POST-NEWTONIAN HYDRODYNAMICS AND POST-NEWTONIAN GRAVITATIONAL WAVE GENERATION FOR NUMERICAL RELATIVITY*

L. Blanchet, T. Damour, G. Schäfer

Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 Bures-sur-Yvette (France)

September 1989

IHES/P/89/59

---

* to appear in: Monthly Notices of the Royal Astronomical Society

POST–NEWTONIAN HYDRODYNAMICS AND POST–NEWTONIAN GRAVITATIONAL WAVE GENERATION FOR NUMERICAL RELATIVITY.

Luc Blanchet[1], Thibault Damour[1,2] and Gerhard Schäfer[3]

[1] DARC, Observatoire de Paris-CNRS
92195 Meudon Cedex, France.

[2] Institut des Hautes Etudes Scientifiques
91440 Bures sur Yvette, France.

[3] Max-Planck-Institut für Physik und Astrophysik
8046 Garching bei München, FRG.

Abstract

We present an approximate formalism for the equations of evolution of a moderately relativistic self-gravitating fluid which takes into account the dominant (quantitative and qualitative) new effects entailed by Einstein's theory. This (first plus second-and-a-half) post-Newtonian scheme is equivalent to known results of the literature, but casts them in an explicitly "quasi-Newtonian" form, so that it becomes as easy to implement on a computer as a Newtonian-gravity code for nonrelativistic hydrodynamics (in particular, all the relativistic nonlocalities are reduced to Poisson equations with compact-supported sources). We complete this scheme by a correspondingly accurate post-Newtonian gravitational waveform extraction formalism which goes beyond the "standard quadrupole equation" in including the first relativistic corrections to the emission of gravitational radiation in a form which can be easily implemented numerically. We view our formalisms as simple and robust tools allowing one, with only a minimal computer investment, to study the most important quantitative and qualitative characteristics of the three-dimensional gravitational dynamics, and wave generation, of a wide range of semi-relativistic sources, such as the collapse of a star to the neutron-star stage, or the coalescence of neutron star binaries.

§1 Introduction

In view of the rapid progress in the development of a world-wide network of gravitational wave detectors, it is an urgent task to perform detailed general relativistic calculations of the generation of gravitational waves by catastrophic events such as the gravitational collapse of a star, or the coalescence of a binary system. The completion of the program of computing the emission of gravitational waves by such highly dynamical, strongly self-gravitating, material sources will require the development of fully general relativistic three dimensional (3D) numerical codes for the evolution of the combined matter + gravitational field system. At present, only 2D general relativistic codes have been fully implemented (Stark & Piran 1986a,b ; Nakamura et al. 1987). On the other hand, we have learned from the work of the last decade that, even in the Newtonian regime, the dynamics of the collapse, and of the various bounces and ejections, is very difficult to follow with precision because it is highly sensitive to many of the physical parameters of the problem (initial state, equation of state, neutrino transfer), as well as to the precision of the numerical simulation of the physics involved in the collapse. Therefore, one expects that in many cases the main difficulties and uncertainties will come from a detailed treatment of the hydrodynamical aspects of the problem (equation of state, choice of hydrodynamical variables, bounces, shocks, ...), rather than from its gravitational aspects.

Having this in mind, we propose in this paper an approximate relativistic formalism which :

(i) is as easy to implement on a computer as a Newtonian-gravity (3D) code, but which :

(ii) takes into account the dominant new effects entailed by Einstein's theory, both in the evolution of the matter and in the generation of gravitational radiation.

We hope that this formalism will constitute a simple and robust ready-to-use tool allowing the owners of Newtonian 3D codes to study the gravitational dynamics of many semi-relativistic sources without having to invest in a sophisticated fully fledged general relativistic 3D code.

We shall not consider in the following gravitational collapses leading to the formation of a black hole. This is a very interesting, but very difficult, problem which, up to now, has been amenable only to a simplified treatment involving axisymmetric configurations, and a polytropic equation of state (Stark & Piran, 1986a,b). We shall rather have in mind gravitational collapses leading to the formation of neutron stars, through some complicated dynamical evolution involving, may be, rapid rotation, fission, bounces, wild oscillations, etc... . Such intricate, and fully three dimensional, evolutions are likely to be efficient, and information-rich, generators of gravitational waves. In such cases the gravitational fields created by the collapse can be considered as being always "moderately strong", the typical relativistic gravitational parameter GM/c^2R never exceeding, say, 10%, so that one expects the orbital velocities to stay also mildly relativistic : $v/c \sim (GM/c^2R)^{1/2} \leq$ 30%. Now, if this is the case, the effects of order $(GM/c^2R)^2$ or $(v/c)^4$ will stay smaller than 1% and one will often be entitled to neglect them in view of the many other uncertainties present in the other aspects of the modelisation (e.g. the equation of state).

Now, such an approximation to General Relativity, which neglects terms of relative order $(GM/c^2R)^2 \sim (v/c)^4$ in the equations of motion, is known under the name of (first) post-Newtonian (1PN) approximation. It has been developed by many authors, starting with the basic work of Fock (1959), and found its way in several textbooks (Weinberg 1972 ; Misner, Thorne & Wheeler 1973). The 1PN approximation already takes into account, at their lowest significant order, most of the specifically general relativistic effects, namely :

(i) the "gravito-magnetic" effects (interaction between mass currents),

(ii) the "gravito-tensorial" effects (linked to the curvature of space),

(iii) and the nonlinear effects (gravity generates gravity).

The main incompleteness of the 1PN approximation concerns the propagation effects (linked to the finite velocity of propagation of gravity). Indeed, it misses the gravitational radiation damping effects which start at order $O(v^5/c^5)$. These effects are quantitatively much smaller, but they are <u>qualitatively</u> new, and should be included because they cause cumulative effects which have an important influence on the overall dynamical evolution of the matter. Therefore we shall add to the 1PN approximation the effects of gravitational radiation damping which come from the so-called second-and-a-half post-Newtonian (2 1/2 PN) approximation (Thorne 1969 ; Chandrasekhar & Esposito 1970 ; Burke 1971 ; see Damour 1987 for a review and references). Note that, for convenience, we use here a slightly inconsistent terminology in which "1PN" refers to the sum of Newtonian and first-post-Newtonian ($O(v^2/c^2)$) terms, while "2 1/2 PN" (or 2.5 PN) refers only to the $O(v^5/c^5)$ terms. On the other hand we shall not try to explicitly include the intermediate effects ($O(v^4/c^4)$) coming from the second post-Newtonian (2PN) approximation (Chandrasekhar & Nutku 1969) because they can be considered as bringing only some quantitatively small corrections to the effects already included at the 1PN level, without giving rise to any qualitatively new physical effects (see however the recent study of orbital 2PN effects in binary pulsars by Damour & Schäfer 1987, 1988).

Recently a promising new approach to the numerical simulation of gravitational collapse has been proposed (Bonazzola & Marck 1986, 1989a). It is based on the use of pseudo-spectral methods for dealing with the spatial dependence of the various variables (while the time evolution is still treated by finite differencing). The use of such numerical schemes allows one to get a high precision in the computation of <u>spatial</u> derivatives, and in the inversion of the Laplacian. In view of this, the specific aims of the present work will be first to reformulate the system of equations describing the <u>evolution of the matter</u> in the 1PN + 2 1/2 PN approximation of General Relativity in a form :

a) which is explicitly that of a first-order evolution system,

$$\frac{\partial u_A(\vec{x},t)}{\partial t} = F_A\left[u_B(\vec{y},t)\right], \tag{1.1}$$

where the (spatially non-local) functionals $F_A[u_B]$ may involve the spatial derivatives of the $u_B(\vec{y},t)$'s, but not their time derivatives (nor any non-locality in time) ;

b) such that the computation of the spatially non-local functionals F_A involves nothing more complicated than solving usual Poisson equations,

$$\Delta U = (\text{localized source}) \quad ; \tag{1.2}$$

c) and, which never involves taking the inverse of the matter density.

Then, our second task will be to complete the preceding post-Newtonian matter-evolution system by a correspondingly accurate <u>post-Newtonian gravitational wave generation</u> formalism, written in a form which can be easily numerically implemented.

As will be seen in the course of this work the preceding tasks are far from being trivial to perform, in spite of the existence in the literature of several post-Newtonian hydrodynamics, and post-Newtonian wave generation, formalisms. The principal tools that will allow us to meet all the requirements listed above are : (1) the use of suitable matter variables (playing the role of a matter density and a velocity field) ; (2) the choice of an adapted coordinate system ; (3) the systematic use of a mathematical trick to reduce the gradient of an iterated Poisson operator, $\partial_i \Delta^{-2}$, to a combination of usual Poisson operators, Δ^{-1}, acting on localized sources ; and, finally, (4) the use of a new post-Newtonian gravitational wave generation formalism which has been recently developed within the multipolar post-Minkowskian approximation method for radiative gravitational fields (Blanchet & Damour 1986, 1989).

Concerning the interplay between the use of an adapted coordinate system and the choice of suitable matter variables, it should be pointed out from the start that the optimal presentation turns out (even in the general, possibly dissipative, case) to be closely related to the general relativistic Hamiltonian approach of Arnowitt, Deser & Misner (1960, 1962), by which we mean the core of their work which was to develop a reduced canonical formalism describing only the "true dynamical degrees of freedom" of the matter + gravitational field system (note however that, in order to prevent the appearance of the inverse of the matter density, we shall use a specific linear momentum variable which has been found useful in several other investigations of relativistic hydrodynamics (Lichnerowicz, 1955 ; Carter & Gaffet 1988 ; Carter 1989)). This close link is not surprising as the Arnowitt-Deser-Misner (ADM) Hamiltonian approach leads, by definition, to first-order evolution systems, like equation (1.1), and, favours, by construction, elliptic equations, like equation (1.2). The usefulness of the ADM approach (and of the ADM coordinate conditions) when dealing with gravitational radiation damping effects had been already pointed out by one of us in the context of the general relativistic N-body problem (Schäfer 1985).

Finally, let us make it clear that the formulation proposed here is not supposed to be the final word on 3D general relativistic hydrodynamics. Clearly a lot of effort should continue to be directed towards including the fully fledged general relativistic description of the gravitational field in numerical simulations of stellar collapse. However, we do hope, not only that the formulation proposed here can be useful in the short term (before the availability of fully relativistic 3D schemes), but also that, even in the long term, it will continue to be a useful and versatile tool : e.g. as a testing bench for numerical codes, as an economical way to see the effects on the evolution of a change in the equation of state, and also as a short cut to get the main characteristics of the evolution, together with a first estimate of the most significant physical quantities (e.g. the emitted gravitational waveform).

The plan of this paper is as follows : in § 2 we show how the space-time metric and the equations of relativistic hydrodynamics at the first post-Newtonian (1PN) approximation can be written in a conveniently compact explicit form when introducing some suitable variables ; in § 3 we add in the effects of gravitational radiation reaction (2.5PN level) ; in § 4 we motivate the choice of a preferred set of matter variables, (r_*, S, w_i), and we show how to reduce all the spatial nonlocalities of the 1PN + 2.5PN approximation to Poisson equations with compact-supported sources ; then we present in a fully explicit and logically ordered way our complete set of evolution equations for a perfect relativistic fluid in § 5 ; finally we complete this evolution scheme by giving in § 6 a corresponding first post-Newtonian gravitational waveform extraction formalism. Appendix A gathers some useful general formulas of the thermodynamics of relativistic fluids, while Appendix B and Appendix C discuss, respectively, the 3+1 split of various quantities, and the reduced Hamiltonian formalism for relativistic perfect fluids.

§2 Post-Newtonian hydrodynamics

Let us consider the general relativistic gravitational field generated by some material source, i.e. the solution $g_{\mu\nu}$ (with signature $-+++$) of the Einstein equations,

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \tag{2.1}$$

(where $T^{\mu\nu}$ denotes the stress-energy tensor of the matter), satisfying some suitable no-incoming-radiation condition. The post-Newtonian expansion of $g_{\mu\nu}$ is a combined weak-field-near-zone expansion in powers of the dimensionless number $\varepsilon \sim (U/c^2)^{1/2} \sim R/cT$, where $U \sim GM/R$ is a characteristic gravitational potential, R a characteristic dimension and T a characteristic time of variation of the source. It is convenient to use the powers of c^{-1} for ordering the successive post-Newtonian approximations. The so-called first post-Newtonian (1PN) approximation consists in keeping the following terms in the various components of the space-time metric $g_{\mu\nu}$: $g_{00} = -1 + c^{-2}(...) + c^{-4}(...)$, $g_{0i} = c^{-3}(...)$, $g_{ij} = \delta_{ij} + c^{-2}(...)$. The knowledge of these various terms allows one to compute the first relativistic corrections $\big(O(\varepsilon^2) = O(v^2/c^2)\big)$ in the equations of motion of the matter (written in terms of the coordinate time, $t = x^0/c$), and to interpret them in operational terms through the space-time metric.

The 1PN approximation has been investigated by many authors (see e.g. Chandrasekhar 1965, Will 1981, Caporali 1981). However its results are often presented in a not very transparent way, and for the restricted case of a perfect fluid source. As a useful starting point for the present work, we shall now present a compact formulation of the 1PN approximation, valid for any kind of material source (i.e. any structure of $T^{\mu\nu}$). We shall work in the "standard post-Newtonian gauge" (Will, 1981), i.e. in coordinates such that

$$\partial_j g_{0j} - \frac{1}{2} \partial_0 g_{jj} = O(c^{-5}), \tag{2.2a}$$

$$\partial_j g_{ij} - \frac{1}{2} \partial_i (g_{jj} - g_{00}) = O(c^{-4}). \tag{2.2b}$$

In such coordinates, the 1PN metric can be written as

$$g_{00} = -e^{-2U/c^2} + O\left(\frac{1}{c^6}\right) = -1 + \frac{2}{c^2} U - \frac{2}{c^4} U^2 + O\left(\frac{1}{c^6}\right), \tag{2.3a}$$

$$g_{0i} = -\frac{1}{c^3}\, A_i \; + \; O\left(\frac{1}{c^5}\right), \tag{2.3b}$$

$$g_{ij} = \delta_{ij}\left(1 + \frac{2}{c^2}U\right) + O\left(\frac{1}{c^4}\right). \tag{2.3c}$$

In eqs (2.3), the "scalar potential" U is a Newtonian-like potential,

$$U(\vec{x}, t) = +G \int d^3\vec{x}\,'\; \frac{\sigma(\vec{x}\,', t)}{|\vec{x} - \vec{x}\,'|}, \tag{2.4}$$

satisfying a Poisson equation with compactly supported source,

$$\Delta U = -4\pi G \sigma, \tag{2.5}$$

where the source σ is the following combination of components of the stress-energy tensor of the material source :

$$\sigma = c^{-2}(T^{00} + T^{ss}) + O(c^{-4}) = c^{-2}\sqrt{g}\left(-T^0_0 + T^s_s\right) + O(c^{-4}), \tag{2.6}$$

(where g denotes $-det\; g_{\mu\nu}$ and where $T^{00} = O(c^{+2})$, $T^{0i} = O(c^{+1})$ and $T^{ij} = O(c^0)$ so that σ is defined to post-Newtonian accuracy, $(...) + c^{-2}(...)$, by eq.(2.6)). The quantity σ plays here the role of an "active gravitational mass density" which generates the scalar part of the <u>near-zone</u> gravitational field. As shown recently (Blanchet & Damour, 1989), it plays also an important role in the generation of the gravitational <u>wave</u> field (see §6 below). The usual post-Newtonian results, to be found in the literature, get more complicated expressions for g_{00} because of their use of a different basic "mass density".

In eqs (2.3), the "vector potential" A_i satisfies

$$\Delta A_i = -16\pi\, \frac{G}{c}\, T^{0i} + \frac{\partial^2 U}{\partial t \partial x^i} + O\left(\frac{1}{c^2}\right) \tag{2.7a}$$

(where $c^{-1}T^{0i} = O(c^0)$ when $c^{-1} \to 0$) so that

$$A_i = 4U_i + \frac{1}{2}\, \partial_{ti} X + O(c^{-2}), \tag{2.7b}$$

where

$$\Delta U_i = -4\pi\; G\; c^{-1}\; T^{0i} + O(c^{-2}), \tag{2.8}$$

and where the non compactly supported source term $\partial_{ti}U$ has been dealt with by introducing the superpotential X (also denoted $-\chi$) defined by

$$X(\vec{x},t) = +G\int d^3\vec{x}\,'|\vec{x}-\vec{x}\,'|\,\sigma(\vec{x}\,',t), \tag{2.9a}$$

and satisfying a Poisson equation with non compact source :

$$\Delta X = +2U. \tag{2.9b}$$

We shall however see below how to compute A_i by means only of Poisson equations with sources having compact support. Before dealing with the equations of motion of the matter, let us note that the formulation (2.3) is closely related to the (harmonic gauge) one used by Blanchet & Damour (1989) : they differ only by a shift of order c^{-4} (proportional to $\partial_t X$) in the time coordinate, a shift which leaves unchanged the 1PN equations of motion. However the Newtonian-like structure of the standard-1PN-gauge metric coefficients, eqs (2.5), (2.7), turns out to be better suited to our present purpose than the Minkowskian-like structure of the harmonic-gauge metric (this feature will become even more important at the 2.5PN level, see §3 below).

The Einstein equations (2.1) imply the covariant conservation of energy and momentum,

$$\nabla_\nu\, T^\nu_\mu = 0. \tag{2.10}$$

It is convenient to write the spatial ($\mu = i$) components of (2.10) as,

$$\partial_t\left(c^{-1}\sqrt{g}\,T^0_i\right) + \partial_j\left(\sqrt{g}\,T^j_i\right) = \mathcal{F}^{\rm grav}_i, \tag{2.11}$$

where

$$\mathcal{F}^{\rm grav}_i = \frac{1}{2}\sqrt{g}\,T^{\mu\nu}\partial_i g_{\mu\nu}. \tag{2.12}$$

Using the formulas (2.3), together with

$$\sqrt{g} = 1 + \frac{2}{c^2}U + O\left(\frac{1}{c^4}\right), \tag{2.13}$$

it is immediate to find that the "gravitational force density" reads

$$\mathcal{F}_i^{\mathrm{grav}} = c^{-2}(T^{00} + T^{ss})\partial_i U - c^{-3} T^{0j} \partial_i A_j + O\left(\frac{1}{c^4}\right). \tag{2.14}$$

Note that $\sigma = c^{-2}(T^{00} + T^{ss})$ appears again in eq.(2.14), playing now the role of a "passive gravitational mass density". Note also that the space integral of $\mathcal{F}_i^{\mathrm{grav}}$ vanishes to 1PN order,

$$\int d^3\vec{x}\ \mathcal{F}_i^{\mathrm{grav}} = \int d^3\vec{x}\left(\sigma\partial_i U - c^{-3} T^{0j}\partial_i A_j\right) = 0 + O\left(\frac{1}{c^4}\right), \tag{2.15}$$

as is easily checked from the definitions (2.5) and (2.7) of U and A_j, and from the lowest-order conservation of energy,

$$\partial_t \sigma\ + \partial_i\left(c^{-1} T^{0i}\right) = O(c^{-2}), \tag{2.16}$$

which follows from the time component of eq.(2.10).

If we now define a "momentum density" of the matter by

$$\pi_i := c^{-1}\sqrt{g}\ T_i^0 \tag{2.17}$$

(where the symbol $a := b$ means that a is defined as being b), we can write the equations of motion of the matter as (see also §4 and Appendix C) :

$$\partial_t \pi_i + \partial_j\left(\sqrt{g}\ T_i^j\right) = \sigma\partial_i U - c^{-2}\pi_j\partial_i A_j + O(c^{-4}). \tag{2.18}$$

The usefulness of the quantity π_i shows up not only in the simplicity of eq.(2.18), but also in the fact that eq.(2.15) implies the conservation to post-Newtonian order of the "total momentum" :

$$\int d^3\vec{x}\ \pi_i = \mathrm{const.} + O\left(\frac{1}{c^4}\right). \tag{2.19}$$

It can be checked that, in the special case of a perfect fluid, and at the 1PN approximation, the "momentum density" (2.17) coïncides with the quantity denoted G_i by Fock (1959 ; eq. (79.18)), and π_α by Chandrasekhar (1965), Chandrasekhar & Nutku (1969) and Caporali (1981).

Up to this point, our results have been valid for an arbitrary structure of the source. However, in order to deduce from eq.(2.18) some explicit evolution equation for the matter distribution we need to choose a specific matter model. Let us consider a perfect fluid, corresponding to a stress-energy tensor having the structure

$$T^{\mu\nu} = r(c^2 + h)u^\mu u^\nu + pg^{\mu\nu}, \tag{2.20}$$

with

$$g_{\mu\nu}u^\mu u^\nu = -1, \tag{2.21}$$

and

$$\nabla_\mu(r\ u^\mu) = 0 \tag{2.22}$$

(conservation of rest-mass ; r being, say, the proper baryon density, n, times some baryon mass m).
In eq.(2.20) the coefficient of $u^\mu u^\nu$, usually written as the sum, $e + p$ of the proper relativistic energy density, e, and of the proper pressure, p, has been decomposed in a rest-mass contributions, rc^2, and an enthalpy part, with

$$h = \frac{e+p}{r} - c^2 = \frac{[e - rc^2] + p}{r}, \tag{2.23}$$

denoting the (proper) specific enthalphy (enthalpy per unit rest- mass, minus the constant rest-mass contribution).

An equation of state for the fluid can be defined by giving, e.g., the energy e as a function of the rest-mass density, r, (or baryon density, n) and of the specific entropy, S (entropy per unit rest-mass, i.e. m^{-1} the entropy per baryon),

$$e = e\ (r, S). \tag{2.24}$$

As discussed in more detail in Appendix A, the first law of thermodynamics and the additivity property of extensive quantities imply

$$p(r, S) = r\left(\frac{\partial e}{\partial r}\right)_S - e, \tag{2.25}$$

so that one can write

$$de = (c^2 + h)dr + rTdS, \tag{2.26}$$

or, equivalently,

$$dp = r dh - rT dS. \tag{2.27}$$

As is well known (see also Appendix A), the projection along u^μ of the covariant conservation law (2.10), together with the "first law of thermodynamics" (2.26), implies that S is constant along each flow line :

$$u^\mu \partial_\mu S = 0. \tag{2.28}$$

Moreover, having now in hand a specific structure for the stress-energy tensor, eq.(2.20), we can render more precise the equations of motion (2.18) by writing down the structure of $\sqrt{g}\, T_i^j$. It is easily seen from eq.(2.20) that

$$\sqrt{g}\, T_i^j = v^j \pi_i + \sqrt{g}\, p\, \delta_i^j, \tag{2.29}$$

where

$$v^i := c\frac{u^i}{u^0} = \frac{dx^i}{dt} \tag{2.30}$$

is the coordinate-time 3-velocity of the fluid.
Therefore the relativistic Euler equations (2.18) can be written as

$$\partial_t \pi_i = -\partial_j \left(v^j \pi_i\right) + \mathcal{F}_i^{\mathbf{press}} + \mathcal{F}_i^{\mathbf{grav}}, \tag{2.31a}$$

with a "pressure force density",

$$\mathcal{F}_i^{\mathbf{press}} = -\partial_i(\sqrt{g}\, p), \tag{2.31b}$$

and a "gravitational force density",

$$\mathcal{F}_i^{\mathbf{grav}} = \sigma \partial_i U - \frac{1}{c^2}\pi_j \partial_i A_j + O\left(\frac{1}{c^4}\right). \tag{2.31c}$$

The rather simple equation(2.31) (to be completed by the link between v^i and π_i, see below) brings out in a compact manner the structure of 1PN hydrodynamics. The matter conservation law (2.22) can also be more explicitly written as an evolution equation by introducing the "coordinate rest-mass density"

$$r_* := \sqrt{g} u^0 r. \tag{2.32}$$

One gets

$$\partial_t r_* = -\partial_i \left(r_* v^i \right). \tag{2.33}$$

Similarly the entropy transport equation (2.28) reads

$$\partial_t S = -v^i \partial_i S. \tag{2.34}$$

Equations (2.31), (2.33) and (2.34) constitute formally a system of 5 evolution equations for the 5 degrees of freedom of a fluid : 3 velocity components, mass density and entropy. However, they must still be cast in the form of eq.(1.1), i.e. we must still express the right-hand sides of eqs(2.31), (2.33) and (2.34) as explicit functionals of the matter variables. Moreover, we wish, for numerical purposes, to satisfy also the requirements b) and c) of §1 (see eq.(1.2)). As all these requirements will have to be still satisfied after inclusion of gravitational radiation damping effects, let us first tackle the latter effects before implementing the requirements a), b) and c) of §1.

§3 Gravitational radiation reaction

The gravitational radiation damping effects enter at the second and a half post-Newtonian (2.5PN) approximation, i.e. through terms $O(c^{-5})$ in the equations of motion. However, the explicit expression of the c^{-5} terms to be added in, say, eq.(2.18) depend on the coordinate system. The expressions derived by Burke (1971), Thorne (1969) or Chandrasekhar & Esposito (1970) are not suited to our present purpose because they add, in the right-hand side of eq.(2.31a), time derivatives of the quadrupole moment of the system of such a high-order (the fifth) that it becomes very difficult to satisfy the requirements a) and b) of §1 (even after reducing to only fourth-order derivatives by shuffling some terms in the left-hand side of eq.(2.31a)). However, it has been shown by Schäfer (1983, 1985, 1989) that there existed some coordinate systems where the radiation reaction force was taking a form involving lower-order time derivatives of the quadrupole moment, which will turn out to be better adapted to our purpose (for the links between the various forms of the radiation reaction, and for references to other works on this problem, see the review Damour, 1987). More precisely, we shall assume that we are using, everywhere in this work, the coordinate system introduced by Arnowitt, Deser & Misner (1960), as defined by the conditions :

$$\partial_j g_{ij} - \frac{1}{3}\,\partial_i g_{jj} = 0, \tag{3.1a}$$

$$\pi^{ii} = 0, \tag{3.1b}$$

where π^{ij} denotes the canonical conjugate to the 3-metric g_{ij},

$$\pi^{ij} = -\sqrt{\gamma}\,\gamma^{ia}\gamma^{jb}(K_{ab} - K g_{ab})\,. \tag{3.2a}$$

In eq.(3.2a) γ denotes the determinant, and γ^{ij} the inverse, of the 3-metric g_{ij}, and K_{ij} denotes the extrinsic curvature tensor of the 3-surface t = const. (and $K = \gamma^{ij}K_{ij}$) :

$$K_{ij} = -\frac{1}{2}(-g^{00})^{1/2}\left[c^{-1}\partial_t g_{ij} - D_i g_{0j} - D_j g_{0i}\right], \tag{3.2b}$$

D_i being the spatially covariant derivative defined by the 3-metric g_{ij} (see Appendices B and C for more details about, respectively, the 3+1 split of the metric, and the ADM Hamiltonian formalism).

The gauge conditions (3.1) are well tuned to the ADM description of the true dynamical degrees of freedom of the gravitational field (Arnowitt, Deser & Misner, 1962). Recent work by Schäfer (1985, 1989) has shown that the ADM Hamiltonian formalism was quite convenient for tackling the post-Newtonian approximation, and especially the gravitational damping effects. Note that, at the 1PN order, the gauge conditions (3.1) imply eqs(2.2), so that the ADM gauge (3.1) is a generalization (to all higher orders) of the "standard 1PN gauge".

Gravitational radiation reaction effects in ADM gauge can be incorporated in two ways. The simple-minded way consists of completing eqs(2.3) by the first time-asymmetric ("reaction") contributions to the metric coefficients in the ADM gauge. To be precise about what we mean by "time-asymmetric", let us take for the present purpose r_*, S and v^i as basic matter variables. Then, the conservation laws (2.33) and (2.34) are independent of the metric, and the solution (containing no incoming radiation) of Einstein equations (2.1), with (2.20), (2.24), (2.30) and (2.32), expressed as a functional of r_*, S and v^i, can be expanded, in the near-zone, in powers of c^{-1}. The coefficients of the lowest orders in c^{-1} (1PN and 2PN approximations) are found to be symmetric under time reversal, while the first time-asymmetric contributions appear at the 2.5PN level and are found to be (Schäfer 1985, 1989) :

$$g_{00}^{\text{reac}} = +\frac{4}{5}\,\frac{G}{c^7}\,Q_{ij}^{(3)}(t)\,U_*^{ij} \;+\; O\left(\frac{1}{c^9}\right), \tag{3.3a}$$

$$g_{0i}^{\text{reac}} = 0 + O\left(\frac{1}{c^8}\right), \tag{3.3b}$$

$$g_{ij}^{\text{reac}} = -\frac{4}{5}\,\frac{G}{c^5}\,Q_{ij}^{(3)}(t) \;+\; O\left(\frac{1}{c^7}\right), \tag{3.3c}$$

where $Q_{ij}^{(3)}$ is the third time-derivative of the (traceless) quadrupole moment of the matter (see eq.(3.11) below) and U_*^{ij} the Newtonian "tensor potential" of r_*,

$$U_*^{ij}(\vec{x},t) = G\int d^3\vec{x}\,'r_*(\vec{x}\,',t)\;\frac{(x^i-x'^i)(x^j-x'^j)}{|\vec{x}-\vec{x}\,'|^3}, \tag{3.4a}$$

which can be expressed in terms of the superpotential X_* as

$$U_*^{ij} = U_*\delta_{ij} - \partial_{ij}X_* \tag{3.4b}$$

(U_* and X_* being defined by eqs(2.4) and (2.9a) with σ replaced by r_*).

The time-asymmetric metric coefficients have two effects on the evolution of the matter : they contribute a time-asymmetric ($O(c^{-5})$) term to the "gravitational force density" (2.12), namely

$$\mathcal{F}_i^{\rm reac} = \frac{1}{2}\sqrt{g}\, T^{\mu\nu}\, \partial_i g_{\mu\nu}^{\rm reac} = \frac{2}{5}\,\frac{G}{c^5}\, r_* Q_{jk}^{(3)} \partial_i U_*^{jk} + O\left(\frac{1}{c^7}\right), \tag{3.5}$$

and they contribute also a time-asymmetric ($O(c^{-5})$) term in the relation expressing π_i (defined by eq.(2.17)) as a functional of r_*, v^i and S, namely

$$\pi_i[r_*, S, v^j]^{\rm reac} = g_{ij}^{\rm reac} r_* v^j + O\left(\frac{1}{c^7}\right) = -\frac{4}{5}\,\frac{G}{c^5}\, Q_{ij}^{(3)} r_* v^j + O\left(\frac{1}{c^7}\right). \tag{3.6}$$

On the other hand, the effective pressure in eq.(2.31b),

$$\sqrt{g}\; p(r,S) = \sqrt{g}\; p\left(\frac{r_*}{\sqrt{g}u^0}, S\right),$$

does not contribute any $O(c^{-5})$ reaction force. Indeed, as all the "scalar parts" of $g_{\mu\nu}^{\rm reac}$ (i.e. $g_{00}^{\rm reac}$ and the trace $g_{ii}^{\rm reac}$) are zero at the c^{-5} level, the same is true of $\sqrt{g}^{\rm reac}$ and $u^0[r_*, S, v_j]^{\rm reac}$. (For the same reason, there are no $O(c^{-5})$ contributions to $\mathcal{F}_i^{\rm reac}$ in eq.(3.5) of the form $(c^{-2}\sqrt{g}T^{00})^{\rm reac}\partial_i U_*$).

An equivalent but more sophisticated way, which will turn out to be better adapted to our needs, of introducing the radiation reaction effects, is to place ourselves fully within the (reduced) Hamiltonian formalism of Arnowitt, Deser & Misner (1962). In this canonical approach the basic independent dynamical variables are r_*, S and π_i (linked by definition to the velocity in ADM gauge by eq.(2.17)) for the matter, and h_{ij}^{TT} and π^{ijTT} for the gravitational field (transverse-traceless parts of g_{ij} and π^{ij} in ADM gauge). Then, the solution of the (t=const)- hypersurface constraint equations,

$$\sqrt{g}\; R_\mu^0 - \frac{1}{2}\sqrt{g}\; R\delta_\mu^0 - \frac{8\pi G}{c^4}\sqrt{g}\; T_\mu^0 = 0, \tag{3.7}$$

together with the coordinate conditions (3.1), determine the <u>reduced</u> Hamiltonian,

$$H_{\rm red} = H\left[r_*, S, \pi_i, h_{ij}^{TT}, \pi^{ijTT}\right], \tag{3.8}$$

which governs the evolution of both the matter and the (true degrees of freedom of the) gravitational field. For more details about this approach, and references to recent developments, see the Appendix C. In this formalism the radiation reaction effects come from the coupling between the matter, and the gravitational field, degrees of freedom, which leads to a "reaction" contribution to the Hamiltonian equal to,

$$H^{\rm reac} = -\frac{1}{2}\int d^3\vec{x}\; h_{ij}^{TT\rm reac}\left(\frac{\pi_i\pi_j}{r_*} + \frac{1}{4\pi G}\partial_i U_* \partial_j U_*\right), \tag{3.9}$$

in which $h_{ij}^{TT\rm reac}$ should be replaced by eq.(3.3c) only <u>after</u> having varied $H^{\rm reac}[r_*, S, \pi_i]$. In other words, the equations of motion of the matter are derived from eq.(3.9) by assuming zero Poisson brackets between the matter variables and $h_{ij}^{TT\rm reac}$ (indeed, the Hamiltonian (3.9) describes the interaction of the material system with any long-wavelength gravitational wave (in TT gauge), while the replacement (3.3c) means that one considers the back action of the wave generated by the material system itself). Using the (Lie-)Poisson brackets of Appendix C, one finds that, in this formalism, the radiation reaction effects in the dynamics of the matter (i.e. in the evolution equations for r_*, S and π_i) consists of :

1) adding the term $\mathcal{F}_i^{\rm reac}$, eq.(3.5), in the right-hand side of $\partial_t\pi_i = ...$, and

2) adding a term in the relation, $v^i = \delta H/\delta\pi_i$, expressing v^i as a functional of π_i :

$$v^i[r_*, S, \pi_j]^{\rm reac} = \frac{\delta H^{\rm reac}}{\delta\pi_i} = -h_{ij}^{TT\rm reac}\,\frac{\pi_j}{r_*} = +\frac{4}{5}\,\frac{G}{c^5}Q_{ij}^{(3)}(t)\,\frac{\pi_j}{r_*}. \tag{3.10}$$

These results are completely equivalent to the ones obtained above (taking into account $\pi_i = r_* v^i + O(c^{-2})$). The sign difference between (3.6) and (3.10) comes from the fact that one is considering two inverse functions, $\pi[v]$ versus $v[\pi]$.

Finally, let us note that the appearance in the right-hand sides of the evolution equations for r_*, S and π_i of the third time-derivative of the Newtonian quadrupole moment of the matter distribution,

$$Q_{ij}(t) = \int d^3\vec{x}\; r_*(\vec{x},t)\left(x^i x^j - \frac{1}{3}\vec{x}^{\,2}\delta^{ij}\right) + O\left(\frac{1}{c^2}\right), \tag{3.11}$$

is susceptible of introducing numerical instabilities (Bonazzola & Marck, private communication) because the evolution would no longer be first-order in time (in the sense of eq.(1.1)). However, as we shall explicitly exhibit below, one can get rid of these three derivatives, i.e. express $Q_{ij}^{(3)}$ in terms of the instantaneous state of the matter distribution, by using both the equations of motion and the relation

$$\partial_t U_* + \partial_i U_i = O(c^{-2}) \tag{3.12}$$

between the Newtonian potentials of r_* and π_i.
Note however that this reduction would not work as simply for four derivatives, in which case it would be necessary to introduce new Newtonian-like potentials. This shows that the ADM-gauge description of radiation damping is better adapted to our aim than, e.g., the Burke-Thorne description.

§4 Choice of matter variables and reduction to compact-support Poisson equations

At this stage it seems that our preferred choice of matter variables would be r_*, S and π_i, which satisfy simple-looking evolution equations at the 1PN+2.5PN approximation. By going through the definitions of the various objects involved, it is easy to see that we can satisfy the requirement a) (eq.(1.1)) of §1. However, we wish also to satisfy the requirements b) and c). Now, the choice of π_i as basic momentum variable is not compatible with requirement c). Indeed, as $\pi_i = r_* v^i + O(c^{-2})$, the inverse relation $v[\pi]$ introduces many inverse factors of r_* which can be quite annoying in a numerical calculation of objects whose mass is essentially concentrated in bounded domains. To solve this problem we shall choose as fundamental matter variables r_*, S and the "momentum per unit rest-mass" (in ADM coordinates),

$$w_i := \frac{\pi_i}{r_*}. \tag{4.1}$$

In the Newtonian limit w_i becomes equal to the 3-velocity v^i. See Appendix B for the exact expression of v^i in terms of w_i (which will be needed for use in the right-hand sides of the evolution equations (2.33), (2.34) and (4.12)).

Using the definitions (2.17) and (2.32) for, respectively, π_i and r_*, and the perfect fluid structure (2.20), one finds that w_i can also be written as

$$w_i = \left(1 + \frac{h}{c^2}\right) c u_i, \tag{4.2}$$

i.e. as the spatial components of the space-time covector

$$w_\mu = \left(1 + \frac{h}{c^2}\right) c u_\mu. \tag{4.3}$$

The quantity w_μ was introduced by Lichnerowicz (1955) under the name of "current vector" (the relativistic enthalpy factor $1 + h/c^2$ being called by him "index of the fluid"). It has been emphasized recently by Carter (1989) (see also Carter & Gaffet 1988) that the co-vectorial character of w_μ was important, that w_μ was to be thought of as a "momentum-energy covector" per unit rest-mass, and that its use as a one-form, together with the Cartan calculus, was leading to simple and elegant results in fluid dynamics. Let us only quote the covariant formulation of the evolution equation for w_μ (under our perfect fluid assumptions (2.20), (2.22), (2.25), and hence (2.28)) :

$$c\, u^\nu(\partial_\nu w_\mu - \partial_\mu w_\nu) = T\ \partial_\mu S, \tag{4.4a}$$

or in Cartan language

$$i_{\vec{u}}\ d\ \underline{w} = c^{-1} T\ dS. \tag{4.4b}$$

The formulation (4.4) could also constitute a starting point for deriving our looked for evolution system. However, it happens that the formal simplicity of eq.(4.4) hides some internal complexities, which show up in the fact that when going from eq.(4.4a), for $\mu = i$, to our result (2.31) many cancellations take place (the same remark applies to the formally elegant Hamiltonian formulation of Appendix C).

Before tackling the issue of the reduction of the post-Newtonian "superpotentials" to a numerically more tractable Newtonian-potential form, some consequences of the use of the variable w_i, eq.(4.1), have to be discussed. Indeed, when using the variable π_i, there appeared in the evolution equation for π_i, eq.(2.31a), a "pressure force density", eq.(2.31b), corresponding to the effective pressure,

$$p^{\text{eff}} = \sqrt{g}\ p(r,S) = \sqrt{g}\ p\left(\frac{r_*}{\sqrt{g}\ u^0}, S\right). \tag{4.5}$$

To 1PN order one has (with $\vec{w}^{\ 2} := w_i w_i$)

$$\sqrt{g}\ = 1 + \frac{2}{c^2}\, U_* + O\left(\frac{1}{c^4}\right), \tag{4.6}$$

$$r = \frac{r_*}{\sqrt{g}\ u^0} = \left[1 - \frac{1}{c^2}\left(\frac{1}{2}\vec{w}^{\ 2} + 3U_*\right) + O\left(\frac{1}{c^4}\right)\right] r_*, \tag{4.7}$$

so that

$$p^{\text{eff}} = \left[1 + \frac{\alpha}{c^2} + O\left(\frac{1}{c^4}\right)\right] p_*, \tag{4.8}$$

where we recall that the $O(c^{-4})$ terms are free of any time-asymmetric $O(c^{-5})$ contributions, and where

$$\alpha = 2U_* - \gamma_* \left(\frac{1}{2} \vec{w}^{\,2} + 3U_* \right), \tag{4.9}$$

$$\gamma_* := \frac{r_*}{p_*} \left(\frac{\partial p_*}{\partial r_*} \right)_S = \left(\frac{\partial \,\log\, p_*}{\partial \,\log\, r_*} \right)_S, \tag{4.10}$$

p_* denoting $p(r_*, S)$.

Now, because the rest-mass conservation law (2.33) holds exactly, the replacement $\pi_i \equiv r_* w_i$ leads to

$$\partial_t \pi_i + \partial_j \left(v^j \pi_i \right) = r_* \left[\partial_t w_i + v^j \partial_j w_i \right]. \tag{4.11}$$

Therefore, that the evolution equation (2.31a) (in which $\mathcal{F}_i^{\text{grav}}$ contains $\mathcal{F}_i^{\text{reac}}$ as given by eq.(3.5)) becomes

$$\partial_t w_i = -v^j \partial_j w_i - \frac{1}{r_*} \partial_i p^{\text{eff}} + \frac{\mathcal{F}_i^{\text{grav}}}{r_*}. \tag{4.12}$$

The r_*^{-1} factor in front of the gravitational force density is of no concern as each term of $\mathcal{F}_i^{\text{grav}}$ is proportional to the density. As for the pressure force term,

$$F_i^{\text{press}} := \frac{\mathcal{F}_i^{\text{press}}}{r_*} = -\frac{1}{r_*} \partial_i p^{\text{eff}}, \tag{4.13}$$

it reads

$$F_i^{\text{press}} = -\left(1 + \frac{\alpha}{c^2} \right) \frac{\partial_i p(r_*, S)}{r_*} - \frac{1}{c^2} \frac{p_*}{r_*} \partial_i \alpha + O\left(\frac{1}{c^4} \right). \tag{4.14}$$

The most delicate term (when $r_* \to 0$) in F_i^{press} is the one proportional to $\partial_i p_* / r_*$. However, the first law of thermodynamics in the form (2.27) yields

$$\frac{\partial_i p_*}{r_*} = \partial_i h_* - T_* \partial_i S, \tag{4.15}$$

where the index $*$ means that the corresponding thermodynamic functions of r and $S, p(r, S), h(r, S), T(r, S)$ (as derived from the equation of state $e = e(r, S)$; see e.g. eq.(2.25)) are to be evaluated for $r = r_*$:

$$p_* := p(r_*, S), \quad h_* := h(r_*, S), \quad T_* := T(r_*, S). \tag{4.16a, b, c}$$

This shows the important role played by the enthalpy h_* in the dynamics of the matter. In the Newtonian case (Bonazzola & Marck, 1989b), it has been found useful to replace the equation of state for h,

$$h = h(r, S), \tag{4.17}$$

(which is susceptible of bringing extra numerical errors because of the wide range of variation of r, of the power-law type dependence of h on r, and of the spatially global nature of the pseudo-spectral expansions) by a propagation equation. As our use of the variables r_* and $h_* = h(r_*, S)$ allows us to be technically very close to the usual Newtonian equations, it is easy to implement this idea in the post-Newtonian case. Namely, we can replace the equation(4.16b) by

$$\partial_t h_* = -v^i \partial_i h_* - \eta_* h_* \partial_i v^i, \tag{4.18}$$

where the information contained in the (power-law type) equation of state (4.17) is replaced by giving oneself the "index" of h_* versus r_*,

$$\eta_* := (\partial \log\ h_* / \partial \log\ r_*)_S, \tag{4.19}$$

as a function of r_* and S, or of h_* and S,

$$\eta_* = \eta(r_*, S) \quad \text{or} \quad \eta_* = \eta(h_*, S). \tag{4.20}$$

In the simple case of a polytropic equation of state of index γ ($= \gamma_*$ of eq.(4.10)), the index η_* of eq.(4.19) is equal to $\gamma - 1$ (see Appendix A).

Having taken care of requirements a) and c) of §1 by a combined optimum choice of gauge and matter variables (as shown by our explicit implementation below), it remains to take care of our requirement b) : i.e. to reduce all the spatial non-localities to Poisson equations with compact-support sources. The quantities that create problems are the "vector potential" A_i, eq.(2.7), and the "tensor potential" U_*^{ij}, eq.(3.4b), appearing in the reaction force density (3.5). The troublesome parts of these potentials are, respectively, proportional to $\partial_t \partial_i X$ and $\partial_i \partial_j X$ (the distinction between X and X_*, i.e. using as basic matter density σ or r_*, is not important in our discussion), where we recall that (for any σ with spatially compact support)

$$U(\vec{x},t) = +G \int d^3\vec{x}\,' \frac{\sigma(\vec{x}\,',t)}{|\vec{x}-\vec{x}\,'|}, \tag{4.21a}$$

$$\Delta U = -4\pi G\sigma, \tag{4.21b}$$

while

$$X(\vec{x},t) = +G \int d^3\vec{x}\,' |\vec{x}-\vec{x}\,'| \sigma(\vec{x}\,',t), \tag{4.22a}$$

$$\Delta X = +2U. \tag{4.22b}$$

Let us now show how one can reduce the evaluation of the spatial gradient of X, $\partial_i X$ (which, a priori, is the solution of a non-compact- supported Poisson equation, $\Delta\partial_i X = 2\partial_i U$) to compact-supported Poisson equations. This aim is achievable by means of the identity

$$\partial_i X \equiv (x^i - a^i)U - \Delta^{-1}\left[(x^i - a^i)\Delta U\right], \tag{4.23}$$

where a^i are any space-independent (but possibly time-dependent) quantities. Eq.(4.23) is easily proven by differentiating the right-hand side of eq.(4.22a) with respect to x^i (or, formally, by taking the Laplacian of both sides of eq.(4.23)). Note that both terms on the right-hand side of eq.(4.23) depend on the choice of an arbitrary origin $x^i = a^i(t)$ in space, although their sum is independent of this choice. In the following, we shall take for simplicity $a^i = 0$, although one should keep in mind the freedom of using $a^i(t) \neq 0$, in the case for instance where the material source consists of two well separated blobs (e.g. binary coalescence) and where it might be advantageous to use two different a^i's corresponding roughly to the centers of mass of the two nearly disconnected supports of ΔU.

Finally, let us mention that, for disposing of the time-derivative appearing in A_i (i.e. $\partial_t\partial_i X_*$), it is sufficient to notice the fact that the source of $\partial_t U_*$, namely $\partial_t r_*$ can be replaced by $-\partial_i\left(r_* v^i\right)$, or with sufficient precision (because A_i enters only at 1PN order) by $-\partial_i \pi_i \equiv -\partial_i(r_* w_i)$.

§5 Evolution system for 1PN + 2.5PN hydrodynamics.

We have introduced in the §§2-4 all the ideas and tools necessary for implementing satisfactorily the requirements listed in the introduction. Putting together the results of §§2-4, and of Appendix B, we shall now present in a logically ordered way (adapted to numerical implementation) our optimal complete set of evolution equations for a perfect relativistic fluid, at the combined first post-Newtonian plus second and a half post-Newtonian approximation. For the sake of clarity, we shall separate our presentation in several subsections.

5.1 Basic independent matter variables

We propose two possible choices : a minimal set consisting of,

. r_* = the coordinate rest-mass density (eq.(2.32)),

. S = the entropy per unit rest-mass,

. w_i = the linear momentum per unit rest-mass (eq.(4.2)),

or an extended set consisting of,

r_*, S, w_i and

. h_* = the enthalpy per unit rest-mass (eq.(2.23)).

In the second case the (power-law like) equation of state giving (in the first case) h_* as a function of r_* and S, is replaced by a (tamer) equation of state giving the index $\eta_* = (\partial \log h_* / \partial \log r_*)_S$, together with an evolution equation for h_*. Note that, as in the second case h_* is considered as an independent variable on the same footing as r_* and S, there is some redundancy freedom in the way of writing the remaining necessary equations of state.

The evolution system will be written explicitly in the form :

$$\frac{\partial r_*(\vec{x},t)}{\partial t} = F_r\left[r_*(\vec{y},t),\ S(\vec{y},t),\ w_i(\vec{y},t),\ h_*(\vec{y},t)\right],$$

$$\frac{\partial S(\vec{x},t)}{\partial t} = F_S\left[r_*(\vec{y},t),\ S(\vec{y},t),\ w_i(\vec{y},t),\ h_*(\vec{y},t)\right],$$

$$\frac{\partial w_i(\vec{x},t)}{\partial t} = F_i\left[r_*(\vec{y},t),\ S(\vec{y},t),\ w_i(\vec{y},t),\ h_*(\vec{y},t)\right],$$

$$\frac{\partial h_*(\vec{x},t)}{\partial t} = F_h\left[r_*(\vec{y},t),\ S(\vec{y},t),\ w_i(\vec{y},t),\ h_*(\vec{y},t)\right],$$

where the functionals appearing in the right-hand sides will be computable from the knowledge all over space of the values of the basic matter variables at time t, by means either of algebraic operations or by solving some Poisson equations with sources of compact support. When using the minimal set of matter variables, it will be sufficient to ignore the evolution equation for h_* and to use instead the algebraic relation, $h_* = h(r_*, S)$.

5.2 Algebraic equations.

One will need to know, in the following equations,

$$T_* := T(r_*, S) = \frac{1}{r_*}\ \frac{\partial e(r_*, S)}{\partial S}, \tag{5.1}$$

$$\pi_* := \frac{p(r_*, S)}{r_*} = \frac{\partial e(r_*, S)}{\partial r_*} - \frac{e(r_*, S)}{r_*}, \tag{5.2}$$

$$\gamma_* := \frac{\partial \ \log\ p(r_*, S)}{\partial\ \log\ r_*}, \tag{5.3}$$

$$\eta_* := \frac{\partial\ \log\ h(r_*, S)}{\partial\ \log\ r_*}, \tag{5.4}$$

the last equation being replaced by

$$h_* = h(r_*, S) = \frac{\partial e(r_*, S)}{\partial r_*} - c^2, \tag{5.4$'$}$$

if one uses the minimal set of matter variables.
In the simple case of a polytrope, i.e. an equation of state

$$e(r, S) = r\ c^2\ +\ \frac{k(S)}{\gamma - 1}\ r^\gamma,$$

the above quantities read

$$\pi_* = k(S)\ r_*^{\gamma-1} = \frac{\gamma-1}{\gamma} h_*, \tag{5.2p}$$

$$\gamma_* = \gamma, \tag{5.3p}$$

$$\eta_* = \gamma - 1, \tag{5.4p}$$

or

$$h_* = k(S)\ \frac{\gamma}{\gamma - 1}\ r_*^{\gamma-1}. \tag{5.4p$'$}$$

5.3 Primary Poisson equations

$$\Delta U_* = -4\pi\ Gr_*, \tag{5.5}$$

$$\Delta U_i = -4\pi\ Gr_* w_i, \tag{5.6}$$

$$\Delta C_i = -4\pi\ Gx^i\ \partial_s(r_* w_s). \tag{5.7}$$

See eqs (5.11) and (5.14) below for the two remaining secondary Poisson equations.

5.4 1PN quantities (in which $\vec{w}^{\ 2}$ denotes $\delta^{ij} w_i w_j$)

$$\alpha = 2U_* - \gamma_* \left(\frac{1}{2}\vec{w}^{\ 2} + 3U_*\right), \tag{5.8}$$

$$\beta = \frac{1}{2}\vec{w}^{\ 2} + h_* + 3U_*, \tag{5.9}$$

$$\delta = \frac{3}{2}\vec{w}^{\ 2} + h_* + 2\pi_* - U_*, \tag{5.10}$$

$$\Delta U_2 = -4\pi\ Gr_*\delta, \tag{5.11}$$

$$A_i = 4U_i + \frac{1}{2}\ C_i - \frac{1}{2}\ x^i \partial_s U_s. \tag{5.12}$$

5.5 2.5 PN quantities

$$P_{ij} = 2 \int d^3\vec{x}\ r_* \left\{3w_i \partial_j U_* - 2w_i\left(\partial_j h_* - T_* \partial_j S\right) + \right.$$
$$\left. + x^i w_s \partial_{sj} U_* - x^i \partial_{sj} U_s \right\}, \tag{5.13a}$$

$$Q_{ij}^{[3]} = \frac{1}{2}\ P_{ij} + \frac{1}{2}\ P_{ji} - \frac{1}{3}\ \delta_{ij} P_{ss}, \tag{5.13b}$$

(Alternatively the STF operation of eq.(5.13b) could be done on the integrand of eq.(5.13a)). When the equations of motion are satisfied, it can be checked that the third time derivative of the Newtonian quadrupole moment of the system (defined, say, by eq.(6.1) below) differs from the quantity $Q_{ij}^{[3]}$ introduced here only by corrections of order $O(1/c^2)$ (see eq.(6.6) below). However, one must be careful, in the following, to distinguish $Q_{ij}^{[3]}(t)$, which is a functional of the instantaneous state of the matter defined as being exactly the right-hand-side of eq.(5.13b) (with eq.(5.13a)), from $Q_{ij}^{(3)} := d^3 Q_{ij}/dt^3$ (they differ both in their functional nature, and in their precise numerical value). The quantity $Q_{ij}^{[3]}$ is the one which enters in the 2.5 PN quantities of our scheme :

$$\Delta R = -4\pi\, G Q_{ij}^{[3]} x^i \partial_j r_*, \tag{5.14}$$

$$U_5 = \frac{2}{5}\, G \left[R - Q_{ij}^{[3]} x^i \partial_j U_* \right]. \tag{5.15}$$

5.6 Velocity and forces

$$v^i = w_i - \frac{1}{c^2}\, \beta w_i + \frac{1}{c^2}\, A_i + \frac{4}{5}\, \frac{G}{c^5}\, w_s Q_{is}^{[3]}, \tag{5.16}$$

$$F_i^{\text{press}} = -\left(1 + \frac{\alpha}{c^2}\right) \left[\partial_i h_* - T_* \partial_i S\right] - \frac{1}{c^2}\, \pi_* \partial_i \alpha, \tag{5.17}$$

$$F_i^{1\text{PN}} = \left(1 + \frac{\delta}{c^2}\right) \partial_i U_* + \frac{1}{c^2}\, \partial_i U_2 - \frac{1}{c^2}\, w_s \partial_i A_s, \tag{5.18}$$

$$F_i^{\text{reac}} = \frac{1}{c^5}\, \partial_i U_5. \tag{5.19}$$

5.7 Evolution system

$$\partial_t r_* = -\partial_i (r_* v^i), \tag{5.20a}$$

$$\partial_t S = -v^i \partial_i S, \tag{5.20b}$$

$$\partial_t w_i = -v^s \partial_s w_i + F_i^{\text{press}} + F_i^{1\text{PN}} + F_i^{\text{reac}}, \tag{5.20c}$$

$$\partial_t h_* = -v^i \partial_i h_* - \eta_* h_* \partial_i v^i. \tag{5.20d}$$

5.8 Exactly conserved quantities

The above defined evolution system guarantees (if it is exactly satisfied) the constancy of the total rest-mass,

$$M_* := \int d^3\vec{x}\; r_* = \text{const.}, \tag{5.21}$$

and of the total linear momentum,

$$P_i := \int d^3\vec{x}\; r_* w_i = \text{const.}. \tag{5.22}$$

The conservation of P_i follows from the separate exact vanishing of the space integral of each of the four terms appearing in the right-hand side of the evolution equation for $\pi_i \equiv r_* w_i$,

$$\partial_t \pi_i = -\partial_s (v^s \pi_i) + \; r_* F_i^{\text{press}} \; + \; r_* F_i^{\text{1PN}} \; + \; r_* F_i^{\text{reac}}. \tag{5.23}$$

This can be checked by explicit calculations for $\mathcal{F}_i^{1PN}$ and $\mathcal{F}_i^{\text{reac}}$, using straightforward techniques (explicit introduction of the Poisson kernels, $|\vec{x}-\vec{x}\,'|^{-1}$, and/or integration by parts). For $\mathcal{F}_i^{press}$ this follows immediately from eqs(4.13)-(4.14) (under the condition that the thermodynamic identity (4.15) is indeed satisfied if one evolves separately h_*). Another way to prove (and understand) the exact conservation of $P_i = \int d^3\vec{x}\; \pi_i$ is to remark that the Hamiltonian $H'[r_*, S, \pi_i, t]$ exhibited in Appendix C (eq.(C.15)) is invariant under spatial translations. On the other hand, it is neither invariant under spatial rotations nor under time translations, because of the coupling to $Q_{ij}^{[3]}(t)$. Instead, one finds that the total angular momentum,

$$J_{ij}(t) := \int d^3\vec{x}\,\left(x^i \pi_j - x^j \pi_i\right), \tag{5.24}$$

and the total 1PN energy,

$$E_1(t) := H_1\left[r_*(t),\; S(t),\; \pi_i(t)\right], \tag{5.25}$$

with H_1 defined in eq.(C.12a,d) of Appendix C, will slowly evolve according to

$$\frac{dJ_{ij}}{dt} = \frac{2}{5}\,\frac{G}{c^5}\,\left(Q_{is}^{[3]}\; I_{js} - Q_{js}^{[3]}\; I_{is}\right), \tag{5.26}$$

$$\frac{dE_1}{dt} = -\frac{1}{5}\,\frac{G}{c^5}\,Q_{ij}^{[3]}\;\frac{d}{dt}\,I_{ij}, \tag{5.27}$$

where $Q_{ij}^{[3]}$ is the quantity defined by eqs(5.13) and where

$$I_{ij} := STF\left\{2\int d^3\vec{x}\; r_*\left[w_i w_j + x^i\partial_j U_*\right]\right\}. \tag{5.28}$$

When the equations of motion are satisfied, the integral I_{ij} is equal to $d^2Q_{ij}/dt^2 + O(1/c^2)$ (see eq.(6.5) below), while $Q^{[3]}_{ij} = d^3Q_{ij}/dt^3 + O(1/c^2)$ (eq.(6.6)). One then recovers, modulo fractional corrections of order $O(1/c^2)$, the familiar "quadrupole formulas" for the losses of energy and angular momentum. (Beware of a misprint interverting the two and three dots on the quadrupole moments in the angular momentum loss formulae of Schäfer 1985). Note, however, that, within our precisely defined scheme, equations (5.26) and (5.27) must hold exactly (and not only within $O(c^{-2})$ fractional corrections).

The conservation of rest-mass and linear momentum, eqs(5.21) and (5.22), as well as the loss equations, (5.26) and (5.27), for J_{ij} and E_1, can constitute useful checks of the numerical accuracy with which the evolution system (5.20) is integrated. Other useful, and stronger, checks come from the exact local identity,

$$\partial_i A_i - 3\;\partial_i U_i \equiv 0, \tag{5.29}$$

or from the approximate one,

$$\partial_t U_* + \partial_i U_i = O\left(\frac{1}{c^2}\right). \tag{5.30}$$

5.9 Alternative possibilities

The (1PN) "gravitational mass density" σ, which is given in terms of the above defined quantities by

$$\sigma = r_*\left(1 + \frac{\delta}{c^2} + O\left(\frac{1}{c^4}\right)\right) = r_*\left[1 + \frac{1}{c^2}\left(\frac{3}{2}\vec{w}^{\,2} - U_* + h_* + 2\pi_*\right) + O\left(\frac{1}{c^4}\right)\right], \tag{5.31}$$

plays an important role, both in the 1PN gravitational force density, eq.(5.31c), and in the generation of gravitational waves (see next section). Therefore, it might be worth pointing out that if one uses as basic matter variables σ, S and π_i, one obtains an evolution system which is, by some aspects, simpler than the one written above. We shall not write it down explicitly here as it can be straightforwardly obtained from the formulas given above. In such a $\sigma - \pi_i$ scheme only one "Newtonian" potential is required,

$$\Delta U = -\ 4\pi\ G\sigma, \tag{5.32}$$

instead of U_* and U_2, and the 1PN gravitational force density reads simply

$$\mathcal{F}_i^{1\mathrm{PN}}[\sigma, \pi_i] = \sigma\partial_i U - \frac{1}{c^2}\pi_s\partial_i A_s. \tag{5.33}$$

However the simple Newtonian-like rest-mass conservation equation (2.33) is replaced by the more complicated propagation equation

$$\partial_t\sigma = -\partial_i(v^i\sigma) + \frac{\sigma}{c^2}\left[\partial_i U_i + 2v^i\partial_i U - 3v^i\ \frac{\partial_i p}{\sigma}\ + (2-3\gamma)\pi\partial_i v^i\right], \tag{5.34}$$

where the thermodynamic quantities can be evaluated at $r = \sigma$ (e.g. $\pi := p(\sigma, S)/\sigma$). Also in this scheme our requirement c) of §1 is not fulfilled as there are many $1/\sigma$ factors. This last remark suggests that it might also be interesting to introduce the variable $\tilde{w}_i := \pi_i/\sigma$, and to consider a $\sigma - S - \tilde{w}_i$ scheme.

§6 Post-Newtonian generation of gravitational waves.

6.1 The standard quadrupole formalism.

An often employed estimation of the generation of gravitational waves by a self-gravitating fluid consists of computing the Newtonian quadrupole of the matter distribution,

$$Q_{ij}(t) := \int d^3\vec{x}\, r_*(\vec{x},\, t) \left(x^i x^j - \frac{1}{3}\, \delta^{ij} \vec{x}^{\,2} \right), \tag{6.1}$$

for use into the "standard Einstein-Landau-Lifshitz quadrupole equation",

$$h_{ij}^{\text{quad}}(t, \vec{x}) = \frac{2G}{c^4 r}\, P_{ijk\ell}(\vec{n})\, \frac{d^2}{dt^2}\, Q_{k\ell}(t - r/c), \tag{6.2a}$$

giving the leading ($O(r^{-1})$) term in the wave zone expansion ($r \to \infty,\ t - r/c$ fixed) of the radiative gravitational field. In eq.(6.2a) $r = |\vec{x}| = \left(\delta_{ij} x^i x^j\right)^{1/2}$, $\vec{n} = \vec{x}/r$ and $P_{ijk\ell}(\vec{n})$ denotes the transverse- traceless projection operator onto the plane orthogonal to the outgoing wave direction, $\vec{n}$, acting on symmetric cartesian tensors, namely

$$P_{ijk\ell}(\vec{n}) = (\delta_{ik} - n_i n_k)(\delta_{j\ell} - n_j n_\ell) - \frac{1}{2}(\delta_{ij} - n_i n_j)(\delta_{k\ell} - n_k n_\ell). \tag{6.2b}$$

This standard quadrupole equation gives the <u>lowest-order</u> term in the slow-motion expansion of the radiative field, and is consistent with computing the dynamics of the fluid at the lowest-order slow-motion approximation, i.e. at the <u>Newtonian approximation</u>. At the same approximation, it is possible to evaluate directly the first three time-derivatives of $Q_{ij}(t)$ in terms of compact support integrals of Newtonian functionals of the source, by using the conservation of rest-mass and the Newtonian equations of motion. Indeed, by repeated use of the formula (which is an exact consequence of eq.(2.33)),

$$\frac{d}{dt} \int d^3\vec{x}\, r_*(\vec{x}, t) F(\vec{x}, t) = \int d^3\vec{x}\, r_*(\vec{x}, t) \left(\partial_t F + v^i \partial_i F\right), \tag{6.3}$$

together with some of the results obtained in the previous sections, and some integrations by parts, one finds successively

$$\frac{dQ_{ij}}{dt} = STF\left\{2\int d^3\vec{x}\; r_* x^i v^j\right\}, \tag{6.4}$$

$$\frac{d^2Q_{ij}}{dt^2} = STF\left\{2\int d^3\vec{x}\; r_*\left(v^i v^j + x^i \partial_j U\right)\right\} + O\left(\frac{1}{c^2}\right), \tag{6.5}$$

$$\frac{d^3Q_{ij}}{dt^3} = STF\left\{2\int d^3\vec{x}\; r_*\left[3v^i\partial_j U - 2v^i\frac{\partial_j p}{r_*} + x^i v^k \partial_{kj}U - \right.\right.$$
$$\left.\left. - x^i \partial_{kj} U_k\right]\right\} + O\left(\frac{1}{c^2}\right), \tag{6.6}$$

where the notation STF means : "take the symmetric trace-free part of", i.e. explicitly for any (in general non symmetric) two-index object A^{ij},

$$STF\{A^{ij}\} := \frac{1}{2}\,A^{ij} + \frac{1}{2}\,A^{ji} - \frac{1}{3}\,\delta^{ij}A^{ss}, \tag{6.7}$$

and where, in eq.(6.6), U_k denotes, as above the Newtonian potential of the mass current $r_* v^k$.

Note that eq.(6.5) allows one to evaluate the wave field (6.2) directly in terms of the state of the material source at only one time (the retarded time $u = t - r/c$), without having to evaluate numerically any time derivatives, and that, moreover, the integral appearing in its right-hand side has a compact support. It could be useful to the practitioners of the "all Newtonian" gravitational wave generation formalism recalled here. (The formula (6.5) has not been considered by Finn (1989) who compared the accuracy of several ways of numerically computing d^2Q_{ij}/dt^2).

Equation(6.6) (which has been already implemented in §5, eq.(5.13)) has played an important role in our general scheme in allowing us to eliminate all time derivatives in the right-hand side of the evolution equations with gravitational radiation damping. As the application of a further time-derivative to eq.(6.6) would generate terms not expressible by means of the potentials introduced above, we see why it has been important to use a convenient coordinate system where it was sufficient to evaluate explicitly (at Newtonian order) only the third time-derivative of the quadrupole moment.

6.2 Post-Newtonian wave generation formalisms

While the just recalled standard quadrupole formalism is consistent with Newtonian hydrodynamical codes, our post-Newtonian hydrodynamical formalism requires for consistency to be completed by a corresponding post-Newtonian gravitational wave generation formalism, i.e. a scheme taking into account the relativistic corrections to eq.(6.2) up to the relative order v^2/c^2 (1PN order). Epstein & Wagoner (1975) and Thorne (1980), have derived such 1PN wave generation formalisms. However their schemes make an essential use of the effective stress-energy distribution of the gravitational field, i.e. a pseudo-tensor $\tau^{\mu\nu}$ which is non-zero outside the material source, and which has a rather slow fall-off ($\sim r^{-4}$) at spatial infinity. As a consequence their final results contain divergent (i.e. non absolutely convergent) integrals which, besides casting doubt on the mathematical soundness of the type of formal expansions used, make them totally unsuited to numerical implementation (however, when handling them analytically with sufficient care, it has been possible to extract finite answers from these ill-defined integrals when applying them to particular systems, see e.g. Wagoner & Will, 1976 ; Turner & Will, 1978 ; Wagoner, 1979). However, a new post-Newtonian gravitational wave generation formalism involving only compact-support integrals has been recently developed (Blanchet & Damour, 1989). In this formalism the first post-Newtonian ($O(v^2/c^2)$) corrections to the standard quadrupole equation (6.1)-(6.2) are explicitly given as integrals over the stress-energy distribution, $T^{\mu\nu}$, of the material source alone (see Blanchet & Schäfer 1989, for a recent application of this formalism to binary systems). We shall now show how to numerically implement this 1PN gravitational wave generation formalism which will consistently complete the 1PN matter evolution formalism presented above.

First, the leading term in the wave-zone expansion of the gravitational wave amplitude, $h_{ij}^{\rm rad}$, in some suitable "radiative coordinate system", is decomposed in "radiative multipoles" (Thorne, 1980) :

$$\begin{aligned} h_{ij}^{\rm rad}(T,\vec{X}) = \frac{2G}{c^4 R}\, P_{ijk\ell}(\vec{N}) \Bigg\{ & \overset{(2)}{\mathrm{I}}{}^{\rm rad}_{k\ell} + \frac{1}{3c}\, N_a \overset{(3)}{\mathrm{I}}{}^{\rm rad}_{ak\ell} + \frac{4}{3c}\, \varepsilon_{ab(k} \overset{(2)}{\mathrm{J}}{}^{\rm rad}_{\ell)a} N_b + \\ & + \frac{1}{12c^2} N_a N_b \overset{(4)}{\mathrm{I}}{}^{\rm rad}_{abk\ell} + \frac{1}{2c^2}\, \varepsilon_{ab(k} \overset{(3)}{\mathrm{J}}{}^{\rm rad}_{\ell)ac} N_b N_c + O\left(\frac{1}{c^3}\right) \Bigg\} (T - R/c) + \\ & + O\left(\frac{1}{R^2}\right), \end{aligned} \tag{6.8}$$

where $\overset{(n)}{\mathrm{I}}(t) \equiv d^n I/dt^n$, $X_{(k\ell)} \equiv \frac{1}{2}(X_{k\ell} + X_{\ell k})$, and $P_{ijk\ell}$ is given by eq.(6.2b). For the definition of the wave-zone coordinate system $X^\mu = (cT, X^i)$ see Blanchet (1987), in the 1PN application below we can, for simplicity, mentally identify X^μ with x^μ, the coordinate system used above in the near-zone (indeed, the difference between X^μ and x^μ starts formally only at the 1.5PN level, i.e. $O(c^{-3})$). Similarly, although Blanchet & Damour (1989) were using harmonic coordinates for the near-zone description of the source, their results can be transcribed without changes in the ADM x^μ coordinates used above because the two coordinate systems differ only by a $O(c^{-4})$ shift (proportional to the time derivative of the superpotential (2.9a)) in the time coordinate.

A generation formalism consists in giving the explicit expression of each of the "radiative multipole moments", $I^{\rm rad}_{ij}$, $I^{\rm rad}_{ijk}$, $I^{\rm rad}_{ijk\ell}$, $J^{\rm rad}_{ij}$, $J^{\rm rad}_{ijk}$, appearing in eq.(6.8) as a functional of the matter variables. For a 1PN generation formalism, it is sufficient to know the radiative mass quadrupole moment, $I^{\rm rad}_{ij}$, with 1PN accuracy, and the other moments (mass octupole, $I^{\rm rad}_{ijk}$, mass 2^4-pole, $I^{\rm rad}_{ijk\ell}$, current quadrupole, $J^{\rm rad}_{ij}$, and current octupole, $J^{\rm rad}_{ijk}$) with "Newtonian" accuracy. The latter 0PN-accurate functionals, written in terms of our basic matter variables, r_* and w_i, are (Thorne 1980) :

$$I^{\rm rad}_{ijk}(t) = \mathop{STF}_{ijk}\left\{\int d^3\vec{x}\; r_*(\vec{x},t)x^ix^jx^k + O\left(\frac{1}{c^2}\right)\right\}, \tag{6.9a}$$

$$I^{\rm rad}_{ijk\ell}(t) = \mathop{STF}_{ijk\ell}\left\{\int d^3\vec{x}\; r_*(\vec{x},t)x^ix^jx^kx^\ell + O\left(\frac{1}{c^2}\right)\right\}, \tag{6.9b}$$

$$J^{\rm rad}_{ij}(t) = \mathop{STF}_{ij}\left\{\int d^3\vec{x}\; r_*(\vec{x},t)x^i\varepsilon^{jab}x^aw_b + O\left(\frac{1}{c^2}\right)\right\}, \tag{6.10a}$$

$$J^{\rm rad}_{ijk}(t) = \mathop{STF}_{ijk}\left\{\int d^3\vec{x}\; r_*(\vec{x},t)x^ix^j\varepsilon^{kab}x^aw_b + O\left(\frac{1}{c^2}\right)\right\}, \tag{6.10b}$$

in which the symbol STF means "taking the symmetric and trace-free part with respect to the indices below". Consistently with the 0PN accuracy, one could also use as mass density in eqs(6.9) either σ, eq. (5.31), or $c^{-2}T^{00}$, eq.(6.24), which both have more gravitational physics content than the plain coordinate rest-mass density, r_* (by contrast, eqs(6.10) are already expressed in terms of the momentum density $\pi_i \equiv r_*w_i$). The STF projection can be decomposed in symmetrizing and trace-subtracting. The symmetrization, say S, must be performed first. For the mass moments, eqs(6.9), this first operation is not necessary as $x^ix^jx^k$, etc..., are already symmetric. For the current moments, eqs(6.10), one must use, with $y^i = \varepsilon^{iab}x^aw_b$,

$$S(x^i y^j) = \frac{1}{2}(x^i y^j + x^j y^i), \tag{6.11a}$$

$$S(x^i x^j y^k) = \frac{1}{3}(x^i x^j y^k + x^j x^k y^i + x^k x^i y^j). \tag{6.11b}$$

Then, the trace-free part of a symmetric tensor, $S^{ij\cdots}$, is given by

$$TF(S^{ij}) = S^{ij} - \frac{1}{3}\,\delta^{ij} S^{ss}, \tag{6.12a}$$

$$TF(S^{ijk}) = S^{ijk} - \frac{1}{5}\,(\delta^{ij} S^{kss} + \delta^{jk} S^{iss} + \delta^{ki} S^{jss}), \tag{6.12b}$$

$$\begin{aligned} TF(S^{ijk\ell}) = S^{ijk\ell} - \frac{1}{7}\,(&\delta^{ij} S^{k\ell ss} + \delta^{ik} S^{\ell jss} + \delta^{i\ell} S^{jkss} + \delta^{k\ell} S^{ijss} + \delta^{\ell j} S^{ikss} + \\ &+ \delta^{jk} S^{i\ell ss}) + \frac{1}{35}\,(\delta^{ij}\delta^{k\ell} + \delta^{ik}\delta^{\ell j} + \delta^{i\ell}\delta^{jk}) S^{sstt}. \end{aligned} \tag{6.12c}$$

The two operations $TF \circ S$ can be either performed on the integrands, or on the resulting integrals, in eqs(6.9)-(6.10). An alternative way of proceeding would be to use the tensorial spherical harmonics representation of the multipole moments (see Thorne 1980). Note that, when using the cartesian representation of symmetric and trace-free (STF) tensors the number of independent components of an ℓ-index STF tensor, $I_{i_1 i_2 \ldots i_\ell}$, is (in 3 dimensions) only $2\ell + 1$. A convenient way of tabulating only the minimal independent algebraic content of $I_{i_1 i_2 \ldots i_\ell}$ (with $i_1, i_2 \ldots = 1, 2, 3$) can be to tabulate the independent components of the following two dimensionally reduced sub-tensors of $I : I_{a_1 a_2 \ldots a_\ell}$ and $I_{3 a_1 a_2 \ldots a_{\ell-1}}$, where $a_1, a_2, \ldots$ run only over 1,2. As these 2D tensors are symmetric (but not trace-free in 2D) they have, respectively, $\ell + 1$ and ℓ independent components which are very easily listed, e.g.

$$I_{abcd} : I_{1111}\,, I_{1112}\,, I_{1122}\,, I_{1222}\,, I_{2222}\,.$$

All the 3D components of $I_{i_1 \ldots i_\ell}$ containing more than one index 3 will be then obtained from the above irreducible set by using the zero 3D-trace condition, e.g.

$$I_{33i} = -I_{11i} - I_{22i}\,.$$

Let us now turn to the 1PN-accurate mass quadrupole equation for $I^{\rm rad}_{ij}$. It has been obtained (Blanchet & Damour 1989) as the following rather simple functional of the 1PN-accurate gravitational mass density, $\sigma \equiv c^{-2}(T^{00}+T^{ss})(1+O(c^{-4}))$, and of the (0PN) momentum density $\pi_i = c^{-1}T^{0i}(1+O(c^{-2}))$:

$$I^{\rm rad}_{ij}(t) = \int d^3\vec{x}\;\sigma(\vec{x},t)\hat{x}^{ij} + \frac{1}{14c^2}\,\frac{d^2}{dt^2}\int d^3\vec{x}\;\sigma(\vec{x},t)\vec{x}^{\,2}\hat{x}^{ij} - $$
$$-\frac{20}{21c^2}\,\frac{d}{dt}\int d^3\vec{x}\;\pi_k(\vec{x},t)\hat{x}^{ijk} + O\left(\frac{1}{c^3}\right), \tag{6.13}$$

where

$$\hat{x}^{ij} \equiv STF(x^ix^j) = x^ix^j - \frac{\vec{x}^{\,2}}{3}\,\delta^{ij}, \tag{6.14a}$$

$$\hat{x}^{ijk} \equiv STF(x^ix^jx^k) = x^ix^jx^k - \frac{\vec{x}^{\,2}}{5}\,(\delta^{ij}x^k + \delta^{jk}x^i + \delta^{ki}x^j). \tag{6.14b}$$

As the numerical computation of time derivatives can be less accurate (and more time consuming) than the computation of an "instantaneous" functional of the matter variables at the time t, we shall now show how to transform away the explicit time derivatives in the result (6.13). This can be achieved by, first, letting the time derivatives act onto the integrands, according to $(\partial_0 = c^{-1}\partial_t)$

$$\frac{1}{c^2}\,\frac{\partial^2}{\partial t^2}\int d^3\vec{x}\;\frac{T^{00}(\vec{x},t)}{c^2}\;\vec{x}^{\,2}\hat{x}^{ij} = \frac{1}{c^2}\int d^3\vec{x}\;\partial_{00}T^{00}\vec{x}^{\,2}\hat{x}^{ij} \tag{6.15a}$$

(where σ is sufficiently accurately given by $c^{-2}T^{00}$ in the 1PN corrections), and

$$-\frac{1}{c^2}\,\frac{\partial}{\partial t}\int d^3\vec{x}\;\frac{T^{0k}(\vec{x},t)}{c}\;\hat{x}^{ijk} = -\frac{1}{c^2}\int d^3\vec{x}\;\partial_0T^{0k}\hat{x}^{ijk}, \tag{6.15b}$$

and, then, by using the Newtonian (0PN) balance equations of energy and momentum

$$\partial_0T^{00} + \partial_kT^{0k} = O\left(\frac{1}{c}\right), \tag{6.16a}$$

$$\partial_0T^{k0} + \partial_\ell T^{k\ell} = \mathcal{F}^k_{\rm grav} + O\left(\frac{1}{c^2}\right), \tag{6.16b}$$

which imply,

$$\partial_{00}T^{00} = \partial_{k\ell}T^{k\ell} - \partial_k\mathcal{F}^k_{\rm grav}\,. \tag{6.16c}$$

Note that, in eq.(6.16b), $\mathcal{F}^k_{\rm grav}$ contains only the (Newtonian) gravitational force density (eq.(2.31c)), the pressure force density being incorporated in $\partial_\ell T^{k\ell}$ (see eq.(2.29)). The relations (6.16) lead to decompose $I^{\rm rad}_{ij}$ into three parts,

$$I^{\rm rad}_{ij}(t) = E_{ij}(t) + S_{ij}(t) + G_{ij}(t) + O\left(\frac{1}{c^3}\right), \tag{6.17}$$

where E_{ij} is the contribution of the energy density,

$$E_{ij}(t) := \int d^3\vec{x}\, \frac{T^{00}(\vec{x},t)}{c^2} \hat{x}^{ij}, \tag{6.18}$$

S_{ij} the contribution of the (kinetic + internal) stresses (including the 1PN term $c^{-2}T^{ss}$ contained in σ),

$$S_{ij}(t) := \frac{1}{c^2} \int d^3\vec{x}\, \left\{ T^{ss}\hat{x}^{ij} + \frac{1}{14}\, \partial_{k\ell}T^{k\ell}\vec{x}^{\,2}\hat{x}^{ij} + \frac{20}{21}\, \partial_\ell T^{k\ell}\hat{x}^{ijk} \right\} \tag{6.19a}$$

$$= \frac{1}{c^2} \int d^3\vec{x}\, \left\{ T^{ss}\hat{x}^{ij} + \frac{1}{14}\, T^{k\ell}\partial_{k\ell}(\vec{x}^{\,2}\hat{x}^{ij}) - \frac{20}{21}\, T^{k\ell}\partial_\ell\hat{x}^{ijk} \right\} \tag{6.19b}$$

(the second form being obtained by integrating by parts some of the compact-support integrals of eq.(6.19a)), and where $G_{ij}(t)$ denotes the contribution of the gravitational force density :

$$G_{ij}(t) := \frac{1}{c^2} \int d^3\vec{x}\, \left\{ -\frac{1}{14}\, \partial_k\mathcal{F}^k_{\rm grav}\,\vec{x}^{\,2}\hat{x}^{ij} - \frac{20}{21}\, \mathcal{F}^k_{\rm grav}\, \hat{x}^{ijk} \right\} \tag{6.20a}$$

$$= \frac{1}{c^2} \int d^3\vec{x}\, \left\{ + \frac{1}{14}\mathcal{F}^k_{\rm grav}\,\partial_k(\vec{x}^{\,2}\hat{x}^{ij}) - \frac{20}{21}\, \mathcal{F}^k_{\rm grav}\,\hat{x}^{ijk} \right\}. \tag{6.20b}$$

Straightforward calculations (using eqs(6.14), or the general formulas of appendix A of Blanchet & Damour 1986) lead then to

$$S_{ij} = \mathop{STF}_{ij} \left\{ \frac{1}{c^2} \int d^3\vec{x}\, \left[\frac{11}{21}\, \vec{x}^{\,2}\hat{T}^{ij} - \frac{4}{7}\, x^i x^k \hat{T}^{kj} \right] \right\}, \tag{6.21}$$

where

$$\hat{T}^{ij} := STF(T^{ij}) = T^{ij} - \frac{1}{3}\, T^{ss}\delta^{ij}, \tag{6.22}$$

and

$$G_{ij} = \underset{ij}{STF} \left\{ \frac{1}{c^2} \int d^3\vec{x} \left[\frac{11}{21}\, \vec{x}^{\,2} x^i \mathcal{F}^j_{\rm grav} - \frac{17}{21}\, x^i x^j x^k \mathcal{F}^k_{\rm grav} \right] \right\}. \tag{6.23}$$

It is interesting to note that the trace of T^{ij} (which means in particular the pressure) does not contribute to S_{ij}.

6.3 Application to the (1PN) hydrodynamics case.

All the results (6.13)-(6.23) are valid for an arbitrary structure of the material source. If we now specialize to the perfect fluid case, the stress- energy tensor (2.20) yields (using the notation of §5) :

$$\frac{1}{c^2}\, T^{00} = r_* \left[1 + \frac{1}{c^2} \left(\frac{1}{2} \vec{w}^{\,2} - U_* + h_* - \pi_* \right) \right] + O\left(\frac{1}{c^4}\right), \tag{6.24}$$

$$T^{ij} = r_* w_i w_j + p_* \delta_{ij} + O\left(\frac{1}{c^2}\right), \tag{6.25}$$

so that $\hat{T}^{ij}$ comes only from the tracefree part of the kinetic-energy tensor :

$$\hat{T}^{ij} = r_* \hat{w}_{ij} + O\left(\frac{1}{c^2}\right), \tag{6.26a}$$

where

$$\hat{w}_{ij} := STF(w_i w_j) = w_i w_j - \frac{1}{3}\, \vec{w}^{\,2} \delta_{ij}, \tag{6.26b}$$

and

$$\mathcal{F}^i_{\rm grav} = r_* \partial_i U_* + O\left(\frac{1}{c^2}\right). \tag{6.27}$$

In summary, one can complete the 1PN hydrodynamical formalism of §5 by a corresponding 1PN gravitational wave extraction formalism defined by eq.(6.8), with a 1PN radiative quadrupole moment given by,

$$I^{\rm rad}_{ij} = E_{ij} + S_{ij} + G_{ij} + O\left(\frac{1}{c^3}\right), \tag{6.28}$$

$$E_{ij} = \underset{ij}{STF} \left\{ \int d^3\vec{x}\; r_* x^i x^j \left[1 + \frac{1}{c^2} \left(\frac{1}{2} \vec{w}^{\,2} - U_* + h_* - \pi_* \right) \right] \right\}, \tag{6.28a}$$

$$S_{ij} = \underset{ij}{STF} \left\{ \int d^3\vec{x}\; \frac{r_*}{c^2} \left[\frac{11}{21}\, \vec{x}^{\,2} \hat{w}_{ij} - \frac{4}{7} x^i x^k \hat{w}_{kj} \right] \right\}, \tag{6.28b}$$

$$G_{ij} = \underset{ij}{STF} \left\{ \int d^3\vec{x}\; \frac{r_*}{c^2} \left[\frac{11}{21}\, \vec{x}^{\,2} x^i \partial_j U_* - \frac{17}{21}\, x^i x^j x^k \partial_k U_* \right] \right\}, \tag{6.28c}$$

(with the notation (6.26b)) and with 0PN mass (2^3- and 2^4-pole) and current (2^2- and 2^3-pole) radiative moments given by eqs(6.9) and (6.10). The outgoing gravitational wave amplitude is obtained from the radiative moments by several time differentiations which must be done numerically. Note however that, thanks to eq.(6.3), the first time differentiation acting on the dominant "Newtonian" quadrupole can be replaced by

$$\frac{d}{dt} \left\{ STF \int d^3\vec{x}\; r_* x^i x^j \right\} \equiv STF \left\{ 2 \int d^3\vec{x}\; r_* x^i v^j \right\},$$

with v^i given by the formulas of §5 in terms of r_* and w_i. The same eq.(6.3) can be used to effect several other time differentiations, along the lines of eqs (6.4)-(6.6) (as for the latter equations, one can make use of the Newtonian equations of motion in all the 1PN correction terms).

ACKNOWLEDGMENTS

It is a pleasure to thank Silvano Bonazzola and Jean-Alain Marck for many stimulating and informative discussions.

Appendix A Thermodynamics of relativistic fluids

For convenience, let us gather here the main formulas of the relativistic thermodynamics of ideal fluids. Following Carter (1989), we shall consider here the general multi-species case, as defined by an equation of state of the form

$$e = e(r, s, y_a), \tag{A.1}$$

where r is a proper total rest-mass density (i.e. the baryon number density multiplied by a reference baryon mass,m), s the proper entropy density, and y_a ($a = 2, 3, ...$) a set of (number or rest-mass) densities of, say, some nuclear species. A formal simplification in the formulas would be reached by defining $y_0 = s$ and $y_1 = r$ (or $y_1 = r/m$), and letting $a = 0, 1, 2, 3, ...$, but we wish here to be as explicit as possible.

Writing that the variation of the energy contained in a proper volume $V, d(eV)$, is linked to the corresponding variations of the rest-mass, entropy, species number and volume by the first law of thermodynamics,

$$d(eV) = \mu d(rV) + Td(sV) + \mu^a d(y_a V) - pdV, \tag{A.2}$$

where T is the temperature, p the pressure and μ and μ^a some chemical potentials. Eq.(A.2) implies

$$\mu = \frac{\partial e(r, s, y_a)}{\partial r}, \; T = \frac{\partial e(r, s, y_a)}{\partial s}, \; \mu^a = \frac{\partial e(r, s, y_a)}{\partial y_a}, \tag{A.3}$$

and

$$e + p = \mu r + Ts + \mu^a y_a. \tag{A.4}$$

Hence,

$$de = \mu dr + Tds + \mu^a dy_a, \tag{A.5}$$

$$dp = r d\mu + sdT + y_a d\mu^a. \tag{A.6}$$

If one now considers e as a function of r, of the specific entropy, $S := s/r$, and of the specific numbers of species (or fractional rest-mass) $Y_a := y_a/r$,

$$e = e(r, S, Y_a), \tag{A.7}$$

the above thermodynamical relations take the form,

$$de = H dr + rTdS + r\mu^a dY_a, \tag{A.8}$$

$$dp = rdH - rTdS - r\mu^a dY_a, \tag{A.9}$$

where we have introduced the relativistic enthalpy

$$H := \frac{e+p}{r} = \mu + TS + \mu^a Y_a = c^2 + h \tag{A.10}$$

($h = H - c^2$ being the enthalpy minus the rest-mass contribution used in the text).

The simplest case is that of a polytropic equation of state defined by

$$e(r, S) = rc^2 + \frac{k(S)}{\gamma - 1}\, r^\gamma, \tag{A.11a}$$

$$p = k(S) r^\gamma = (\gamma - 1)(e - rc^2), \tag{A.11b}$$

$$h = H - c^2 = k(S)\, \frac{\gamma}{\gamma - 1}\, r^{\gamma - 1}, \tag{A.11c}$$

with some constant "polytropic index" γ (which could also be taken as a function of S).

Let us now consider the perfect-fluid contribution to the total stress-energy tensor of a general (possibly non-perfect, or acted upon by external forces) relativistic fluid,

$$T^{\mu\nu} = (e + p) u^\mu u^\nu + p g^{\mu\nu}, \tag{A.12}$$

where $g_{\mu\nu} u^\mu u^\nu = -1$, and where p is assumed to satisfy eq.(A.4). The divergence of (A.12) can be decomposed according to,

$$\nabla_\nu T^{\mu\nu} = \mathcal{E} u^\mu + \mathcal{F}^\mu, \quad \text{with} \quad u_\mu \mathcal{F}^\mu = 0. \tag{A.13}$$

The local energy creation rate, $\mathcal{E} = -u_\mu \nabla_\nu T^{\mu\nu}$, is then given by

$$\begin{aligned}\mathcal{E} &= \mu\nabla_\nu(ru^\nu) + T\nabla_\nu(su^\nu) + \mu^a\nabla_\nu\left(y_a u^\nu\right)\\ &= H\nabla_\nu(ru^\nu) + rTu^\nu\nabla_\nu S + r\mu^a u^\nu\nabla_\nu Y_a,\end{aligned} \tag{A.14}$$

while the relativistic force, $\mathcal{F}^\mu = (\delta^\mu_\lambda + u^\mu u_\lambda)\nabla_\nu T^{\lambda\nu}$, is given by

$$\mathcal{F}^\mu = (e+p)u^\nu\nabla_\nu u^\mu + (g^{\mu\nu} + u^\mu u^\nu)\nabla_\nu p. \tag{A.15}$$

Using eq.(A.9), i.e.

$$\frac{\nabla_\nu p}{r} = \nabla_\nu H - T\nabla_\nu S - \mu^a\nabla_\nu Y_a, \tag{A.16}$$

one can rewrite eq.(A.15) in the form

$$u^\nu\left[\nabla_\nu(Hu_\mu) - \nabla_\mu(Hu_\nu)\right] = (\delta^\nu_\mu + u_\mu u^\nu)\left[T\nabla_\nu S + \mu^a\nabla_\nu Y_a\right] + \frac{\mathcal{F}_\mu}{r}. \tag{A.17}$$

The set of equations presented here show how to modify the formalism presented in the text when considering a perfect fluid ($\mathcal{E} = \mathcal{F}_\mu = 0$) having a multi-variable equation of state, $e = e(r, S, Y_a)$.

Appendix B 3 + 1 split

The 4-dimensional metric, $g_{\mu\nu}$, is conveniently split in 3-dimensional objects, $(\alpha, \beta_i, \gamma_{ij})$ relative to the slicing of the space-time in the t = const. space-like hypersurfaces,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 (dx^0)^2 + \gamma_{ij}(dx^i + \beta^i dx^0)(dx^j + \beta^j dx^0). \qquad (B.1)$$

Explicitly :

$$g_{00} = -\left[\alpha^2 - \beta_i \beta^i\right], \quad g^{00} = -\frac{1}{\alpha^2}, \qquad (B.2a)$$

$$g_{0i} = +\beta_i, \qquad g^{0i} = +\frac{\beta^i}{\alpha^2}, \qquad (B.2b)$$

$$g_{ij} = \gamma_{ij}, \qquad g^{ij} = \gamma^{ij} - \frac{\beta^i \beta^j}{\alpha^2}, \qquad (B.2c)$$

where

$$\beta^i = \gamma^{ij} \beta_j , \quad \gamma^{is} \gamma_{js} = \delta^i_j. \qquad (B.3)$$

The determinant $g = -det(g_{\mu\nu})$ is then given in terms of $\gamma = +det(\gamma_{ij})$ by

$$\sqrt{g} = \alpha \sqrt{\gamma}. \qquad (B.4)$$

In terms of this 3+1 split the normalization condition $g^{\mu\nu} u_\mu u_\nu = -1$ reads

$$-\frac{1}{\alpha^2}\left[u_0 - \beta^i u_i\right]^2 + \gamma^{ij} u_i u_j = -1, \qquad (B.5)$$

in which one can insert

$$u^0 = g^{0\mu} u_\mu = -\frac{u_0 - \beta^i u_i}{\alpha^2}, \qquad (B.6)$$

to get

$$u^0 = \frac{(1 + \gamma^{ij} u_i u_j)^{1/2}}{\alpha}. \qquad (B.7)$$

Using now (eqs (2.32) and (4.2))

$$r_* = r\sqrt{g}u^0, \tag{B.8}$$

$$w_i = (1 + c^{-2}h(r,S))cu_i, \tag{B.9}$$

together with (B.4) and (B.7), we see that r is defined implicitly as a function of r_*, S and w_i (assuming the metric known) by

$$r = r_*\gamma^{-1/2}\left\{1 + \frac{\gamma^{ij}w_iw_j}{c^2\left[1 + c^{-2}h(r,S)\right]^2}\right\}^{-1/2}. \tag{B.10}$$

Then the 3-velocity, $v^i = cu^i/u^0$, is obtained from $h = h[r(r_*, S, w_i), S]$, and from the relation

$$u_i = g_{i\mu}u^\mu = u^0\left[\gamma_{ij}\frac{v^j}{c} + \beta_i\right] = c^{-1}w_i/(1 + c^{-2}h), \tag{B.11}$$

as the following function of r_*, S and w_i,

$$v^i = \frac{\alpha\gamma^{ij}w_j}{\left[(1 + c^{-2}h)^2 + c^{-2}\gamma^{k\ell}w_kw_\ell\right]^{1/2}} - c\beta^i. \tag{B.12}$$

The equations presented here, together with the values of the metric coefficients discussed in the text, allow one to compute the 1PN + 2.5PN functionals of r_*, S and w_i needed in §5.

Appendix C Reduced Hamiltonian formalism for relativistic fluids.

Several recent works (Künzle & Nester, 1984 ; Holm, 1985 ; Bao, Marsden & Walton, 1985) have investigated the constrained Hamiltonian formalism for relativistic fluids. We are here interested in the corresponding "reduced" Hamiltonian formalism, obtained à la Arnowitt-Deser-Misner (1962) by solving the constraints in terms of the true dynamical degrees of freedom of the full system : fluid + gravitational field. The latter quantities consist of the matter variables, r_*, S, π_i, and of the gravitational field ones, h_{ij}^{TT}, π^{ijTT}. The matter variables denote the same quantities as introduced in the text (eqs(2.32) and (2.17)), and in the Hamiltonian approach they have the following Poisson brackets

$$\{r_*(\vec{x},t), r_*(\vec{x}\,',t)\} = 0, \tag{C.1a}$$

$$\{r_*(\vec{x},t), S(\vec{x}\,',t)\} = 0, \tag{C.1b}$$

$$\{S(\vec{x},t), S(\vec{x}\,',t)\} = 0, \tag{C.1c}$$

$$\{\pi_i(\vec{x},t), S(\vec{x}\,',t)\} = \frac{\partial S(\vec{x}\,',t)}{\partial x'^i}\ \delta(\vec{x}-\vec{x}\,'), \tag{C.1d}$$

$$\{\pi_i(\vec{x},t), r_*(\vec{x}\,',t)\} = \frac{\partial}{\partial x'^i}\left[r_*(\vec{x}\,',t)\delta(\vec{x}-\vec{x}\,')\right], \tag{C.1e}$$

$$\{\pi_i(\vec{x},t), \pi_j(\vec{x}\,',t)\} = \pi_i(\vec{x}\,',t)\ \frac{\partial}{\partial x'^j}\ \delta(\vec{x}-\vec{x}\,') - \pi_j(\vec{x},t)\ \frac{\partial}{\partial x^i}\ \delta(\vec{x}-\vec{x}\,'). \tag{C.1f}$$

Note that the brackets C.1 have a universal structure independent of the metric, and even of any special relativistic effects. A simple way to see why, and to derive them, is to deduce them from the canonical brackets of some underlying Lagrange-variables description of the fluid :

$$r_*(\vec{x},t) = r_L(q^A(\vec{x},t))\ det\left(\frac{\partial q^A(\vec{x},t)}{\partial x^i}\right), \tag{C.2a}$$

$$S(\vec{x},t) = S_L(q^A(\vec{x},t)), \tag{C.2b}$$

$$\pi_i(\vec{x},t) = -\ \frac{\partial q^A(\vec{x},t)}{\partial x^i} P_A(\vec{x},t), \tag{C.2c}$$

where q^A, $A=1,2,3$ are Lagrange-fields ($\partial_t q^A + v^i\partial_i q^A = 0$) and P_A their associated canonical momenta :

$$\{q^A(\vec{x},t),\ P_B(\vec{x}\,',t)\} = \delta^A_B \delta(\vec{x}-\vec{x}\,'). \tag{C.2d}$$

The Poisson brackets (C.1) are closely linked with the structure of the three-dimensional diffeomorphism group. This shows up, e.g., in the following general formula (considered at a fixed time)

$$\left\{ \int d^3\vec{x}\ \xi^i \pi_i, F[r_*, S, \pi_j] \right\} = \int d^3\vec{x} \left[\frac{\delta F}{\delta r_*}\ \mathcal{L}_\xi r_* + \frac{\delta F}{\delta S}\ \mathcal{L}_\xi S + \frac{\delta F}{\delta \pi_i}\ \mathcal{L}_\xi \pi_i \right], \tag{C.3}$$

where F is a functional of $r_*(\vec{x}), S(\vec{x}), \pi_j(\vec{x})$ (with Fréchet derivatives $\delta F/\delta r_*$, etc ...) and where $\mathcal{L}_\xi$ denotes the Lie derivative along the 3D vector $\xi^i \partial/\partial x^i$ of, respectively, a (spatial) density, a scalar, and a co-vectorial density. For fuller discussions of the geometric significance of the "Lie-Poisson" (-"Kirillov-Kostant") brackets (C.1) see e.g. Arnold (1966), Zakharov & Kuznetsov (1984), Holm (1985), Bao, Marsden & Walton (1985), and references therein.

The gravitational variables, h^{TT}_{ij} and π^{ijTT}, are the transverse and traceless (TT) parts of, respectively, $\gamma_{ij} - \delta_{ij} (= g_{ij} - \delta_{ij})$ and

$$\pi^{ij} = -\sqrt{\gamma}\gamma^{ia}\gamma^{jb}(K_{ab} - \gamma_{ab}\gamma^{cd}K_{cd}), \tag{C.4a}$$

$$K_{ij} = -\alpha\Gamma^0_{ij} = -\ \frac{1}{2\alpha}\ \left[c^{-1}\partial_t\gamma_{ij} - D_i\beta_j - D_j\beta_i\right], \tag{C.4b}$$

(with $D_i\gamma_{jk} = 0$) in the ADM (1960) coordinate system defined by

$$\partial_j \left(\gamma_{ij} -\ \frac{1}{3}\ \delta_{ij}\delta^{k\ell}\gamma_{k\ell}\right) = 0, \tag{C.5a}$$

$$\delta_{ij}\pi^{ij} = 0. \tag{C.5b}$$

The gravitational variables are canonically conjugate (modulo the (flat-space) TT projection), i.e. their only non-zero brackets are

$$\{h^{TT}_{ij}(\vec{x},t), \pi^{k\ell TT}(\vec{x}\,',t)\} = \delta^{TTk\ell}_{ij}(\vec{x}-\vec{x}\,'). \tag{C.6}$$

The general canonical evolution equation for possibly explicitly time-dependent phase-space functionals,

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \{F, H\}, \tag{C.7}$$

together with the Poisson brackets, (C.1), implies the following equations of motion for the fluid :

$$\frac{\partial r_*}{\partial t} = -\partial_i \left(\frac{\delta H}{\delta \pi_i} r_* \right), \tag{C.8a}$$
$$\frac{\partial S}{\partial t} = - \frac{\delta H}{\delta \pi_i} \partial_i S, \tag{C.8b}$$
$$\frac{\partial \pi_i}{\partial t} = -\partial_s \left(\frac{\delta H}{\delta \pi_s} \pi_i \right) - \partial_i \left(\frac{\delta H}{\delta \pi_s} \right) \pi_s - \partial_i \left(\frac{\delta H}{\delta r_*} \right) r_* + \frac{\delta H}{\delta S} \partial_i S. \tag{C.8c}$$

To obtain the reduced Hamiltonian appearing in eqs(C.8) one must solve the energy and momentum constraints that appear in the 3 + 1 split of the 4D action (using the Lagrange-variables description (C.2) (Künzle & Nester, 1984), and choosing units such that $16\pi G = 1 = c$)

$$\begin{aligned} A &= \int d^4x \sqrt{g}\, [R(g) - e(r, S)] \\ &= \int dt \int d^3\vec{x} \left[\pi^{ij} \partial_t \gamma_{ij} + P_A \partial_t q^A - \alpha \mathcal{H} - \beta^i \mathcal{H}_i \right] + \\ &\quad + \text{ surface terms.} \end{aligned} \tag{C.9}$$

Note that $P_A \partial_t q^A$ can be rewritten, using eq.(C.2c) and the Lagrange-transport equation, $\partial_t q^A + v^i \partial_i q^A = 0$, as $\pi_i v^i$. This shows that the canonical conjugate of π_i is the coordinate 3-velocity (a result, $v^i = \delta H / \delta \pi_i$, already apparent in eqs(C.8)).

The energy and momentum constraints read

$$\begin{aligned} 0 = \mathcal{H} &= -\gamma^{1/2} R(\gamma) + \gamma^{-1/2} \left(\pi^{ij} \pi_{ij} - \frac{1}{2} (\pi^s_s)^2 \right) + \\ &\qquad + \mathcal{H}_{\text{matter}} (r_*, S, \pi_i, \gamma^{jk}), \end{aligned} \tag{C.10a}$$
$$0 = \mathcal{H}_i = -2 D_j \pi^j_i - \pi_i, \tag{C.10b}$$

where $\mathcal{H}_{\text{matter}} = \alpha^2 \gamma^{1/2} T^{00}$ can be expressed in terms of r_*, S, π_i and γ^{jk} by means of eqs(B.7)-(B.10). The Hamiltonian H is obtained by solving the constraints (C.10) together with the coordinate conditions (C.5), and by inserting the result in the surface term giving the ADM energy (Regge & Teitelboim, 1974) :

$$E = \oint d^2 S_i \left(\partial_s g_{is} - \partial_i g_{ss}\right). \tag{C.11}$$

Keeping the 1PN terms in the matter Hamiltonian, and the lowest-order terms involving the gravitational wave degrees of freedom, the result is found to be (Schäfer, 1989) (with c = 1) :

$$H = H_1\left[r_*, S, \pi_i\right] + H_2\left[r_*, \pi_i, h_{ij}^{TT}\right] + H_3\left[h_{ij}^{TT}, \pi^{ijTT}\right], \tag{C.12}$$

$$\begin{aligned} H_1 = & \int d^3\vec{x}\; e[r_*, S] - \int d^3\vec{x}\; \frac{1}{2}\, r_* U_* \left(1 + 2h_* + 4\,\frac{p_*}{r_*} - U_*\right) + \\ & + \int d^3\vec{x}\; \frac{1}{2}\,\frac{\vec{\pi}^{\,2}}{r_*}\left(1 - h_* - 3U_* - \frac{1}{4}\,\frac{\vec{\pi}^{\,2}}{r_*^2}\right) + \\ & + \frac{G}{4}\iint d^3\vec{x} d^3\vec{x}\,'\left[7\,\frac{\pi_i(\vec{x},t)\pi_i(\vec{x}\,',t)}{|\vec{x}-\vec{x}\,'|} + \right. \\ & \left. + \frac{(x^i - x'^i)(x^j - x'^j)\pi_i(\vec{x},t)\pi_j(\vec{x}\,',t)}{|\vec{x}-\vec{x}\,'|^3}\right], \end{aligned} \tag{C.12a}$$

$$H_2 = -\frac{1}{2}\int d^3\vec{x}\; h_{ij}^{TT}\left[\frac{\pi_i\pi_j}{r_*} + \frac{1}{4\pi G}\,\partial_i U_* \partial_j U_*\right], \tag{C.12b}$$

$$H_3 = \frac{1}{16\pi G}\int d^3\vec{x}\left[\frac{1}{4}\left(\partial_k h_{ij}^{TT}\right)^2 + (\pi^{ijTT})^2\right]. \tag{C.12c}$$

The notation used in eqs(C.12) is the same as in §5, notably $h_* = \partial e(r_*, S)/\partial r_* - c^2$, $\Delta U_* = -4\pi G r_*$. It is straightforward to check that the double integral $(G/4 \iint d^3\vec{x} d^3\vec{x}\,'[...])$ appearing in H_1, eq.(C.12a), can be written as

$$\frac{1}{2}\int d^3\vec{x}\; \pi_i(\vec{x},t) A_i(\vec{x},t), \tag{C.12d}$$

with A_i defined by eqs(5.6),(5.7) (where $r_* w_i \equiv \pi_i$) and (5.12). For the terms of order higher than the ones written in eqs(C.12) (2PN terms and higher-order coupling to h_{ij}^{TT}) see Schäfer (1989).

The equations of motion for h_{ij}^{TT} and π^{ijTT} deduced from eqs(C.12) (with (C.6) and (C.7)) lead to the following inhomogeneous wave equation for h_{ij}^{TT} :

$$\Box\, h_{ij}^{TT} = -\frac{16\pi G}{c^4} \left\{ \frac{\pi_i \pi_j}{r_*} + \frac{1}{4\pi G}\, \partial_i U_* \partial_j U_* \right\}^{TT}, \qquad (C.13)$$

where TT means taking the transverse-trace-free projection. The solution of this equation, in the case where one assumes the absence of any free incoming radiation impinging on the system, is simply the retarded integral of its right-hand side. When considered in the near-zone of the system, the latter retarded integral for h_{ij}^{TT} can be expanded in powers of the light-crossing delay $\sim |\vec{x} - \vec{x}\,'|/c$. This leads to a post-Newtonian expansion whose first term is $O(c^{-4})$ and time-symmetric, and whose second term ($O(c^{-5})$ and time-asymmetric) represents the lowest order effect of the radiation reaction. The latter "reaction" term is only a function of time, and can be written as (Schäfer 1985)

$$\left(h_{ij}^{TT}\right)^{\rm reac} = -\frac{4G}{5c^5}\, Q_{ij}^{[3]}(t), \qquad (C.14)$$

where use has been made of the Newtonian equations of motion to express $h^{\rm reac}$, originally given in terms of the time derivative of the integral I_{ij} of eq.(5.28), in terms of the quantity introduced in eqs(5.13).

As the gravitational degrees of freedom were held fixed in the fluid equations of motion (C.8), we see that the fluid motion, at the 1PN + radiation reaction level, is deducible (via the same equations (C.8)) from the time-dependent Hamiltonian obtained by inserting (C.14) into (C.12b) and by discarding (C.12c), namely

$$\begin{aligned} H'[r_*, S, \pi_i, t] = {} & H_1[r_*, S, \pi_i] + \\ & + \frac{2G}{5c^5} Q_{ij}^{[3]}(t) \int d^3\vec{x} \left[\frac{\pi_i \pi_j}{r_*} + \frac{1}{4\pi G}\, \partial_i U_* \partial_j U_* \right]. \end{aligned} \qquad (C.15)$$

A long but straightforward calculation allows one to check that inserting H', eq.(C.15), into eqs(C.8) leads exactly (and not only modulo $O(c^{-4})$ terms) to the 1PN + 2.5PN results obtained in the text, §5 (see Schäfer (1989) for extension to higher-order approximations). The conservation laws of the fluid are most conveniently discussed from this Hamiltonian point of view, starting from the symmetries (or lack thereof) of H' under space translations, time translations and space rotations, and using, e.g., eqs(C.3) and (C.7) (see subsection 5.6 for the results ; note that twice the trace-free part of the integral appearing in the second term of the right-hand-side of eq.(C.15) is exactly equal to I_{ij}, as defined by eq.(5.28) of the text).

REFERENCES

Arnold, V. 1966. *Ann. Inst. Fourier, Grenoble,* **16**, 319.

Arnowitt, R., Deser, S. & Misner, C.W. 1960. *Phys. Rev.,* **120**, 313.

Arnowitt, R., Deser, S. & Misner, C.W. 1962. In : *Gravitation : An Introduction to Current Research,* ed. L. Witten, Wiley, New York, Chap.7.

Bao, D., Marsden, J. & Walton, R. 1985. *Comm. math. Phys.,* **99**, 319.

Blanchet, L. 1987. *Proc. R. Soc. London. A* **409**, 383.

Blanchet, L. & Damour, T. 1986. *Phil. Trans. R. Soc. London. A* **320**, 379.

Blanchet, L. & Damour, T. 1989. *Ann. Inst. H. Poincaré (Phys. Théor.),* in press.

Blanchet, L. & Schäfer, G. 1989. *Mon. Not. R. Astro. Soc.,* in press.

Bonazzola, S. & Marck, J.A. 1986. *Astron. Astrophys.,* **164**, 300.

Bonazzola, S. & Marck, J.A. 1989a. *Journ. Computational Phys.,* in press.

Bonazzola, S. & Marck, J.A. 1989b. In preparation.

Burke, W.L. 1971. *J. Math. Phys.,* **12**, 401.

Caporali, A. 1981. *Nuov. Cim.,* **61B**, 181.

Carter, B. 1989. In *Relativistic Hydrodynamics,* ed. M. Anile, Springer, in press.

Carter, B., & Gaffet, B. 1988. *J. Fluid Mech.,* **186**, 1.

Chandrasekhar, S. 1965. *Astrophys. J.,* **142**, 1488.

Chandrasekhar, S. & Nutku, Y. 1969. *Astrophys. J.,* **158**, 55.

Chandrasekhar, S. & Esposito, F.P. 1970. *Astrophys. J.,* **160**, 153.

Damour, T. 1987. In : *Gravitation in Astrophysics,* eds B. Carter and J.B. Hartle, Plenum, New York, p.3.

Damour, T. & Schäfer, G. 1987. *C. R. Acad. Sci. Paris,* **305**, série II, 839.

Damour, T. & Schäfer, G. 1988. *Nuovo Cim.,* **101B**, 127.

Epstein, R. & Wagoner, R.V. 1975. *Astrophys. J.,* **197**, 717.

Finn, L.S. 1989. In : *Frontiers in Numerical Relativity,* eds C.R. Evans, L.S. Finn and D.W. Hobill, Cambridge University Press, Cambridge, p. 126.

Fock, V. 1959. *The Theory of Space, Time and Gravitation,* Pergamon, London.

Holm, D.D. 1985. *Physica,* **17D**, 1.

Künzle, H.P. & Nester, J.M. 1984. *J. Math. Phys.,* **25**, 1009.

Lichnerowicz, A. 1955. *Théories Relativistes de la Gravitation et de l'Electromagnétisme,* Masson, Paris.

Misner, C.W., Thorne, K.S. & Wheeler, J.A. 1973. *Gravitation*, Freeman, San Francisco.

Nakamura, T., Oohara, K. & Kojima, Y. 1987. *Progr. Theor. Phys. Supplement*, **90**, 1.

Regge, T. & Teitelboim, C. 1974. *Ann. Phys. (N.Y.)*, **88**, 286.

Schäfer, G. 1983. *Lett. Nuov. Cim.*, **36**, 105.

Schäfer, G. 1985. *Ann. Phys. (N.Y.)*, **161**, 81.

Schäfer, G. 1989. In : *Proceedings of the Fifth Marcel Grossmann Meeting,* eds. D.G. Blair and M.J. Buckingham, World Scientific, Singapore ; and paper in preparation.

Stark, R.F. & Piran, T. 1986a. In : *Proceedings of the Fourth Marcel Grossmann Meeting on General Relativity,* ed. R. Ruffini, Elsevier, Amsterdam, p. 327.

Stark, R.F. & Piran, T. 1986b. In : *Gravitational Collapse and Relativity*, eds H. Sato and T. Nakamura, World Scientific, Singapore, p.249.

Thorne, K.S. 1969. *Astrophys. J.*, **158**, 997.

Thorne, K.S. 1980. *Rev. Mod. Phys.*, **52**, 299.

Turner, M. & Will, C.M. 1978. *Astrophys. J.*, **220**, 1107.

Wagoner, R.V. 1979. *Phys. Rev. D*, **19**, 2897.

Wagoner, R.V. & Will, C.M. 1976. *Astrophys. J.*, **210**, 764.

Weinberg, S. 1972. *Gravitation and Cosmology*, Wiley, New York.

Will, C.M. 1981. *Theory and Experiment in Gravitational Physics* , Cambridge University Press, Cambridge.

Zakharov, V.E. & Kuznetsov, E.A. 1984. *Mathematical Physics Reviews*, **4**, 167.