

**RENORMALIZATION IN QUANTUM FIELD THEORY
AND THE RIEMANN-HILBERT PROBLEM I:
THE HOPF ALGEBRA STRUCTURE OF GRAPHS
AND THE MAIN THEOREM**

Alain CONNES and Dirk KREIMER



Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Décembre 1999

IHES/M/99/91

Renormalization in quantum field theory and the Riemann-Hilbert problem I: the Hopf algebra structure of graphs and the main theorem

Alain Connes and Dirk Kreimer

Institut des Hautes Etudes Scientifiques
connes@ihes.fr, kreimer@ihes.fr

December 1999

Abstract *This paper gives a complete selfcontained proof of our result announced in [6] showing that renormalization in quantum field theory is a special instance of a general mathematical procedure of extraction of finite values based on the Riemann-Hilbert problem.*

We shall first show that for any quantum field theory, the combinatorics of Feynman graphs gives rise to a Hopf algebra \mathcal{H} which is commutative as an algebra. It is the dual Hopf algebra of the envelopping algebra of a Lie algebra \underline{G} whose basis is labelled by the one particle irreducible Feynman graphs. The Lie bracket of two such graphs is computed from insertions of one graph in the other and vice versa. The corresponding Lie group G is the group of characters of \mathcal{H} .

We shall then show that, using dimensional regularization, the bare (unrenormalized) theory gives rise to a loop

$$\gamma(z) \in G, \quad z \in C$$

where C is a small circle of complex dimensions around the integer dimension D of space-time. Our main result is that the renormalized theory is just the evaluation at $z = D$ of the holomorphic part γ_+ of the Birkhoff decomposition of γ .

We begin to analyse the group G and show that it is a semi-direct product of an easily understood abelian group by a highly non-trivial group closely tied up with groups of diffeomorphisms. The analysis of this latter group as well as the interpretation of the renormalization group and of anomalous dimensions are the content of our second paper with the same overall title.

1 Introduction

This paper gives a complete selfcontained proof of our result announced in [6] showing that renormalization in quantum field theory is a special instance of a general mathematical procedure of extraction of finite values based on the Riemann-Hilbert problem. In order that the paper be readable by non-specialists we shall begin by giving a short introduction to both topics of renormalization and of the Riemann-Hilbert problem, with our apologies to specialists in both camps for recalling well-known material.

Perturbative renormalization is by far the most successful technique for computing physical quantities in quantum field theory. It is well known for instance that it accurately predicts the first ten decimal places of the anomalous magnetic moment of the electron.

The physical motivation behind the renormalization technique is quite clear and goes back to the concept of effective mass in nineteenth century hydrodynamics. Thus for instance when applying Newton's law,

$$(1) \quad F = m a$$

to the motion of a spherical rigid balloon B, the inertial mass m is not the mass m_0 of B but is modified to

$$(2) \quad m = m_0 + \frac{1}{2} M$$

where M is the mass of the volume of air occupied by B.

It follows for instance that the initial acceleration a of B is given, using the Archimedean law, by

$$(3) \quad -Mg = \left(m_0 + \frac{1}{2} M\right) a$$

and is always of magnitude less than $2g$.

The additional inertial mass $\delta m = m - m_0$ is due to the interaction of B with the surrounding field of air and if this interaction could not be turned off there would be no way to measure the mass m_0 .

The analogy between hydrodynamics and electromagnetism led (through the work of Thomson, Lorentz, Kramers... [10]) to the crucial distinction between the bare parameters, such as m_0 , which enter the field theoretic equations, and the observed parameters, such as the inertial mass m .

A quantum field theory in $D = 4$ dimensions, is given by a classical action functional,

$$(4) \quad S(A) = \int \mathcal{L}(A) d^4x$$

where A is a classical field and the Lagrangian is of the form,

$$(5) \quad \mathcal{L}(A) = (\partial A)^2/2 - \frac{m^2}{2} A^2 + \mathcal{L}_{\text{int}}(A)$$

where $\mathcal{L}_{\text{int}}(A)$ is usually a polynomial in A and possibly its derivatives.

One way to describe the quantum fields $\phi(x)$, is by means of the time ordered Green's functions,

$$(6) \quad G_N(x_1, \dots, x_N) = \langle 0 | T \phi(x_1) \dots \phi(x_N) | 0 \rangle$$

where the time ordering symbol T means that the $\phi(x_j)$'s are written in order of increasing time from right to left.

The probability amplitude of a classical field configuration A is given by,

$$(7) \quad e^{i \frac{S(A)}{\hbar}}$$

and if one could ignore the renormalization problem, the Green's functions would be computed as,

$$(8) \quad G_N(x_1, \dots, x_N) = \mathcal{N} \int e^{i \frac{S(A)}{\hbar}} A(x_1) \dots A(x_N) [dA]$$

where \mathcal{N} is a normalization factor required to ensure the normalization of the vacuum state,

$$(9) \quad \langle 0 | 0 \rangle = 1.$$

It is customary to denote by the same symbol the quantum field $\phi(x)$ appearing in (6) and the classical field $A(x)$ appearing in the functional integral. No confusion arises from this abuse of notation.

If one could ignore renormalization, the functional integral (8) would be easy to compute in perturbation theory, i.e. by treating the term \mathcal{L}_{int} in (5) as a perturbation of

$$(10) \quad \mathcal{L}_0(\phi) = (\partial\phi)^2/2 - \frac{m^2}{2} \phi^2.$$

With obvious notations the action functional splits as

$$(11) \quad S(\phi) = S_0(\phi) + S_{\text{int}}(\phi)$$

where the free action S_0 generates a Gaussian measure $\exp(i S_0(\phi)) [d\phi] = d\mu$.

The series expansion of the Green's functions is then given in terms of Gaussian integrals of polynomials as,

$$(12) \quad G_N(x_1, \dots, x_N) = \left(\sum_{n=0}^{\infty} i^n/n! \int \phi(x_1) \dots \phi(x_N) (S_{\text{int}}(\phi))^n d\mu \right) \left(\sum_{n=0}^{\infty} i^n/n! \int S_{\text{int}}(\phi)^n d\mu \right)^{-1}.$$

The various terms of this expansion are computed using integration by parts under the Gaussian measure μ . This generates a large number of terms $U(\Gamma)$, each being labelled by a Feynman graph Γ , and having a numerical value $U(\Gamma)$ obtained as a multiple integral in a finite number of space-time variables. We shall come back later in much detail to the precise definition of Feynman graphs and of the corresponding integrals. But we now know enough to formulate the problem of renormalization. As a rule the unrenormalized values $U(\Gamma)$ are given by nonsensical divergent integrals.

The conceptually really nasty divergences are called ultraviolet¹ and are associated to the presence of arbitrarily large frequencies or equivalently to the unboundedness of momentum space on which integration has to be carried out. Equivalently, when one attempts to integrate in coordinate space, one confronts divergences along diagonals, reflecting the fact that products of field operators are defined only on the configuration space of distinct spacetime points.

The physics resolution of this problem is obtained by first introducing a cut-off in momentum space (or any suitable regularization procedure) and then by cleverly making use of the unobservability of the bare parameters, such as the bare mass m_0 . By adjusting, term by term of the perturbative expansion, the dependence of the bare parameters on the cut-off parameter,

¹The challenge posed by the infrared problem is formidable though. Asymptotic expansions in its presence quite generally involve decompositions of singular expression similar to the methods underlying renormalization theory. One can reasonably hope that in the future singular asymptotic expansions will be approachable by the methods advocated here.

it is possible for a large class of theories, called renormalizable, to eliminate the unwanted ultraviolet divergences. This resolution of divergences can actually be carried out at the level of integrands, with suitable derivatives with respect to external momenta, which is the celebrated BPHZ approach to the problem.

The soundness of this physics resolution of the problem makes it doubtful at first sight that it could be tied up to central parts of mathematics.

It was recognized quite long ago [11] that distribution theory together with locality were providing a satisfactory formal approach to the problem when formulated in configuration space, in terms of the singularities of (6) at coinciding points, formulating the BPHZ recursion in configuration space. The mathematical program of constructive quantum field theory [13] was first completed for superrenormalizable models, making contact with the deepest parts of hard classical analysis through the phase space localization and renormalization group methods. This led to the actual rigorous mathematical construction of renormalizable models such as the Gross-Neveu model in 2-dimensions [12]. The discovery of asymptotic freedom, which allows to guess the asymptotic expansion of the bare parameters in terms of the cut-off leads to the partially fulfilled hope that the rigorous construction can be completed for physically important theories such as QCD.

However neither of these important progresses does shed light on the actual complicated combinatorics which has been successfully used by particle physicists for many decades to extract finite results from divergent Feynman graphs, and which is the essence of the experimentally confirmed predictive power of quantum field theory.

We shall fill this gap in the present paper by unveiling the true nature of this seemingly complicated combinatorics and by showing that it is a special case of a general extraction of finite values based on the Riemann-Hilbert problem.

Our result was announced in [6] and relies on several previous papers [7, 16, 17, 18, 4] but we shall give below a complete account of its proof.

The Riemann-Hilbert problem comes from Hilbert's 21st problem which he formulated as follows:

“Prove that there always exists a Fuchsian linear differential equation with given singularities and given monodromy.”

In this form it admits a positive answer due to Plemelj and Birkhoff (cf. [1] for a careful exposition). When formulated in terms of linear systems of the

form,

$$(13) \quad y'(z) = A(z) y(z), \quad A(z) = \sum_{\alpha \in S} \frac{A_\alpha}{z - \alpha},$$

where S is the given finite set of singularities, $\infty \notin S$, the A_α are complex matrices such that

$$(14) \quad \sum A_\alpha = 0$$

to avoid singularities at ∞ , the answer is not always positive [2], but the solution exists when the monodromy matrices M_α (fig.1) are sufficiently close to 1. It can then be explicitly written as a series of polylogarithms [19].

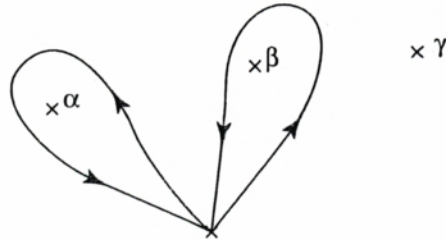


Figure 1

Another formulation of the Riemann-Hilbert problem, intimately tied up to the classification of holomorphic vector bundles on the Riemann sphere $P_1(\mathbb{C})$, is in terms of the Birkhoff decomposition

$$(15) \quad \gamma(z) = \gamma_-(z)^{-1} \gamma_+(z) \quad z \in C$$

where we let $C \subset P_1(\mathbb{C})$ be a smooth simple curve, C_- the component of the complement of C containing $\infty \notin C$ and C_+ the other component. Both γ and γ_\pm are loops with values in $GL_n(\mathbb{C})$,

$$(16) \quad \gamma(z) \in G = GL_n(\mathbb{C}) \quad \forall z \in \mathbb{C}$$

and γ_\pm are boundary values of holomorphic maps (still denoted by the same symbol)

$$(17) \quad \gamma_\pm : C_\pm \rightarrow GL_n(\mathbb{C}).$$

The normalization condition $\gamma_-(\infty) = 1$ ensures that, if it exists, the decomposition (15) is unique (under suitable regularity conditions).

The existence of the Birkhoff decomposition (15) is equivalent to the vanishing,

$$(18) \quad c_1(L_j) = 0$$

of the Chern numbers $n_j = c_1(L_j)$ of the holomorphic line bundles of the Birkhoff-Grothendieck decomposition,

$$(19) \quad E = \oplus L_j$$

where E is the holomorphic vector bundle on $P_1(\mathbb{C})$ associated to γ , i.e. with total space:

$$(20) \quad (C_+ \times \mathbb{C}^n) \cup_{\gamma} (C_- \times \mathbb{C}^n).$$

The above discussion for $G = \mathrm{GL}_n(\mathbb{C})$ extends to arbitrary complex Lie groups.

When G is a simply connected nilpotent complex Lie group the existence (and uniqueness) of the Birkhoff decomposition (15) is valid for any γ . When the loop $\gamma : C \rightarrow G$ extends to a holomorphic loop: $C_+ \rightarrow G$, the Birkhoff decomposition is given by $\gamma_+ = \gamma$, $\gamma_- = 1$. In general, for $z \in C_+$ the evaluation,

$$(21) \quad \gamma \rightarrow \gamma_+(z) \in G$$

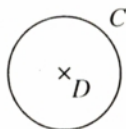
is a natural principle to extract a finite value from the singular expression $\gamma(z)$. This extraction of finite values coincides with the removal of the pole part when G is the additive group \mathbb{C} of complex numbers and the loop γ is meromorphic inside C_+ with z as its only singularity.

We shall first show that for any quantum field theory, the combinatorics of Feynman graphs gives rise to a Hopf algebra \mathcal{H} which is commutative as an algebra. It is the dual Hopf algebra of the envelopping algebra of a Lie algebra \underline{G} whose basis is labelled by the one particle irreducible Feynman graphs. The Lie bracket of two such graphs is computed from insertions of one graph in the other and vice versa. The corresponding Lie group G is the group of characters of \mathcal{H} .

We shall then show that, using dimensional regularization, the bare (unrenormalized) theory gives rise to a loop

$$(22) \quad \gamma(z) \in G, \quad z \in C$$

where C is a small circle of complex dimensions around the integer dimension D of space-time



Our main result is that the renormalized theory is just the evaluation at $z = D$ of the holomorphic part γ_+ of the Birkhoff decomposition of γ .

We begin to analyse the group G and show that it is a semi-direct product of an easily understood abelian group by a highly non-trivial group closely tied up with groups of diffeomorphisms. The analysis of this latter group as well as the interpretation of the renormalization group and of anomalous dimensions are the content of our second paper with the same overall title [8].


2 The Hopf algebra of Feynman graphs


The Hopf algebra structure of perturbative quantum field theory is by now well established [7, 16, 17, 18, 4]. To the practitioner its most exciting aspect is arguably that it is represented as a Hopf algebra of decorated rooted trees [7, 18, 4] or, equivalently, as a Hopf algebra of parenthesized words on an alphabet provided by the skeleton expansion of the theory [16]. The relation to rooted trees exhibits most clearly the combinatorics imposed on Feynman graphs so that overlapping subdivergences can be resolved to deliver local counterterms [17]. The Hopf algebra structure can be directly formulated on graphs though [17, 4]. It is this latter representation to which we will turn here, to make the contact with the notation of Collins' textbook as close as possible.


Feynman graphs Γ are combinatorial labels for the terms of the expansion (12) of Green's functions in a given quantum field theory. They are graphs consisting of vertices joined by lines. The vertices are of different kinds corresponding to terms in the Lagrangian of the theory.

We shall require that the theory we start with is renormalizable and include all corresponding vertices in the diagrams.

Thus, and if we start, for notational simplicity,² with φ^3 in $D = 6$ dimensions, we shall have three kinds of vertices:³

the three line vertex  corresponding to the φ^3 term of the Lagrangian;

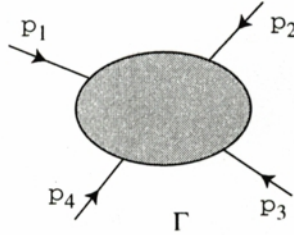
the two line vertex  corresponding to the φ^2 term of the Lagrangian;

the two line vertex  corresponding to the $(\partial\varphi)^2$ term of the Lagrangian.

In general the number of lines attached to a vertex is the degree of the corresponding monomial in the Lagrangian of the theory. A line joining two vertices is called internal. The others are attached to only one vertex and are called external. To specify a Feynman graph one needs to specify the values of parameters which label the external lines. When working in configuration space these would just be the space-time points x_j of (12) associated to the corresponding external vertices (i.e. those vertices attached to external lines). It will be more practical to work in momentum space and to specify the external parameters of a diagram in terms of external momenta. It is customary to orient the momenta carried by external lines so that they all go inward. The law of conservation of momentum means that for each connected graph Γ ,

$$(1) \quad \sum p_i = 0$$

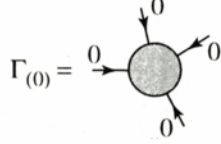
where the p_i are the external momenta,



We shall use the following notation to indicate specific external structures of a graph Γ . We let $\Gamma_{(0)}$ be the graph with all its external momenta nullified, i.e.

²Our results extend in a straightforward manner to any theory renormalizable by local counterterms.

³In the case of a massless theory, there will be only two kinds, as the two-line vertex corresponding to the φ^2 term is missing.



For self energy graphs, i.e. graphs Γ with just two external lines, we let ⁴

$$(2) \quad \Gamma_{(1)} = \left(\frac{\partial}{\partial p^2} \Gamma(p) \right)_{p=0}$$

where p is the momentum flowing through the diagram. Note that the sign of p is irrelevant in (2). This notation might seem confusing at first sight but becomes clear if one thinks of the external structure of graphs in terms of distributions. This is necessary to have space-time parameters x_j on the same footing as the momentum parameters p_j using Fourier transform. We shall return to this point in greater detail after the proof of theorem 1.

A Feynman graph Γ is called one particle irreducible (1PI) if it is connected and cannot be disconnected by removing a single line. The following graph is one particle reducible:



The diagram $\text{---}\times\text{---}$ is considered as not 1PI.⁵

Let us now define the Hopf algebra \mathcal{H} .⁶ As a linear space \mathcal{H} has a basis labelled by all Feynman graphs Γ which are disjoint unions of 1PI graphs.

$$(3) \quad \Gamma = \bigcup_{j=1}^n \Gamma_j.$$

The case $\Gamma = \emptyset$ is allowed.

The product in \mathcal{H} is bilinear and given on the basis by the operation of disjoint union:

$$(4) \quad \Gamma \cdot \Gamma' = \Gamma \cup \Gamma'.$$

⁴in the case of a massless theory, we take $\Gamma_{(1)} = \left(\frac{\partial}{\partial p^2} \Gamma(p) \right)_{p^2=\mu^2}$

⁵This is a conveniently chosen but immaterial convention.

⁶We work with any field of coefficients such as \mathbb{Q} or \mathbb{C} .

Note also that the external structure of $\Gamma/\gamma_{(i)}$ is identical to that of Γ .
To get familiar with (6) we shall now give a few examples of coproducts.

$$(9) \quad \Delta(-\bigcirc-) = -\bigcirc- \otimes 1 + 1 \otimes -\bigcirc-$$

$$(10) \quad \left\{ \begin{array}{l} \Delta(-\bigcirc\bigcirc-) = -\bigcirc\bigcirc- \otimes 1 + 1 \otimes -\bigcirc\bigcirc- + \\ 2 \text{ } \bigtriangleleft \otimes -\bigcirc- \end{array} \right.$$

$$(11) \quad \left\{ \begin{array}{l} \Delta(-\bigcirc\bigcirc\bigcirc-) = -\bigcirc\bigcirc\bigcirc- \otimes 1 + 1 \otimes -\bigcirc\bigcirc\bigcirc- \\ + 2 \text{ } \bigtriangleleft \otimes -\bigcirc- + 2 \text{ } \bigtriangleleft \otimes -\bigcirc\bigcirc- \\ + \text{ } \bigtriangleleft \text{ } \bigtriangleleft \otimes -\bigcirc- \end{array} \right.$$

$$(12) \quad \left\{ \begin{array}{l} \Delta(-\bigcirc\bigcirc\bigcirc\bigcirc-) = -\bigcirc\bigcirc\bigcirc\bigcirc- \otimes 1 + 1 \otimes -\bigcirc\bigcirc\bigcirc\bigcirc- \\ + \text{ } \bigcirc_{(i)} \otimes \bigcirc_i \end{array} \right.$$

where one sums over $i = 0, 1$.

Example (9) shows what happens in general for graphs with no subgraph fulfilling (7), i.e. graphs without subdivergences. Example (10) shows an example of overlapping subdivergences. Example (11) also, but it illustrates an important general feature of the coproduct $\Delta(\Gamma)$ for Γ a 1PI graph, namely that $\Delta(\Gamma) \in \mathcal{H} \otimes \mathcal{H}_{(1)}$ where $\mathcal{H}_{(1)}$ is the subspace of \mathcal{H} generated by 1 and the 1PI graphs. Similar examples were given in [7, 17].

The coproduct Δ defined by (6) on 1PI graphs extends uniquely to a homomorphism from \mathcal{H} to $\mathcal{H} \otimes \mathcal{H}$. The main result of this section is ([16, 17]):

THEOREM 1. *The pair (\mathcal{H}, Δ) is a Hopf algebra.*

Proof. Our first task will be to prove that Δ is coassociative, i.e. that,

$$(13) \quad (\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta.$$

Since both sides of (13) are algebra homomorphisms from \mathcal{H} to $\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}$, it will be enough to check that they give the same result on 1PI graphs Γ . Thus we fix the 1PI graph Γ and let \mathcal{H}_Γ be the linear subspace of \mathcal{H} spanned by 1PI graphs with the same external structure as Γ . Also we let \mathcal{H}_c be the subalgebra of \mathcal{H} generated by the 1PI graphs γ with two or three external legs and with external structure of type (i) , $i = 0, 1$. One has by (6) that

$$(14) \quad \Delta \Gamma - \Gamma \otimes 1 \in \mathcal{H}_c \otimes \mathcal{H}_\Gamma$$

while

$$(15) \quad \Delta \mathcal{H}_c \subset \mathcal{H}_c \otimes \mathcal{H}_c.$$

Thus we get,

$$(16) \quad (\Delta \otimes \text{id}) \Delta \Gamma - \Delta \Gamma \otimes 1 \in \mathcal{H}_c \otimes \mathcal{H}_c \otimes \mathcal{H}_\Gamma.$$

To be more specific we have the formula,

$$(17) \quad (\Delta \otimes \text{id}) \Delta \Gamma - \Delta \Gamma \otimes 1 = \sum_{\substack{\gamma \subset \Gamma \\ \neq}} \Delta \gamma_{(i)} \otimes \Gamma / \gamma_{(i)}$$

where $\gamma = \emptyset$ is allowed now in the summation of the right hand side.

We need a nice formula for $\Delta \gamma_{(i)}$ which is defined as

$$(18) \quad \Pi \Delta \gamma_{(i_j)}^j$$

where the γ^j are the components of γ .

For each of the graphs $\gamma'' = \gamma_{(i_j)}^j$ the formula (6) for the coproduct simplifies to give

$$(19) \quad \Delta \gamma'' = \sum_{\gamma' \subset \gamma''} \gamma'_{(k)} \otimes \gamma'' / \gamma'_{(k)}$$

where the subset γ' of the set $\gamma''^{(1)}$ of internal lines of γ'' is now allowed to be empty (which gives $1 \otimes \gamma''$) or full (which gives $\gamma'' \otimes 1$). Of course the sum (19) is restricted to γ' such that each component is 1PI and satisfies (7) relative to $\gamma'' = \gamma_{(i_j)}^j$. But this is equivalent to fulfilling (7) relative to Γ since one has,

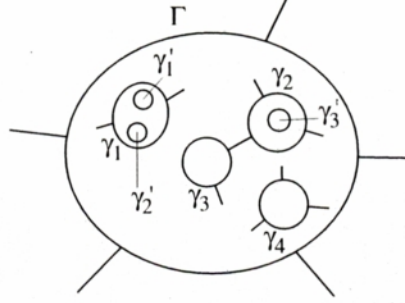
$$(20) \quad \varepsilon_{\gamma''}(\gamma') = \varepsilon_\Gamma(\gamma').$$

Indeed since γ'' is a subgraph of Γ one has $\varepsilon_{\gamma''}(\gamma') \subset \varepsilon_{\Gamma}(\gamma')$ but every line $\ell \in \varepsilon_{\Gamma}(\gamma')$ is a line of Γ which meets $\gamma''^{(1)}$ and hence belongs to γ'' which gives equality.

The equality (20) also shows that the symbol $\gamma'_{(k)}$ in (19) can be taken relative to the full graph Γ . We can now combine (17) (18) and (19) to write the following formula:

$$(21) \quad (\Delta \otimes \text{id}) \Delta \Gamma - \Delta \Gamma \otimes 1 = \sum_{\substack{\gamma' \subset \gamma \subset \Gamma \\ \neq}} \gamma'_{(k)} \otimes \gamma_{(i)}/\gamma'_{(k)} \otimes \Gamma/\gamma_{(i)},$$

where both γ and γ' are subsets of the set of internal lines $\Gamma^{(1)}$ of Γ and γ' is a subset of γ . Both γ and γ' satisfy (7). Since γ is not necessarily connected we need to define the symbol $\gamma_{(i)}/\gamma'_{(k)}$ as the replacement in the not necessarily connected graph $\gamma_{(i)}$ of each of the component of γ' by the corresponding vertex. The only point to remember is that if a component of γ' is equal to a component of γ the corresponding index k_j is equal to i_j (following (19)) and the corresponding term in γ/γ' is equal to 1.



Let us now compute $(\text{id} \otimes \Delta) \Delta \Gamma - \Delta \Gamma \otimes 1$ starting from the equality,

$$(22) \quad \Delta \Gamma = \Gamma \otimes 1 + \sum_{\substack{\gamma' \subset \Gamma \\ \neq}} \gamma'_{(k)} \otimes \Gamma/\gamma'_{(k)}$$

where, unlike in (6), we allow $\gamma' = \emptyset$ in the sum. Let us define $\Delta' : \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{H}$ by

$$(23) \quad \Delta' X = \Delta X - X \otimes 1 \quad \forall X \in \mathcal{H}.$$

One has $(\text{id} \otimes \Delta') \Delta' X = (\text{id} \otimes \Delta') \Delta X$ since $\Delta' 1 = 0$, and $(\text{id} \otimes \Delta') \Delta X = (\text{id} \otimes \Delta) \Delta X - (\text{id} \otimes \text{id} \otimes 1) \Delta X = (\text{id} \otimes \Delta) \Delta X - \Delta X \otimes 1$. Thus,

$$(24) \quad (\text{id} \otimes \Delta) \Delta \Gamma - \Delta \Gamma \otimes 1 = (\text{id} \otimes \Delta') \Delta' \Gamma.$$

Moreover (22) gives the formula for Δ' on $\mathcal{H}_{(1)}$ which is thus enough to get,

$$(25) \quad (\text{id} \otimes \Delta) \Delta \Gamma - \Delta \Gamma \otimes 1 = \sum_{\gamma', \gamma''} \gamma'_{(k)} \otimes \gamma''_{(j)} \otimes (\Gamma / \gamma'_{(k)}) / \gamma''_{(j)}.$$

In this sum γ' varies through the (possibly empty) admissible subsets of $\Gamma^{(1)}$, $\gamma' \neq \Gamma^{(1)}$, while γ'' varies through the (possibly empty) admissible subsets of the set of lines $\Gamma'^{(1)}$ of the graph

$$(26) \quad \Gamma' = \Gamma / \gamma'_{(k)}.$$

To prove that the sums (21) and (25) are equal it is enough to prove that for any admissible subset $\gamma' \subsetneq \Gamma^{(1)}$ the corresponding sums are equal. We also fix the multi index k . The sum (25) then only depends upon the graph $\Gamma' = \Gamma / \gamma'_{(k)}$. We thus need to show the equality

$$(27) \quad \sum_{\gamma' \subsetneq \gamma \subsetneq \Gamma} \gamma_{(i)} / \gamma'_{(k)} \otimes \Gamma / \gamma_{(i)} = \sum_{\gamma''} \gamma''_{(j)} \otimes \Gamma' / \gamma''_{(j)}.$$

Let $\pi : \Gamma \rightarrow \Gamma'$ be the continuous projection,

$$(28) \quad \pi : \Gamma \rightarrow \Gamma / \gamma'_{(k)} = \Gamma'.$$

Let us show that the map ρ which associates to every admissible subset γ of Γ containing γ' its image $\gamma / \gamma'_{(k)}$ in Γ' gives a bijection with the admissible subsets γ'' of Γ' .

Since each connected component of γ' is contained in a connected component of γ , we see that $\rho(\gamma)$ is admissible in Γ' . Note that when a connected component γ^0 of γ is equal to a connected component of γ' its image in Γ' is defined to be empty. For the components of γ which are not components of γ' their image is just the contraction $\gamma / \gamma'_{(k)}$ by those components of γ' which it contains. This does not alter the external leg structure, so that

$$(29) \quad \gamma_{(i)}^0 / \gamma'_{(k)} = (\gamma^0 / \gamma'_{(k)})_{(i)}.$$

The inverse of the map ρ is obtained as follows. Given γ'' an admissible subset of Γ' , one associates to each component of γ'' its inverse image by π which gives a component of γ . Moreover to each vertex v of Γ' which does not belong to γ'' but which came from the contraction of a component of γ' , one associates this component as a new component of γ . It is clear then that $\gamma' \subset \gamma \subsetneq \Gamma$ and that γ is admissible in Γ with $\rho(\gamma) = \gamma''$.

To prove (27) we can thus fix γ and $\gamma'' = \rho(\gamma)$. For those components of γ equal to components of γ' the index i_0 is necessarily equal to k_0 so they contribute in the same way to both sides of (27). For the other components there is freedom in the choice of i_0 or j_0 but using (29) the two contributions to (27) are also equal. This ends the proof of the coassociativity of the coproduct Δ and it remains to show that the bialgebra \mathcal{H} admits an antipode.

This can be easily proved by induction but it is worthwhile to discuss various gradings of the Hopf algebra \mathcal{H} associated to natural combinatorial features of the graphs. Any such grading will associate an integer $n(\Gamma) \in \mathbb{Z}$ to each 1PI graph, the corresponding grading of the algebra \mathcal{H} is then given by

$$(30) \quad \deg(\Gamma_1 \dots \Gamma_e) = \sum n(\Gamma_j), \quad \deg(1) = 0$$

and the only interesting property is the compatibility with the coproduct which means, using (6), that

$$(31) \quad \deg(\gamma) + \deg(\Gamma/\gamma) = \deg(\Gamma)$$

for any admissible γ .

The first two natural gradings are given by

$$(32) \quad I(\Gamma) = \text{number of internal lines in } \Gamma$$

and

$$(33) \quad v(\Gamma) = V(\Gamma) - 1 = \text{number of vertices of } \Gamma - 1.$$

An important combination of these two gradings is

$$(34) \quad L = I - v = I - V + 1$$

which is the loop number of the graph or equivalently the rank of its first homology group. It governs the power of \hbar which appears in the evaluation of the graph.

Note that the number of external lines of a graph is not a good grading since it fails to fulfill (31). For any of the three gradings I, v, L one has the following further compatibility with the Hopf algebra structure of \mathcal{H} ,

LEMMA. a) *The scalars are the only homogeneous elements of degree 0.*

b) For any non scalar homogeneous $X \in \mathcal{H}$ one has

$$\Delta X = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$$

where X', X'' are homogeneous of degree strictly less than the degree of X .

Proof. a) Since the diagram $\text{---}\star\text{---}$ is excluded we see that $I(\Gamma) = 0$ or $v(\Gamma) = 0$ is excluded. Also if $L(\Gamma) = 0$ the Γ is a tree but a tree diagram cannot be 1PI unless it is equal to $\text{---}\star\text{---}$ which is excluded.

b) Since X is a linear combination of homogeneous monomials it is enough to prove b) for $\Pi \Gamma_i$ and in fact for 1PI graphs Γ . Using (6) it is enough to check that the degree of a non empty γ is strictly positive, which follows from a). ■

We can now end the proof of Theorem 1 and give an inductive formula for the antipode S .

The counit $\bar{\epsilon}$ is given as a character of \mathcal{H} by

$$(35) \quad \bar{\epsilon}(1) = 1, \quad \bar{\epsilon}(\Gamma) = 0 \quad \forall \Gamma \neq \emptyset.$$

The defining equation for the antipode is,

$$(36) \quad m(S \otimes \text{id}) \Delta(a) = \bar{\epsilon}(a) 1 \quad \forall a \in \mathcal{H}$$

and its existence is obtained by induction using the formula,

$$(37) \quad S(X) = -X - \sum S(X') X''$$

for any non scalar homogeneous $X \in \mathcal{H}$ using the notations of Lemma 2. This antipode also fulfills the other required identity [7, 16],

$$m(\text{id} \otimes S) \Delta(a) = \epsilon(a) 1 \quad \forall a \in \mathcal{H}.$$

This completes the proof of Theorem 1. ■

Let us now be more specific about the external structure of diagrams. Given a 1PI graph, with n external legs labelled by $i \in \{1, \dots, n\}$ we specify its external structure by giving a distribution σ defined on a suitable test space \mathcal{S} of smooth functions on

$$(38) \quad \left\{ (p_i)_{i=1, \dots, n} ; \sum p_i = 0 \right\} = E_n.$$

Thus σ is a continuous linear map,

$$(39) \quad \sigma : \mathcal{S}(E) \rightarrow \mathbb{C}.$$

To a graph Γ with external structure σ we have associated above an element of the Hopf algebra \mathcal{H} and we require the linearity of this map, i.e.

$$(40) \quad \delta_{(\Gamma, \lambda_1 \sigma_1 + \lambda_2 \sigma_2)} = \lambda_1 \delta_{(\Gamma, \sigma_1)} + \lambda_2 \delta_{(\Gamma, \sigma_2)}.$$

One can easily check that this relation is compatible with the coproduct.

There is considerable freedom in the choice of the external structure of the 1PI graphs which occur in the left hand side of the last term of the coproduct formula (6). We wrote the proof of Theorem 1 in such a way that this freedom is apparent. The only thing which matters is that, say for self energy graphs, the distributions σ_0 and σ_1 satisfy,

$$(41) \quad \sigma_0(a m^2 + b p^2) = a, \quad \sigma_1(a m^2 + b p^2) = b,$$

where $p = p_1$ is the natural coordinate on E_2 and m is a mass parameter.

This freedom in the definition of the Hopf algebra \mathcal{H} is the same as the freedom in the choice of parametrization of the corresponding QFT and it is important to make full use of it, for instance for massless theories in which the above choice of nullified external momenta is not appropriate and one would rely only on $\sigma_1(b p^2) = b$.

For simplicity of exposition we shall keep the above choice, the generalization being obvious.

We shall now apply the Milnor-Moore theorem to the bigraded Hopf algebra \mathcal{H} . The two natural gradings are v and L , the grading I is just their sum.

This theorem first gives a Lie algebra structure on the linear space,

$$(42) \quad \bigoplus_{\Gamma} \mathcal{S}(E_{\Gamma}) = L$$

where for each 1PI graph Γ , we let $\mathcal{S}(E_{\Gamma})$ be the test space associated to the external lines of Γ as in (38). Given $X \in L$ we consider the linear form Z_X on \mathcal{H} given, on the monomials Γ , by

$$(43) \quad \langle \Gamma, Z_X \rangle = 0$$

unless Γ is a (connected) 1PI, in which case,

$$(44) \quad \langle \Gamma, Z_X \rangle = \langle \sigma_{\Gamma}, X_{\Gamma} \rangle$$

where σ_Γ is the distribution giving the external structure of Γ and where X_Γ is the corresponding component of X . By construction Z_X is an infinitesimal character of \mathcal{H} and the same property holds for the commutator,

$$(45) \quad [Z_{X_1}, Z_{X_2}] = Z_{X_1} Z_{X_2} - Z_{X_2} Z_{X_1}$$

where the product in the right hand side is given by the coproduct of \mathcal{H} , i.e. by

$$(46) \quad \langle Z_1 Z_2, \Gamma \rangle = \langle Z_1 \otimes Z_2, \Delta \Gamma \rangle.$$

The computation of the Lie bracket is straightforward as in [9] p. 207 or [7] and is given as follows. One lets Γ_j , $j = 1, 2$ be 1PI graphs and $\varphi_j \in \mathcal{S}(E_{\Gamma_j})$ corresponding test functions.

For $i \in \{0, 1\}$, we let $n_i(\Gamma_1, \Gamma_2; \Gamma)$ be the number of subgraphs of Γ which are isomorphic to Γ_1 while

$$(47) \quad \Gamma/\Gamma_1(i) \simeq \Gamma_2.$$

We let (Γ, φ) be the element of L associated to $\varphi \in \mathcal{S}(E_\Gamma)$, the Lie bracket of (Γ_1, φ_1) and (Γ_2, φ_2) is then,

$$(48) \quad \sum_{\Gamma, i} \sigma_i(\varphi_1) n_i(\Gamma_1, \Gamma_2; \Gamma) (\Gamma, \varphi_2) - \sigma_i(\varphi_2) n_i(\Gamma_2, \Gamma_1; \Gamma) (\Gamma, \varphi_1).$$

What is obvious in this formula is that it vanishes if $\sigma_i(\varphi_j) = 0$ and hence we let L_0 be the subspace of L given by,

$$(49) \quad L_0 = \oplus \mathcal{S}(E_\Gamma)_0, \mathcal{S}(E_\Gamma)_0 = \{\varphi; \sigma_i(\varphi) = 0, i = 0, 1\}.$$

It is by construction a subspace of finite codimension in each of the $\mathcal{S}(E_\Gamma)$. We need a natural supplement and in view of (41) we should choose the obvious test functions,

$$(50) \quad \varphi_0(p) = m^2, \varphi_1(p) = p^2.$$

We shall thus, for any 1PI self energy graph, let

$$(51) \quad (\Gamma^{(i)}) = (\Gamma, \varphi_i).$$

Similarly for a vertex graph with the constant function 1. One checks using (48) that the $\Gamma^{(i)}$ for 1PI graphs with two or three external legs, do generate a Lie subalgebra

$$(52) \quad L_c = \left\{ \sum \lambda (\Gamma^{(i)}) \right\}.$$

We can now state the following simple fact,

THEOREM 2. *The Lie algebra L is the semi-direct product of L_0 by L_c . The Lie algebra L_0 is abelian, while L_c has a canonical basis labelled by the $\Gamma^{(i)}$ and such that*

$$[\Gamma, \Gamma'] = \sum_v \Gamma \circ_v \Gamma' - \sum_{v'} \Gamma' \circ_{v'} \Gamma$$

where $\Gamma \circ_v \Gamma'$ is obtained by grafting Γ' on Γ at v .

Proof. Using (48) it is clear that L_0 is an abelian Lie subalgebra of L . The Lie bracket of $(\Gamma^{(i)}) \in L_c$ with $(\Gamma, \varphi) \in L_0$ is given by

$$(53) \quad \sum n_i (\Gamma^{(i)}, \Gamma; \Gamma') (\Gamma', \varphi)$$

which belongs to L_0 so that $[L_c, L_0] \subset L_0$.

To simplify the Lie bracket of the $(\Gamma^{(i)})$ in L_c we introduce the new basis,

$$(54) \quad \Gamma^{(i)} = -S(\Gamma) (\Gamma)$$

where $S(\Gamma)$ is the symmetry factor of a Feynman graph, i.e. the cardinality of its group of automorphisms. In other words if another graph Γ' is isomorphic to Γ there are exactly $S(\Gamma)$ such isomorphisms. From the definition (47) of $n_i(\Gamma_1, \Gamma_2; \Gamma)$ we see that $S(\Gamma_1) S(\Gamma_2) n_i(\Gamma_1, \Gamma_2; \Gamma)$ is the number of pairs j_1, j_2 of an embedding

$$(55) \quad j_1 : \Gamma_1 \rightarrow \Gamma$$

and of an isomorphism

$$(56) \quad j_2 : \Gamma_2 \simeq \Gamma / \Gamma_1(i).$$

Giving such a pair is the same as giving a vertex v of type (i) of Γ_2 and an isomorphism,

$$(57) \quad j : \Gamma_2 \circ_v \Gamma_1 \rightarrow \Gamma.$$

Since there are $S(\Gamma)$ such isomorphisms when $\Gamma \sim \Gamma_2 \circ_v \Gamma_1$ we get the formula of Theorem 2 using (48). ■

It is clear from Theorem 2 that the Lie algebra L_c is independent of the choice of the distributions σ_j fulfilling (41). The same remark applies to L using (53).

By the Milnor-Moore theorem the Hopf algebra \mathcal{H} is the dual of the envelopping algebra $\mathcal{U}(L)$. The linear subspace $\mathcal{H}_{(1)}$ of \mathcal{H} spanned by 1 and the 1PI graphs give a natural system of affine coordinates on the group G of characters of \mathcal{H} , i.e. of homomorphisms,

$$(58) \quad \varphi : \mathcal{H} \rightarrow \mathbb{C}, \quad \varphi(XY) = \varphi(X) \varphi(Y) \quad \forall X, Y \in \mathcal{H}$$

from the algebra \mathcal{H} to complex numbers.

We shall only consider homomorphisms which are continuous with respect to the distributions labelling the external structure of graphs.

The group operation in G is given by the convolution,

$$(59) \quad (\varphi_1 * \varphi_2)(X) = \langle \varphi_1 \otimes \varphi_2, \Delta X \rangle \quad \forall X \in \mathcal{H}.$$

The Hopf subalgebra \mathcal{H}_c of \mathcal{H} generated by the 1PI with two or three external legs and external structure given by the σ_i , is the dual of the envelopping algebra $\mathcal{U}(L_c)$ and we let G_c be the group of characters of \mathcal{H}_c .

The map $\varphi \rightarrow \varphi/\mathcal{H}_c$ defines a group homomorphism,

$$(60) \quad \rho : G \rightarrow G_c$$

and as in Theorem 2 one has,

PROPOSITION 3. *The kernel G_0 of ρ is abelian and G is the semi-direct product $G = G_0 \rtimes G_c$ of G_0 by the action of G_c .*

Proof. A character φ of \mathcal{H} belongs to G_0 iff its restriction to \mathcal{H}_c is the augmentation map. This just means that for any 1PI graph γ one has $\varphi(\gamma_{(i)}) = 0$. Thus the convolution of two characters $\varphi_j \in G_0$ is just given by,

$$(61) \quad (\varphi_1 * \varphi_2)(\Gamma) = \varphi_1(\Gamma) + \varphi_2(\Gamma)$$

for any 1PI graph Γ . This determines the character $\varphi_1 * \varphi_2$ uniquely, so that $\varphi_1 * \varphi_2 = \varphi_2 * \varphi_1$.

Let us now construct a section $\rho' : G_c \rightarrow G$ which is a homomorphism. We just need to construct a homomorphism,

$$(62) \quad \varphi \rightarrow \tilde{\varphi} = \varphi \circ \rho'$$

from characters of \mathcal{H}_c to characters of \mathcal{H} , such that

$$(63) \quad \tilde{\varphi}/\mathcal{H}_c = \varphi.$$

It is enough to extend φ to all 1PI graphs (Γ, σ) where σ is the external structure, this then determines $\tilde{\varphi}$ uniquely as a character and one just has to check that,

$$(64) \quad (\varphi_1 * \varphi_2)^\sim = \tilde{\varphi}_1 * \tilde{\varphi}_2.$$

We let $\tilde{\varphi}(\Gamma, \sigma) = 0$ for any 1PI graph with $n \neq 2, 3$ external legs or any 1PI graph with 2 or 3 external legs such that,

$$(65) \quad \sigma((p^2)^i) = 0 \quad i = 0, 1 \quad \text{if } n = 2, \quad i = 0 \quad \text{if } n = 3.$$

By (41) this gives a natural supplement of $\mathcal{H}_{c(1)}$ in $\mathcal{H}_{(1)}$ and determines $\tilde{\varphi}$ uniquely.

To check (64) it is enough to test both sides on 1PI graphs (Γ, σ) . One uses (6) to get,

$$(66) \quad (\tilde{\varphi}_1 * \tilde{\varphi}_2)(\Gamma, \sigma) = \tilde{\varphi}_1(\Gamma, \sigma) + \tilde{\varphi}_2(\Gamma, \sigma) + \sum \varphi_1(\gamma_{(i)}) \tilde{\varphi}_2(\Gamma/\gamma_{(i)}, \sigma).$$

For any (Γ, σ) in the above supplement of $\mathcal{H}_{c(1)}$ the right hand side clearly vanishes. The same holds for $(\varphi_1 * \varphi_2)^\sim(\Gamma, \sigma)$ by construction. For any $(\Gamma, \sigma_i) \in \mathcal{H}_{c(1)}$ one simply gets the formula for $(\varphi_1 * \varphi_2)(\Gamma, \sigma_i) = (\varphi_1 * \varphi_2)^\sim(\Gamma, \sigma_i)$. This gives (64). We have thus proved that ρ is a surjective homomorphism from G to G_c and that $\rho' : G_c \rightarrow G$ is a group homomorphism such that

$$(67) \quad \rho \circ \rho' = \text{id}_{G_c}.$$

■

Let us now compute explicitly the action of G_c on G_0 given by inner automorphisms,

$$(68) \quad \alpha(\varphi_1) \varphi = \tilde{\varphi}_1 * \varphi * \tilde{\varphi}_1^{-1}.$$

One has

$$(69) \quad (\varphi * \tilde{\varphi}_1^{-1})(\Gamma, \sigma) = \varphi(\Gamma, \sigma)$$

for any (Γ, σ) in the supplement of $\mathcal{H}_{c(1)}$ in $\mathcal{H}_{(1)}$. Thus,

$$(70) \quad \alpha(\varphi_1)(\varphi)(\Gamma, \sigma) = \varphi(\Gamma, \sigma) + \sum \varphi_1(\gamma_{(i)}) \varphi(\Gamma/\gamma_{(i)}, \sigma).$$

This formula shows that the action of G_c on G_0 is just a linear representation of G_c on the vector group G_0 .

A few remarks are in order. First of all, the Hopf algebra of Feynman graphs as presented in [17] is the same as \mathcal{H}_c . As we have seen in theorem 2 and proposition 3, the only nontrivial part in the Hopf algebra \mathcal{H} is \mathcal{H}_c . For instance, the Birkhoff decomposition of a loop $\gamma(z) \in G$, is readily obtained from the Birkhoff decomposition of its homomorphic image, $\gamma_c(z) \in G_c$ since the latter allows to move back to a loop $\gamma_0(z) \in G_0$ with G_0 abelian. Dealing with the Hopf algebra \mathcal{H} allows however to treat oversubtractions and operator product expansions in an effective manner.

Second, one should point out that there is a deep relation between the Hopf algebra \mathcal{H}_c and the Hopf algebra of rooted trees [7, 16, 17]. This is essential for the practitioner of QFT [4]. The relation was first established using the particular structures imposed on the perturbation series by the Schwinger-Dyson equation [16]. Alternative reformulations of this Hopf algebra confirmed this relation [15] in full agreement with the general analysis of [17].

A full description of the relation between the Hopf algebras of graphs and of rooted trees in the language established here will be given in the second part of this paper [8].

3 Renormalization and the Birkhoff decomposition

We shall show that given a renormalizable quantum field theory in D space-time dimensions, the bare theory gives rise, using dimensional regularization, to a loop γ of elements in the group G associated to the theory in section 2. The parameter z of the loop $\gamma(z)$ is a complex variable and $\gamma(z)$ makes sense for $z \neq D$ in a neighborhood of D , and in particular on a small circle C centered at $z = D$. Our main result is that the renormalized theory is just the evaluation at $z = D$ of the holomorphic piece γ_+ in the Birkhoff decomposition,

$$(1) \quad \gamma(z) = \gamma_-(z)^{-1} \gamma_+(z)$$

of the loop γ as a product of two holomorphic maps γ_{\pm} from the respective components C_{\pm} of the complement of the circle C in the Riemann sphere $\mathbb{C}P^1$. As in section 2 we shall, for simplicity, deal with φ^3 theory in $D = 6$

dimensions since it exhibits all the important general difficulties of renormalizable theories which are relevant here.

The loop $\gamma(z)$ is obtained by applying dimensional regularization (Dim. Reg.) in the evaluation of the bare values of Feynman graphs Γ , and our first task is to recall the Feynman rules which associate an integral

$$(2) \quad U_{\Gamma}(p_1, \dots, p_N) = \int d^d k_1 \dots d^d k_L I_{\Gamma}(p_1, \dots, p_N, k_1, \dots, k_L)$$

to every graph Γ .

For convenience we shall formulate the Feynman rules in Euclidean space-time to eliminate irrelevant singularities on the mass shell and powers of $i = \sqrt{-1}$. In order to write these rules directly in d space-time dimensions it is important ([14]) to introduce a unit of mass μ and to replace the coupling constant g which appears in the Lagrangian as the coefficient of $\varphi^3/3!$ by $\mu^{3-d/2} g$. The effect then is that g is dimensionless for any value of d since the dimension of the field φ is $\frac{d}{2} - 1$ in a d -dimensional space-time.

The integrand $I_{\Gamma}(p_1, \dots, p_N, k_1, \dots, k_L)$ contains L internal momenta k_j , where L is the loop number of the graph Γ , and results from the following rules,

$$(3) \quad \text{Assign a factor } \frac{1}{k^2 + m^2} \text{ to each internal line.}$$

$$(4) \quad \text{Assign a momentum conservation rule to each vertex.}$$

$$(5) \quad \text{Assign a factor } \mu^{3-d/2} g \text{ to each 3-point vertex.}$$

$$(6) \quad \text{Assign a factor } m^2 \text{ to each 2-point vertex}_{(0)}.$$

$$(7) \quad \text{Assign a factor } p^2 \text{ to each 2-point vertex}_{(1)}.$$

Again, the 2-point vertex₍₀₎ does not appear in the case of a massless theory. There is moreover an overall normalization factor which depends upon the conventions for the choice of the Haar measure in d -dimensional space. We shall normalize the Haar measure so that,

$$(8) \quad \int d^d p \exp(-p^2) = \pi^{d/2}.$$

This introduces an overall factor of $(2\pi)^{-dL}$ where L is the loop number of the graph, i.e. the number of internal momenta.

The integral (2) makes sense using the rules of dimensional regularization (cf. [5] Chap. 4) provided the complex number d is in a neighborhood of $D = 6$ and $d \neq D$.

If we let σ be the external momenta structure of the graph Γ we would like to define the bare value $U(\Gamma)$ simply by evaluating σ on the test function (2) but we have to take care of two requirements. First we want $U(\Gamma)$ to be a pure number, i.e. to be a dimensionless quantity. In order to achieve this we simply multiply $\langle \sigma, U_\Gamma \rangle$ by the appropriate power of μ to make it dimensionless.

The second requirement is that, for a graph with N external legs we should divide by g^{N-2} where g is the coupling constant.

We shall thus let:

$$(9) \quad U(\Gamma) = g^{(2-N)} \mu^{-B} \langle \sigma, U_\Gamma \rangle$$

where $B = B(d)$ is the dimension of $\langle \sigma, U_\Gamma \rangle$.

Using (3)-(7) this dimension is easy to compute. One can remove all 2-point vertices from the graph Γ without changing the dimension of U_Γ since removing such a vertex removes an internal line and a factor (6), (7) which by (3) does not alter the dimension. Thus let us assume that all vertices of Γ are 3-point vertices. Each of them contributes by $(3 - d/2)$ to the dimension and each internal line by -2 , and each loop by d (because of the integration on the corresponding momenta). This gives

$$(10) \quad \dim(U_\Gamma) = \left(3 - \frac{d}{2}\right) V - 2I + dL.$$

One has $L = I - V + 1$, so the coefficient of d in (10) is $I - \frac{3}{2}V + 1 = 1 - \frac{N}{2}$. (The equality $N = 3V - 2I$ follows by considering the set of pairs (x, y) where $x \in \Gamma^{(0)}$ is a vertex, $y \in \Gamma^{(1)}$ is an internal line and $x \in y$.) The constant term is $3V - 2I = N$ thus,

$$(11) \quad \dim(U_\Gamma) = \left(1 - \frac{N}{2}\right) d + N.$$

We thus get,

$$(12) \quad B = \left(1 - \frac{N}{2}\right) d + N + \dim \sigma.$$

Thus (11) and (12) are valid for arbitrary 1PI graphs (connected) since both sides are unchanged by the removal of all two point vertices.

To understand the factor g^{2-N} in (9) let us show that the integer valued function,

$$(13) \quad \text{Order } \Gamma = V_3 - (N - 2)$$

where V_3 is the number of 3-point vertices of the 1PI connected graph Γ , and N the number of its external lines, does define a grading in the sense of (31) section 2. We need to check that,

$$(14) \quad \text{Order } (\gamma) + \text{Order } (\Gamma/\gamma) = \text{Order } \Gamma$$

using the notations of section 2, with γ connected. If γ is a self energy graph one has $\text{Order } \gamma = v_3$ which is the number of 3-point vertices removed from Γ in passing to Γ/γ , thus (14) follows. If γ is a vertex graph, then $\text{Order } \gamma = v_3 - 1$ which is again the number of 3-point vertices removed from Γ in passing to Γ/γ since γ is replaced by a 3-point vertex in this operation. Thus (14) holds in all cases.

Thus we see that the reason for the convention for powers of g in (9) is to ensure that $U(\Gamma)$ is a monomial of degree $\text{Order } (\Gamma)$ in g .

We extend the definition (9) to disjoint unions of 1PI graphs Γ_j by,

$$(15) \quad U(\Gamma = \cup \Gamma_j) = \prod U(\Gamma_j).$$

One can of course write simple formulas involving the number of external legs and the number of connected components of Γ to compare $U(\Gamma)$ with $\langle \sigma, U_\Gamma \rangle$ as in (9).

Before we state the main result of this paper in the form of Theorem 4 below let us first recall that if it were not for the divergences occurring at $d = D$, one could give perturbative formulas for all the important physical observables of the QFT in terms of sums over Feynman graphs. It is of course a trivial matter to rewrite the result in terms of the $U(\Gamma)$ defined above and we shall just give a few illustrative examples.

The simplest example is the effective potential which is an ordinary function,

$$(16) \quad \phi_c \rightarrow V(\phi_c)$$

of one variable traditionally noted ϕ_c .

The n^{th} derivative $V^{(n)}(0)$ is just given as the sum, with $S(\Gamma)$ the symmetry factor of Γ ,

$$(17) \quad V^{(n)}(0) = \sum_{\Gamma} 1/S(\Gamma) \langle \sigma_0, U_{\Gamma} \rangle$$

where σ_0 is evaluation at 0 external momenta, and where Γ varies through all 1PI graphs with n external momenta and only 3-point interaction vertices.¹⁰ We thus get

$$(18) \quad V^{(n)}(0) = g^{n-2} \mu^{n-d} \left(\frac{n}{2}-1\right) \sum_{\Gamma} \frac{1}{S(\Gamma)} U(\Gamma)$$

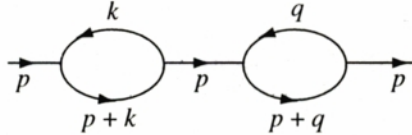
but this expression is meaningless at $d = D$ which is the case of interest.

If instead of evaluating at 0 external momenta one keeps the dependence on p_1, \dots, p_n one obtains the expression for the effective action,

$$(19) \quad \Lambda = \sum_n \frac{1}{n!} \int d^d x_1 \dots d^d x_n \Lambda^{(n)}(x_1, \dots, x_n) \phi_c(x_1) \dots \phi_c(x_n)$$

where $\Lambda^{(n)}(p_1, \dots, p_n)$ is given by (18) evaluated at external momenta given by the p_j 's.

For expressions which involve connected diagrams which are not 1PI, such as the connected Green's functions one just needs to express the bare value of a graph $U_{\Gamma}(p_1, \dots, p_N)$ in terms of the bare values of the 1PI components which drop out when removing internal lines which carry a fixed value (depending only on p_1, \dots, p_N) for their momenta.



In this example we get $U_{\Gamma}(p, -p) = U_{\Gamma_1}(p, -p) U_{\Gamma_2}(p, -p)$ where Γ_j is the one loop self energy graph.

Similarly for expressions such as the Green's functions which involve diagrams which are not connected one simply uses the equality

$$(20) \quad U_{\Gamma_1 \cup \Gamma_2} = U_{\Gamma_1} U_{\Gamma_2}$$

¹⁰The reader should not forget that we committed ourselves to an Euclidean metric, so that appropriate Wick rotations are necessary to compare with results obtained in Minkowski space.

where $\Gamma_1 \cup \Gamma_2$ is the disjoint union of Γ_1 and Γ_2 . In all cases the graphs involved are only those with 3-point interaction vertices and the obtained expressions only contain finitely many terms of a given order in terms of the grading (13). If it were not for the divergences at $d = D$ they would be the physical meaningful candidates for an asymptotic expansion in terms of g for the value of the observable.

Let us now state our main result:

THEOREM 4. a) *There exists a unique loop $\gamma(z) \in G$, $z \in \mathbb{C}$, $|z - D| < 1$, $z \neq D$ whose Γ -coordinates are given by $U(\Gamma)_{d=z}$.*

b) *The renormalized value of a physical observable \mathcal{O} is obtained by replacing $\gamma(D)$ in the perturbative expansion of \mathcal{O} by $\gamma_+(D)$ where*

$$\gamma(z) = \gamma_-(z)^{-1} \gamma_+(z)$$

is the Birkhoff decomposition of the loop $\gamma(z)$ on any circle with center D and radius < 1 .

Proof. To specify the renormalization we use the graph by graph method [3] using dimensional regularization and the minimal subtraction scheme. We just need to concentrate on the renormalization of 1PI graphs Γ . We shall use the notations of [5] to make the proof more readable. Our first task will be to express the Bogoliubov, Parasiuk and Hepp recursive construction of the counterterms $\Gamma \rightarrow C(\Gamma)$ and of the renormalized values of the graphs $\Gamma \rightarrow R(\Gamma)$, in terms of the Hopf algebra \mathcal{H} .

We fix a circle C in \mathbb{C} with center at $D = 6$ and radius $r < 1$ and let T be the projection

$$(21) \quad T : \mathcal{A} \rightarrow \mathcal{A}_-$$

where \mathcal{A} is the algebra of smooth functions on C which are meromorphic inside C with poles only at $D = 6$, while \mathcal{A}_- is the subalgebra of \mathcal{A} given by polynomials in $\frac{1}{z-6}$ with no constant term. The projection T is uniquely specified by its kernel,

$$(22) \quad \text{Ker } T = \mathcal{A}_+$$

which is the algebra of smooth functions on C which are holomorphic inside C .

Thus T is the operation which projects on the pole part of a Laurent series according to the MS scheme. It is quite important (cf. [5] p. 103 and 6.3.1

p. 147) that T is only applied to dimensionless quantities which will be ensured by our conventions in the definition (9) of $U(\Gamma)$.

Being the projection on a subalgebra, \mathcal{A}_- , parallel to a subalgebra, \mathcal{A}_+ , the operation T satisfies an equation of compatibility with the algebra structure of \mathcal{A} , the multiplicativity constraint ([18]),

$$(23) \quad T(xy) + T(x)T(y) = T(T(x)y + xT(y)) \quad \forall x, y \in \mathcal{A}.$$

(By bilinearity it is enough to check it for $x \in \mathcal{A}_\pm, y \in \mathcal{A}_\pm$.)

We let U be the homomorphism,

$$(24) \quad U : \mathcal{H} \rightarrow \mathcal{A}$$

given by the unrenormalized values of the graphs as defined in (9) and (15). It is by construction a homomorphism from \mathcal{H} to \mathcal{A} , both viewed as algebras.

Let us start the inductive construction of C and R . For 1PI graphs Γ without subdivergences one defines $C(\Gamma)$ simply by,

$$(25) \quad C(\Gamma) = -T(U(\Gamma)).$$

The renormalized value of such a graph is then,

$$(26) \quad R(\Gamma) = U(\Gamma) + C(\Gamma).$$

For 1PI graphs Γ with subdivergences one has,

$$(27) \quad C(\Gamma) = -T(\overline{R}(\Gamma))$$

$$(28) \quad R(\Gamma) = \overline{R}(\Gamma) - T(\overline{R}(\Gamma))$$

where the \overline{R} operation of Bogoliubov, Parasiuk and Hepp prepares the graph Γ by taking into account the counterterms $C(\gamma)$ attached to its subdivergences. It is at this point that we make contact with the coproduct (II.6) of the Hopf algebra \mathcal{H} and claim that the following holds,

$$(29) \quad \overline{R}(\Gamma) = U(\Gamma) + \sum_{\substack{\gamma \subset \Gamma \\ \neq}} C(\gamma) U(\Gamma/\gamma).$$

The notations are the same as in II.6. This formula is identical to 5.3.8 b) in [5] p. 104 provided we carefully translate our notations from one case to

the other. The first point is that the recursive definition (27) holds for 1PI graphs and $\gamma = \cup \gamma_j$ is a union of such graphs so in (29) we let,

$$(30) \quad C(\gamma) = \Pi C(\gamma_j).$$

This agrees with (5.5.3) p. 110 in [5].

The second point is that our $C(\gamma)$ is an element of \mathcal{A} , i.e. a Laurent series, while the counterterms used in [5] are in general functions of the momenta which flow through the subdivergence. However since the theory is renormalizable we know that this dependence corresponds exactly to one of the three terms in the original Lagrangian. This means that with the notations of II.6 we have,

$$(31) \quad C_\gamma(\Gamma) = \sum_{(i)} C(\gamma_{(i)}) U(\Gamma/\gamma_{(i)})$$

where $C_\gamma(\Gamma)$ is the graph with counterterms associated to the subdivergence γ as in Collins 5.3.8 b).

To check (31) we have to check that our convention (9) is correct. As we already stressed the power of the unit of mass μ is chosen uniquely so that we only deal for U , C and R with dimensionless quantities so this is in agreement with [5] (2)p. 136. For the power of the coupling constant g it follows from (14) that it defines a grading of the Hopf algebra so that all terms in (31) have the same overall homogeneity. There is still the question of the symmetry factors since it would seem at first sight that there is a discrepancy between the convention (4) p. 24 of [5] and our convention in (9). However a close look at the conventions of [5] (cf. p. 114) for the symmetry factors of $C_\gamma(\Gamma)$ shows that (31) holds with our conventions. We have thus checked that (29) holds and we can now write the BPH recursive definition of C and R as follows, replacing \overline{R} by its value (29) for a 1PI graph Γ in (27), (28),

$$(32) \quad C(\Gamma) = -T \left(U(\Gamma) + \sum_{\substack{\gamma \subset \Gamma \\ \neq}} C(\gamma) U(\Gamma/\gamma) \right)$$

$$(33) \quad R(\Gamma) = U(\Gamma) + C(\Gamma) + \sum_{\substack{\gamma \subset \Gamma \\ \neq}} C(\gamma) U(\Gamma/\gamma).$$

We now rewrite both formulas (32), (33) in terms of the Hopf algebra \mathcal{H} without using the generators Γ . Let us first consider (32) which together with (30) uniquely determines the homomorphism $C : \mathcal{H} \rightarrow \mathcal{A}$. We claim that for any $X \in \mathcal{H}$ belonging to the augmentation ideal

$$(34) \quad \tilde{\mathcal{H}} = \text{Ker } \bar{e}$$

one has the equality,

$$(35) \quad C(X) = -T(U(X) + \sum C(X') U(X''))$$

where we use the following slight variant of the Sweedler notation for the coproduct ΔX ,

$$(36) \quad \Delta X = X \otimes 1 + 1 \otimes X + \sum X' \otimes X'', \quad X \in \tilde{\mathcal{H}}$$

and where the components X', X'' are of degree strictly less than the degree of X . To show that (35) holds for any $X \in \tilde{\mathcal{H}}$ using (30) and (32) one defines a map $C' : \mathcal{H} \rightarrow \mathcal{A}$ using (35) and one just needs to show that C' is multiplicative. This is done in [6] [18] but we repeat the argument here for the sake of completeness. One has, for $X, Y \in \tilde{\mathcal{H}}$

$$(37) \quad \Delta(XY) = XY \otimes 1 + 1 \otimes XY + X \otimes Y + Y \otimes X + XY' \otimes Y'' + Y' \otimes XY'' + X'Y \otimes X'' + X' \otimes X''Y + X'Y' \otimes X''Y''.$$

Thus using (35) we get

$$(38) \quad C'(XY) = -T(U(XY)) - T(C'(X) U(Y) + C'(Y) U(X) + C'(XY') U(Y'') + C'(Y') U(XY'') + C'(X'Y) U(X'') + C'(X') U(X''Y) + C'(X'Y') U(X''Y'')).$$

Now U is a homomorphism and we can assume that we have shown C' to be multiplicative, $C'(AB) = C'(A) C'(B)$ for $\deg A + \deg B < \deg X + \deg Y$. This allows to rewrite (38) as,

$$(39) \quad C'(XY) = -T(U(X) U(Y) + C'(X) U(Y) + C'(Y) U(X) + C'(X) C'(Y') U(Y'') + C'(Y') U(X) U(Y'') + C'(X') C'(Y) U(X'') + C'(X') U(X'') U(Y) + C'(X') C'(Y') U(X'') U(Y'')).$$

Let us now compute $C'(X) C'(Y)$ using the multiplicativity constraint (23) fulfilled by T in the form,

$$(40) \quad T(x) T(y) = -T(xy) + T(T(x) y) + T(x T(y)).$$

We thus get,

$$(41) \quad \begin{aligned} C'(X) C'(Y) = & -T((U(X) + C'(X') U(X'')) (U(Y) + \\ & C'(Y') U(Y'')) + T(T(U(X) + C'(X') U(X'')) (U(Y) + \\ & C'(Y') U(Y'')) + T((U(X) + C'(X') U(X'')) T(U(Y) + C'(Y') U(Y'')))) \end{aligned}$$

by applying (40) to $x = U(X) + C'(X') U(X'')$, $y = U(Y) + C'(Y') U(Y'')$. Since $T(x) = -C'(X)$, $T(y) = -C'(Y)$ we can rewrite (41) as,

$$(42) \quad \begin{aligned} C'(X) C'(Y) = & -T(U(X) U(Y) + C'(X') U(X'') U(Y) \\ & + U(X) C'(Y') U(Y'') + C'(X') U(X'') C'(Y') U(Y'')) \\ & -T(C'(X)(U(Y) + C'(Y') U(Y'')) - T((U(X) + C'(X') U(X'')) C'(Y))). \end{aligned}$$

We now compare (39) with (42), both of them contain 8 terms of the form $-T(a)$ and one checks that they correspond pairwise which yields the multiplicativity of C' and hence the validity of (35) for $C = C'$. We now have a characterization of C independently of any choice of generators of \mathcal{H} and we can rewrite (33) in intrinsic form too,

$$(43) \quad R(X) = \langle C \otimes U, \Delta(X) \rangle \quad \forall X \in \mathcal{H}$$

which can be checked using (33) and the multiplicativity of both sides of (43). It is convenient to use the notation $C \star U$ for the homomorphism $\mathcal{H} \rightarrow \mathcal{A}$ given by,

$$(44) \quad (C \star U)(X) = \langle C \otimes U, \Delta(X) \rangle \quad \forall X \in \mathcal{H}.$$

We are now ready to check that (43) gives the Birkhoff decomposition of the loop $\gamma(z)$, $z \in C$ of elements of the group G of section 2, associated to the homomorphism,

$$(45) \quad U : \mathcal{H} \rightarrow \mathcal{A}.$$

The precise definition of γ is as follows. Each complex number $z \in C$ defines a character of the algebra \mathcal{A} given by,

$$(46) \quad \chi_z(f) = f(z) \quad \forall f \in \mathcal{A}$$

which makes sense since f is smooth on the curve C . Thus $\chi_z \circ U$ is a character of \mathcal{H} and hence (cf. section 2) an element of G ,

$$(47) \quad \gamma(z) = \chi_z \circ U \quad \forall z \in C.$$

Next we can similarly define two other loops with values in G , namely,

$$(48) \quad \gamma_-(z) = \chi_z \circ C, \quad \gamma_+(z) = \chi_z \circ R \quad \forall z \in C.$$

The multiplicativity of both C and R , $\mathcal{H} \rightarrow \mathcal{A}$ ensures that we are dealing with G -valued loops. The equality (43) just means,

$$(49) \quad \gamma_+(z) = \gamma_-(z) \gamma(z) \quad \forall z \in C$$

since (44) is the same as the operation of pointwise product in G for G -valued loops. It remains to check that γ_{\pm} extends to a G -valued map holomorphic in C_+ . By (35) one has $C(\tilde{\mathcal{H}}) \subset \mathcal{A}_-$ and every $z \in C_-$ defines using (46) a character on \mathcal{A}_- with $\chi_{\infty} = 0$ the trivial character. It thus follows that γ_- extends to a G -valued map holomorphic on C_- and such that,

$$(50) \quad \gamma_-(\infty) = 1.$$

Similarly, by (35) one has $T((C \star U)(X)) = 0$ so that $R(\tilde{\mathcal{H}}) \subset \mathcal{A}_+ = \text{Ker } T$. Since every $z \in C_+$ defines using (46) a character of \mathcal{A}_+ we see that γ_+ extends to a G -valued map holomorphic in C_+ . This shows that (49) gives the Birkhoff decomposition of γ and that the renormalized value $R(\Gamma)$ of any 1PI graph is simply obtained by replacing the ill defined evaluation $\gamma(D)$ by $\gamma_+(D)$. ■

Again, a few remarks are in order. First, the above decomposition singles out the use of minimal subtraction together with Dim.Reg. as a favoured approach. From here, one can find the relation to other schemes using the methods of [18]. In [8] we will discuss the global nature of the group G_c , its relation with diffeomorphism groups and with the renormalization group. We shall also discuss the relation between the quantized calculus and the reduction to first order poles implicitly allowed by the above combinatorial structure.

Finally, the reader might expect that the relation to the Riemann-Hilbert problem indicates the presence of a differential equation in z which relates the z dependence of counterterms and renormalized Green's functions to derivations on the Hopf algebra. Such a differential equation can be given and the relation of monodromy to anomalous dimensions will be discussed in [8] as well.

Acknowledgements

D.K. thanks the Clay Mathematics Institute for support during a stay at Lyman Laboratories, Harvard University, and is grateful to the DFG for a Heisenberg Fellowship.

References

- [1] A. Beauville, *Monodromie des systèmes différentiels linéaires à pôles simples sur la sphère de Riemann*, *Séminaire Bourbaki* 45ème année, 1992-1993, n.**765**.
- [2] A. Bolibruch, *Fuchsian systems with reducible monodromy and the Riemann-Hilbert problem*, *Lecture Notes in Math.***1520**, 139-155(1992).
- [3] N. N. Bogoliubov, O. Parasiuk *Acta Math.* **97**, 227 (1957);
K. Hepp, *Commun.Math.Phys.* **2**, 301 (1966).
- [4] D.J. Broadhurst, D. Kreimer, *J. Symb. Comput.***27**, 581 (1999); hep-th/9810087;
D. Kreimer, R. Delbourgo, *Phys. Rev.* **D60**, 105025 (1999); hep-th/9903249.
- [5] J. Collins, *Renormalization*, *Cambridge monographs in math. physics*, Cambridge University Press (1984).
- [6] A. Connes, D. Kreimer, *J.High Energy Phys.***09**, 024 (1999); hep-th/9909126.
- [7] A. Connes, D. Kreimer, *Commun. Math. Phys.* **199**, 203 (1998); hep-th/9808042.
- [8] A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem II: the renormalization and diffeomorphism groups and anomalous dimensions*, in preparation,

- [9] A. Connes and H. Moscovici, *Hopf Algebras, Cyclic Cohomology and the Transverse Index Theorem*, *Commun. Math. Phys.* **193** (1998), 199-246.
- [10] M. Dresden, *Renormalization in historical perspective - The first stage*, in *Renormalization*, ed. L. Brown, Springer-Verlag, New York, Berlin, Heidelberg (1994).
- [11] H. Epstein, V. Glaser, *The role of locality in perturbation theory*, *Ann. Inst. H. Poincaré A* **19** (1973) 211-295.
- [12] K. Gawedzki, A. Kupiainen, *Exact renormalization of the Gross-Neveu model of quantum fields*, *Phys. Rev. Lett.* **54**(1985);
J. Feldman, J. Magnen, V. Rivasseau, R. Seneor, *Massive Gross-Neveu model: a rigorous perturbative construction*, *Phys. Rev. Lett.* **54**(1985).
- [13] J. Glimm, A. Jaffe, *Quantum Physics*, Springer Verlag, New York, Berlin, Heidelberg (1987).
- [14] G. 't Hooft, *Nuclear Physics B* **61** 455 (1973).
- [15] T. Krajewski, R. Wulkenhaar, *Eur.Phys.J.* **C7**, 697-708 (1999); hep-th/9805098.
- [16] D. Kreimer, *Adv. Theor. Math. Phys.* **2.2**, 303 (1998); q-alg/9707029;
A. Connes, D. Kreimer, *Lett. Math. Phys.* **48**, 85 (1999); hep-th/9904044.
- [17] D. Kreimer, *Commun. Math. Phys.* **204**, 669 (1999); hep-th/9810022.
- [18] D. Kreimer, *Adv. Theor. Math. Phys.* **3.3** (1999); hep-th/9901099.
- [19] I. Lappo-Danilevskii, *Mémoires sur la théorie des systèmes des équations différentielles linéaires*, Chelsea, New York (1953).