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In his late years, Tullio Levi–Civita (29 March 1873 – 29 December 1941) became interested in the problem of the motion of several bodies in general relativity. He was well aware of the earlier investigations by Droste and de Sitter (Droste 1916a, de Sitter 1916)[1], but (rightly) felt the incompleteness of these works, notably with respect to the treatment of the self–field contributions of the bodies. Around the year 1936, he started an ambitious research program on the general relativistic problem of motion aimed at : (i) clarifying the mathematical and physical assumptions underlying the (now–called) post–Newtonian approximation method ; (ii) studying the self–field effects, and (iii) deriving the observable relativistic effects (periastron advance and acceleration of the center of mass) in binary systems comprising objects of comparable masses[2]. However, in spite of his very careful presentation of the issues at hand, the final results of Levi–Civita's calculations obviously show a diversity of self–contradictory features.

Regarding the self–field contributions of the bodies, Levi–Civita at first (Levi–Civita 1937a) claimed that they can be absorbed into the mass parameters of the bodies only in the two–body case (with Brillouin, Levi–Civita called this phenomenon of renormalizability the "effacing principle", see Damour 1987 for a recent discussion of this issue). In his later book (Levi–Civita 1950)[3], he claims to show that the absorption of the self–field contributions can be extended to the n–body case.

Regarding the relativistic periastron advance, Levi–Civita has obtained three different results. At the Harvard Tercentenary Conference of Arts and Sciences, September 1936, he gave (Robertson 1938) a preliminary report on his work and announced an incorrect result of $(1 + 5p/3)$ times the correct one ; $p = m_1 m_2/(m_1 + m_2)^2$ where m_1 and m_2 are the mass parameters of the bodies. In his second paper (Levi–Civita 1937b) one can read the correct result. In his book[4] he gets $(1 + p/3)$ times the correct result. Strangely enough, in his book he does not comment on these differences nor on the difference with papers (Robertson 1938, Eddington and Clark 1938) which were well known to him and where two independent derivations of the correct result were given. On the contrary, he felt secure enough about his last found (incorrect) result to comment on the deviation with respect to the familiar (since Einstein's Mercury perihelion calculation) restricted–two–body result (with the replacement "mass of the central body" goes into "sum of the masses of the two bodies"). One can interpret, at the credit of Levi–Civita, this confidence in his results, notwithstanding their discrepancy with other results, from his probable feeling that the other approaches contained some basic methodological obscurities. Indeed, on the one hand the calculation of Robertson was based on a paper (Einstein, Infeld and Hoffmann 1938) which introduced a complicated and ill–justified method[5], and which led to calculations so complex that they could not be published in detail. On the other hand, the calculation of Eddington and Clark, far from following from a consistent method, consisted of grafting onto the previous works of Droste and de Sitter some purely heuristic arguments concerning the relativistic concept of mass and the "effacement" of self–field effects (i.e. precisely two of the issues to which Levi–Civita wanted to provide well–justified answers by means of a logically consistent method).

Regarding the secular acceleration of the center of mass motion of a binary system, Levi–Civita has published two different results. In his second article (Levi–Civita 1937b) one finds a non–zero secular acceleration, and in his later book he claims to prove the absence of any acceleration, in accordance with Robertson, and Eddington and Clark. In his book he is saying that only "une erreur matérielle de calcul" was responsible for his (wrong) first result.

A superficial look at the various results of Levi–Civita that we have quoted above may lead one to believe that Levi–Civita was the first one to derive (in 1937) the correct result for the relativistic periastron advance and that he had confirmed (in his book), after Robertson, and Eddington and Clark, that there is no secular acceleration of the center of mass of a binary system. Furthermore, he seems to have proven, in his book, the effacing principle for Einstein's theory at the first post–Newtonian order of approximation. However, in view of the diversity of Levi–Civita's results a thorough investigation of his contributions is obviously called for to clarify what has been correctly derived by him and what are the kind of errors involved, calculational or methodological. The two articles (Levi–Civita 1937a, 1937b) do not give enough details for enabling one to trace back the origin of his different results, therefore one must make use of his book.

Levi–Civita thought first that it was possible to absorb the self–field contributions into the mass parameters of the bodies only for the two–body problem. This seems to have been the consequence of a calculational error which has not been done in the book (compare the correct equation (V.47) of the book with the equation (16) of the first article). In his book, he shows formally that one can efface self–field effects for a system of pressure–less bodies. However, as had previously been remarked by Einstein (see Levi–Civita 1937a), a dust model is unable to yield dynamically stable bodies [6]. The first physically satisfactory treatment of self–field effects is to be found in the works by Fock and coworkers (Fock 1939, Petrova 1940)[7] who worked throughout with an ideal fluid model for the matter.

In the Chapter VI, Section 3.2 of his book, Levi–Civita concludes in favour of the absence of secular acceleration. A careful examination shows that this conclusion is inconsistent with the equations of motion he writes down in the book which still predict a secular acceleration. There are in fact several misprints, and forgotten coefficients, in his crucial sequence of equation (VI.80), but, both as they stand, and, as they should read, according to his previous result (VI.39), they lead to a non–zero secular acceleration. Obviously, these are calculational errors. The main error is located between equation (VI.37) and equation (VI.38) where Levi–Civita is computing $B_{h|i}D_h^{(2)}$. There he omits the term proportional to the spatial gradient of $\beta_r^2 = (\vec{n}\cdot\vec{v})^2$; possibly because of the notation which makes it ressemble a pure velocity–dependent term[8]. In our opinion, this is a methodological error, linked to the difficulties of implementing his way[9] of finding a "constant–dependent" Lagrangian. Indeed, he is looking for a function $L(\vec{r},\vec{v},e,a)$ such that $(\partial_{\vec{r}} - d/dt\,\partial_{\vec{v}})\ L$ gives the correct equations of motion when, after differentiation, e and $1/a$ are equated to the energy and the angular momentum (modulo some coefficients). In such an approach it is easy to make mistakes by replacing too soon $\vec{v}^2$ or $(\vec{n}\cdot\vec{v})^2$ in terms of e, $1/a$, and $1/r$. Moreover, the term proportional to $r^{-1}\partial_i\beta_r^2$ that he is omitting is the most difficult to compute correctly within his approach. No wonder he never got it right. It is certainly possible, by treating differently (and still incorrectly), this term, and by making some slight calculational errors, to get also his earlier Lagrangian

(Levi–Civita 1937b). On the other hand, by treating correctly this term we found (Damour and Schäfer 1988) that the first post–Newtonian correction to the Newtonian relative–motion Lagrangian should read (in his notation)

$$D^{\text{correct}} = \Pi - \left(\frac{1}{2}\beta^2 - \gamma - e\right) p\gamma ,$$

where Π is given in equation (5) of the 1937b–paper. This expression differs also from the equation (VI.42) in his book. A further effect related to this correction is the absence of any secular acceleration of the center of mass motion. The surprising result that Levi–Civita coincidentally obtained the correct periastron advance in his 1937b–paper (and a correct potential function for the relative spatial trajectory, as already verified by Eddington and Clark) lies in the fact that the correction to Π in our equation above happens to drop out for this effect (however it does contribute to the computation of the relative motion as a function of the time).

In conclusion, we may say that Levi–Civita was the first to stress the need of setting up a mathematically consistent framework for the general relativistic problem of the motion of n weakly self–gravitating bodies, including a clear treatment of self–field effects. Moreover, he emphasized also first the physical importance of deriving several observable relativistic effects in binary systems comprising objects of comparable masses : namely the relativistic periastron advance and an a priori possible acceleration of the center of mass motion. Concerning the general framework, he has given in his book a careful presentation of the many subtle (mathematical and physical) issues involved in the relativistic problem of motion. This beautiful model of what a relativistic theory of the n–body problem should look like has, alas, rarely been matched by subsequent works in this field. Particularly remarkable is Levi–Civita's clear recognition of the importance of the "effacement" properties of self–field effects in general relativity.

However, as far as explicit new results are concerned, one must admit that Levi–Civita fell short of achieving the ambitious aims of his research program. He went beyond Droste and de Sitter in expressing correctly an n–body metric in terms of a coordinate–conserved mass density (page 55 of his book ; where the density μ is defined so that $\int \mu d^3x$ is constant). If he had been willing to ignore heuristically self–field effects, he could probably have preceeded Eddington and Clark both in deriving explicitly the correct metric for n point masses, and their equations of motion. However, his (self–contradictory) use of a pressure–less model for the matter rendered physically unsatisfactory his proof of the effacement of self–field effects. The merit for a satisfactory demonstration must be credited to Fock and his school (Fock 1939, Petrova 1940) although there exists a long forgotten forerunner (Lorentz and Droste 1917a, 1917b) which had implicitly achieved this (for a particular incompressible fluid model). On the other hand, the priority for the correct derivation of the relativistic periastron advance, and the absence of secular acceleration of the Newtonian center of mass, of a binary system belongs to Robertson (Eddington and Clark coming about five months later than Robertson). Fock proved, within his approach, the absence of acceleration of a suitably defined relativistic center of mass in 1941 (Fock 1941). The explicit, fully "effaced", form of the n–body Lagrangian has been first published in 1950, the year of posthumous publication of Levi–Civita's book (Fihtengol'c 1950). It must however be remarked that this Lagrangian is, after a small substitution for the "renormalized" masses, identical to the 1917 result of Lorentz and Droste[10].

As a final cautionary remark, and as a tribute to the vision of Levi–Civita, it must be said that no complete, and logically consistent, treatment of the relativistic theory of n extended bodies at the first post–Newtonian approximation level has been published to date.

NOTES

1. In the thirties, these works were considered as the standard approach to the relativistic problem of motion and were, e.g., discussed in detail in the book of Chazy (Chazy 1930). The fact that (because of an error which originated in Droste 1916a,b) they lead to incorrect equations of motion was apparently first noted by Eddington and Clark in 1938 (Eddington and Clark 1938). The early correct treatment of Lorentz and Droste (1917) stayed forgotten in the literature until much later.

2. In all previous applications one had considered only the "restricted" relativistic two–body (or n–body) problem, i.e. the case where one of the masses is much bigger than the other one(s).

3. Apart from some misprints, Levi–Civita 1964 is, as we have checked, a mere translation of Levi–Civita 1950.

4. The completion of this book took place between 1939 (last quoted reference) and 1941 (Levi–Civita's death).

5. In effect, the method used by Einstein, Infeld and Hoffmann was logically flawed. In subsequent publications, Einstein and Infeld admitted this and more or less corrected the method, but introduced, so doing, a technical mistake. In fact, it is only through the recent researches on the motion of strongly self–gravitating bodies that the justification and realm of applicability of the Einstein–Infeld–Hoffmann approach have been clarified (see Damour 1987 and references therein).

6. Both in his first article and in the Chapter V, Section 4 (in particular, Hypothesis II) of his book, Levi–Civita is assuming an internal structure for his bodies that he can never dynamically realize within his dust model.

7. The work by Petrova was published only much later (Petrova 1949).

8. On the other hand, his treatment of the $\dot{\beta}_j$–acceleration term (of the other body) as constant is correct because, at this stage, he is dealing with the geodesic Lagrangian of a test–particle in the field of the other body.

9. In his book Levi–Civita is saying (Chapter VI, Section 2.2) that unfortunately he did not "succeed" otherwise.

10. Lorentz and Droste do not try to make contact with the previous (incompatible) results of Droste and de Sitter, nor on the difference in one coefficient of g_{44} between Droste 1916a and Droste 1916b.

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