source $\Delta\, \Sigma(\tau', \sigma) \equiv e^{\phi/2}\, \tilde{S}(s)$ (so that the stringy spacetime source $\int d^2\sigma' \Sigma^{\rm eff.}(\tau', \sigma')\, \delta^4(x - z(\sigma'))$ transforms into the standard point-particle source $\int ds\, \tilde{S}(s)\, \delta^4(x - z(s))$), we recovered from Eqs. (A17), (A18) known point-particle results [26]. This check is powerful enough to verify the correctness of all the coefficients in Eqs. (A17), (A18).

In order to compare directly our expressions with what derived by Battye and Shellard in [14], [15], let us write Eq. (A18) for the axion field. We get

$$H^{\lambda\mu\nu} = \frac{4G\,\lambda\,\Delta}{(\dot{z}^2)^2} \left[\frac{1}{3}\, \dddot{z}^{[\lambda}\, V^{\mu\nu]} + \ddot{z}^{[\lambda}\, \dot{V}^{\mu\nu]} + \dot{z}^{[\lambda}\, \ddot{V}^{\mu\nu]} - 4\dot{z}^{[\lambda}\, \dot{V}^{\mu\nu]} \left(\frac{\dot{z} \cdot \ddot{z}}{\dot{z}^2} \right) - 2\ddot{z}^{[\lambda}\, V^{\mu\nu]} \left(\frac{\dot{z} \cdot \ddot{z}}{\dot{z}^2} \right) \right], \tag{A19}$$

where $K^{[\lambda\mu\nu]} = K^{\lambda\mu\nu} + K^{\mu\nu\lambda} + K^{\nu\lambda\mu}$. Note that, when identifying the basic *contravariant* tensors z^μ and $V^{\mu\nu}$, the tensor $H^{\lambda\mu\nu}$ (and the force density $\mathcal{F}^\mu$) must be identical in our conventions and in the ones of Refs. [14], [15] (who use the opposite signature). However, our result Eq. (A19) differs, after the substitution $G\lambda \to f_a/8$, in many terms from the second Eq. (31) of Ref. [15]. Whatever be the corrections we could think of doing on the second term in their Eq. (31) (which is dimensionally wrong, probably by a copying error leading to a forgotten overdot on one of the two terms), we saw no way of reconciling their result with ours (even after expanding explicitly $V^{\mu\nu} = \dot{z}^\mu\, z^{\nu\prime} - \dot{z}^\nu\, z^{\mu\prime}$).

REFERENCES

[1] M. Hindmarsh and T.W.B. Kibble, Rept. Prog. Phys. **58** (1995) 477.

[2] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and other Topological Defects*, Cambridge University Press, Cambridge, 1994.

[3] Y.B. Zel'dovich, Mon. Not. R. Astron. Soc. **192** (1980) 663.

[4] A. Vilenkin, Phys. Rev. Lett. **46** (1980) 1169.

[5] A. Vilenkin, Phys. Lett. **B117** (1981) 47.

[6] R.L. Davis, Phys. Rev. **D32** (1985) 3172.

[7] R.L. Davis, Phys. Lett. **B180** (1986) 225.

[8] R.L. Davis and E.P.S. Shellard, Nucl. Phys. **B324** (1989) 167.

[9] T. Damour and A. Vilenkin, Phys. Rev. Lett. **78** (1997) 2288

[10] D.P. Bennett and F.R. Bouchet, Phys. Rev. Lett. **60** (1988) 257.

[11] J.M. Quashnock and D.N. Spergel, Phys. Rev. **D42** (1990) 2505.

[12] B. Allen and R.R. Caldwell, Phys. Rev. **D43** (1991) 2457.

[13] D. Austin, E. Copeland and T.W.B. Kibble Phys. Rev. **D48** (1993) 5594.

[14] R.A. Battye and E.P.S. Shellard, Phys. Rev. Lett. **75** (1995) 4354.

[15] R.A. Battye and E.P.S. Shellard, Phys. Rev. **D53** (1996) 1811.

[16] E. Copeland, D. Haws and M. Hindmarsh, Phys. Rev. **D42** (1990) 727.

[17] T. Damour and G. Esposito-Farèse, Class. Quant. Grav. **9** (1992) 2093.

[18] P.A.M. Dirac, Proc. Roy. Soc. **A 167** (1938) 148.

[19] A. Dabholkar and J.M. Quashnock, Nucl. Phys. **B333** (1990) 815.

[20] A. Dabholkar and J.A. Harvey, Phys. Rev. Lett. **63** (1989) 478.

[21] A. Dabholkar, G. Gibbons, J.A. Harvey and F.R. Ruiz, Nucl. Phys. **B340** (1990) 33.

[22] J. Hadamard, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, Hermann, Paris, 1932;

L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1978, p. 38.

[23] M. Riesz, Acta Mathematica **81** (1949) 1.

[24] N.E. Fremberg, Proc. Roy. Soc. **A 188** (1946) 18.

[25] S.T. Ma, Phys. Rev. **71** (1947) 787.

[26] T. Damour, Nuovo Cimento **26B** (1975) 157.

[27] T. Damour, in *Gravitational Radiation* , ed. by N. Deruelle and T. Piran, North-Holland, Amsterdam, 1983, pp. 59–144.

[28] C.J. Burden, Phys. Lett. **B164** (1985) 277.

[29] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, Pergamon Press, Oxford, 1962.

[30] T. Damour, in *300 Years of Gravitation*, ed. by S.W. Hawking and W. Israel, Cambridge University Press, Cambridge, 1987, pp. 128–198.