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ABSTRACT. A new approach to the problem of the motion, in General Relativity, of systems of N weakly self–gravitating rotating deformable bodies, with arbitrary inner structures, is presented. This approach splits the general relativistic N–body problem into three sub–problems, namely : (i) the "external problem" (motion of N center–of–mass wordlines), (ii) the "internal problem" (motion of each body in its local center–of–mass reference system), and (iii) the general relativistic theory of local and global reference systems. The main tools used in our formalism are : a certain "exponential" parametrization of the metric coefficients which linearizes both the field equations and the transformation laws under a change of reference system, a specific definition of the relativistic multipole moments (and of the center of mass) of each body which make them both technically useful and operationally measurable, and an algebraic way of rigidly fixing (in each reference system) the spatial coordinate freedom while leaving a convenient post–Newtonian gauge freedom in the time coordinate.

1. INTRODUCTION

The N–body problem, i.e. the problem of describing the dynamics of N gravitationally interacting extended bodies, is the central problem of any theory of gravity. Within the framework of Newton's theory this problem, called "celestial mechanics", has been thoroughly investigated (see e.g. Tisserand, 1960). Very shortly after the discovery of Einstein's theory of gravity, Einstein (1915), Droste (1916), De Sitter (1916) and Lorentz and Droste (1917) devised an approximation method (called "post–Newtonian") which allowed them

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to compare General Relativity with solar system observations, and to predict several "relativistic effects" in celestial mechanics, such as the relativistic advance of the perihelion of planets, and the relativistic precession of the Moon's orbit. This post–Newtonian approach to general relativistic celestial mechanics was subsequently developed (and completed) by many authors, notably by Levi–Civita (1950), Fock (1959), Papapetrou (1951), Chandrasekhar and colleagues (1965, 1969, 1970), Caporali (1981), Grishchuk and Kopejkin (1986) and others (for a review of the development of the problem of motion in General Relativity see e.g. Damour, 1987). However, the great increase in precision of current, and foreseeable, observational techniques in the solar system makes it now necessary to reconsider this traditional "post–Newtonian" way of tackling the gravitational N–body problem.

In order to match the high precision which is already achieved by means of space techniques such as Satellite Laser Ranging (SLR), Lunar Laser Ranging (LLR) or Very Long Baseline Interferometry (VLBI), one needs a correspondingly accurate relativistic theory of celestial mechanics able to describe, at one and the same time, the global gravitational dynamics of a system of N extended bodies, the local gravitational structure of each, arbitrarily composed and shaped, rotating deformable body, and the way each of these N local structures meshes into the global one. The traditional post–Newtonian approach to relativistic celestial mechanics, that uses only one global coordinate system $x^\mu = (ct, x, y, z) \equiv (ct, x^i)$, $i = 1, 2, 3$, to describe an N–body system, fails in this task, for both conceptual and technical reasons. Concepts like "center of mass", "multipole moments", "mass–centered frames" are used while they are ill defined. Usually such "mass–centered frames" with spatial coordinates X^i, e.g. given by

$$X^i = x^i - z^i(t),$$

where z^i denotes the global coordinates of the "center of mass" of the body under consideration are not dynamically useful in the sense that they do not *efface* the external gravitational field down to tidal effects but, instead, introduce into the description of the internal dynamics of the bodies many external "relativistic" effects proportional to the square of the orbital velocity or to the external gravitational potential. This is because the external (global) description of each body contains many "apparent deformations" (Lorentz contraction, Einstein contraction, etc.) which are not intrinsic to the body itself. As emphasized by Damour (1987), those technical shortcomings of the usual global post–Newtonian approach are partly rooted in the surreptitious introduction of a kind of neo–Newtonian interpretation of General Relativity (Eisenstaedt, 1986) by which the global coordinates, (t, x, y, z), are implicity identified with the absolute time and absolute space of Newtonian theory.

In recent years, several authors have tried to remedy some of the defects of the traditional post–Newtonian approach to the N–body problem. For instance, Martin et al. (1985) and Hellings (1986) have tried, in an essentially heuristic manner, to explicitly take into account the main apparent deformations due to the use of an external coordinate representation. Other authors, notably Ashby and Bertotti (1984, 1986), have attempted a consistent constructive approach to the definition of "good" relativistic local reference systems, able to describe each body without introducing (as far as possible) non–intrinsic relativistic effects. More recently a notable progress in the theory of such local relativistic frames (at the post–Newtonian approximation, relevant to systems of N weakly self–gravitating bodies) has been achieved by Brumberg and Kopejkin (1988a,b) (Kopejkin, 1988; Brumberg, 1990) in a series of publications (see also Voinov, 1988). Their approach combines the usual post–Newtonian–type expansions with the multipole expansion formalisms for internally generated (Thorne, 1980; Blanchet and Damour, 1986, 1989) and externally generated (Thorne and Hartle, 1985), gravitational fields, and with the asymptotic matching techniques used in the theory of the motion of strongly self–gravitating bodies (D'Eath, 1975, Damour, 1983). We believe, however, that the approach by Brumberg and Kopejkin has several drawbacks: ad hoc assumptions about the structure of various expansions (as e.g. in the coordinate transformation between global and local coordinates) are made, which are only partially justified by some later consistency checks; the scheme is confined to a particular model for the matter (isentropic perfect fluid) and rigidly restricts itself to considering only some special (harmonic) coordinate conditions; moreover, their approach is basically incomplete in that it neither describes the full multipole moment structure of the bodies with post–Newtonian accuracy, nor gets (translational or rotational) equations of motion with full post–Newtonian accuracy.

2. A NEW APPROACH TOWARDS RELATIVISTIC CELESTIAL MECHANICS

We have introduced (Damour, Soffel and Xu, 1990a) a new formalism for treating the relativistic celestial mechanics of systems of N, arbitrarily composed and shaped, weakly self–gravitating, rotating, deformable bodies. This formalism yields, at one and the same time, a complete description (at the first post–Newtonian level) of, (i) the global dynamics of such N–body systems ("external problem"), (ii) the local gravitational structure of each body ("internal problem"), and, (iii) the way they fit together ("relativistic theory of reference systems"). This new scheme successfully overcomes, in our opinion, the problems encountered by previous approaches (notably the one of Brumberg and Kopejkin): only very general assumptions are made for the structure of the formalism which is developed in a constructive way by proving a number of theorems; the structure of the stress–energy tensor of the matter is left completely open; the scheme is formulated in a certain "gauge–invariant" way which leaves a convenient flexibility in the choice of the time gauge (at the order $\delta t = O(c^{-4})$); the scheme describes with full post–Newtonian accuracy the gravitational structure of each body by means of a set of multipole moments which are linked in an operational way to what can be observed in the local gravitational environment of each body; finally, the scheme succeeds in getting translational and rotational equations of motion with full post–Newtonian accuracy, and inclusion of all multipole moments, for the N–body system.

Our approach does not use any asymptotic matching technique, for instance the only assumption made about the coordinate transformation linking global coordinates ($x^\mu; \mu = 0,1,2,3$) to local ones ($X^\alpha; \alpha = 0,1,2,3$) is that it is twice differentiable so that one can, without loss of generality, write it in the form

$$x^\mu = f^\mu(X^\alpha) = z^\mu(X^0) + e^\mu_a(X^0)X^a + \xi^\mu(X^0, X^a) \ , \tag{1a}$$

with

$$\xi^\mu = O((X^a)^2) \ , \quad \text{as } X^a \to 0 \ ; \ a = 1,2,3 \ . \tag{1b}$$

On the other hand, our scheme takes advantage of two different recent progresses in the first post–Newtonian approximation method: (i) linearization of Einstein's field equations by means of certain "exponential parametrization" of the metric tensor (introduced by Blanchet and Damour (1989), and Blanchet, Damour and Schäfer (1990)), and (ii) definition, by Blanchet and Damour (1989), of a set of post–Newtonian multipole moments of an isolated body given as compact support integrals of the stress–energy tensor of the matter. A third basic element of the present approach is our way of restricting (without fixing completely) the coordinate freedom inherent to the theory of General Relativity. We do that not by imposing one of the two *differential* coordinate conditions generally used in the post–Newtonian literature (namely "harmonic gauge" versus "standard post–Newtonian gauge") but by imposing, in all coordinate systems, some *algebraic* conditions on the metric coefficients, which can be written as $(i,j = 1,2,3)$

$$g_{00}\, g_{ij} = -\delta_{ij} + O(1/c^4) \ . \tag{2}$$

This condition can be described by saying that the spatial coordinates are "conformally cartesian" or "isotropic". This condition is compatible with both usual choices but is, at once, more flexible (for the time gauge) and more rigid (for the space gauge) than either one of them. It plays an important technical role in freezing down the coordinate freedom to a level which is nearly the usual freedom in Newtonian celestial mechanics (arbitrary choice of a time–dependent spatial origin and of a time–dependent rotation matrix).

3. THEORY OF REFERENCE SYSTEMS

For our problem of N gravitationally interacting extended bodies we employ $N+1$ coordinate charts (reference systems) : one "global" chart with coordinates $x^\mu = (ct, x^i)$ and N "local" charts with coordinates $X^\alpha = (cT, X^a)$. Each one of the local charts is defined in the vicinity of some body, and "comoving" with it (in a sense made precise below). In each of these reference systems we use an exponential representation for the metric tensor of the form

$$g_{00} = -\, e^{-2w/c^2} + O(6) \tag{3a}$$

$$g_{0i} = -\,\frac{4}{c^3} w_i + O(5) \tag{3b}$$

$$g_{ij} = +\, e^{+2w/c^2}\, \gamma_{ij} + O(4) \tag{3c}$$

where $O(n) \equiv O(c^{-n})$ indicates the post–Newtonian order of magnitude. One finds that the Riemann curvature tensor of γ_{ij} is of order $O(4)$, i.e. to post–Newtonian (PN) order the spatial metric γ_{ij} is flat. Hence, there exists a preferred class of spatial coordinates, where

$$\gamma_{ij} = \delta_{ij} + O(4) \ , \tag{4}$$

or, equivalently, where condition (2) holds. In the formulation of our framework we systematically use such preferred spatially isotropic coordinates. Then, for each reference system, the information in the metric tensor is fully contained in the scalar field w and the vector field w_i. The Einstein field equations remarkably become *linear* in terms of these variables:

$$\Delta w + \frac{3}{c^2}\partial_t^2 w + \frac{4}{c^2}\partial_{ti}^2 w_i = -4\pi G\sigma + O(4) \tag{5a}$$

$$\Delta w_i - \partial_{ij}^2 w_j - \partial_{ti}^2 w = -4\pi G\sigma^i + O(2) \; , \tag{5b}$$

where

$$\sigma \equiv \frac{T^{00} + T^{ss}}{c^2} \tag{6a}$$

$$\sigma^i \equiv \frac{T^{0i}}{c} \; . \tag{6b}$$

Though, for each reference system, our spatial coordinates are fixed (modulo a choice of origin and rigid rotation) by our spatial isotropy condition, we do not fix completely our time coordinate, but keep a certain flexibility linked to a gauge invariance of the 1PN field equations: if $w_\mu \equiv (w, w_i)$ is a solution of eqs. (5) with some given source terms $\sigma^\mu \equiv (\sigma, \sigma^i)$ so is $w'_\mu = (w', w'_i)$ (modulo PN error terms) with

$$w' = w - \frac{1}{c^2}\partial_t \lambda \; , \tag{7a}$$

$$w'_i = w_i + \frac{1}{4}\partial_i \lambda \; , \tag{7b}$$

where $\lambda(x^\mu)$ is an arbitrary (differentiable) function. This gauge invariance corresponds to a shift of the time variable according to

$$\delta t = \frac{1}{c^4}\lambda(t, \boldsymbol{x}) \; , \tag{8}$$

which affects none of the physical quantities at the 1PN level. Note, that our gauge freedom encompasses both the choice of the "harmonic gauge" as well as of the "standard post–Newtonian gauge". E.g., the harmonic–gauge solution of the field equations in the global reference system is essentially unique (when imposing boundary conditions appropriate to isolated systems), and reads

$$w = G I_{-1}[\sigma] + \frac{G}{2c^2}\partial_t^2 I_1[\sigma] + O(4) \tag{9a}$$

$$w_i = G I_{-1}[\sigma^i] + O(2) \; , \tag{9b}$$

where

$$I_\alpha[f](t, \boldsymbol{x}) \equiv \int d^3x' |\boldsymbol{x} - \boldsymbol{x}'|^\alpha f(t, \boldsymbol{x}') \; . \tag{10}$$

On the other hand, this uniqueness is lost for the solutions in the local reference systems, because of the possible addition of arbitrary homogeneous solutions of the (linear) local–frame field equations.

Similarly to what is done in Maxwell's theory of electromagnetism, we can introduce gauge–invariant (gravito–electric and gravito–magnetic) fields $\boldsymbol{e}$ and $\boldsymbol{b}$ by

$$\boldsymbol{e} \equiv \nabla w + \frac{4}{c^2}\partial_t \boldsymbol{w} \tag{11a}$$

$$\boldsymbol{b} \equiv -4\nabla \times \boldsymbol{w}, \tag{11b}$$

satisfying (in each system) "Maxwell–like" equations of the form

$$\nabla \cdot \boldsymbol{b} = 0 \ , \tag{12a}$$

$$\nabla \times \boldsymbol{e} = -\frac{1}{c^2}\,\partial_t \boldsymbol{b} \ , \tag{12b}$$

$$\nabla \cdot \boldsymbol{e} = -\frac{3}{c^2}\,\partial_t^2 w - 4\pi G\sigma + O(4) \ , \tag{12c}$$

$$\nabla \times \boldsymbol{b} = 4\,\partial_t \boldsymbol{e} - 16\pi G\boldsymbol{\sigma} + O(2) \ . \tag{12d}$$

For the coordinate transformation between each of the local coordinates X^α and the global coordinates x^μ we start with the completely general ansatz

$$x^\mu = f^\mu(X^\alpha) = z^\mu(X^0) + e^\mu_a(X^0)X^a + \xi^\mu(X^0, X^a) \ . \tag{13}$$

Here, $a = 1, 2, 3$ labels the spatial coordinates in the local system, $z^\mu(X^0)$ (which is just $f^\mu(X^0, 0, 0, 0)$) describes the "global" motion of some "central worldline" of the body under consideration (which will later be chosen as the worldline of the "BD center of mass" of the body) and ξ^μ is assumed to be at least quadratic in X^a. Now, our PN assumptions plus spatially isotropic coordinates essentially determine $f^\mu(X^\alpha)$ completely, modulo the choices of some arbitrary central worldline and of some (slowly varying) rotation matrix R^j_a in $e^i_a(T)$ (see eq.(23) below). E.g., one finds uniquely

$$\xi^i(T, \boldsymbol{X}) = \frac{1}{c^2} e^i_a(T) \left[\frac{1}{2} A_a \boldsymbol{X}^2 - X^a(\boldsymbol{A}\boldsymbol{X})\right] + O(4) \ , \tag{14}$$

where

$$A_a \equiv f_{\mu\nu}\, e^\mu_a \frac{d^2 z^\nu}{d\tau_f^2} \tag{15}$$

(the Minkowskian 4–acceleration of the central worldline projected into the corresponding local system). Here, $f_{\mu\nu}$ denotes the usual flat Minkowski metric in Cartesian coordinates ($f_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$), and τ_f the Minkowskian proper time along the central worldline.

Because of the linearity of the field equations, in each local system, we can uniquely split the metric potentials† $W_\alpha \equiv (W, W_a)$ into a "self–" and an "external–part"

$$W_\alpha = \overset{+}{W}_\alpha + \overline{W}_\alpha \ . \tag{16}$$

Here, the self–part ($W^+{}_\alpha$) describes the gravitational influence of the central body itself as recorded in its associated reference frame, while the external–part describes the action of all the *other* bodies of the system (plus *inertial* terms).

As central results of our scheme we find the following transformation rules of potentials between some local and the global system to PN order :

$$w = \left(1 + \frac{2}{c^2} V^a V^a\right) W + \frac{4}{c^2} V^a W_a + \frac{c^2}{2} \ln \left[A^0_0 A^0_0 - A^0_a A^0_a\right] \tag{17a}$$

$$w^i = R^i_a W^a + v^i W + \frac{c^3}{4} \left[A^0_0 A^i_0 - A^0_a A^i_a\right] \ , \tag{17b}$$

where the A's are the coefficients of the Jacobian matrix,

$$A^\mu_\alpha \equiv \frac{\partial x^\mu}{\partial X^\alpha} \ , \tag{18}$$

and v^i is the velocity of the "central point" in the global system ($V^a = R^a_i v^i$). Hence, not only are the field equations linear, but so are the various $w \leftrightarrow W$ relationships ! Writing this affine transformation in the form

$$w^\mu(x) = \mathcal{A}^\mu_\alpha(T) W^\alpha(X) + \mathcal{B}^\mu(X) \ , \tag{19}$$

we further prove that the transformation of the self–parts take the simple form

$$\overset{+}{w}{}^\mu(x) = \mathcal{A}^\mu_\alpha(T) \, \overset{+}{W}{}^\alpha(X) \ , \tag{20}$$

a remarkable result indeed. Hence, in contrast to previous works on relativistic reference systems, we obtain the various transformation laws in closed, i.e., non–expanded form (we do *not* use a matched asymptotic expansion technique like, e.g, Brumberg and Kopejkin), which has in fact many advantages for practical applications.

We introduce the following (BD) mass (M_L) and spin (S_L) multipole moments of each body defined by (L is a multi spatial index, $L \equiv a_1 \dots a_\ell$)

$$M_L = \int d^3X \hat{X}_L \Sigma + \frac{1}{2(2\ell+3)c^2} \frac{d^2}{dT^2} \left[\int d^3X \hat{X}_L \boldsymbol{X}^2 \Sigma\right]$$
$$- \frac{4(2\ell+1)}{(\ell+1)(2\ell+3)c^2} \frac{d}{dT} \left[d^3X \hat{X}_{aL} \Sigma_a\right] \ , \tag{21a}$$

$$S_L = \int d^3X \epsilon^{ab<c_\ell} \hat{X}^{L-1>a} \Sigma^b \ , \tag{21b}$$

† We use capital letters for local quantities.

where all quantities are considered in the local system of the considered body, where the caret, or equivalently the bracket <>, indicates that the symmetric and trace–free (STF) part should be taken (see e.g. Thorne, 1980) and where the integration extends over the support of the body under consideration. These BD–moments are called "physical" by us because the self–part of the local gravitational potentials of the considered body can be expanded in terms of these multipole moments (modulo an irrelevant gauge transformation). Blanchet and Damour (1989) have introduced these multipole moments for *isolated* bodies and proven, in that case, that the BD mass (M) agrees with the constant ADM mass to PN order, and that the time–derivatives of the BD moments (for $\ell \geq 2$) give the PN–accurate multipole expansion of the gravitational wave amplitude emitted at future null infinity (see Damour and Iyer, 1990, for an extension of this work to PN–accurate *spin* moments).

Analogously to the self–potentials $W^{+}{}_{\alpha}$ that are expanded in terms of (STF) moments, we "skeletonize" the external–potentials $\overline{W}_{\alpha}$, or rather the corresponding $\overline{E}$ and $\overline{B}$–fields, by defining (for each local system) two corresponding (gravito–electric and gravito–magnetic) sets of post–Newtonian *tidal moments* :

$$(\ell \geq 1) \quad G_L \equiv \left[\partial_{<L-1}\overline{E}_{a_l>}(T,\boldsymbol{X})\right]_{X^a=0} \;, \tag{22a}$$

$$(\ell \geq 1) \quad H_L \equiv \left[\partial_{<L-1}\overline{B}_{a_l>}(T,\boldsymbol{X})\right]_{X^a=0} \;. \tag{22b}$$

Using the various expansions of the self–potentials and the transformation laws we can get the external tidal moments explicitly as functions of the intrinsic moments M_L, S_L of all the *other* bodies, plus some inertial contributions.

If we require (as we may) the quantities e^{μ}_{α} ($e^{\mu}_{0} \equiv c^{-1} dz^{\mu}(T)/dT$) to represent an orthonormal tetrad with respect to the "external metric" defined by $\overline{w}_{\mu}$, our theory of reference systems is completely specified up to the choice of:

– the time gauge

– the central worldlines, $z^i(T)$

and a slowly time dependent rotation matrix $R^j_a(T)$ appearing in

$$e^i_a(T) = \left(1 - \frac{1}{c^2}\overline{w}|_{X^a=0}\right)\left(\delta^{ij} + \frac{1}{2c^2}v^i v^j\right) R^j_a(T) \;. \tag{23}$$

4. EQUATIONS OF MOTION

In our approach, global equations of motion are derived by combining the local energy–momentum balance equations

$$T^{\beta}_{\alpha;\beta} = 0 \tag{24}$$

with conditions chosen to relate the central worldline of a body with the corresponding energy–momentum distribution. We find that a theorem of the following form holds in each local frame

Theorem. *The energy–momentum conservation equations (24) in each local frame imply constraints on the time–evolution of the three lowest BD multipole moments of the form :*

$$\frac{dM}{dT} = \frac{1}{c^2}\mathcal{E}^{1\mathrm{PN}}(\overset{(p)}{M}_L, \overset{(p')}{G}_{L'}) + O(4)\ , \tag{25a}$$

$$\frac{d^2 M_a}{dT^2} = \sum_{\ell\geq 0}\frac{1}{\ell!}M_L G_{aL} + \frac{1}{c^2}\mathcal{F}_a^{(1\mathrm{PN})}(\overset{(p)}{M}_L, \overset{(q)}{S}_L; \overset{(p')}{G}_{L'}, \overset{(q')}{H}_{L'}) + O(4)\ , \tag{25b}$$

$$\frac{dS_a}{dT} = \sum_{\ell\geq 0}\frac{1}{\ell!}\epsilon_{abc}M_{bL}G_{cL} + \frac{1}{c^2}\mathcal{G}_a^{(1\mathrm{PN})}(\overset{(p)}{M}_L, \overset{(q)}{S}_L; \overset{(p')}{G}_{L'}, \overset{(q')}{H}_{L'}) + O(2/4)\ , \tag{25c}$$

where

$$\overset{(p)}{M} \equiv \frac{d^p}{dT^p}M \qquad \text{etc.}$$

and all the right–hand sides of eqs. (25) are bilinear in the BD multipole moments and in the above–introduced tidal moments, and their time derivatives.

More explicitly, the right–hand sides of eqs. (25) consist of an infinite series of terms, each having the form

$$\overset{(p)}{M}\overset{(q)}{G},\ \overset{(p)}{M}\overset{(q)}{H},\ \overset{(p)}{S}\overset{(q)}{G},\quad \text{or}\quad \overset{(p)}{S}\overset{(q)}{H}.$$

The special notation $O(2/4)$ in eq. (25c) means that, when one is working strictly within the 1PN approximation, it is sufficient to know S_a to Newtonian accuracy and therefore the explicitly written Newtonian torque is enough. However, it is possible to define a local spin vector for body A (differing from the Newtonian spin moment (21b) by $O(c^{-2})$ additional terms) whose time evolution is given, modulo $O(4)$, by an equation of the form (25c).

At this point, we *define a center–of–mass worldline* for each body by requiring the vanishing of the BD dipole moment computed in the corresponding local reference system : $M_a = 0$. From eq. (25b) this then implies

$$0 = \frac{d^2 M_a}{dT^2} = \sum_{\ell\geq 0}\frac{1}{\ell!}M_L G_{La} \quad + (c^{-2} - \text{terms})\ . \tag{26}$$

Since one finds that the inertial contribution to the *tidal dipole moment* reads

$$G_a = -\frac{d^2 z^a(t)}{dt^2} + \overline{w}_{,a}|_{X^a=0} \quad + (c^{-2} - \text{terms}), \tag{27}$$

we see that the "local equation of motion" (26) can be rewritten in the looked for global form for the equation of translational motion :

$$M\,\frac{d^2 z^a(t)}{dt^2} = M\,\overline{w}_{,a}|_{X^a=0} + \sum_{\ell\geq 2}\frac{1}{\ell!}M_L G_{La} + (c^{-2} - \text{terms})\ , \tag{28}$$

where we have derived the complete PN expression of the right hand side of eq. (28) for arbitrary mass– and spin–moments of the individual bodies (Damour, Soffel and Xu, 1990b). We have explicitly verified (Damour, Soffel and Xu, 1990a) that in the monopole limit without spins ("spherical, non–rotating" bodies) one recovers the usual Lorentz–Droste–-Einstein–Infeld–Hoffmann equations of motion used for modern numerical ephemeris programs (such as the DE programs from JPL). Work for the PN–spin motion is still in progress.

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