NON COMMUTATIVE GEOMETRY

AND PHYSICS

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This set of lectures is an introduction to non-commutative geometry, a theory which is fully developed in the forthcoming book [Co]. The general purpose of non-commutative geometry is to adapt the traditional tools of geometry such as measure theory, topology, differential calculus and Riemannian geometry to spaces X which can no longer be comprehended as point sets because of their intrinsically quantum nature. As it turns out this adaptation of traditional tools also leads to some remarkable improvements. In particular, the new quantum calculus, when employed in the context of an ordinary space, turns out to be significantly more powerful than distribution theory.

The purpose of this course however, is not only to provide an introduction to non commutative geometry, but also to survey the interplay between a selection of the latest developments with theoretical physics. In particular, we shall see how experimental physics leads us to accept not only the quantisation of phase space, but also that of the Brillouin zone in solid sate physics and even of space-time itself! Whereas the former is tied up with the remarkable work of Bellissard on the *quantum Hall effect*, the latter involves the present day phenomenological conclusion of particle physics. The achievements to date have been succinctly encapsulated by theorists in a form known as the standard model of particle physics. Based on nothing more than this experimentally driven model, we will discover a certain fine structure for space-time which is neither continuous or discrete, but a subtle mixture of both. Moreover, in the new arena a certain duality becomes apparent between the *strong* and *electroweak* sectors of the model. We shall see in the last section that this duality is precisely that required by the definition of a manifold in the quantum setting.

Whereas the discovery of a quantum structure to space-time already at the electroweak scale might seem surprising to most physicists, it was largely expected—and considered by many to be inevitable—that a radical rethinking of space-time structure is necessary at the Planck scale. Indeed, one of the major points of debate in quantum gravity has always been whether the construction of a successful theory is possible within the general framework of existing physics, or whether it necessarily entails a radical reappraisal of the fundamental concepts of space, time and matter. Because of the specific mathematical structure of general relativity, many physicists believe that the inherent limitations of our current conception of a geometric space lie at the root of some of the most serious technical and conceptual difficulties encountered in its quantisation. As a first step towards the resolution of these problems, we examine in detail the natural notions of metric space and Riemannian manifold offered within the framework of non commutative geometry. Whereas Riemann's conception of a metric space seemed at loggerheads with the basic principles of quantum mechanics from the outset, its new operator algebraic replacement appears well adapted to the quantum regime. This is clear from the computation of the distance between points which now involves functions on the space rather than paths. Moreover, it becomes natural to split up the conformal and metric aspects of the geometry of a space in a way already found essential by string theorists. To take up this suggestive route, we first formulate the Polyakov action of string theory in the framework of non-commutative geometry. Very remarkably, and quite distinct from the classical versions, this reformulation continues to make sense as a conformal action in higher dimensions. In particular, we discover that in dimension four the resulting dynamical theory is closely related to Einstein's theory of gravity. Full details of the involved calculations which led to this tantalizing result are explained within.

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We shall now list a few basic principles of non-commutative geometry which will be developed and tied up with examples from physics in the course of the lectures.

1) *A* space X *corresponds to a (not necessarily commutative) involutive algebra* $\mathcal{A}$*.*

The algebra should be thought of as the algebra of complex valued functions $f : X \to \mathbf{C}$, with involution $f \mapsto \overline{f}$, where

$$\overline{f}(x) = \overline{f(x)}$$

for all $x \in X$. Equivalently $\mathcal{A}$ is the algebra generated by the coordinates on X. The need for allowing the coordinates to fail to commute was discovered by W. Heisenberg, whose motivation drew from the experimental results of spectroscopy. In this case the space X under consideration is the phase space of a simple atomic system.

2) *The* measure theory *of a space* X *is understood through the unitary representations of the associated algebra* $\mathcal{A}$ *in Hilbert space.*

In the classical case a measure μ on X gives rise to the Hilbert space $L^2(X,\mu)$ of square integrable functions on X and to the representation π of the algebra of functions $f \in \mathcal{A}$ given by:

$$(\pi(f)\xi)(x) = f(x)\,\xi(x) \qquad \forall x \in X.$$

In other words each function f on X is represented as a multiplication operator in $\mathcal{H} = L^2(X,\mu)$.

3) *The* topology *of a space X is given by the C^* algebra norm $f \to \|f\|$, $\forall f \in \mathcal{A}$. Such a norm satisfies the fundamental equality:*

$$\|f\|^2 = \|f^* f\| \qquad \forall f \in \mathcal{A}$$

and, when $\mathcal{A}$ is completed, is given by the algebraic formula:

$$\|f\|^2 = \text{ Spectral radius } f^* f = \operatorname{Sup}\{|\lambda|, \lambda \in \mathbf{C}, f^* f - \lambda \text{ is not invertible}\}.$$

The prototype of such a norm for a *classical* space X is:

$$\|f\| = \operatorname{Sup}\{|f(p)|, p \in X\} \quad , \quad \forall f \in C(X)$$

where $C(X)$ is the algebra of continuous functions on the compact space X.

Note that a unitary representation π of $\mathcal{A}$ defines a corresponding norm by:

$$\|f\|_\pi = \|\pi(f)\| \qquad \forall f \in \mathcal{A}$$

where the right hand side is the operator norm:

$$\|T\| = \operatorname{Sup}\{\|T\xi\|; \xi \in \mathcal{H}, \|\xi\| \leq 1\}.$$

One has $\|f^* f\|_\pi = \|f\|_\pi^2 \quad \forall f \in \mathcal{A}$.

4) *Complex vector bundles* over X are given by finite projective modules $\mathcal{E}$ over $\mathcal{A}$.

In other words $\mathcal{E}$ is a $\mathbf{C}$-vector space on which $\mathcal{A}$ acts linearly by $(\xi, a) \in \mathcal{E} \times \mathcal{A} \to \xi a \in \mathcal{E}$ and which after addition of another such module $\mathcal{E}'$ becomes isomorphic to the following "trivial n dimensional module":

$$\mathcal{E}'' = \mathcal{A}^n = \{(\xi_1, \ldots, \xi_n); \xi_j \in \mathcal{A}\},$$

with action

$$(\xi_1, \ldots, \xi_n)a = (\xi_1 a, \ldots, \xi_n a) \qquad \forall \xi_j, a \in \mathcal{A}.$$

If X is a classical space and E a complex vector bundle over X the corresponding projective module $\mathcal{E}$ over the algebra of functions on X is the module of sections of E.

5) *The differential calculus* over X is quantized by the following redefinition of the differential df of a function $f \in \mathcal{A}$:

$$df = [F, f].$$

Here the involutive algebra $\mathcal{A}$ is represented in the Hilbert space $\mathcal{H}$ and F is a selfadjoint operator of square one, $F^2 = 1$, in $\mathcal{H}$. The differential df is thus an *operator* in $\mathcal{H}$, given by the commutator $[F, f] = Ff - fF$.

Let for instance $\mathcal{A}$ be the algebra of functions of one real variable. Take $\mathcal{H} = L^2(\mathbf{R})$ in which $\mathcal{A}$ acts by multiplication, and let F be the Hilbert transform. Then $[F, f]$ is given by the kernel:

$$k(x,y) = \frac{f(x) - f(y)}{x - y} \qquad \forall x, y \in \mathbf{R}.$$

With the new calculus one can perform operations, such as raising $|df|$ to some power, which are not available in distribution theory.

6) The *dimension* of X is governed by the growth of the characteristic values of the differentials:

$$\mu_n = n^{th} \text{ eigenvalue of } |df| \ , \ f \in \mathcal{A}.$$

With p a positive real number, X is of dimension p means

$$\mu_n = O(n^{-1/p}) \ \text{ for any } \ f \in \mathcal{A}.$$

Thus for instance one recovers the usual dimension of manifolds, but the Julia sets of iteration theory inherit their natural fractal dimension.

With p as above, we say that X is of dimension p_- when

$$\sum \mu_n^p < \infty.$$

We shall get examples of spaces $X \subset \mathbf{R}^2$ with positive 2 dimensional Lebesgue measure but of dimension 2_-.

7) *The characteristic classes of* X, are obtained from the homotopy class of F. There is a choice involved in the quantization 5) of the calculus. One chooses a *Fredholm module* $(\mathcal{H}, F)$ over the algebra $\mathcal{A}$. The knowledge of the homotopy class of this Fredholm module is governed by a group $K^*(A)$, the K homology group of the C^*-algebra A completion of $\mathcal{A}$. The knowledge of this group is essential in the construction of $(\mathcal{H}, F)$. Similarly one forms the group $K_*(A)$ generated by stable isomorphism classes of finite projective modules over A and the Fredholm index gives a pairing:

$$K_*(A) \times K^*(A) \to \mathbf{Z}$$

which allows in many cases to detect non zero elements of either groups.

8) *The Chern character* computation of characteristic classes is done thanks to *cyclic cohomology*.

Thus for instance the following formula gives a *cyclic cocycle* on the algebra $\mathcal{A}$, with $p \leq 2k+1$: (p as in 6))

$$\tau(f^0, \dots, f^{2k+1}) = \ \mathrm{Trace}(f^0 \ df^1 \dots df^{2k+1}) \qquad \forall f^j \in \mathcal{A}.$$

Here the ordinary operator trace is applied to the product of the operators $df^j = [F, f^j]$ in $\mathcal{H}$.

This cyclic cocycle computes the index pairing with K theory by formulae which, because they yield *integers*, are remarkably stable under deformations. Let X be an n-dimensional manifold and $\mathcal{A}$ be the algebra of smooth functions on X. Then any cyclic cocycle τ of dimension q on $\mathcal{A}$ is, up to trivial cocycles called coboundaries, of the form:

$$\tau = C_q + SC_{q-2} + S^2 C_{q-4} + \cdots + 0$$

where C_ℓ is a de Rham current of dimension ℓ viewed as the associated multilinear form on $\mathcal{A}$. (Recall that a de Rham current C is a linear form on the space of differential forms of degree ℓ, it thus defines a multilinear form:

$$\widetilde{C}(f^0, f^1, \ldots, f^\ell) = \langle C, f^0\, df^1 \wedge \ldots \wedge df^\ell \rangle \qquad \forall f^j \in \mathcal{A}.$$

In the above formula the operator S is a periodicity operation which associates to a cyclic cocycle of dimension q another one $S\tau$ of dimension $q+2$.

9) The *metric aspect* of the geometry of X is given by the formula:

$$d(\varphi, \psi) = \operatorname{Sup}\{|\varphi(f) - \psi(f)| \;;\; f \in \mathcal{A}, \|[D, f]\| \leq 1\}$$

where φ, ψ are states on the algebra $\mathcal{A}$ and in the simplest case, i.e. when $\mathcal{A}$ is commutative, can be specialized as *points* of X, i.e. as characters of $\mathcal{A}$. (A character χ of a commutative algebra is a homomorphism $\chi : \mathcal{A} \to \mathbb{C}$, the characters of $\mathcal{A} = C(X)$ are exactly given by the maps $f \to f(p)$ where p is a point in X.) In the formula 9) the operator D is a selfadjoint operator in $\mathcal{H}$, with sign equal to F: $\operatorname{Sign}(D) = D|D|^{-1} = F$. The knowledge of F only gives the *conformal* aspect, the full knowledge of D is required for the metric aspect. When X is a Riemannian manifold D is the Dirac operator associated to a Spin structure, and the above formula gives the geodesic distance:

$$d(p, q) = \operatorname{Inf}\{\text{Length } \gamma \;,\; \gamma \text{ a path from } p \text{ to } q\}.$$

The passage from F to $D = F|D|$ is in general given by the formal equality:

$$|D|^{-2} = \tfrac{1}{p-1} \sum (dx^\mu)^*\, g_{\mu\nu}\, dx^\nu$$

where p is the dimension, $dx = [F, x] \quad \forall x \in \mathcal{A}$, the x^μ are generators of $\mathcal{A}$ and the matrix $[g_{\mu\nu}]$ is a positive element of $M_d(\mathcal{A})$, the algebra of matrices over $\mathcal{A}$.

10) The *Lebesgue measure*, or volume form of a Riemannian space, is replaced by the following *trace* on $\mathcal{A}$:

$$f \in \mathcal{A} \to \operatorname{Tr}_\omega (f|D|^{-p}).$$

Here Tr_ω is the Dixmier trace of operators in Hilbert space which belong to the following ideal:

$$\mathcal{L}^{(1,\infty)} = \left\{ T \text{ compact operator }, \sum_0^N \mu_n(T) = 0(\log N) \right\}$$

where $\mu_n(T)$ means the n-th characteristic value of T. Applying this formula for D the Dirac operator, in the Riemannian case, one gets the measure

$$f \mapsto \int f \, dv$$

where $dv = \sqrt{g}\, dx^1 \wedge \ldots \wedge dx^n$ is the volume form. The above formula however makes sense when the space X is a fractal such as a Julia set of Hausdorff dimension $p \in]1,2[$. It yields in that case the following formula for the ***Hausdorff measure*** $d\mu$ on X:

$$\int f \, d\mu = \mathrm{Tr}_\omega \left(f(Z) |dZ|^p \right)$$

where Z is the variable in $\mathbb{C} \supset X$, while the Fredholm module $(\mathcal{H}, F)$ on the algebra $\mathcal{A}$ of functions on X, is defined using boundary values of holomorphic $\frac{1}{2}$ differentials.

11) The ***gauge group*** of second kind is the unitary group $\mathcal{U} = \{u \in \mathcal{A} \; ; \; uu^* = u^*u = 1\}$. In the presence of a "vector bundle over X", i.e. of a finite projective module $\mathcal{E}$ over $\mathcal{A}$ (cf. 4) it becomes:

$$\mathcal{U} = \{u \in \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \; ; \; uu^* = u^*u = 1\}$$

where $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ denotes the (involutive) algebra of endomorphisms of the module $\mathcal{E}$, i.e. of linear maps $T : \mathcal{E} \to \mathcal{E}$ which commute with the right action of $\mathcal{A}$.

12) ***Gauge connections*** are given, (in the simplest case of the module $\mathcal{E} = \mathcal{A}$), by selfadjoint operators in $\mathcal{H}$, of the form:

$$\nabla = D + \sum a_i \, [D, b_i] \qquad a_i, b_i \in \mathcal{A}.$$

They form an affine space on which the gauge group $\mathcal{U}$ acts in an affine manner.

A similar notion is defined for arbitrary hermitian finite projective modules. In the special case of a Riemannian manifold X, with D the Dirac operator, one recovers the usual notion of gauge potential.

The ***curvature*** of a gauge connection is given, with the above notations, by the operator:

$$\theta = \sum \, [D, a_i][D, b_i] + \left(\sum a_i \, [D, b_i] \right)^2.$$

It varies covariantly under gauge transformations, i.e. the latter replaces θ by the operator $u\theta u^*$.

13) ***The Yang-Mills action functional*** is given by

$$YM(\nabla) = \mathrm{Tr}_\omega(\theta^2\ D^{-p})$$

where p is the dimension of X and θ the curvature of ∇. As above Tr_ω is the Dixmier trace.

The action functional of electrodynamics is given by the following $\mathcal{U}$ invariant expression where ∇ is a gauge connection and $\psi \in \mathcal{H}$ is a vector:

$$I(\nabla, \psi) = \mathrm{Tr}_\omega(\theta^2\ D^{-p}) + \langle \nabla\psi, \psi \rangle.$$

The above theory makes sense for many spaces X in which the ordinary concepts of Riemannian geometry do not apply. In particular it applies to the non commutative Brillouin zone of the quantum Hall effect as shown by J. Bellissard. It also applies to spaces X which are mixtures of continuum and discrete spaces, such as products:

$$X = \text{ Continuum } \times \text{ Discrete.}$$

The action of electrodynamics on such product spaces, (with a discrete space formed of 2 points) gives the Glashow Weinberg Salam action of the standard model of particle physics. The strong forces appear from the understanding of the *fundamental class* of X in the K homology of the product $X \times X'$, with X' Poincaré dual to X.

14) The notion of a *manifold* is given by the fundamental class in K-homology which is represented by a specific unbounded Fredholm module over the algebra of coordinates. This is discussed in great detail in section seven of these notes.
Finally the action functional for gravity is simply given by

$$\mathrm{Tr}_\omega(D^{2-p}).$$

For the special case of dimension 4, i.e. $p = 4$ we shall use the above formula, $D^{-2} = \Sigma(dx^\mu)^*\ g_{\mu\nu}\ dx^\nu$, and compute the following action for a map X of a 4-manifold Σ in $\mathbf{R}^d$ with the metric $\eta_{\mu\nu}$:

$$\mathrm{Tr}_\omega(\eta_{\mu\nu}\ dX^\mu\ dX^\nu).$$

As we shall see this action, which reduces to the Polyakov action in dimension 2, will be conformally invariant and intimately related to the Einstein action of gravity in dimension 4.

We shall now review the above points in more detail, neglecting technical or historical matters which are discussed at great length in [Co].

1. The involutive algebra of coordinates on a quantum space X.

We shall begin with our favorite example, that of Heisenberg's discovery of matrix mechanics. Despite the long history of this result, the impact of Heisenberg's message is still not fully appreciated. The real power of this example lies in the intimate relation it ties between experimental results and the need for abandoning the commutativity of coordinates on the phase space of an atomic system. The later discovery of the Schrödinger equation and the illusory relation between the latter and space coordinates for the case of the Hydrogen atom considerably weakened the strength and novelty of Heisenberg's message. With a few exceptions such as Heisenberg's or Tomonaga's book, the courses of quantum mechanics have always emphasized the Hilbert space aspect of the theory instead of the *non commutative* nature of the phase space. Let us now review in modern terminology, how the experimental results of spectroscopy forced Heisenberg to abandon the commutative for the non-commutative, and thereby drove him to his theory of matrix mechanics. (cf. also the deformation theory aspect of quantization [Li]).

In classical mechanics the observable quantities are just functions f on the phase space X. Each point in X corresponds to a pure state of the system and an observable f takes a definite value $f(p)$ at any such point. The fundamental equation is Hamilton's evolution equation:

$$\frac{df}{dt} = \{H, f\}$$

which gives the *time evolution* of any observable quantity f. This equation involves a specific observable H which measures the *energy*, and the Poisson brackets $\{\ \}$ which are derived form the structure of the phase space X.

In the simplest possible case the model obtained is totally integrable. This means that there are sufficiently many constants of motion so that on their specification, the dynamics is reduced to an almost periodic motion. One can thus perform a canonical transformation to a set of *action angle* variables in which the description of the system is considerably simplified. This yields an invariant torus on which the solutions appear as winding orbits and the algebra of observable quantities as the commutative algebra of almost periodic series

$$q(t) = \sum q_{n_1,\dots n_k}\ \exp(2\pi i \langle n, \nu\rangle t).$$

Here $n_i \in \mathbb{Z}$, $\langle n, \nu\rangle = \sum n_i\ \nu_i$ where the ν_i are positive real numbers called the fundamental frequencies.

Let our system X describe a simple atom. Thanks to spectroscopy we know how to analyse the light emitted by such an atom in interaction with the electromagnetic field. The classical model for this interaction will however, as we shall see, yield results which are contradicted by the experimental results of spectroscopy. Indeed with the classical model, the interaction of the atomic system with the electromagnetic field generates an electromagnetic wave whose radiative part is a superposition of plane waves W_n, with frequencies:

$$\langle n, \nu \rangle = \sum n_i \, \nu_i$$

and whose intensity can be computed from the coefficients

$$q^x_{n_1,\dots,n_k} \quad , \quad q^y_{n_1,\dots,n_k} \quad , \quad q^z_{n_1,\dots,n_k}$$

of the three components $q = (q^x, q^y, q^z)$ of a vector valued observable called the *dipole moment* of the atomic system. The formula for the intensity $I_n = dE/dt$ is:

$$I_n = \frac{2}{3c^3} \; |2\pi \langle \nu, n \rangle|^4 \; \left(|q^x_n|^2 + |q^y_n|^2 + |q^z_n|^2 \right).$$

Now this formula implies that the subset of $\mathbf{R}$ given by the emitted frequencies is the subgroup

$$\Gamma = \left\{ \sum n_i \, \nu_i \; ; \; n_i \in \mathbf{Z} \right\} \subset \mathbf{R}.$$

In particular note that the sum of emitted frequencies is still allowed, as well as all the integral multiples or harmonics of a given frequency.

The experimental results of spectroscopy were different however. One of the most important observational rules that emerged from this study is the Ritz-Rydberg combination principle which can be stated as follows.

1. The rays of a spectrum can be labelled by pairs of indices (i, j) (these letters "i" should not be thought of as numbers but mere indices).

2. The set of rays (or spectrum) is naturally endowed with a partially defined law of composition; given three indices (i, j, k) the frequencies ν_{ik} and ν_{kj} combine to yield a new frequency $\nu_{ij} = \nu_{ik} + \nu_{kj}$.

This experimental fact, the Ritz-Rydberg combination principle, was the guiding force behind Heisenberg's discovery of matrix mechanics. In particular this result completely contradicts what one would predict from the above naive planetary model of the atom based on Maxwell's theory of electromagnetism and the dynamics of classical mechanics.

Indeed the emitted frequencies are not parametrized by the group Γ but rather by the set Δ of pairs of indices (i, j). This set is almost a group since we can compose *certain* pairs:

$$(i, k) \circ (k, j) = (i, j)$$

and every element $(i, j) \in \Delta$ has an inverse (j, i). Such a set Δ with a partially defined law of composition that is *associative*, and for which every element has an inverse, is called a *groupoid*.

Hence from the examination of atomic spectra we see the set of observed frequencies is not a group, but rather the *groupoid* specified by the Ritz-Rydberg combination principle. That is to say the set Δ of pairs (i, j) of indices used above in labeling the spectral rays does not form a group, but is nevertheless endowed with a partially defined associative law of composition in which every element (i,j) has an inverse (j,i). Heisenberg's great insight was to call into question the theory of classical mechanics to account for this discrepancy. In particular he proceeded to derive from the Ritz-Rydberg combination principle the precise modifications it entailed for the algebra of physical observables.

To recover his results one must first understand how the commutative algebra of observables of classical mechanics is obtained from the abelian group Γ. Afterall an observable in this case is just a function on the torus and as we have seen can be expanded as an almost periodic series

$$q(t) = \sum q_{n_1\ldots n_k} \exp(2\pi i(n,\nu)t).$$

The coefficients $q_{n_1,\ldots,n_k}$ are labelled by the elements $n = n_1 \ldots n_k$ of the group Γ. This Fourier transform takes the ordinary product of functions into the convolution product, and hence one recovers the algebra of classical observables as the convolution algebra of the group Γ of frequencies. Since Γ is commutative, this algebra is also commutative.

In quantum physics however Γ is not a group but rather the groupoid

$$\Delta = \{(i,j) : i,j \in I\},$$

whose composition rule is specified by experiment to be given by the Ritz-Rydberg combination principle

$$(i,j) \circ (j,k) = (i,k).$$

To recover the algebra of physical observables we must therefore construct the "convolution algebra" of this groupoid. To see what this means, consider the ordinary convolution product abstractly for a moment

$$(ab)(g) = \sum_{g_1 \in G} a(g_1)\, b(g_1^{-1} \cdot g).$$

By setting $g_2 = g_1^{-1} \cdot g$ we may rewrite the formula as:

$$(ab)(g) = \sum_{g_1 g_2 = g} a(g_1)\, b(g_2).$$

From this form, one can make a straightforward generalization to the case of our groupoid Δ.

$$(ab)_{(i,k)} = \sum_{(i,j)\circ(j,k)=(i,k)} a_{(i,j)}\, b_{(j,k)}.$$

In our case it gives exactly the product of matrices and in particular fails to be commutative. Hence, from nothing but the results of experiments, Heisenberg had derived the product of observables to be *non commutative*. In replace of the classical formula

$$q_{n_1\ldots n_k}(t) = q_{n_1\ldots n_k} \exp(2\pi i(n,\nu)t)$$

the evolution law in matrix mechanics is given by

$$q_{(i,j)}(t) = q_{(i,j)} \ \exp(2\pi i \nu_{(i,j)} t)$$

or equivalently:

$$\frac{d}{dt}\, q(t) = \ \frac{i}{\hbar}\, [H, q] \qquad (\hbar = h/2\pi)$$

where H denotes the diagonal matrix

$$H_{(i,i)} = h\nu_i \ \ (\nu_i = \nu_{ij} + \nu_j : i, j \in I),$$

which replaces the Hamiltonian observable in the *quantum theory*. By analogy with classical mechanics one requires the observables, q of position and p of momentum, satisfy

$$[p, q] = i\hbar$$

according to the general rule of replacing Poisson brackets of functions by commutators of matrices. Hence Heisenberg's analysis shows the first concrete instance of a (phase) space turning non commutative.

We shall now give two more examples from physics. The first is the work of J. Bellissard on the Quantum Hall effect.

Let us describe the experimental facts pertaining to the Quantum Hall effect, starting with the classical Hall effect which goes back to 1880 ([Hal]). One considers a very thin strip S of pure metal and a strong magnetic field B, uniform and perpendicular to the strip. Under these conditions elementary classical electrodynamics shows that particles of mass m and charge e in the plane S must move in circular orbits with angular frequency given by the "cyclotron frequency"

$$\omega_c = e\ B/m.$$

Let us view the charge carriers in S as a two dimensional gas of classical charged particles with density N and charge e. Then if an additional electric field E is applied in the plane S, there is a drift of the above circular orbits with velocity E/B in the direction perpendicular to E. The resulting current density j perpendicular to both E and B is such that, in the stationary state the resulting force vanishes, i.e.

$$NeE + j \wedge B = 0.$$

In practice however, the charge carriers are scattered in a time which is short with respect to the cyclotron period. Thus the observed current is mostly in the direction of the electric field E. Nevertheless, a small component in the perpendicular direction called the *Hall current* remains, and is given using the above formula by:

$$j = NeB \wedge E/|B|^2$$

which shows that

$$|j| = (Ne/B)|E|.$$

In other words the Hall conductivity σ_H, i.e. the ratio of the Hall current to the electric potential is to first approximation given by a linear function of N:

$$\sigma_H = Ne/B.$$

As early as 1880 Hall observed the above drift current j and showed that the sign of the charge carriers may be negative or positive depending on the metal considered. This was the first evidence of what is now understood as electron or hole conduction.

In the regime of very low temperatures, $T \sim 1^0\ K$, the effects of quantum mechanics become predominant and can with a gross oversimplification be described as follows. First the two dimensional gas of charge carriers, say of electrons, has a one particle Hamiltonian given by the Landau formula:

$$H = (p - eA)^2/2m$$

where p is the quantum mechanical momentum operator and A is a (classical) vector potential solution of $\mathrm{rot}(A) = B$. Also m is the effective mass of the charge carrier. It is immediate that the operators $K_j = p_j - eA_j$ satisfy the commutation relation:

$$[K_1, K_2] = i\hbar\ e\ B$$

while $H = K^2/2m$. Thus the energy levels of the charge carrier have discrete values which are all integer multiples of Planck's constant times the cyclotron frequency:

$$E_n = n\hbar\ \omega_c.$$

Each of these "Landau levels" is highly degenerate, due to the translation invariance of the system, and can be filled by $\sim eB/h$ charge carriers per unit area.

A naive argument combining the filling of Landau levels with the drift velocity E/B thus allows one to expect that the Hall current density should be given in the quantum regime by

$$j = \frac{eB}{h} \times e \times n\frac{E}{B} = n\ e^2/h\ E$$

where n is the number of filled Landau levels and j is in a direction perpendicular to the electric field. In particular this implies that the Hall conductivity σ_H is an integer multiple $\sigma_H = n\ e^2/h$ of e^2/h provided the Fermi level is just in between two Landau levels. This argument however does not, in any way, account for the existence of the plateaux of conductivity which were discovered experimentally by K. von Klitzing, G. Dorda and M. Pepper ([Kl-D-P]). In their paper the above three authors exhibited the quantization of the Hall conductivity thus giving the possibility of determining the fine structure constant $\alpha = \frac{e^2}{\hbar c}$ with an accuracy comparable to the best available methods.

A first explanation for the integrality of σ_H on the plateaux of vanishing direct conductivity was given by Laughlin in 1981 using the gauge invariance of the one electron Hamiltonian and a special topology for the sample. Then Avron and Seiler put the argument of Laughlin in rigorous mathematical form assuming that, on the plateau, the Fermi level (cf. below) belongs to a gap in the spectrum of the one particle Hamiltonian. This approach is however unsatisfactory in that it not only uses the special topology of the sample, but also fails to account for the role of localized electron states, which are tied up with disorder, and imply that the plateaux cannot correspond to gaps in the spectrum of the one particle Hamiltonian. To explain this more carefully we need to introduce another parameter besides the charge carrier density N, it is called the ***Fermi level*** μ and plays the role of a chemical potential. In the approximation of a free Fermi gas, the thermal average of any observable quantity A at inverse temperature $\beta = 1/kT$ and chemical potential μ is given by:

$$\langle A\rangle_{\beta,\mu} = \lim_{V\to\infty} \frac{1}{|V|} \operatorname{Trace}_V(f(H)A)$$

where Trace_V denotes the local trace of the operators in the finite volume $V \subset S$ and where f is the Fermi weight function:

$$f(H) = \left(1 + e^{\beta(H-\mu)}\right)^{-1}.$$

In general the Fermi level μ is adjusted so as to give the correct value to the charge carrier density:

$$N = \lim_{V\to\infty} \frac{1}{|V|} \operatorname{Trace}_V(f(H)).$$

Note that the right hand side $N(\beta,\mu)$ of this formula is in the limit of 0 temperature, i.e. $\beta \to +\infty$, dependent only on the spectral projection E_μ of H on the interval $]-\infty,\mu]$ and is thus insensitive to the variation of μ in a spectral gap.

In our lectures we shall explain the results of J. Bellissard [Bel] on the existence of the plateaux of conductivity and the integrality of σ_H. For the time being we shall just explain why the Brillouin zone becomes non commutative and restrict our discussion to the case of a periodic crystal.

In their paper [Tho-K-N-dN], Thouless, Kohimoto, Den Nijs and Nightingale investigated the case of a perfectly periodic crystal with the hypothesis that the magnetic flux in units h/e is ***rational***, an obviously unwanted assumption. Their argument shows clearly that the origin of the integrality of σ_H is not the shape of the sample but rather the topology of the so called Brillouin zone in momentum space. When the magnetic flux is irrational this Brillouin zone becomes a non commutative torus $\mathbf{T}^2_\theta$.

Let us take as a model of the metallic strip S the plane $\mathbf{R}^2$ with atoms at each vertex of a periodic lattice $\Gamma \subset \mathbf{R}^2$. The interaction of these atoms with the charge carriers, let us say the electron, modifies the one particle Hamiltonian to:

$$H = H_0 + V \quad , \quad H_0 = (p - eA)^2/2m$$

where the potential V is a Γ-periodic function on $\mathbf{R}^2$. The whole set up is invariant under the group Γ of plane translations belonging to Γ so that we should get a corresponding projective representation of Γ on the one particle quantum mechanical Hilbert space $\mathcal{H}$. We should normally write $\mathcal{H}$ as the space of L^2 sections of a complex line bundle L on $\mathbf{R}^2$ with constant curvature, but this just means that viewing $\mathcal{H}$ as $L^2(\mathbf{R}^2)$ the correct action of the translation group is given by the following unitaries, called magnetic translations:

$$U(X) = \exp\ i(p - eA) \cdot X \qquad \forall X \in \mathbf{R}^2.$$

For $X \in \Gamma$ this unitary commutes with H, but due to the curvature the $U(X)$ do not commute with each other. For the generators e_1, e_2 of Γ we get the commutation relation:

$$U_2 U_1 = \lambda\ U_1 U_2\ ;\ \lambda = \exp\ 2\pi i\theta$$

where $U_j = U(e_j)$ and where θ is the flux of the magnetic field B through a fundamental domain for the lattice Γ, in dimensionless units. The role of the rationality of θ in the Thouless et al paper thus appears clearly since we know that when θ is irrational the von Neumann algebra W in $\mathcal{H}$ of operators which have the symmetries U_ℓ, $\ell \in \Gamma$:

$$W = \{T \in \mathcal{L}(\mathcal{H})\ ;\ U_\ell\, T\, U_\ell^{-1} = T \quad \forall \ell \in \Gamma\}$$

is the hyperfinite factor of type II_∞ namely $R_{0,1}$ (cf. below, measure theory).

In other words if we investigate the operators which obey the natural invariance of the problem we are not in a type I but in a type II_∞ situation. From the measure theory point of view the Brillouin zone is of type II. Moreover the canonical trace τ on the factor W is given, using an averaging sequence V_j of compact subsets of $\mathbf{R}^2$ by:

$$\tau(T) = \lim_{V \to \infty} \frac{1}{|V|}\ \mathrm{Trace}_V(T) \tag{1}$$

so that this part of the thermodynamic limit has a clear interpretation.

Since we need to understand the topology of the non commutative Brillouin zone we need (cf. below, topology) a C^*-algebra $A \subset W$ of observables for our system. By construction any bounded function $f(H)$ belongs to W and in view of the formula giving the statistical average of observables it is natural to require that A contains $f(H)$ for any $f \in C_0(\mathbf{R})$. The obtained algebra so far is commutative and is too small to perform the computation on the Hall conductivity. For that purpose we need another observable which is the *current* j associated to the motion of the charge carrier. This current is a vector, given classically by

$$j = e\ \dot{X}$$

where X is the position of the charge carrier. Thus in quantum mechanics we have:

(2) $$J = e\ i/\hbar\ [H, X]$$

where it is understood that both sides are pairs of operators; i.e. given by their components on a basis of $\mathbf{R}^2$:

$$J_j = e\ i/\hbar\ [H, X_j]$$

with X_j the multiplication operator by the coordinate.

To understand clearly why J is invariant under the symmetries U_ℓ, $\ell \in \Gamma$ we can rewrite the formula (2) as:

(3) $$J = e/\hbar(\partial\ \alpha_s(H))_{s=0}$$

where the group $(\mathbf{R}^2)^\wedge$ dual of $\mathbf{R}^2$ acts by automorphisms α_s, $s \in (\mathbf{R}^2)^\wedge$ on the von Neumann algebra W by:

$$\alpha_s(T) = e^{is\cdot X}\ T\ e^{-is\cdot X} \qquad \forall T \in W.$$

We thus can take J as an observable (except for the trivial fact that since J is unbounded, as for H we need to use $f(J)$, $f \in C_0(\mathbf{R}^2)$).

But in order to compute the Hall conductivity we also need to turn on an electric field E and see how our quantum statistical system reacts. This means that we replace the time evolution given by H, $\sigma_t(a) = e^{itH}\ a\ e^{-itH}$ by the time evolution associated to the perturbed Hamiltonian:

$$H' = H + eE \cdot X$$

or equivalently by the differential equation:

$$\frac{d}{dt}\ \sigma'_t(a) = i/\hbar\ [H, a] + e/\hbar\ \frac{d}{dt}\ \alpha_{tE}(a).$$

This makes it clear that the smallest C^*-algebra of observables appropriate for the computation of the Hall conductivity, besides containing $f(H)$, $f \in C_0(\mathbf{R})$ and $f(J)$, $f \in C_0(\mathbf{R}^2)$, should be invariant under the automorphism group α_s of W. In fact (cf. [Bel]) it is not difficult to see that the C^*-algebra A generated by the $\alpha_s(f(H))$ does contain the functions of the current and is thus the natural algebra of observables for our problem. On $A \subset W$ we have the (semifinite semicontinuous) trace τ coming from the von Neumann algebra W (formula (1)) and the automorphism group (α_s) with generators the derivations:

$$\delta_j = (\partial_j\ \alpha_s)_{s=0}.$$

We shall come back to this algebra later on but we already mention that it corresponds to a very nice and simple non commutative space: the non commutative two torus $\mathbf{T}^2_\theta$.

We shall now pass to an example of a somewhat different nature. Rather than phase space or the Brillouin zone coordinates, we now consider a modification of the algebra of coordinates of space-time itself. For reasons of convenience it will be better to deal with imaginary time, i.e. with Euclidean space time. In many parts of quantum field theory the space time points $x \in X$ only play the role of labels or indices for the quantum fields $\varphi(x)$. This is not so however in two fundamental instances:

α) The actual *local* form of the Lagrangian

β) The invariance of the theory under the group $\mathcal{G}$ of gauge transformations of second kind (local gauge transformations).

We shall discuss α) in more detail below, but let us concentrate on β). Here the group $\mathcal{G}$ appears as a natural symmetry group of the quantum theory, the question we shall answer is:

Given $\mathcal{G}$ as an abstract (topological) group, what information do we have about space time X.

Now recall that $\mathcal{G}$ is the group of local gauge transformations, i.e. of maps from X to a fixed compact group G, (the global one):

$$\mathcal{G} = \text{Map}(X, G) \qquad G \text{ compact Lie group.}$$

In particular the constant maps give us a natural inclusion:

$$G \subset \mathcal{G}.$$

We shall now show that if $G = U(n)$, $n \geq 2$, then the knowledge of the above inclusion of groups: $G \subset \mathcal{G}$ gives back in a natural manner the algebra $\mathcal{A}$ of functions on X, from which (as we shall see later) the space X is uniquely recovered. We shall ignore the technical details on which exact smoothness is assumed on elements of $\mathcal{G} = \text{Map}(X, G)$ and proceed as follows. We take $n = 2$ for simplicity. The Lie algebra $\text{Lie}(\mathcal{G}) = \mathcal{L}$ is a linear space which contains by hypothesis the Lie algebra L of $U(2)$. Thus $\mathcal{L}_{\mathbf{C}} \supset L_{\mathbf{C}}$ and we identify the complexified Lie algebra $L_{\mathbf{C}}$ with the Lie algebra of 2×2 complex matrices with bracket $[X, Y] = XY - YX$. In particular we let $e_{ij} \in M_2(\mathbf{C})$ be the natural matrix units, thus $e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ for instance. Let $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in L_{\mathbf{C}}$. Then one checks that the linear space

$$\mathcal{A} = \{\xi \in \mathcal{L}_{\mathbf{C}}\ ;\ [\gamma, \xi] = 2\xi\}$$

with the product given by the equality:

$$\xi_1 \cdot \xi_2 = [[[\xi_1, e_{21}], e_{11}], \xi_2]$$

is an algebra isomorphic to the algebra of functions on X with the pointwise product:

$$(f_1 f_2)(p) = f_1(p)\ f_2(p).$$

Thus the gauge group of second kind contains a great deal of information about the structure of space time, since the latter can be recovered as the spectrum of the algebra $\mathcal{A}$. Now a group (or equivalently a Lie algebra) is a mathematical object quite different from an associative algebra. For instance given two representations π_1, π_2 of a group G (or of a Lie algebra) we can form their tensor product $\pi_1 \otimes \pi_2$ given by

$$(\pi_1 \otimes \pi_2)(g) = \pi_1(g) \otimes \pi_2(g) \qquad \forall g \in G$$

and this yields a representation of G.

On the contrary, if $\mathcal{A}$ is an algebra and π_1, π_2 are two representations of $\mathcal{A}$ then $\pi_1 \otimes \pi_2$ is a representation of $\mathcal{A} \otimes \mathcal{A}$ but not in general a representation of $\mathcal{A}$ itself. There is a natural manner of converting an associative algebra $\mathcal{A}$ into a group, one replaces $\mathcal{A}$ by the group

$$GL_1(\mathcal{A}) = \{a \in \mathcal{A}\ ;\ a \text{ is invertible}\}.$$

When $\mathcal{A}$ is involutive its unitary group is given by:

$$\mathcal{U}_1(\mathcal{A}) = \{u \in \mathcal{A}\ ;\ uu^* = u^*u = 1\}.$$

More generally, using the algebra $M_n(\mathcal{A})$ of matrices over $\mathcal{A}$ yields the groups $GL_n(\mathcal{A})$ and $\mathcal{U}_n(\mathcal{A})$ for every integer n. Of course not all groups G are of the form GL_1 or $\mathcal{U}$ for some algebra $\mathcal{A}$. When they are they inherit a lot of interesting properties from $\mathcal{A}$ as shown by algebraic K theory.

It is the above relation between the gauge group of second kind and the algebra of coordinates on space time which will, together with the very specific Lagrangian of the standard model, dictate the following modification of the latter algebra:

$$\mathcal{A} = \text{ algebra of functions on 4 dim continuum } \otimes (\mathbf{C} \oplus \mathbf{H}).$$

Here $\mathbf{C} \oplus \mathbf{H}$ is the algebra of pairs (λ, q) where $\lambda \in \mathbf{C}$ is a complex number, $q \in \mathbf{H}$ is a quaternion and where the product of two such pairs is given by:

$$(\lambda_1, q_1)\ (\lambda_2, q_2) = (\lambda_1 \lambda_2, q_1 q_2).$$

2. Measure theory and representations.

Let X be a quantum space and $\mathcal{A}$ the involutive algebra which plays the role of the algebra of coordinates on X. The first step in the analysis of X is the understanding of its measure theory which as we shall see, is intimately tied up with the involutive representations of $\mathcal{A}$ as an algebra of operators in Hilbert space.

The notion of a ***probability measure*** on X carries over to the non commutative case as follows:

Definition 1. *A state on $\mathcal{A}$ is a linear form $\varphi : \mathcal{A} \to \mathbb{C}$ such that*

α) $\varphi(a^* a) \geq 0 \ \ \forall a \in \mathcal{A}$

β) $\varphi(1) = 1$.

The first condition is called ***positivity*** and the second is just a normalization condition.

As a first example take $\mathcal{A} = C(X)$ the algebra of continuous functions on a compact space X with the pointwise multiplication and the involution defined by $f^*(x) = \overline{f(x)}$ $\forall x \in X$. Then the states on $\mathcal{A}$ correspond exactly to the probability measures μ on X by the equality:

$$\varphi(f) = \int_X f \, d\mu \qquad \forall f \in C(X).$$

As a next example take $\mathcal{A} = M_N(\mathbb{C})$ the matrix algebra as obtained in matrix mechanics from the discovery of Heisenberg. Then the states on $\mathcal{A}$ correspond exactly to ***density matrices*** ρ, i.e. to selfadjoint matrices with positive eigenvalues and trace 1, by the equality:

$$\varphi(T) = \text{Trace}(\rho T) \qquad \forall T \in M_N(\mathbb{C}).$$

A state is called ***pure*** iff it cannot be written as a non trivial mixture $\varphi = \lambda_1 \varphi_1 + \lambda_2 \varphi_2$, $\lambda_i > 0$ of two distinct states. Thus in the first example $\mathcal{A} = C(X)$ the pure states correspond to the points of X, to each $x \in X$ one assigns the Dirac mass $\delta_x = \mu$ at this point. They are already more interesting in the second example: $\mathcal{A} = M_N(\mathbb{C})$. In this case they correspond exactly to the rays in the Hilbert space $\mathbb{C}^N$ in which $M_N(\mathbb{C})$ acts. To any such ray $\mathbb{C}\xi$, $\xi \in \mathbb{C}^N$, $\xi \neq 0$, there corresponds an orthogonal projection e of rank one with range $\mathbb{C}\xi$. The operator e is a density matrix of trace one and hence a state φ on $\mathcal{A}$:

$$\varphi(T) = \text{Trace}(eT) = \langle T\xi, \xi \rangle \qquad (\text{with } \|\xi\| = 1).$$

In the formalism of quantum mechanics the rays in Hilbert space correspond to quantum mechanical states of the system. The more general mixed states (i.e. not pure) correspond to quantum statistical mechanics.

We shall now explain first what to do once one has a state on $\mathcal{A}$ and then how to choose interesting states on a given involutive algebra.

The GNS construction.

Let $\mathcal{A}$ be an involutive algebra (with unit) and φ a state on $\mathcal{A}$. Then the ***positivity*** condition α) immediately gives us a Hilbert space $\mathcal{H}_\varphi$, namely the completion of the linear space $\mathcal{A}$ for the inner product:

$$\langle x, y\rangle = \varphi(y^* x) \qquad \forall x, y \in \mathcal{A}.$$

Next every element $a \in \mathcal{A}$ defines an operator $\pi(a)$ in $\mathcal{H}_\varphi$ where the domain of $\pi(a)$ is $\mathcal{A} \subset \mathcal{H}_\varphi$ and where

$$\pi(a)x = ax \qquad \forall x \in \mathcal{A}.$$

The equality:

$$\langle \pi(a)x, y\rangle = \langle x, \pi(a^*)y\rangle \qquad \forall x, y \in \mathcal{A}$$

follows from $\varphi(y^* ax) = \varphi((a^* y)^* x)$. It shows that the densely defined operator $\pi(a)$ has a densely defined adjoint: $\pi(a^*)$ and hence is closable. We could then already define a von Neumann algebra $M = \pi(\mathcal{A})''$ in $\mathcal{H}_\varphi$ as the commutant of the group $\mathcal{U}$ of unitaries in $\mathcal{H}_\varphi$ which commutes with all the operators $\overline{\pi(a)}$, closures of $\pi(a)$.

But in fact it is a natural hypothesis that the operators $\pi(a)$ are all bounded in $\mathcal{H}_\varphi$. This is implied for instance by the following condition:

$$\forall a \in \mathcal{A}\ ,\ \exists b \in \mathcal{A} \text{ such that } a^* a + b^* b = \lambda 1,$$

for some $\lambda \in \mathbf{R}_+$.

This condition means that we are dealing with bounded observables or equivalently bounded functions on our space X. There are simple general formulae such as

$$f \to e^{itf} \quad , \quad f \to f(1 + f^* f)^{-1/2}$$

which allow to pass from a selfadjoint unbounded element to a bounded one.

We shall thus let $\pi(\mathcal{A})$ be the involutive algebra of operators in $\mathcal{H}_\varphi$ of the form $\pi(a)$ for some $a \in \mathcal{A}$.

Definition 2. *Let $\mathcal{H}$ be a Hilbert space. A von Neumann algebra M in $\mathcal{H}$ is an involutive subalgebra of $\mathcal{L}(\mathcal{H})$, the algebra of bounded operators in $\mathcal{H}$, closed in the* weak topology.

We refer to the lectures by O. Lanford in Les Houches 1970 for a thorough introduction to this notion. We just recall that a net T_α of operators converges *weakly* to T iff

$$\langle T_\alpha\, \xi, \eta\rangle \to \langle T\xi, \eta\rangle \qquad \forall \xi, \eta \in \mathcal{H}.$$

The bicommutant theorem of von Neumann characterizes von Neumann algebras as those sets of operators in $\mathcal{H}$ which obey a given symmetry group $\mathcal{U}$:

$$M = \{T \in \mathcal{L}(\mathcal{H}) \ , \ VTV^* = T \quad \forall V \in \mathcal{U}\}.$$

When $\mathcal{U}$ varies among subsets equivalently (subgroups) of the unitary group one obtains all von Neumann algebras in $\mathcal{H}$.

Now any involutive subalgebra $\mathcal{B}$ of $\mathcal{L}(\mathcal{H})$ generates a von Neumann algebra M, the latter is equivalently the weak closure of $\mathcal{B}$ or the double commutant of $\mathcal{B}$:

$$B'' = \{T \in \mathcal{L}(\mathcal{H}) \ ; \ VTV^* = T \quad \forall V \text{ unitary such that } VxV^* = x \quad \forall x \in B\}.$$

Passing from $\mathcal{B}$ to $\mathcal{B}''$, one looses some information.

Proposition 3. (GNS) *Let $M = \pi(\mathcal{A})''$ be the weak closure of $\pi(\mathcal{A})$, then M is a von Neumann algebra, $\pi : \mathcal{A} \to M$ a $*$ homomorphism with dense range, and one has a vector $\xi_0 \in \mathcal{H}_\phi = \mathcal{H}$ such that:*

$$\varphi(a) = \langle \pi(a)\xi_0, \xi_0 \rangle \qquad \forall a \in \mathcal{A}.$$

Now on M we can define a state $\widetilde{\varphi}$ by the equality

$$\widetilde{\varphi}(T) = \langle T\xi_0, \xi_0 \rangle \qquad \forall T \in M.$$

This state is by construction continuous in the *weak topology*. We shall now show that in the *commutative* case we are back to the set up of the Lebesgue measure theory, a necessary step in order to understand the meaning of the general non commutative case.

The commutative case.

Let us assume that $\mathcal{A}$ is commutative. Thus $M = \pi(\mathcal{A})''$ is also commutative and we need to understand all the triples $(\mathcal{H}, M, \xi_0)$ where

a) $\mathcal{H}$ is a Hilbert space

b) M is a commutative von Neumann algebra

c) ξ_0 is a unit vector in $\mathcal{H}$ such that $\overline{M\xi_0} = \mathcal{H}$.

We assume for simplicity that $\mathcal{H}$ has a *countable* basis. The classical result (cf. [Di$_2$]) describes all triples satisfying the conditions a) b) c') where the weaker condition c'), implied by c) using the commutativity of M (i.e. $M \subset M'$) is:

c') $\overline{M'\xi_0} = \mathcal{H}$.

Theorem 4. *Let $I = [0,1]$ be the standard Borel space. Then any such triple $(\mathcal{H}, M, \xi_0)$ is of the following form:*

$$\mathcal{H} = L^2(I, \mu, N) \ \ , \ \ M = L^\infty(I, \mu) \ , \ \xi_0 = 1$$

where μ is a Borel probability measure on I and N a measurable map from I to $\{1, 2, \ldots, \infty\}$.

We need to explain the meaning of our notations. We denote as usual by $L^2(I,\mu)$ the Hilbert space of square integrable functions on I for the measure μ, but we use the integer valued function N to introduce multiplicity. In other words on the subset $E_k = \{t \in I \,;\, N(t) = k\}$ we replace $L^2(E_k,\mu)$ by the direct sum of k copies of $L^2(E_k,\mu)$. Equivalently $\mathcal{H} = L^2(I,\mu,N)$ is the space of measurable sections of the measurable bundle of Hilbert spaces whose fiber is constant on E_k and equal to $\ell^2(\{1,\ldots,k\})$. One only takes square integrable sections, i.e. such that:

$$\|\xi\|^2 = \int_I \|\xi(x)\|^2 \, d\mu(x) < \infty.$$

The bounded measurable functions $f \in L^\infty(I,\mu)$ act by multiplication:

$$(f\xi)(x) = f(x)\ \xi(x) \qquad \forall \xi \in \mathcal{H}\ ,\ x \in I.$$

Finally ξ_0 is any measurable section such that $\|\xi_0(x)\| = 1$ for any $x \in I$, thus:

$$\langle f\xi_0, \xi_0\rangle = \int f\, d\mu \qquad \forall f \in L^\infty(I,\mu).$$

In the statement of the theorem one can replace the interval I by any standard Borel space X, which shows that measure theory gives very little information about a space X.

The multiplicity function N appears algebraically from the commutant $M' = \{T \in \mathcal{L}(\mathcal{H}) \,;\, Tf = fT \quad \forall f \in M\}$ and one easily checks that over E_k the commutant of $L^\infty(E_k,\mu)$ is $L^\infty(E_k,\mu) \otimes M_k(\mathbb{C})$ (with $\mathcal{L}(\ell^2)$ instead of $M_k(\mathbb{C})$ for $k = \infty$). Thus the commutant M' of M is the direct sum of von Neumann algebras:

$$M' = \bigoplus_{1}^{\infty} L^\infty(E_k,\mu) \otimes M_k(\mathbb{C}).$$

Under the stronger hypothesis c) one has $M' = M$ and the multiplicity function N is equal to 1.

The above classical result has many corollaries such as the spectral theorem, the Borel functional calculus i.e. the (unique) definition of $f(T)$ for any Borel function $f : \mathbb{C} \to \mathbb{C}$ where T is a normal operator: $TT^* = T^*T$. It also shows that the von Neumann algebra M generated by such an operator T is:

$$M = \{f(T) \,;\, f \text{ Borel bounded}\} =$$
$$\{S \in \mathcal{L}(\mathcal{H}) \,;\, USU^* = S \text{ for any unitary } U \text{ such that } UTU^* = T\}.$$

The modular theory.

Let us now pass to the general case. We let $M = \pi(\mathcal{A})''$. It is a von Neumann algebra in $\mathcal{H}$, and the triple $(\mathcal{H}, M, \xi_0)$ satisfies a) b) c) except for the commutativity of M.

Let M' be the commutant of M and consider the subspace $\mathcal{H}_0 = \overline{M'\xi_0}$ spanned by the vectors $T\xi_0$, $T \in M'$. It contains ξ_0 and by construction the orthogonal projection P on $\mathcal{H}_0$ belongs to the commutant of M', i.e. to M. The state:

$$\varphi(a) = (a\xi_0, \xi_0) \qquad \forall a \in M$$

vanishes on $a - PaP$ for any $a \in M$ and thus, to understand it, one can focus on the ***reduced*** von Neumann algebra M_p:

$$M_p = \{a \in M \ ; \ aP = Pa = a\}.$$

One then lets M_p act in $\mathcal{H}_0$, in which the original vector ξ_0 is *cyclic* and *separating*:

cyclic means $M_p \ \xi_0$ is dense in $\mathcal{H}_0$

separating means $(M_p)'\xi_0$ is dense in $\mathcal{H}_0$.

For notational simplicity we now drop the suffix P and the 0. The theory starts with the following key result of M. Tomita ([T₂]):

Theorem 5. *Let M be a von Neumann algebra in $\mathcal{H}$ and ξ_0 a cyclic and separating vector for M. Then the operator with domain $M\xi_0$ which transforms $x\xi_0$ to $x^*\xi_0 \quad \forall x \in M$, is* closable. *Its closure S has a polar decomposition $S = J\Delta^{1/2}$ with Δ positive and J antilinear of square* 1 *and one has:*

1) $\Delta^{it} \ M \ \Delta^{-it} = M \quad \forall t \in \mathbf{R}$

2) $JMJ = M'$.

The operator $\Delta = S^*S$ is called the *modular* operator and the one parameter group of automorphisms of M given by

$$\sigma_t(x) = \Delta^{it} \ x \ \Delta^{-it} \qquad \forall x \in M \ , \ t \in \mathbf{R}$$

is called the modular automorphism group of M. It only depends upon the state φ and makes sense for any *faithfull normal* state on M where:

faithful means that $\varphi(x^*x) > 0$ if $x \neq 0$

normal means that the restriction of φ to the unit ball of M is weakly continuous. Equivalently it means that if $e_\alpha \in M$ is an increasing net of projections, $e_\alpha = e_\alpha^2 = e_\alpha^* \in M$ one has $\varphi(\vee e_\alpha) = \ \mathrm{Sup} \ \varphi(e_\alpha)$ where $\vee e_\alpha$ is the least projection $e \in M$, $e \geq e_\alpha$ for all α.

One then lets σ_t^φ be the modular automorphism group associated to φ.

The next result, due to M. Takesaki and M. Winnink characterizes the one parameter group σ_t^φ as the unique time evolution of M satisfying the Kubo Martin Schwinger condition (KMS) as formulated by Haag Hugenholtz and Winnink ([H-H-W]):

Theorem 6. *Let M be a von Neumann algebra, φ a faithful normal state on M then σ_t^φ is the unique one parameter group of automorphisms of M satisfying the KMS condition relative to φ.*

We have used the following notion:

Definition 7. *Let $\mathcal{A}$ be an involutive algebra, φ a state on $\mathcal{A}$ and $\alpha_t \in Aut\ \mathcal{A}$ a one parameter group of automorphisms of $\mathcal{A}$. Then the pair (α, φ) satisfies the KMS condition iff for any $a, b \in \mathcal{A}$ there exists a bounded holomorphic function $F_{a,b}(z)$, $0 \leq Im\ z \leq 1$ such that:*

$$F_{a,b}(t) = \varphi(a\ \alpha_t(b))\ ,\ F_{a,b}(t+i) = \varphi(\alpha_t(b)a) \qquad \forall t \in \mathbf{R}.$$

We shall come back later to the role of the KMS condition in the choice of states on $\mathcal{A}$.

The next key result is the analogue, in this general non commutative measure theory context, of the classical Radon Nikodym theorem. In the usual Radon Nikodym theorem the derivative $d\mu/d\nu$ of one measure with respect to another is a, generally unbounded, positive function h. If moreover both μ and ν are equivalent the function h does not vanish so that the one parameter group of unitaries

$$u_t = h^{it} \qquad \forall t \in \mathbf{R}$$

is well defined and characterizes h uniquely.

The non commutative case is more subtle and involves the following 1-cocycle condition:

$$u_{t_1+t_2} = u_{t_1}\ \sigma_{t_1}^{\varphi}(u_{t_2}) \qquad \forall t_1, t_2 \in \mathbf{R}.$$

Theorem 8. ([Co]) *Let M be a von Neumann algebra and φ, ψ be faithful normal states on M. There exists a canonical unitary 1-cocycle $u_t \in M$, such that*

$$\sigma_t^{\psi}(x) = u_t\ \sigma_t^{\varphi}(x)\ u_t^* \qquad \forall t \in \mathbf{R}\ ,\ \forall x \in M.$$

This canonical cocycle is denoted by $u_t = (D\psi : D\varphi)_t$.

Moreover, $\sqrt{-1}\,\left(\frac{d}{dt}\ u_t\right)_{t=0}$ coindices

1) in the commutative case, with the logarithm of the Radon-Nikodym derivative $(d\psi/d\varphi)$;

2) in the case of statistical mechanics, with the difference of the Hamiltonians corresponding to two equilibrium states, or the relative Hamiltonian of H. Araki [Ar$_2$].

It follows that, given a von Neumann algebra M, there exists a canonical homomorphism δ of $\mathbf{R}$ into the group $\mathrm{Out}M = \mathrm{Aut}M/\mathrm{Int}M$ (the quotient of the automorphism group by the normal subgroup of inner automorphisms), given by the class of σ_t^{φ} independently of the choice of φ. Thus, $\mathrm{Ker}\ \delta = T(M)$ is an invariant of M, as is $Sp\ \delta = S(M) = \bigcap_{\varphi}\ Sp\ \Delta\varphi$.

Thus von Neumann algebras are *dynamical* objects. Such an algebra possesses a group of automorphism classes parametrized by $\mathbf{R}$. This group, which is completely canonical, is a manifestation of the non commutativity of the algebra M. It has no counterpart in the commutative case and attests to the originality of non commutative measure theory with respect to the usual theory.

The classification of factors.

We now have a complete classification of hyperfinite von Neumann algebras. I refer to [Co] for a detailed history of this work. I shall just give here the relevant notion and a drawing of the result.

A von Neumann algebra M is of type I iff it is isomorphic to the commutant of a commutative von Neumann algebra. Thus Theorem 4 gives us the list of type I von Neumann algebras, namely the algebras M' of this theorem.

The class of ***hyperfinite*** von Neumann algebras is obtained by ***monotone closure*** from the type I. In other words, the class of hyperfinite von Neumann algebras is stable under:

α) Decreasing intersection $\cap M_\alpha$

β) Increasing union $\overline{\cup M_\alpha}$ (weak closure)

and it is the smallest such class containing type I. Moreover all representations of nuclear C^*-algebras, of connected locally compact groups, of amenable discrete groups generate only hyperfinite von Neumann algebras. The commutant of a hyperfinite von Neumann algebra is hyperfinite.

Moreover, the classification of hyperfinite von Neumann algebras reduces to that of the hyperfinite factors on writing $M = \int M_t \, d\mu(t)$, where each M_t is a factor, that is, having center equal to $\mathbb{C}$. Finally, the list of hyperfinite factors is as follows:

I_n $M = M_n(\mathbb{C})$.

I_∞ $M = \mathcal{L}(\mathcal{H})$, the algebra of all operators on an infinite-dimensional Hilbert space.

II_1 $R =$ Cliff(E), the Clifford algebra of an infinite-dimensional Euclidean space E.

II_∞ $R_{0,1} = R \otimes \mathrm{I}_\infty$.

III_λ $R_\lambda =$ the Powers factors ($\lambda \in \,]0,1[$).

III_1 $R_\infty = R_{\lambda_1} \otimes R_{\lambda_2}$ ($\forall \lambda_1, \lambda_2, \lambda_1/\lambda_2 \notin \mathbb{Q}$), the Araki-Woods factor.

III_0 R_W, the Krieger factor associated with an ergodic flow W.

Since we are in a region of great mountains I shall try a drawing of the results as follows:

The KMS condition and Quantum statistical mechanics.

A cubic centimeter of water contains a considerable number of molecules of water agitated by an incessant movement. The detailed description of the motion of each molecule is not necessary, any more than is the precise knowledge of the microscopic state of the system, for determining the results of macroscopic observations. In classical statistical mechanics, a microscopic state of the system is represented by a point of the phase space, which is of dimension $6N$ for N point molecules. A statistical state is described not by a point of the phase space but by a measure μ on that space, a measure that associates with each observable f its mean value:

$$\int f \, d\mu.$$

For a system that is maintained at fixed temperature by means of a thermostat, the measure μ is called the Gibbs' canonical ensemble: it is given by a formula that invokes the Hamiltonian H of the system and the Liouville measure that arises from the symplectic structure of the phase space. One sets

$$(1) \qquad d\mu = \frac{1}{Z} \, e^{-\beta H} \cdot \text{ Liouville measure.}$$

where $\beta = 1/kT$, T being the absolute temperature and k the Boltzmann constant, whose value is approximately 1.38×10^{-23} joules per degree Kelvin, and where Z is a normalization factor.

The thermodynamic quantities, such as the entropy or the free energy, are calculated as functions of β and a small number of macroscopic parameters introduced in the formula that gives the Hamiltonian H. For a finite system, the free energy is an analytic function of these parameters. For an infinite system, some discontinuities appear that correspond to the phase transition phenomenon. The rigorous proof, starting with the mathematical formula that specifies H, of the absence or existence of these discontinuities is a difficult branch of mathematical analysis.

However, as we have seen, the microscopic description of matter cannot be carried out without quantum mechanics. Let us consider, to fix the ideas, a solid having an atom at each vertex of a crystal lattice $\mathbf{Z}^3$. The algebra of observable physical quantities associated with each atom $x = (x_1, x_2, x_3)$ is a matrix algebra Q_x, and if we assume for simplicity that these atoms are of the same nature and can only occupy a finite number n of quantum states, then $Q_x = M_n(\mathbf{C})$ for every x. Now let Λ be a finite subset of the lattice. The algebra Q_Λ of observable physical quantities for the system formed by the atoms contained in Λ is given by the tensor product $Q_\Lambda = \otimes_{x\in\Lambda} Q_x$.

The Hamiltonian H_Λ of this finite system is a self-adjoint matrix that is typically of the form

$$H_\Lambda = \sum_{x\in\Lambda} H_x + \lambda H_{\text{int}},$$

where the first term corresponds to the absence of interactions between distinct atoms and where λ is a coupling constant that governs the intensity of the interaction. A statistical state of the finite system Λ is given by a linear form φ that associates with each observable $A \in Q_\Lambda$ its mean value $\varphi(A)$ and which has the same positivity and normalization properties as a probability measure μ, namely,

a) Positivity: $(\forall A \in Q_\Lambda)$;

b) Normalization: $\varphi(1) = 1$.

If the system is maintained at fixed temperature T, the equilibrium state is given by the quantum analogue of the above formula (1):

$$\varphi_\Lambda(A) = \frac{1}{Z} \operatorname{trace}(e^{-\beta H_\Lambda} A) \quad (\forall A \in Q_\Lambda),$$

where the unique trace on the algebra Q_Λ replaces the Liouville measure.

As in classical statistical mechanics, the interesting phenomena appear when one passes to the thermodynamic limit, that is, when $\Lambda \to \mathbf{Z}^3$. A state of the infinite system being given by the family (φ_Λ) of its restrictions to the finite systems indexed by Λ, one obtains in this way all of the families such that

a) for every Λ, φ_Λ is a state on Q_Λ;

b) if $\Lambda_1 \subset \Lambda_2$ then the restriction of φ_{Λ_2} to Q_{Λ_1} is equal to φ_{Λ_1}.

In general, the family φ_Λ defined above by means of $\exp(-\beta H_\Lambda)$ does not satisfy the condition b) and it is necessary to better understand the concept of state of an infinite system. This is where C^*-algebras make their appearance. In fact, if one takes the inductive limit Q of the finite-dimensional C^*-algebras Q_Λ, one obtains a C^*-algebra that has the following property:

An arbitrary state φ on Q is given by a family (φ_Λ) satisfying the conditions a) *and* b).

Thus, the families (φ_Λ) satisfying a) and b), that is the states of the infinite system, are in natural bijective correspondence with the states of the C^*-algebra Q. Moreover, the family (H_Λ) uniquely determines a one-parameter group (α_t) of automorphisms of the C^*-algebra Q by the equation

$$\frac{d}{dt}\,\alpha_t(A) = \lim_{\Lambda \to \mathbf{Z}^3} \frac{2\pi i}{h}\,[H_\Lambda, A].$$

This one-parameter group gives the time evolution of the observables of the infinite system that are given by the elements A of Q, and is calculated by passing to the limit starting from Heisenberg's formula. For a finite system, maintained at temperature T, the formula gives the equilibrium state in a unique manner as a function of H_Λ, but in the thermodynamic limit one cannot have a simple correspondence between the Hamiltonian of the system, or, if one prefers, the group of time evolution, and the equilibrium state of the system. Indeed, during phase transitions, distinct states can coexist, which precludes uniqueness of the equilibrium state as a function of the group (α_t). It is impossible to give a simple formula that would define in a unique manner the equilibrium state as a function of the one-parameter group (α_t). In compensation, there does exist a relation between a state φ on Q and the one-parameter group α_t that does not always uniquely specify φ from the knowledge of α_t, but which is the analogue of the formula (4). This relation is the ***Kubo-Martin-Schwinger condition*** ([Ku], [Mart-S]) as formulated by Haag, Hugenholtz and Winnink [H-H-W]:

Given T, a state φ on Q and the one-parameter group α_t of automorphisms of Q satisfy the KMS-condition if and only if for every pair A, B of elements of Q there exists a function $F(z)$ holomorphic in the strip $\{z \in \mathbf{C},\ \mathrm{Im}\, z \in [0, \hbar\beta]\}$ such that

$$F(t) = \varphi(A\alpha_t(B))\ ,\ F(t + i\hbar\beta) = \varphi(\alpha_t(B)A) \qquad (\forall t \in \mathbf{R}).$$

Here t is a parameter of time, as is $\hbar\beta = \hbar/kT$ which, for $T = 10^{-3}K$, has value approximately $10^{-8}s$.

This condition allows us to formulate mathematically, in quantum statistical mechanics, the problem of the coexistence of distinct phases at given temperature T, that is, the problem of the uniqueness of φ, given (α_t) and β.

For instance the set C_β of KMS states at inverse temperature β is always a Choquet simplex whose extreme points correspond to the pure phases. We refer to [H] for a more thorough discussion of these points.

3. C^*-algebras and topology.

Let X be a quantum space and $\mathcal{A}$ the associated involutive algebra. We have seen in section 2 how to extend measure theory in the general non commutative case, but measure theory is not sufficient to get a satisfactory notion of a point $p \in X$ of the corresponding quantum space X. Indeed (in the commutative case) a point is usually negligible for any of the relevant probability measures μ on X, and the von Neumann algebra $L^\infty(X,\mu)$ only controls X up to negligible subsets. But given any family π_α of involutive representations of $\mathcal{A}$ we can define a norm on $\mathcal{A}$ by the equality:

$$\|a\| = \operatorname*{Sup}_\alpha \ \|\pi_\alpha(a)\| \qquad \forall a \in \mathcal{A}. \tag{1}$$

We need to know that the right hand side is finite (at least for sufficiently many elements of $\mathcal{A}$), and this finiteness follows if for instance we can, for any $a \in \mathcal{A}$ find $b \in \mathcal{A}$ with $a^*a + b^*b = \lambda 1$, $\lambda < \infty$. It may happen that we do not have enough representations π_α to separate points of $\mathcal{A}$ but $J = \{a \in \mathcal{A}, \|a\| = 0\}$ is by construction a two sided ideal, equal to $\bigcap_\alpha \operatorname{Ker} \pi_\alpha$, and we can replace $\mathcal{A}$ by $\mathcal{A}/J$.

Proposition 1. *The completion of $\mathcal{A}$ for the norm $\| \ \|$ is a C^*-algebra A. All the representations π_α extend by continuity to A and one has $\pi_\alpha(A) = \overline{\pi_\alpha(\mathcal{A})}$ $\forall \alpha$.*

We have used here the following definition:

Definition 2. *A C^*-algebra is an involutive Banach algebra A such that $\|x^*x\| = \|x\|^2$ $\forall x \in A$.*

Before we discuss proposition 1 we need to make a number of mathematical comments about definition 2. First the general notion of a Banach algebra has in common with the general notion of Banach space that there is a lot of arbitrariness in the specific choice of the norm. The norm in a Banach algebra B is supposed to satisfy the inequality:

$$\|xy\| \leq \|x\| \ \|y\| \qquad \forall x, y \in B$$

but, even if $\|1\| = 1$, it is not specified uniquely by the topology of B. This arbitrariness is of the same nature as that of a choice of a convex set and is a good reason to consider general Banach spaces (or algebras) as tools rather than as basic objects. It is fundamental that for C^*-algebras the norm, $\| \ \|$, is uniquely determined by the algebraic structure: one has for any $x \in A$:

$$\|x\| = \{\operatorname{Sup}|\lambda| \ , \ x^*x - |\lambda|^2 \text{ not invertible in } A\}.$$

This shows of course that C^*-algebras have a unique norm satisfying definition 2, and it eliminates the above defect of Banach algebras. The second point we need to explain is the meaning of the above algebraic formula for $\|x\|$. It is essentially the natural algebraic definition of the sup norm. Let us consider the simple example of the algebra $A = C(X)$ of continuous functions on a compact space X with pointwise multiplication $(fg)(p) = f(p)\,g(p) \quad \forall p \in X$, and $f^*(p) = \overline{f(p)} \quad \forall p \in X$. Let $f \in A$, then for $\lambda \in \mathbf{C}$ the function $f^* f - |\lambda|^2$ does vanish somewhere in X iff for some $p \in X$ one has $|f(p)| = |\lambda|$. If $f^* f - |\lambda|^2$ does not vanish then by compactness of X it is bounded below in absolute value and its inverse $(f^* f - |\lambda|^2)^{-1}$ is a continuous function. Thus we get

$$\{\mathrm{Sup}|\lambda| \ , \ f^* f - |\lambda|^2 \text{ not invertible in } A\} = \mathop{\mathrm{Sup}}_{p \in X} \ |f(p)|.$$

Let us now discuss proposition 1. The first important fact is that the norm of an operator in Hilbert space:

$$\|T\| = \mathrm{Sup}\,\{\|T\xi\| \ ; \ \|\xi\| \leq 1\}$$

does satisfy the C^*-algebra condition of definition 2:

$$\|T^* T\| = \|T\|^2 \qquad \forall T \in \mathcal{L}(\mathcal{H}).$$

It follows then immediately that the norm on $\mathcal{A}$ given by the equality (1) still satisfies the C^*-condition.

We however need to *complete* $\mathcal{A}$ in order to get the C^*-algebra A and we need to explain what new elements are adjoined to $\mathcal{A}$ by this procedure. The involutive algebra $\mathcal{A}$ we started with needed to be formed by "bounded" elements, say obtained from unbounded ones by the simple algebraic expressions of section 2. Apart from that, it could have been given in a fairly formal manner, say by generators and relations. As a rule it is formed of fairly regular "functions on the quantum space X". Thus for instance $\mathcal{A}$ could be the algebra of smooth functions on a manifold or of polynomials on a subset of $\mathbf{R}^n$ or of bounded functions of local quantum fields etc.... When we complete $\mathcal{A}$ to the C^*-algebra A using the sup norm (1) we now get *all continuous "functions on the quantum space X"*. To have an idea of what this means consider the commutative case, so that X is an ordinary compact space, say the circle S^1 to fix the ideas. Then the completion $C(X)$ contains all continuous functions, all the sums $\sum f_n$, $f_n \in \mathcal{A}$, $\sum \|f_n\| < \infty$ obtained by modifying at each new scale ε_n, $\sum \varepsilon_n < \infty$, the previous sum $\sum\limits_1^{n-1} f_j$ by any element f_n of sup norm less than ε_n, as in the construction of fractals.

One should be well aware that by piling up such small modifications one can, in the C^0-category i.e. in topology, do things which are not allowed in the smooth (C^∞) category such as filling a square with a Peano's curve, construct a Jordan curve in the plane with > 0 2-dimensional area, or construct homeomorphisms between manifolds which are not diffeomorphic.

The following fundamental result of Gelfand shows that the *commutative* C^*-algebras correspond exactly by the duality

$$\text{Compact space } X \rightarrow *\text{algebra } C(X)$$

to the compact spaces X.

To formulate the theorem we first describe the other map, Spec, from algebras to spaces. Given a commutative $*$ algebra A one defines a ***character*** of A as a ***one dimensional*** involutive representation of A or equivalently as an involutive homomorphism $X : A \to \mathbb{C}$.

Theorem 3. ([G-N$_2$]) *Let A be a unital C^*-algebra and $X = \mathrm{Spec} A$ the set of characters of A endowed with the weak topology as a subset of A^*. Then X is a compact space and the Gelfand transform $a \in A \to$ { the function $\chi \in X \to \chi(a)$ } is an involutive isometric isomorphism $A = C(X)$.*

The weak topology means that $\chi_\alpha \to \chi$ iff for any $a \in A$ one has $\chi_\alpha(a) \to \chi(a)$.

The theorem extends to the case of non unital C^*-algebras, in which case X is now only ***locally compact*** and $C(X)$ is replaced by $C_0(X)$ the algebra of continuous functions on X which vanish at ∞: $f(p) \to 0$ when $p \to \infty$.

If one only has in mind the case where X is given to start with, theorem 3 might seem of little practical use, so we just give the following 3 examples to show its power and significance:

Example 1. Let G be a locally compact abelian group with Haar measure ds. Let $\mathcal{H} = L^2(G, ds)$ and $A = C^*(G)$ be the C^*-algebra norm closure of the operators of convolution $\pi(f)$, $f \in C_c(G)$

$$(\pi(f)\xi)(t) = \int_G f(s)\, \xi(t-s)\, ds.$$

Then the Gelfand space $X = \mathrm{Spec}(A)$ is the Pontrjagin dual group $\widehat{G}$. $\bullet$

Example 2. Let G be a real semi simple Lie group, $K \subset G$ a maximal compact subgroup and A the convolution algebra of K biinvariant functions on G acting in the Hilbert space $L^2(G/K) = \mathcal{H}$. Determine the spectrum of A.

Example 3. Let T be a self adjoint operator in a Hilbert space $\mathcal{H}$ and A the C^*-algebra generated by T. Identify the spectrum of A as a subset of $\mathbb{R}$.

Remarks.

a) The characters of a C^*-algebra are automatically continuous (and of norm one). They are thus uniquely determined by their restriction to a dense subalgebra $\mathcal{A} \subset A$. It is of course not true in general that a character of $\mathcal{A}$ (a purely algebraic notion) extends as a character of the completion A. It does however happen for the following class of involutive algebras:

Definition 4. *A pre C^*-algebra $\mathcal{A}$ is an involutive algebra isomorphic to a subalgebra of a C^*-algebra A which is stable under holomorphic functional calculus in A.*

This stability means that if $a \in \mathcal{A}$ and f is a holomorphic function on $\mathrm{Spec}_A(a) = \{\lambda \in \mathbf{C}\ ,\ \lambda - a \text{ not invertible in } A\}$, then $f(a)$, defined as a Cauchy integral $\frac{1}{2\pi i}\int_C f(s)\,\frac{1}{s-a}\,ds \in A$ does in fact belong to $\mathcal{A}$.

This essentially means that if $a \in \mathcal{A}$ is invertible in A then its inverse is also in $\mathcal{A}$.

For a pre C^*-algebra $\mathcal{A}$ one gets uniquely a norm:

$$\|a\| = \ \mathrm{Sup}\{|\lambda|^{1/2}\ ;\ a^*a - \lambda \text{ not invertible in } \mathcal{A}\}$$

and the completion of $\mathcal{A}$ with respect to this norm is a C^*-algebra. The simplest example of a pre C^*-algebra is given by the algebra of smooth functions $C^\infty(M)$ on a manifold M. Indeed (with M compact) if such a function $f \in C^\infty(M)$ is invertible in $C(M)$ then it does not vanish and its inverse f^{-1} is a smooth function on M.

Proposition 5. *Let $\mathcal{A}$ be a pre C^*-algebra then any involutive representation of $\mathcal{A}$ in Hilbert space is automatically continuous for the norm $\|\ \|$ and extends uniquely to the associated C^*-algebra.*

In particular when $\mathcal{A}$ is commutative the characters of $\mathcal{A}$ are exactly the restrictions to $\mathcal{A}$ of the characters of A and $\mathrm{Spec}(\mathcal{A}) = \mathrm{Spec}(A)$ as compact topological spaces. The same holds in the locally compact case.

b) For compact spaces (or locally compact ones) one has the equivalence:

$$X \text{ is metrisable } \Leftrightarrow C(X) \text{ is norm separable.}$$

In other words the topology of X can be defined by a metric iff the C^*-algebra $A = C(X)$ admits a dense countable subset. It is quite important when one begins to work with C^*-algebras and von Neumann algebras to make clear the distinction between the two topics. The weak topology on $\mathcal{L}(\mathcal{H})$ is considerably weaker than the norm topology, so that it is much harder for an involutive subalgebra of $\mathcal{L}(\mathcal{H})$ to be *weakly closed* (von Neumann algebra) than norm closed (C^*-algebra). It is true, and at first confusing, that any von Neumann algebra is a C^*-algebra but not an interesting one because it is usually not norm separable. For instance let (X, μ) be a diffuse probability space (every point $p \in X$ is μ-negligible), then $L^\infty(X, \mu)$ is a von Neumann algebra (in $L^2(X, \mu)$) but *it is not norm separable* and its spectrum as a C^*-algebra is a pathological space that has little to do with the original standard Borel space X.

c) We have seen above that any involutive representation π of our initial algebra $\mathcal{A}$ gives rise to a norm,

$$\|a\|_\pi = \|\pi(a)\| \qquad \forall a \in \mathcal{A}.$$

Many *inequivalent* representations give rise to the same norm so that some information is lost in keeping only $\|\ \|_\pi$ instead of π.

Let us now go back to the three examples of spaces X discussed in section 1). We start with the easiest and use Gelfand's theorem (Theorem 3) to get back Euclidean space time X from the group inclusion (cf. section 1): (with $G = U(n)$, $n \geq 2$)

$$G \subset \mathcal{U} = \text{ Smooth maps from } X \text{ to } G.$$

Using proposition 5 the exact degree of smoothness is not relevant here as long as it satisfies definition 4 and this is the case for C^0, C^k, $C^\infty, \ldots$. We then recover X as the spectrum of the algebra $\mathcal{A}$ constructed from the group inclusion $G \subset \mathcal{U}$ in section 1.

We thus get, in this simple case of a $U(n)$ gauge group, an interpretation of the role of space time points, not as indices of quantum fields, but as characters of an algebra intimately related to the group $\mathcal{U}$ of symmetries of the theory.

Let us next consider the example of quantum mechanics. We have seen above that for an atomic system with N allowed states the corresponding observable algebra is the matrix algebra $M_N(\mathbb{C})$. When $N = \infty$ the corresponding C^*-algebra is the so called *elementary C^*-algebra k*. It is the C^*-algebra of all compact operators in Hilbert space:

$$k = \{T \in \mathcal{L}(\mathcal{H}) \ ; \ T \text{ is compact}\}.$$

(Recall that a bounded operator T is compact iff the image of the unit ball $T(B) = \{T\xi \ ; \ \xi \in \mathcal{H} \ , \ \|\xi\| \leq 1\}$ is a compact subset of $\mathcal{H}$ in the norm topology. Equivalently the n-th characteristic value of T, $\mu_n(T) = n$-th eigenvalue of $|T| = (T^*T)^{1/2}$, tends to 0 when $n \to \infty$.)

Since all Hilbert spaces with an infinite countable orthonormal basis are pairwise isomorphic, the C^*-algebra k is well defined up to isomorphism. It is non unital and is norm separable and is characterized by the following property.

Proposition 6. [Di3] a) *The representation of k as operators in $\mathcal{H}$ is (up to isomorphism) its only irreducible unitary representation.*

b) *The C^*-algebra k is (up to isomorphism) the only non unital norm separable C^*-algebra which admits only one irreducible representation.*

The statement a) easily implies the Stone von Neumann theorem on the uniqueness of irreducible representations of the canonical commutation relations.

The pure states of the C^*-algebra k correspond exactly to the rays in the quantum mechanical Hilbert space.

We see, with this second example, that in the non commutative case the notion of a point $p \in X$ as given by Gelfand's theorem can be extended to that of a *pure state* of the C^*-algebra together with the notion of *equivalence* given by the unitary equivalence of the associated representation.

Proposition 7. [Di3] *Let A be a C^*-algebra.*

a) *A state φ on A is* pure *iff the associated GNS representation π_φ is irreducible.*

b) *Given a pure state φ on A there is a canonical bijection between rays in the associated Hilbert space $\mathcal{H}_\varphi$ and the equivalence class of φ:*

$$C_\varphi = \{\psi \textit{ pure state on } A \; ; \; \pi_\psi \textit{ equivalent to } \pi_\varphi\}.$$

(We recall that two representations π, π' of A are *equivalent* iff there exists a unitary $U : \mathcal{H} \to \mathcal{H}'$ such that $\pi'(a) = U\pi(a)U^* \quad \forall a \in A$.)

The bijection of proposition 7b) is given explicitly by associating to $\xi \in \mathcal{H}_\varphi$, $\|\xi\| = 1$ the state on A given by:

$$\psi(a) = \langle \pi_\varphi(a)\xi, \xi \rangle \qquad \forall a \in A.$$

Note that, even when A is unital, the space Σ of pure states on A is *not compact* in general. It is however always a G_δ (countable intersection of open sets) in the compact space of A and is always sufficiently large (cf. [Di$_3$]).

As a next and more elaborate example of a non commutative C^*-algebra let us consider the observable algebra Q occurring in quantum statistical mechanics as in section 2.

Q is the natural C^*-algebra whose states φ are exactly the families $(\varphi_\Lambda)_{\Lambda \text{ finite}}$ fulfilling conditions α) β) of section 2. This C^*-algebra Q is the norm closure of the union of the C^*-algebra Q_Λ. The latter union is a pre C^*-algebra in the sense of definition 4 so that its norm is uniquely defined as well as its C^*-completion Q. The local Hamiltonians H_Λ of the finite systems Λ determine an unbounded derivation of Q given formally by:

$$\delta(x) = \lim_{\Lambda \to \mathbb{Z}^3} [H_\Lambda, x].$$

This derivation generates a one parameter group $\sigma_t \in \mathrm{Aut}(Q)$ of automorphisms of Q and the KMS condition (cf. section 2) plays a decisive role in characterizing the equilibrium states of the system at inverse temperature β (cf. [H]).

In this example the C^*-algebra Q corresponds to a non commutative analogue of a totally disconnected Cantor set X. Since we only care about states on Q the fine details of the topology of X do not play an important role.

In the next example, the quantum Hall effect, we shall see how the differential topology of a quantum space X can play a decisive role in the understanding of the main physical quantity in the problem, the Hall conductivity.

Recall that we discussed in section 1 the construction by J. Bellissard of a natural C^*-algebra A generated by the $\alpha_s\, f(H)$, of an action of $\mathbb{R}^2$ on A by automorphisms $\alpha_s \in \mathrm{Aut}\,A$ $\forall s \in \mathbb{R}^2$, and of a trace τ.

Let then δ_j be the unbounded derivations of A given by:

$$\delta_j(x) = \partial_j \; \alpha_s(x)_{s=0} \qquad j = 1, 2.$$

The key formula for the Hall conductivity is the Kubo formula which can be expressed as follows:

Lemma 8. ([Bel]) *Let the Fermi level μ belong to a gap of the Hamiltonian, $\mu \notin \mathrm{Spec}H$, and E_μ be the spectral projection of H corresponding to energies smaller than μ. Then the Hall conductivity σ_H is given by the formula:*

$$\sigma_H = \frac{e^2}{h}\,\frac{1}{2\pi i}\,\tau_2(E_\mu, E_\mu, E_\mu)$$

where τ_2 is the trilinear functional given by:

$$\tau_2(a_0, a_1, a_2) = \tau\left(a_0(\delta_1(a_1)\,\delta_2(a_2) - \delta_2(a_1)\,\delta_1(a_2))\right) \qquad \forall a_j \in \mathrm{Dom}\,\delta.$$

As we shall see later this formula will continue to hold (cf. [Bel]) with the only hypothesis that μ is in the spectrum of localised states, but we need first of all to understand the geometric meaning of the above formula.

The C^*-algebra A can be written quite simply as

$$A = A_\theta \otimes k$$

where k is the elementary C^*-algebra described above and where, with θ the flux of the magnetic field through a fundamental domain of Γ, the C^*-algebra A_θ is the irrational rotation C^*-algebra [Ri$_1$]. It is generated by two unitaries V_1, V_2 which satisfy the relation:

$$(*) \qquad V_2V_1 = \lambda\, V_1V_2 \ , \ \ \lambda = \exp(2\pi i\theta).$$

For any value of $\theta \in \mathbb{R}/\mathbb{Z}$, we let A_θ be the C^*-completion of the involutive algebra generated by V_1, V_2 with the presentation $(*)$ and norm given by:

$$\|a\| = \mathrm{Sup}\,\{\|\pi(a)\|, \pi \text{ a unitary representation of } *\}\,.$$

When $\theta = 0$ one checks that A_0 is the C^*-algebra $C(\mathbb{T}^2)$ of continuous functions on a 2-dimensional torus, $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, with $V_1 = \exp(2\pi i\ \alpha_1)$, $V_2 = \exp(2\pi i\ \alpha_2)$.

When θ is irrational the norm $\|\pi(a)\| = \|a\|_\pi$ on the above involutive algebra is independent of the choice of the unitary representation π and the C^*-algebra A_θ is a simple C^*-algebra: it has no non trivial two sided ideal. Nevertheless we shall continue to think of it as a (non commutative) torus and think of elements $a = \Sigma\ a_{n,m}\ V_1^nV_2^m$ of A_θ as continuous functions $a(\alpha_1, \alpha_2)$ on this torus. We then extend as follows (cf. [Co]) the familiar notions:

1) Partial differentiation $\frac{\partial}{\partial\alpha_j}\,a$, $j = 1, 2$.

This yields the following densely defined commuting derivations δ_j of A_θ:

$$\delta_1\left(\Sigma\ a_{n,m}\ V_1^nV_2^m\right) = \Sigma\ a_{n,m}\ (2\pi i n)\ V_1^nV_2^m$$

$$\delta_2\left(\Sigma\, a_{n,m}\, V_1^n V_2^m\right) = \Sigma\, a_{n,m}\, (2\pi i m)\, V_1^n V_2^m .$$

One checks indeed that in spite of the non commutation of V_1 and V_2 the δ_j are ***derivations***, i.e. satisfy:

$$\delta_j(ab) = \delta_j(a)\; b + a\; \delta_j(b).$$

2) *Smoothness:* a is of class C^∞.

This yields the following dense subalgebra A_θ^∞ of A_θ

$$A_\theta^\infty = \{a \in A_\theta\ ,\ \delta_1^{n_1}\delta_2^{n_2}(a) \in A_\theta \quad \forall n_1, n_2 \in \mathbf{N}\}\,.$$

Equivalently A_θ^∞ is the space of C^∞ vectors for the action of $\mathbf{R}^2$ by automorphisms of A_θ given by

$$\beta_s(a) = \Sigma\ \exp(2\pi i(ns_1 + ms_2))\; a_{n,m}\; V_1^n V_2^m \qquad \forall s = (s_1, s_2) \in \mathbf{R}^2\ ,\ \forall a \in A_\theta .$$

One has $\beta_s \in \mathrm{Aut} A_\theta \quad \forall s \in \mathbf{R}^2$. Moreover the elements of A_θ^∞ have a very simple characterization in terms of the coefficients $a_{n,m}$. One has

Lemma 9. (cf. [Co])

$$A_\theta^\infty = \left\{\Sigma\ a_{n,m}\ V_1^n V_2^m\ ;\ n^k m^{k'} |a_{n,m}|\ \textit{bounded for any}\ k, k' \in \mathbf{N}\right\}.$$

In other words the linear space A_θ^∞ is independent of θ, only the ***product*** rule changes, due to the phase factor introduced by $(*)$.

3) *Integration* $\int a\; d\alpha \wedge d\beta$.

This yields the following ***trace*** τ on A_θ:

$$\tau\left(\Sigma\ a_{n,m}\ V_1^n V_2^m\right) = a_{00}.$$

It is relevant to note that when θ is irrational we get (up to normalisation) the ***only*** trace on A_θ.

Now it is important to make sense more generally of the following expression:

$$(**) \qquad\qquad \int a_0\ da_1 \wedge da_2 = \tau_2(a_0, a_1, a_2)$$

where a_0, a_1, a_2 are arbitrary elements of A_θ.

We can do that by just writing the result $(**)$ in terms of τ and of the partial differentiations δ_1, δ_2 of 1), we get:

$$\tau_2(a_0, a_1, a_2) = \tau\left(a_0(\delta_1(a_1)\,\delta_2(a_2) - \delta_2(a_1)\,\delta_1(a_2))\right) \qquad \forall a_j \in A_\theta^\infty.$$

So far we have shown how to adapt familiar differential geometric expressions to the non commutative torus but we did not see new phenomena occurring. The first new one has to do with *connectedness*.

The ordinary 2 torus is *connected*, equivalently this means that any $f \in C(\mathbf{T}^2)$, any continuous function on $\mathbf{T}^2$ has a connected spectrum: $\mathrm{Spec} f = f(\mathbf{T}^2) \subset \mathbf{R}$ (or $\mathbf{C}$ if f is complex valued).

On the opposite, for θ irrational, the non commutative torus is ***highly disconnected*** and one can find many elements $a = a^* \in A_\theta$ whose spectrum is disconnected. The simplest example of such an a is $a = V_1 + V_1^* + V_2 + V_2^*$ when θ is a Liouville number but the first example was given by R. Powers and M. Rieffel ([Ri$_1$]) and was simply giving an idempotent $E \in A_\theta$, $E = E^*$, $E^2 = E$, $E \neq 0$, $E \neq 1$. In other words E is a selfadjoint element of A_θ whose spectrum is $\{0, 1\}$.

The construction of E depends on an arbitrary choice of a smooth function (to insure that $E \in A_\theta^\infty$) in one variable (cf. [Ri$_1$]) and the first striking result of non commutative differential geometry is the following ([Co]).

Theorem 10. ([Co]) *For any idempotent* $E \in A_\theta^\infty$

$$\frac{1}{2\pi i}\int E\, dE \wedge dE \qquad \textit{is an } \text{integer.}$$

This result holds in the same way for the C^*-algebra $A = A_\theta \otimes k$, the derivations δ_1, δ_2 and the trace τ appearing in the formula (lemma 8) for the Hall conductivity. We shall now explain its geometrical significance. starting with the elementary notions of K-theory and characteristic classes. Both the ***stability***, under deformations of E, of the integral $\frac{1}{2\pi i}\int E\, dE \wedge dE$ and its ***integrality*** will be given a conceptual explanation in sections 4) and 5) below. This will allow in particular (cf. [Bel]) to obtain the integrality result for the Hall conductivity σ_H (in units $\frac{e^2}{h}$) when the Fermi level μ belongs to a gap of extended states which is the correct physical assumption.

4. Vector bundles and cyclic cohomology.

In this section we shall describe the notions of vector bundles and characteristic classes of such bundles and extend them to the non commutative case. This extension plays a role both in the quantum Hall effect, already discussed above, and in gauge theories. We shall begin our discussion with the first instance of an invariant of a vector bundle computed by an integral formula, the Gauss-Bonnet theorem.

The Gauss-Bonnet theorem.

This theorem is easy to visualize, it has to do with the theory of surfaces in the space $\mathbf{R}^3$. To fix the ideas, take a concrete example of such a surface. When one attempts to understand it and to study it, the fundamental concept of curvature is seen to appear. This concept is even easier to understand in one less dimension, where one is interested with curves in $\mathbf{R}^2$. Let us therefore recall what the curvature of a plane curve is, before returning to the space $\mathbf{R}^3$.

Let us consider a plane curve at a point P. Among all of the circles with center lying on the normal to the curve passing through P, there is one with best possible fit to the curve in a neighborhood of P. The radius of this circle is called the radius of curvature of the curve at the point P. Then the curvature of the curve at P, which should be greater the more the curve is curved is defined as the inverse of the radius of curvature at P, i.e., $K = 1/R$.

Let us now return to the case of surfaces. Through a point P of the surface, one can draw the normal to the surface and, to reduce the dimension by 1, cut the surface by a plane that passes through the normal. When the surface is intersected by a plane passing through the normal, one obtains a plane curve. This plane curve has a radius of curvature at the point P, and there is no reason for it to be constant as one varies the plane passing through the normal. For a sphere, which is perfectly symmetrical, one always has the same radius of curvature, but in general one finds different curvatures; in the saddle-shaped surface case, there is a plane passing through the normal for which the curvature is zero. This means that if the curvature is given a sign, it changes sign between the two extremes as the plane is rotated about the normal.

A theorem due to Euler shows that in fact one does not have to know the curvatures for all the planes passing through the normal in order to describe the situation completely. It suffices to know the two extreme curvatures K_1 and K_2. They are attained for two perpendicular planes, and when the surface is cut by a plane that passes through the normal but makes an angle θ with the plane for which the curvature has the extreme value K_1, the following formula due to Euler gives the value of the curvature:

$$K_\theta = K_1 \cos^2 \theta + K_2 \sin^2 \theta.$$

The Gauss-Bonnet theorem may be stated as follows:

Theorem 1. *Let Σ be an oriented surface embedded in $\mathbf{R}^3$. For every $P \in \Sigma$, let $R(P) = K_1 K_2$ be the total curvature of Σ at P. Then*

$$\int_\Sigma R(P)\ d^2\ P = 2\pi(2-2g),$$

where g is an integer, independent of the embedding, called the genus *of Σ.*

Thus, even though the number $\int_\Sigma R(P)d^2P$ is calculated as an integral, it has the altogether extraordinary property of being a stable number. This means that even though this number is defined by means of numerous parameters, it does not depend on their choice.

In fact, we have an infinite number of parameters at our disposal. Let us take the surface and make a small bump on it, the theorem shows that one thereby introduces exactly as much positive curvature as negative curvature. Indeed, positive curvature is introduced at the top of the bump, since the curvatures K_1 and K_2 at that place have the same sign; however, at the bottom of the slopes, the situation is that of a saddle: there the total curvature $R = K_1K_2$ is negative. The number obtained by calculating the integral of the total curvature over the entire surface thus has a remarkable property. It is not the integral of an arbitrary function: when the surface is modified slightly, or even modified radically but without changing its topological nature, this number does not change.

When a number has this quality, it has much greater significance than does an ordinary number. Certain physical quantities of fundamental significance are numbers that can be calculated by a method of this type and that possess the same property of stability with respect to deformations.

Let us return to the statement of the Gauss-Bonnet theorem: the integral, over the entire surface, of the total curvature $R = K_1K_2$ is an integral multiple of 2π of the form $2(1-g)2\pi$, where g is a positive integer called the *genus* of the surface. This number characterizes the topological type of the surface in question. Thus for instance, draw an example of a surface of genus 2. The genus measures, if one likes, the number of holes in the surface. For a sphere there is no hole, for a torus there is just one, and so on.

Our next aim will be to show that theorem 10 of section 3) and the Gauss-Bonnet theorem are both special cases of a simple general algebraic result which constructs invariants of K-theory. For that we first need to cast the notion of *vector bundle* in its natural algebraic framework.

Vector bundles and idempotents.

Let X be a compact topological space. Then a *vector bundle* E over X is given by:

α) A topological space E (the total space of the bundle)

β) A continuous map $p : E \to X$ (the canonical projection)

γ) A vector space structure on each fiber $p^{-1}(x)$, $x \in X$.

These data are moreover supposed to satisfy the *local triviality* condition, which asserts that one can cover X by open sets U for which the above triple when restricted to $p^{-1}(U)$ is isomorphic to $U \times V$ where V is a *finite dimensional* vector space.

As an example consider the oriented surface Σ of theorem 1 and let E be the space of pair (x, ξ) where $x \in \Sigma$ and $\xi \in T_x(\Sigma)$ is a tangent vector to Σ at the point x. It is clear that we can endow E with a natural topology and let $p : E \to \Sigma$ be the projection $p(x,\xi) = x$. To get an interesting vector bundle we want each fiber $T_x(\Sigma)$ to be a complex vector space of dimension one. This is done uniquely if one requires that multiplication by i is just rotation of angle $\pi/2$ in the oriented Euclidean plane $T_x(\Sigma) \subset \mathbf{R}^3$. One then checks directly that the obtained triple satisfies the local triviality condition and hence defines a vector bundle over Σ.

There is an obvious general notion of isomorphism of vector bundles E, E' over the same base X. An isomorphism $T : E \to E'$ is given by an homeomorphism of total spaces, commuting with p, and linear in each fiber. The relation between vector bundles and idempotents is given by the following construction of a vector bundle E on X from an idempotent $e \in M_N(C(X))$ where N is an integer and $M_N(C(X))$ is the C^*-algebra of $N \times N$ matrices over $C(X)$. In other words $M_N(C(X)) = C(X) \otimes M_N(\mathbf{C})$. An element of $M_N(C(X))$ such as e, can be viewed as a continuous map $e \in C(X, M_N(\mathbf{C}))$ from X to $M_N(\mathbf{C})$ and the algebraic operations occurring pointwise ($(ef)(x) = e(x)f(x) \quad \forall x \in X$ for instance) it follows that e is an idempotent:

$$e^2 = e$$

iff $e(x)$ is an idempotent for all $x \in X$.

We can then define a vector bundle $E = \operatorname{Im}(e)$ as follows:

Total space $E = \{(x,\xi) \; ; \; x \in X \; , \; \xi \in \mathbf{C}^N \; , \; e(x)\xi = \xi\}$

Projection $p : E \to X$, $p(x,\xi) = x$

Vector space structure of $E_x = \operatorname{Im}(e(x)) \subset \mathbf{C}^N$.

The local triviality of E follows from a simple property of idempotents $e \in M_N(\mathbf{C})$. If two of them e, f are close enough in norm then the map $e : \operatorname{Im} f \to \operatorname{Im} e$ is an isomorphism.

Note also that without changing the vector bundle $\operatorname{Im}(e)$ we can replace the idempotent e by a selfadjoint idempotent $e = e^*$.

Proposition 2. ([Ser$_3$] [Sw]) *Let X be a compact space.*

1) *Then every vector bundle E on X is isomorphic to a vector bundle* $\operatorname{Im} e$ *for a (selfadjoint) idempotent $e \in M_N(C(X))$, with N large enough.*

2) *Two vector bundles* $\operatorname{Im} e$ *and* $\operatorname{Im} f$ *are isomorphic iff e and f are equivalent in the sense of Murray and von Neumann, i.e. there exists $u, v \in M_N(C(X))$ with $uv = e$, $vu = f$.*

To prove the existence of e one constructs, using compactness of X a finite number N of sections ξ_j of E (i.e. of continuous maps $\xi_j : X \to E$ with $p \circ \xi_j = id_X$) such that at each $x \in X$ the vectors $\xi_j(x)$ span the vector space E_x. Let us see how to do this in our example of the line bundle E over a surface $\Sigma \subset \mathbf{R}^3$. We can directly define the idempotent $e \in M_2(C(\Sigma))$ as follows: note first that the unit sphere $S^2 \subset \mathbf{R}^3$ can be identified with the space of selfadjoint idempotent 2×2 matrices of rank one $e = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$. Indeed such matrices are necessarily of the form $\begin{bmatrix} a & b \\ \bar{b} & 1-a \end{bmatrix}$ with a real, b complex and with vanishing determinant, i.e.

$$a(1-a)+|b|^2=0$$

which gives the 2 spheres S^2. Next the normal map $\nu : \Sigma \to S^2$ thus gives us a specific element:

$$e \in C(\Sigma, M_2(\mathbf{C})) = M_2(C(\Sigma))$$

which is the desired idempotent. It is clear indeed that the vector bundle $\mathrm{Im}(e)$ is isomorphic to E.

Invariants of vector bundles.

By proposition 2 we know how to construct arbitrary vector bundles over X using idempotents $e \in M_N(C(X))$. The same statement holds for smooth vector bundles, replacing $C(X)$ by the algebra $C^\infty(X)$ of smooth functions on X and hence $M_N(C(X))$ by the pre C^*-algebra $M_N(C^\infty(X))$. The remarkable ***stability*** under deformations which occurs both in theorem 10 of section 3 and in the Gauss-Bonnet theorem is a special case of the following:

Lemma 3. *Let $\mathcal{A}$ be an algebra, τ_2 a trilinear form on $\mathcal{A}$ satisfying the following 2 conditions:*

α) $\tau_2(a^1,a^2,a^0)=\tau_2(a^0,a^1,a^2) \quad \forall a^0,a^1,a^2 \in \mathcal{A}$

β) $\tau_2(a_0a_1,a_2,a_3)-\tau_2(a_0,a_1a_2,a_3)+\tau_2(a_0,a_1,a_2a_3)-\tau_2(a_3a_0,a_1,a_2)=0 \quad \forall a_j \in \mathcal{A}$.

Then the quantity $\tau(e,e,e)$, e idempotent, $e \in \mathcal{A}$, is invariant under deformations of e and only depends upon the Murray-von Neumann equivalence class of e. •

By a deformation of e we mean that we have say a one parameter family e_t of idempotents, $e_t \in \mathcal{A}$, with $\frac{d}{dt}\, e_t = \dot{e}_t \in \mathcal{A}$. The proof of the lemma is quite simple, using the fact that the deformation is necessarily isospectral since $\mathrm{Spec}(e_t) \subset \{0,1\}$ so that one can find $a_t \in \mathcal{A}$ such that:

$$\frac{d}{dt}\, e_t = [a_t, e_t].$$

The condition β) above is a natural generalization of the ***tracial*** condition:

$$\tau(a_0a_1)-\tau(a_1a_0)=0 \qquad \forall a_0,a_1 \in \mathcal{A}$$

for a linear form on $\mathcal{A}$. The invariant given by lemma 3 is a natural generalization of the ***trace*** $\tau(e)$ of an idempotent which only depends upon the Murray-von Neumann class of e. Let us now give examples of functionals τ_2 satisfying α) β). They are called ***cyclic 2-cocycles***. The cyclic refers to the cyclic permutation occurring in α), the 2 refers to the indices a_0, a_1, a_2, and the term *cocycle* refers to the general coboundary operation b defined for any $n+1$ linear form by:

$$(b\varphi)(a_0, \ldots, a_{n+1}) =$$
$$\sum_0^n (-1)^j \; \varphi(a_0, \ldots, a_j a_{j+1}, \ldots, a_{n+1}) + (-1)^{n+1} \; \varphi(a_{n+1} a_0, a_1, \ldots, a_n) \qquad \forall a_j \in \mathcal{A}.$$

Example 1. Let $\mathcal{A} = A_\theta^\infty$ be the algebra (occurring in the quantum Hall effect (cf. section 3)) of smooth functions on the non commutative torus. Then the functional τ_2 of theorem 10,

$$\tau_2(a_0, a_1, a_2) = \tau\left(a_0(\delta_1(a_1)\; \delta_2(a_2) - \delta_2(a_1)\; \delta_1(a_2))\right)$$

is a cyclic 2-cocycle.

In particular lemma 3 accounts for the remarkable stability of $\tau_2(e, e, e)$ when applied to the Powers-Rieffel projections e depending on an arbitrary choice of function.

Example 2. Let $\mathcal{A} = M_N\left(C^\infty(\Sigma)\right)$ for Σ an oriented surface. Let us first define, using the orientation of Σ, a natural cyclic 2-cocycle τ_2 on $C^\infty(\Sigma)$. We let:

$$\tau_2(f^0, f^1, f^2) = \int_\Sigma f^0 \; df^1 \wedge df^2 \qquad \forall f^j \in C^\infty(\Sigma).$$

This makes sense because Σ is compact and oriented. Note that α) β) are easy to check for τ_2 and that in fact one has a stronger property:

α') $\tau_2\left(f^{\sigma(0)}, f^{\sigma(1)}, f^{\sigma(2)}\right) = \varepsilon(\sigma)\; \tau_2(f^0, f^1, f^2) \quad \forall f^j \in C^\infty(\Sigma)$

for any permutation σ of $\{0, 1, 2\}$.

There is a general manner to extend a cyclic cocycle from an algebra $\mathcal{A}$ to $M_N(\mathcal{A}) = M_N(\mathbf{C}) \otimes \mathcal{A}$ for any N:

Lemma 4. *Let τ_2 be a cyclic 2-cocycle on $\mathcal{A}$ then the following formula defines a cyclic 2-cocycle on $M_N(\mathcal{A})$ for any N:*

$$\widetilde{\tau}_2(a_0 \otimes \mu_0, a_1 \otimes \mu_1, a_2 \otimes \mu_2) = \tau(a_0, a_1, a_2)\; \mathrm{Trace}(\mu_0 \mu_1 \mu_2) \qquad \forall a_j \in \mathcal{A}\,,\; \mu_j \in M_N(\mathbf{C}).$$

Note that if we apply this formula to get a cyclic 2-cocycle $\widetilde{\tau}_2$ on $M_N\left(C^\infty(\Sigma)\right)$ the stronger property α') of τ_2 is lost and only α) survives.

If we use this cyclic 2-cocycle $\widetilde{\tau}_2$ on $M_N\left(C^\infty(\Sigma)\right)$, and evaluate it on the idempotent $e \in M_2\left(C^\infty(\Sigma)\right)$ associated to the normal map, an immediate check shows that $\widetilde{\tau}_2(e, e, e)$ is exactly the integral curvature:

$$\tilde{\tau}_2(e,e,e) = \int_\Sigma R(P)\, d^2\, P.$$

Thus lemma 3 accounts also for the remarkable stability of this quantity.

In section 5 below we shall account for the *integrality* of the result occurring in these two examples. For the time being we shall give a quick introduction to K-theory and cyclic cohomology.

Vector bundles over quantum spaces.

By proposition 2 we have an obvious way to adapt the notion of vector bundle to the non commutative case. Given the (involutive) algebra $\mathcal{A}$ of coordinates on our quantum space X we should use idempotents $e \in M_N(\mathcal{A})$ to get such vector bundles. There is a small nuance we have to deal with however. Indeed the vector bundle E does not determine e uniquely but only up to equivalence. The answer is given by the following result of Serre and Swan:

Proposition 5. *Let X be a compact space.*

1) *Let E be a vector bundle over X, then the $C(X)$ module $\mathcal{E} = C(X,E)$ of the continuous sections of E is a* finite projective module *over the C^*-algebra $C(X)$.*

2) *Let $\mathcal{E}$ be a finite projective module over $C(X)$, then there is a canonically associated vector bundle E over X such that*

$$\mathcal{E} = C(X,E) \qquad (\textit{as a } C(X) \textit{ module}).$$

The fiber over $x \in X$ of E in 2) is given by the tensor product:

$$E_x = \mathcal{E} \underset{C(X)}{\otimes} \mathbb{C}$$

where $C(X)$ acts on $\mathbb{C}$ by the character associated to x $(f \to f(x))$. We need to define the terminology and we shall do so in the general *non commutative* case:

Definition 6. *Let $\mathcal{A}$ be a unital algebra. A finite projective module $\mathcal{E}$ over $\mathcal{A}$ is a vector space $\mathcal{E}$ with a (right) action of $\mathcal{A}$ on $\mathcal{E}$ by linear maps such that:*

1) *$\mathcal{E}$ is* finitely generated *as an $\mathcal{A}$-module*

2) *$\mathcal{E}$ is a direct summand of a trivial module $\mathcal{A}^N$.*

Condition 2) means that we can find a right module $\mathcal{E}'$ and an isomorphism of right modules $\mathcal{E} \oplus \mathcal{E}' = \mathcal{A}^N$. It is straightforward that any idempotent

$$e \in M_N(\mathcal{A})$$

determines a finite projective module, namely:

$$\mathcal{E} = e\,\mathcal{A}^N.$$

Conversely any finite projective module is of this form, so that the analogue of proposition 2 holds in general. The advantage of the above algebraic translation of the notion of vector bundle is that it now makes good sense in the non commutative case as well (using definition 6).

K-theory.

The purpose of K-theory is, given an algebra $\mathcal{A}$, to classify up to isomorphism the finite projective modules over $\mathcal{A}$. In the special case where $\mathcal{A} = C(X)$ (or $C^\infty(X)$ for a manifold) it means classifying the vector bundles over X up to isomorphism. This problem is handled by the introduction of a *group* $K_0(\mathcal{A})$ obtained using the following notion:

Definition 7. 1) *The direct sum $\mathcal{E} \oplus \mathcal{E}'$ of two finite projective modules over $\mathcal{A}$ is their sum as vector spaces with action:*

$$(\xi, \xi')a = (\xi a, \xi' a) \qquad \forall \xi \in \mathcal{E}\ ,\ \xi' \in \mathcal{E}'\ ,\ a \in \mathcal{A}.$$

2) *Two finite projective modules $\mathcal{E}$, $\mathcal{E}'$ over $\mathcal{A}$ are called* stably isomorphic *iff they become so after addition of a suitable trivial module $\mathcal{E}_0 = \mathcal{A}^N$.*

With the operation of direct sum the set K_0^+ of stable isomorphism classes of finite projective modules over $\mathcal{A}$, forms an abelian semigroup and uniquely generates a group K_0. One can view an element of K_0 as an equivalence class of *virtual* f.p. module, i.e. of pairs $(\mathcal{E}_+, \mathcal{E}_-)$ where $(\mathcal{E}_+, \mathcal{E}_-) \sim (\mathcal{E}'_+, \mathcal{E}'_-)$ iff the following modules are stably isomorphic:

$$\mathcal{E}_+ \oplus \mathcal{E}'_- \simeq \mathcal{E}'_+ \oplus \mathcal{E}_-.$$

(The use of *stable* isomorphism instead of isomorphism is necessary for the transitivity of this relation.)

To any element of K_0^+ corresponds the pair $(\mathcal{E}, 0)$.

The key result of (topological) K-theory is Bott's periodicity result which holds in the more general context of Banach algebras but can be formulated as follows:

Theorem 8. (Bott) *Let A be a unital pre C^*-algebra and $\mathcal{U}_\infty(A)$ be the inductive limit of the unitary groups:*

$$\mathcal{U}_N(A) = \{u \in M_N(A)\ ;\ u^*u = uu^* = 1\}$$

with inclusion of $\mathcal{U}_N$ in $\mathcal{U}_{N+1}$ by $u \to \begin{bmatrix} u & 0 \\ 0 & 1 \end{bmatrix}$ $\forall u \in \mathcal{U}_N$. Then the map ε which to each idempotent $e = e^* \in M_N(A)$ associates the loop $t \to \exp(2\pi i t e)$, $t \in [0,1]$, establishes a group isomorphism:

$$K_0(A) \xrightarrow{\varepsilon} \pi_1(\mathcal{U}_\infty(A)).$$

In fact the Bott periodicity theorem computes all the homotopy groups π_n of $\mathcal{U}_\infty(A)$, these groups only depend upon the *parity* of n. For n even they are equal to $K_1(A)$ which is given by:

Definition 9. a) *Let A be a non unital C^*-algebra, then $K_0(A)$ is defined as the kernel of the augmentation map:*

$$K_0(\widetilde{A}) \to \mathbb{Z}$$

where $\widetilde{A} = A + \mathbb{C}1$ is obtained from A by adjoining a unit.

b) *For any C^*-algebra A, $K_1(A)$ is by definition $K_0(A \otimes C_0(\mathbb{R}))$.*

One denotes by SA the C^*-algebra $A \otimes C_0(\mathbb{R})$, it corresponds in the commutative case $A = C(X)$ to the replacement of X by its suspension. The general Bott periodicity statement is then:

$$\pi_n(\mathcal{U}_\infty(A)) = K_1(A) \qquad n \text{ even}$$

$$\pi_n(\mathcal{U}_\infty(A)) = K_0(A) \qquad n \text{ odd.}$$

For a *pre C^*-algebra $\mathcal{A}$* the group $K_0(\mathcal{A})$ is *equal* to $K_0(A)$ where A is the C^*-completion of $\mathcal{A}$.

When A is norm separable the group $K_0(A)$ is always a ***countable abelian group***. One important corollary of the Bott periodicity theorem is the relation between the K groups of ideals and quotients of C^*-algebras, given by a 6 terms exact sequence:

$$\begin{array}{ccccc} & K_0(J) & \to & K_0(A) & \\ \nearrow & & & & \searrow \\ K_1(B) & & & & K_0(B) \\ \nwarrow & & & & \swarrow \\ & K_1(A) & \leftarrow & K_1(J) & \end{array}$$

associated to any closed ideal J of A with quotient $B = A/J$. We refer to [Co] for a lot more information on K-theory of C^*-algebras.

Cyclic cohomology.

Let $\mathcal{A}$ be an algebra. We shall now see how one of the main tools of K-theory for spaces, namely the Chern character, extends to the non commutative case. This will be done by extending lemma 3 above to cyclic cocycles of arbitrary dimension.

Definition 10. *Let $\mathcal{A}$ be an algebra, $n \in \mathbb{N}$ an integer. A* cyclic n cocycle *on $\mathcal{A}$ is an $n+1$ linear form τ on $\mathcal{A}$ such that:*

α) $\tau(a^1, \dots, a^n, a^0) = (-1)^n \; \tau(a^0, \dots, a^n) \quad \forall a^j \in \mathcal{A}$

β) $\sum\limits_{j=0}^{n} \; (-1)^j \; \tau(a^0, \dots, a^j a^{j+1}, \dots, a^{n+1}) + (-1)^{n+1} \; \tau(a^{n+1} a^0, \dots, a^n) = 0 \quad \forall a^j \in \mathcal{A}.$

The general extension of lemma 3 is given by the following theorem. As in lemma 4 we extend for each N the cocycle τ to a cocycle $\widetilde{\tau}$ on $M_N(\mathcal{A})$ using the formula:

$$\widetilde{\tau}(a^0 \otimes \mu^0, \ldots, a^n \otimes \mu^n) = \tau(a^0, \ldots, a^n) \operatorname{Trace}(\mu^0 \ldots \mu^n) \quad \forall a^j \in \mathcal{A}\,,\ \mu^j \in M_N(\mathbf{C}).$$

Theorem 11. ([Co]) *Let $\mathcal{A}$ be an algebra, n an even integer, τ an n-cyclic cocycle on $\mathcal{A}$. Then there exists a unique additive map $\langle \tau, \cdot \rangle$ from $K_0(\mathcal{A})$ to $\mathbf{C}$ given by*

$$\langle \tau, [e] \rangle = \widetilde{\tau}(e, \ldots, e)$$

for any idempotent $e \in M_N(\mathcal{A})$.

A similar statement, with a suitable form of $K_1(\mathcal{A})$, holds in the odd case, where the pairing with $\pi_0(\mathcal{U}_\infty(\mathcal{A}))$ (with $\mathcal{A}$ involutive) is given by the formula:

$$\langle \tau, [u] \rangle = \widetilde{\tau}(u^{-1}, u, u^{-1}, u, \ldots, u^{-1}, u) \qquad \forall u \in \mathcal{U}_N(\mathcal{A}).$$

This theorem covers the usual Chern Weil construction of the Chern character of ordinary vector bundles over a manifold V. Indeed the Chern character $Ch(E) \in H^*(V, \mathbf{C})$ of a vector bundle is fully determined by its pairing with arbitrary homology classes $x \in H_{2k}(V, \mathbf{C})$, $2k = 0, 2, \ldots$. Any such class can be represented by a closed de Rham current C of dimension $2k$, i.e. as a linear form on the space of differential forms of degree $2k$ on V which is:

a) Continuous in the C^∞ topology

b) Vanishing on coboundaries $d\omega$.

Proposition 12. *Let C be a q-dimensional closed de Rham current on a manifold V. Then the following equality defines a q-dimensional cyclic cocycle on the algebra $C^\infty(V)$ of smooth functions on V:*

$$\tau_C(f^0, \ldots, f^q) = \langle C, f^0\, df^1 \wedge df^2 \wedge \ldots \wedge df^q \rangle.$$

Moreover if $q = 2k$ is even, the corresponding map $\langle \tau_C, \cdot \rangle$ from $K_0(C^\infty(V)) = K^0(V)$ to $\mathbf{C}$ is the Chern character:

$$\langle \tau_C, [E] \rangle = \langle [C], Ch(E) \rangle$$

for any vector bundle E on V.

The first part is easy to check and we urge the reader to do so. The second is also straightforward if one computes the curvature of the canonical connection on the tautological bundle over the Grassmanian

$$\{e \in M_N(\mathbf{C})\ ,\ e = e^* = e^2\}.$$

It is given by the matrix valued form $\theta = e\, de \wedge de$. Cyclic cohomology allows to extend de Rham homology and the Chern character to the general non commutative case. The first important elementary fact is the following:

Proposition 13. *Let $\mathcal{A}$ be an algebra and for each $n \in \mathbf{N}$, let $C_\lambda^n(\mathcal{A})$ be the space of $n+1$ linear forms on $\mathcal{A}$ fulfilling condition 10 α). Then for any $\varphi \in C_\lambda^n(\mathcal{A})$ one has $b\varphi \in C_\lambda^{n+1}(\mathcal{A})$ where $b\varphi$ is given by:*

$$(b\varphi)(a^0,\dots,a^{n+1}) =$$
$$\sum_0^n (-1)^j\ \varphi(a^0,\dots,a^j a^{j+1},\dots,a^{n+1}) + (-1)^{n+1}\ \varphi(a^{n+1}a^0,\dots,a^n) \quad \forall a^j \in \mathcal{A}.$$

Moreover one has $b^2 = 0$.

Definition 14. *The cyclic cohomology $HC^*(\mathcal{A})$ is the cohomology of the complex (C_λ^*, b). In other words:*

$$HC^n(\mathcal{A}) = \operatorname{Ker}\left\{b : C_\lambda^n \to C_\lambda^{n+1}\right\} \big/ \operatorname{Im}\left\{b : C_\lambda^{n-1} \to C_\lambda^n\right\}.$$

The elements of Im b are easy to construct and hence trivial cocycles. In particular they all give 0 when paired with K-theory as in theorem 11.

To compute cyclic cohomology $HC^*(\mathcal{A})$ of an algebra one relates it to Hochschild cohomology which itself can be computed using the tools of homological algebra. By a *bimodule* $\mathcal{M}$ over an algebra $\mathcal{A}$ we mean a vector space equipped with commuting left and right actions of $\mathcal{A}$, noted:

$$\xi \to a\,\xi\,b \ ,\ \ \xi \in \mathcal{M} \ ,\ \ a \in \mathcal{A} \ ,\ \ b \in \mathcal{A}.$$

Let then $(C^n(\mathcal{A},\mathcal{M}), b)$ be the complex where $C^n(\mathcal{A},\mathcal{M})$ is the space of n linear maps T from $\mathcal{A} \times \cdots \times \mathcal{A}$ to $\mathcal{M}$ and where the coboundary map b is:

$$(bT)(a^1,\dots,a^{n+1}) = a^1\ T(a^2,\dots,a^{n+1}) +$$
$$\sum_{j=1}^n (-1)^j\ T(a^1,\dots,a^j a^{j+1},\dots,a^{n+1}) + (-1)^{n+1}\ T(a^1,\dots,a^n)a^{n+1} \quad \forall a^j \in \mathcal{A}.$$

Next, consider the bimodule $\mathcal{A}^* = \{$ space of linear forms on $\mathcal{A}\}$ with:

$$(a\,\varphi\,b)(x) = \varphi(b\,x\,a) \qquad \forall a,b,x \in \mathcal{A}.$$

We identify any $n+1$ linear form φ on $\mathcal{A}$ with the element of $C^n(\mathcal{A},\mathcal{A}^*)$ given by the equality:

$$\varphi(a^1,\dots,a^n)(a^0)=\varphi(a^0,a^1,\dots,a^n) \quad \forall a^j\in\mathcal{A}.$$

Under this identification the coboundary operator b of proposition 13 is just the *restriction* to $C^n_\lambda\subset C^n(\mathcal{A},\mathcal{A}^*)$ of the Hochschild coboundary operator. This shows that the cyclic complex (C^*_λ,b) is a *subcomplex* of the Hochschild complex:

$$(C^*(\mathcal{A},\mathcal{A}^*),b).$$

It gives a natural map from cyclic to Hochschild cohomology:

$$I:HC^n(\mathcal{A})\to H^n(\mathcal{A},\mathcal{A}^*).$$

Before we study this map in general we need to understand the meaning of Hochschild cohomology in the special case of manifolds, i.e. for $\mathcal{A}=C^\infty(V)$ the algebra of smooth functions on a compact manifold V. (The statement easily adapts to the non compact case.) We shall only consider multilinear functionals which are *continuous* for the C^∞ topology and refrain from using a special notation for them.

Proposition 15. [Co] a) *Let $k\in\mathbb{N}$ and C be a k-dimensional de Rham current on V. Then the following formula defines a k-dimensional Hochschild cocycle $\varphi_C\in Z^k(\mathcal{A},\mathcal{A}^*)$:*

$$\varphi_C(f^0,\dots,f^k)=\langle C,f^0\,df^1\wedge\ldots\wedge df^k\rangle \quad \forall f^j\in C^\infty(V).$$

b) *The map $C\to\varphi_C$ is an isomorphism between the space $C^{-\infty}(V,\wedge^k)$ of de Rham currents of dimension k and the (continuous) Hochschild cohomology group $H^k(\mathcal{A},\mathcal{A}^*)$.*

Note that in a) and unlike in proposition 12 we did not assume that the current C was *closed*. In particular the Hochschild cocycle φ_C constructed in a) is not in general a *cyclic* cocycle. It is iff C is closed.

We shall now describe in full generality a natural map

$$B:H^n(\mathcal{A},\mathcal{A}^*)\to HC^{n-1}(\mathcal{A})$$

which will allow to characterize the image $\mathrm{Im}(I)$ for any algebra $\mathcal{A}$.

Notation 16. Let $\mathcal{A}$ be a unital algebra. Let B_0, A, B be the following operators:
$B_0:C^n(\mathcal{A},\mathcal{A}^*)\to C^{n-1}(\mathcal{A},\mathcal{A}^*)$,

$$(B_0\varphi)(a^0,\dots,a^{n-1})=\varphi(1,a^0,\dots,a^{n-1})-(-1)^n\,\varphi(a^0,\dots,a^{n-1},1) \quad \forall a^j\in\mathcal{A}$$

$A:C^n(\mathcal{A},\mathcal{A}^*)\to C^n_\lambda(\mathcal{A})$

$$(A\varphi)(a^0,\dots,a^n)=\sum_{j=0}^{n}(-1)^{nj}\,\varphi(a^j,a^{j+1},\dots,a^n,a^0,\dots,a^{j-1}) \quad \forall a^j\in\mathcal{A}$$

$B : C^n(\mathcal{A}, \mathcal{A}^*) \to C_\lambda^{n-1}(\mathcal{A})$, $B = AB_0$.
We then have:

Proposition 17. a) *For every Hochschild n-cocycle $\varphi \in Z^n(\mathcal{A}, \mathcal{A}^*)$, the functional $B\varphi$ is a* cyclic $n-1$*-cocycle.*

b) *B defines a map: $H^n(\mathcal{A}, \mathcal{A}^*) \to HC^{n-1}(\mathcal{A})$.*

c) *One has* $\mathrm{Im}(I) = \mathrm{Ker}\ B$.

The map $I : HC^n(\mathcal{A}) \to H^n(\mathcal{A}, \mathcal{A}^*)$ is not injective in general. Let us see this in the simplest example: $\mathcal{A} = \mathbf{C}$. Then the cyclicity condition shows immediately that:

$$C_\lambda^n(\mathbf{C}) = \mathbf{C} \qquad \text{if } n \text{ is even}$$

$$C_\lambda^n(\mathbf{C}) = \{0\} \qquad \text{if } n \text{ is odd.}$$

It thus follows that $HC^{2m}(\mathbf{C}) = \mathbf{C}$, $HC^{2m+1}(\mathbf{C}) = \{0\}$. One checks directly moreover that the Hochschild cohomology $H^n(\mathbf{C}, \mathbf{C})$ vanishes for all $n > 0$, so that I is not injective. Let σ be the generator of $HC^2(\mathbf{C})$ given by:

$$\sigma(\lambda_0, \lambda_1, \lambda_2) = \lambda_0\lambda_1\lambda_2 \qquad \forall \lambda_j \in \mathbf{C}.$$

Proposition 18. a) *Given two algebras $\mathcal{A}$, $\mathcal{B}$ there is a natural notion of tensor product:*

$$HC^n(\mathcal{A}) \times HC^m(\mathcal{B}) \to HC^{n+m}(\mathcal{A} \otimes \mathcal{B}).$$

b) *The product by $\sigma \in HC^2(\mathbf{C})$ determines a canonical map*

$$S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A}).$$

To define the product $\varphi \# \psi$ of two cyclic cocycles on $\mathcal{A}$ and $\mathcal{B}$ respectively one proceeds formally as follows. One writes:

$$\varphi(a^0, \ldots, a^n) = \int a^0\, da^1 \ldots da^n \qquad \forall a^j \in \mathcal{A}$$

$$\psi(b^0, \ldots, b^m) = \int b^0\, db^1 \ldots db^m \qquad \forall b^j \in \mathcal{B}$$

and one computes formally the expression:

$$\begin{aligned} &(\varphi \# \psi)(a^0 \otimes b^0, \ldots, a^{n+m} \otimes b^{n+m}) \\ \overset{(\mathrm{def})}{=}\ &\int (a^0 \otimes b^0)\, d(a^1 \otimes b^1)\, d(a^2 \otimes b^2) \ldots d(a^{n+m} \otimes b^{n+m}) \end{aligned}$$

where the computation is done in the tensor product of the formal graded differential algebras $\Omega\mathcal{A}$, $\Omega\mathcal{B}$ generated by the symbols da, db (cf. [Co]).

Let us compute a simple example, say with $n = 1$ so that φ is a cyclic 1-cocycle on $\mathcal{A}$, and with $\mathcal{B} = \mathbf{C}$, $\psi = \sigma$ so that we are computing $S\varphi$, a cyclic 3-cocycle on $\mathcal{A}$. We have:

$$S\varphi(a^0, a^1, a^2, a^3) = \int (a^0 \otimes 1)\, d(a^1 \otimes 1)\, d(a^2 \otimes 1)\, d(a^3 \otimes 1)$$

but $d(a \otimes 1) = (da) \otimes 1 + a \otimes d1$. Note that $d1$ is not 0 in general. Now since φ is a cyclic 1-cocycle we only get 3 relevant terms among the 8 terms of the expansion of

$$(da^1 \otimes 1 + a^1 \otimes d1)(da^2 \otimes 1 + a^2 \otimes d1)(da^3 \otimes 1 + a^3 \otimes d1)$$

namely those which invoke only one da. They give:

$$\int (a^0 \otimes 1)(da^1 \otimes 1)(a^2 \otimes d1)(a^3 \otimes d1) + \int (a^0 \otimes 1)(a^1 \otimes d1)(da^2 \otimes 1)(a^3 \otimes d1)$$
$$+ \int (a^0 \otimes 1)(a^1 \otimes d1)(a^2 \otimes d1)(da^3 \otimes 1) = \varphi(a^2 a^3 a^0, a^1) + \varphi(a^0 a^1 a^2, a^3)$$

which gives the formula for $S\varphi$. We refer to [Co] for more details.

A key result of cyclic cohomology is the following:

Theorem 19. *Let $\mathcal{A}$ be a unital algebra. The following is a long exact sequence of vector spaces (i.e. the kernel of each map is given by the image of the preceding one):*

$$HC^{n-1}(\mathcal{A}) \xrightarrow{S} HC^{n+1}(\mathcal{A}) \xrightarrow{I} H^{n+1}(\mathcal{A}, \mathcal{A}^*) \xrightarrow{B}$$

$$HC^{n}(\mathcal{A}) \xrightarrow{S} HC^{n+2}(\mathcal{A}) \xrightarrow{I} H^{n+2}(\mathcal{A}, \mathcal{A}^*) \to \cdots$$

This long exact sequence reduces the problem of computation of cyclic cohomology $HC^*(\mathcal{A})$ to that of Hochschild cohomology and of the map $I \circ B$ in many examples (cf. [Co]).

For instance for $\mathcal{A} = C^\infty(V)$, V a compact manifold one finds:

$$HC^k(\mathcal{A}) = \{\text{closed currents of dimension } k\} \oplus H_{k-2}(V, \mathbf{C}) \oplus H_{k-4}(V, \mathbf{C}) \oplus \cdots$$

For $k > \dim V$ this gives back the de Rham homology. The finite dimensional corrections $H_{k-2j}(V, \mathbf{C})$ are essential in the understanding of the Chern character in K-homology ([Co]).

The periodicity map $S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A})$ has the remarkable property of leaving unchanged the pairing with K-theory:

$$\langle S\tau, x\rangle = \langle \tau, x\rangle \qquad \forall \tau \in HC^n(\mathcal{A})\ ,\ x \in K(\mathcal{A}).$$

It is thus natural to introduce the stabilized group

$$H^*(\mathcal{A}) = \varinjlim\ (HC^*, S)$$

filtered by the image of HC^n. It is called *periodic cyclic* cohomology (cf. [Co]).

5. The quantized calculus.

In this section we shall describe a quantized version of the differential calculus which we introduced at first as a substitute of calculus for non commutative or quantum spaces. It turns out however that this calculus allows one to make computations not allowed by distribution theory even in the case of functions of one real variable. This will be explained in the first of the three examples below. The second will use the *integrality* of Fredholm indices and get general integrality results as a byproduct of the quantization of the calculus. In the third example we shall compute the 4-dimensional analogue of the 2-dimensional Polyakov action. Each of these examples is tied up with a formula:

$$\int f(Z)\,|dZ|^p \tag{1}$$

$$\int E dE dE \in \mathbb{Z} \tag{2}$$

$$\int \eta_{\mu\nu}\, dX^\mu\, dX^\nu. \tag{3}$$

In order to give meaning to these formulae we need to specify the involutive algebra $\mathcal{A}$ to which the symbols Z, E, X^μ belong to. We also need to define the quantum differential da for $a \in \mathcal{A}$ and the integral sign. The algebra $\mathcal{A}$ will be the algebra of functions on the space X we consider. This space will be the real line or the circle in example 1. It will be the non commutative Brillouin zone of the quantum Hall effect, as understood by J. Bellissard, in example 2. Finally in example 3 the space X will be a 4-dimensional conformal manifold Σ.

We shall now give two definitions, the main formula (definition 2 below) defines the quantum differential da as an *operator* in Hilbert space. The origin of the first definition comes from K-homology, and we refer to [Co] for a detailed discussion of that theory and its origin (cf also section 7).

Definition 1. *Let $\mathcal{A}$ be an involutive algebra. A* Fredholm module $(\mathcal{H}, F)$ *over $\mathcal{A}$ is given by:*

α) An involutive representation $\mathcal{A} \times \mathcal{H} \to \mathcal{H}$ of $\mathcal{A}$ as operators in Hilbert space, noted $(a, \xi) \to a\xi \quad \forall a \in \mathcal{A}, \xi \in \mathcal{H}$.

β) An operator F, $F = F^*$, $F^2 = 1$ in $\mathcal{H}$ such that:

$$[F, a] \text{ is a compact operator for any } a \in \mathcal{A}.$$

We shall use the same notation for an element a of $\mathcal{A}$ and the corresponding operator a in $\mathcal{H}$.

Giving the operator F is equivalent to giving the decomposition of $\mathcal{H}$ as the direct sum of the two orthogonal subspaces:

$$\{\xi \in \mathcal{H}\,;\ F\xi = \xi\} \ ,\ \{\xi \in \mathcal{H}\,;\ F\xi = -\xi\}.$$

Recall finally that an operator T in Hilbert space, is ***compact*** iff it is a norm limit of operators of finite rank. If we let R_n be the set of all operators of rank less than n: $R_n = \{S \in \mathcal{L}(\mathcal{H})\ ,\ \mathrm{Dim}(\mathrm{Im}S) \leq n\}$ then an operator T is compact iff the norm distance:

$$\mu_n(T) = \mathrm{dist}(T, R_n) \qquad \forall n \in \mathbf{N} \tag{4}$$

satisfies $\mu_n(T) \to 0$ when $n \to \infty$.

Definition 2. *Let $\mathcal{A}$ be an involutive algebra, $(\mathcal{H}, F)$ a Fredholm module over $\mathcal{A}$. Then the quantum differential da, $a \in \mathcal{A}$ is defined as:*

$$da = [F, a] = Fa - aF \qquad \forall a \in \mathcal{A}.$$

By hypothesis da is *small* inasmuch as it is a compact operator and compact operators play the same role among bounded operators as do infinitesimal numbers among numbers. In particular they form a two sided ideal, noted $k \subset \mathcal{L}(\mathcal{H})$:

$$\begin{gathered} T_j \in k \Rightarrow T_1 + T_2 \in k \\ T_1 \in k\ ,\ T_2 \in \mathcal{L}(\mathcal{H}) \Rightarrow T_1T_2\ ,\ T_2T_1 \in k. \end{gathered} \tag{5}$$

Exactly as infinitesimals have orders, one can filter the ideal k of compact operators by more refined ideals. Let us say that a compact operator T is of ***order*** α, for $\alpha > 0$, when:

$$\mu_n(T) = O(n^{-\alpha}) \ \text{ for } \ n \to \infty \tag{6}$$

(i.e. there exists a constant C such that $\mu_n \leq Cn^{-\alpha}$). The definition (4) of $\mu_n(T)$ and the obvious inclusions

$$R_n + R_m \subset R_{n+m}\ ,\ R_n\mathcal{L} = \mathcal{L}R_n = R_n$$

show that one has the general inequalities

$$\begin{gathered} \mu_{n+m}(T_1 + T_2) \leq \mu_n(T_1) + \mu_m(T_2) \\ \mu_{n+m}(T_1T_2) \leq \mu_n(T_1)\ \mu_m(T_2) \\ \mu_n(TT_1) \leq \|T\|\ \mu_n(T_1) \\ \mu_n(T_1T) \leq \|T\|\ \mu_n(T_1) \end{gathered} \tag{7}$$

valid for any pair of positive integers, any $T_1, T_2 \in k$ and $T \in \mathcal{L}(\mathcal{H})$.

It thus follows that operators of order α form a two sided ideal, and that moreover:

$$T_1 \text{ of order } \alpha\ ,\ T_2 \text{ of order } \beta \Rightarrow T_1 T_2 \text{ of order } \alpha + \beta. \tag{8}$$

Our main new tool will be the Dixmier trace which will apply to operators of order 1 and neglects all operators of higher order. It is best to keep our analogy with infinitesimals, the Dixmier trace will allow us to integrate infinitesimals of order one and will neglect those of higher order.

It thus follows that the interesting range, for the order α, is the interval $]0,1]$. We are not interested in the higher orders which we shall neglect. This might at first seem surprising since in dimension k one will necessarily invoke integrals of the form:

$$\int f(x)\ |dx|^k.$$

The point is that on a space X of dimension k the order of the quantum differential dx of an element $x \in \mathcal{A}$ is $1/k$. At a more technical level it is important that, for $\alpha \in]0,1[$ the ideal of operators of order α is a ***normed ideal*** (cf. [Go-K]). More precisely, with $\alpha = \frac{1}{p}$, $p \in]1,\infty[$, it is the ideal $\mathcal{L}^{(p,\infty)}$ obtained by real interpolation theory from the ideals k of compact operators and $\mathcal{L}^1$ of trace class operators. A natural norm on $\mathcal{L}^{(p,\infty)}$ is given by:

$$\|T\|_{p,\infty} = \operatorname*{Sup}_N \ N^{\frac{1}{p}-1}\ \sigma_N(T) \tag{9}$$

with $\sigma_N(T) = \sum_0^N \mu_n(T) \quad \forall N \in \mathbf{N}$.

For $p > 1$ one checks that the ideal of operators of order $1/p$ is a Banach space for the norm (9); the triangle inequality holds for any of the norms σ_N thanks to the equality (cf. [Go-K])

$$\sigma_N(T) = \operatorname{Sup}\{\|TE\|_1\ ,\ E \text{ an } N\text{-dimensional subspace}\} \tag{10}$$

where $\|\ \|_1$ is the $\mathcal{L}^1$-norm, $\|T\|_1 = \operatorname{Trace}(|T|)$.

For $p = 1$ one has to be more careful and the correct norm is:

$$\|T\|_{1,\infty} = \operatorname*{Sup}_{N \geq 2}\ (\log N)^{-1}\ \sigma_N(T). \tag{11}$$

It is clear that any operator of order 1 has finite norm for $(1,\infty)$ so that the ideal of operators of order 1 is contained in the normed ideal $\mathcal{L}^{(1,\infty)}$ defined by (11). It is the latter ideal which is the natural domain for the Dixmier trace which we shall now define.

The Dixmier trace Tr_ω is first defined for positive operators, and then extended by linearity to arbitrary elements of $\mathcal{L}^{(1,\infty)}$. Let $T \geq 0$, $T \in \mathcal{L}^{(1,\infty)}$. The characteristic values $\mu_n(T)$ defined by (4) are the eigenvalues of T arranged by decreasing order of size. Thus the partial sums for the *trace* of T are

$$\sigma_N(T) = \sum_0^N \mu_n(T).$$

By (11) the only thing we can assert is that $\sigma_N(T) = O(\log N)$ so that for some constant C one has $\sigma_N(T) \leq C \log N$. In particular the trace of T, trace $(T) = \sum\limits_0^\infty \mu_n(T)$ is in general divergent, but not more than logarithmically. The basic formula for the Dixmier trace is:

$$\mathrm{Tr}_\omega(T) = \lim_{N\to\infty} \frac{1}{\log N} \sum_0^N \mu_n(T) \qquad \forall T \geq 0. \tag{12}$$

Since the eigenvalues $\mu_n(T)$ are unitarily invariant (i.e., $\mu_n(UTU^*) = \mu_n(T)$ for U unitary), so is the sequence $(1/\log N) \sum\limits_{n=0}^{N-1} \mu_n(T)$. There are two problems with the above formula: its linearity and its convergence.

To handle linearity, for $T_i \geq 0$, $T_i \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$ one has to compare

$$\frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T_1 + T_2) = \gamma_N$$

and

$$\frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T_1) + \frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T_2) = \alpha_N + \beta_N.$$

The subadditivity of σ_N shows that $\gamma_N \leq \alpha_N + \beta_N$. Next, by (10) we have, for $T \geq 0$,

$$\sigma_N(T) = \sup\{\mathrm{Trace}(TE)\ .\ \dim E = N\},$$

and it follows ([Di$_4$]) that

$$\sigma_N(T_1) + \sigma_N(T_2) \leq \sigma_{2N}(T_1 + T_2),$$

as one sees by taking the linear span $E = E_1 \vee E_2$ of two N-dimensional subspaces E_1, E_2 of $\mathcal{H}$. We thus have

$$\alpha_N + \beta_N \leq \left(\frac{\log 2N}{\log N}\right) \gamma_{2N} \quad , \quad \gamma_N \leq \alpha_N + \beta_N.$$

Since $\frac{\log 2N}{\log N} \to 1$ as $N \to \infty$, we see that linearity would follow easily if we had convergence. Now, from the hypothesis $T_i \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, the sequences α_N, β_N, γ_N are ***bounded*** and thus, even without the convergence, we get a unitarily invariant positive trace on $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ for each linear form $\lim_\omega = \ell$ on the space $\ell^\infty(\mathbf{N})$ of bounded sequences that satisfies the following conditions:

α) $\lim_\omega(\alpha_n) \geq 0$ if $\alpha_n \geq 0$,

β) $\lim_\omega(\alpha_n) = \lim(\alpha_n)$ if α_n is convergent,

γ) $\lim_\omega(\alpha_1, \alpha_1, \alpha_2, \alpha_2, \alpha_3, \alpha_3, \ldots) = \lim_\omega(\alpha_n)$.

Condition γ) is a crucial condition of *scale invariance*. To get it one uses Cesaro summation for dyadic blocks $2^{N-1} \leq n < 2^N$ (cf. [Di$_4$] and [Co] for more details).

Definition 3. *For $T \geq 0$, $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, we set*

$$\mathrm{Tr}_\omega(T) = \lim_\omega \frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T).$$

The above inequalities show that Tr_ω is additive:

$$\mathrm{Tr}_\omega(T_1 + T_2) = \mathrm{Tr}_\omega(T_1) + \mathrm{Tr}_\omega(T_2) \qquad (\forall T_i \geq 0\ ,\ T_i \in \mathcal{L}^{(1,\infty)}(\mathcal{H})).$$

Thus, Tr_ω extends uniquely by linearity to the entire ideal $\mathcal{L}^{(1,\infty)}(\mathcal{H})$ and has the following properties:

Proposition 4. a) *If $T \geq 0$ then $\mathrm{Tr}_\omega(T) \geq 0$.*

b) *If S is any bounded operator and $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, then $\mathrm{Tr}_\omega(ST) = \mathrm{Tr}_\omega(TS)$.*

c) $\mathrm{Tr}_\omega(T)$ *is independent of the choice of the inner product on $\mathcal{H}$, i.e., it depends only on the Hilbert space $\mathcal{H}$ as a topological vector space.*

d) Tr_ω *vanishes on the ideal $\mathcal{L}_0^{(1,\infty)}(\mathcal{H})$, which is the closure, for the $\| \ \|_{1,\infty}$-norm, of the ideal of finite-rank operators.*

Property c) follows from b) since, for S bounded and invertible,

$$\mathrm{Tr}_\omega(STS^{-1}) = \mathrm{Tr}_\omega(T) \qquad (\forall T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})). \tag{13}$$

Property d) has an obvious corollary:

$$\mathrm{Tr}_\omega(T) = 0 \ \text{ if } \ \mu_n(T) = o\left(\frac{1}{n}\right) \tag{14}$$

i.e. if $n\mu_n(T) \to 0$ when $n \to \infty$.

It is this vanishing property of the Dixmier trace that makes it possible to neglect all operators of order higher than one. We shall now discuss the problem of convergence of the sequence $\frac{1}{\log N} \sum_{n=0}^{N-1} \mu_n(T)$.

For a positive operator $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, the complex powers T^s ($s \in \mathbb{C}$, $\mathrm{Re}(s) > 1$) make sense and are of trace class, so that the equality

$$(15) \qquad \zeta(s) = \operatorname{Trace}(T^s) = \sum_{n=0}^{\infty} \mu_n(T)^s$$

defines a holomorphic function in the half-plane $\mathrm{Re}(s) > 1$. Now, the Tauberian theorem of Hardy and Littlewood can be stated as follows:

Proposition 5. *For $T \geq 0$, $T \in \mathcal{L}^{(1,\infty)}(\mathcal{H})$, the following two conditions are equivalent:*

1) $(s-1)\,\zeta(s) \to L$ *as* $s \to 1+$;

2) $(1/\log N) \sum_{n=0}^{N-1} \mu_n \to L$ *as* $N \to \infty$.

Under these conditions, the value of $\mathrm{Tr}_\omega(T)$ is of course independent of ω, and if $\zeta(s)$ has a simple pole at $s = 1$, this value is just the residue of ζ at $s = 1$.

As an example of a rather general situation in which the above type of convergence is satisfied, let us make the connection between the Dixmier trace Tr_ω and the notion of residue for pseudodifferential operators, introduced by Manin [Mani$_1$], Wodzicki [Wo$_2$] and Guillemin [Gu].

Proposition 6. *Let M be an n-dimensional compact manifold and let $T \in OP^{-n}(M, E)$ be a pseudodifferential operator of order $-n$ acting on sections of a complex vector bundle E on M. Then:*

1) *The corresponding operator T on $\mathcal{H} = L^2(M, E)$ belongs to the ideal $\mathcal{L}^{(1,\infty)}(\mathcal{H})$.*

2) *The Dixmier trace $\mathrm{Tr}_\omega(T)$ is independent of ω and is equal to the residue $\mathrm{Res}(T)$.*

Let us recall that the Wodzicki residue $\mathrm{Res}(T)$ is given by a completely explicit formula from the principal symbol $\sigma_{-n}(T) = \sigma(T)$. The latter is a homogeneous function of degree $-n$ on the cotangent space T^*M of M, consequently the following integral is independent of the choice (using a metric on M) of the unit sphere bundle $S^*(M) \subset T^*(M)$ with its induced volume element:

$$(16) \qquad \mathrm{Res}(T) = \int_{S^*M} \operatorname{trace}_E(\sigma)\, ds.$$

It is quite important for our later purposes that the Wodzicki residue continues to make sense for pseudodifferential operators of arbitrary order [Wo$_2$]. It is the *unique trace* which extends the Dixmier trace and is given by the same formula (16) applied to the symbol of order $-n$ of T. We shall come back to this point in example 3.

In general, for $T \in \mathcal{L}^{(1,\infty)}$, $T \geq 0$, it is not true that the sequence (12) is convergent, so that the value of $\mathrm{Tr}_\omega(T)$ depends, in general, on the limiting procedure ω. This feature of the Dixmier trace is tied up with its *non normality*, (cf. [Di$_4$]), and it is precisely this non normality which we shall use in a positive manner in example 1 to pass, in the context of Julia sets, from the harmonic measure to the Hausdorff measure.

To summarize the above discussion, we now have at our disposal the notions of quantum differential (definition 2), of order of a compact operator (6) and its main properties, together with the notion of integral given by the Dixmier trace (proposition 4) which neglects operators of order higher than one (14).

The general principle that we shall use now is that while the ordinary trace of quantum differential expressions is in general difficult to compute and non local, the Dixmier trace of such expressions, thanks to the simplifications due to (14), is much easier to compute and yields local quantities expressible in classical terms.

Let us now pass to examples of quantized calculus.

Example1. $\int f(Z)\,|dZ|^p$.

In this example we shall apply our calculus to functions of one real variable and show that it gives meaning to expressions which are meaningless in distribution theory.

Our algebra $\mathcal{A}$ is the algebra of functions $f(s)$ of one real variable $s \in \mathbf{R}$, we do not specify their regularity at the moment. To quantize the calculus we need a representation of $\mathcal{A}$ in a Hilbert space $\mathcal{H}$ and an operator F as in definition 1. The representation of $\mathcal{A}$ is given by a measure class on $\mathbf{R}$ and a multiplicity function (cf. section 2 theorem 4). Since we want the calculus to be translation invariant the measure class is necessarily the Lebesgue class and the multiplicity is a constant. We shall take it equal to one, the more general case does not bring anything new. Thus, so far, we have functions on $\mathbf{R}$ acting, by multiplication operators in the Hilbert space $L^2(\mathbf{R})$:

$$(1) \qquad \mathcal{H} = L^2(\mathbf{R})\ ,\ (f\xi)(s) = f(s)\,\xi(s) \qquad \forall s \in \mathbf{R}\ ,\ \xi \in L^2(\mathbf{R}).$$

Any measurable bounded function $f \in L^\infty(\mathbf{R})$ defines a bounded operator in $\mathcal{H}$ by the equality (1).

Since we want the calculus to be translation invariant, the operator F must commute with translations and hence be given by a convolution operator. We shall also require that it commutes with dilatations, $s \to \lambda s$, $\lambda > 0$ and it then follows easily (cf. for instance [Stein]) that the only non trivial choice of F, $F^2 = 1$, is the Hilbert transform, given by:

$$(2) \qquad (F\xi)(s) = \frac{1}{\pi i}\int \frac{\xi(t)}{s-t}\,dt$$

where the integral is taken for $|s-t| > \varepsilon$ and then $\varepsilon \to 0$.

The quantum differential $df = [F, f]$ of $f \in L^\infty(\mathbf{R})$ has the following very simple expression, it is the operator in $L^2(\mathbf{R})$ associated by the equality:

(3) $$T\xi(s) = \int k(s,t)\, \xi(t)\, dt$$

to the following kernel $k(s,t)$; $s, t \in \mathbf{R}$

(4) $$k(s,t) = \frac{f(s) - f(t)}{s - t}.$$

(Up to the factor $\frac{1}{\pi i}$ which we ignore.)

Note that the group $SL(2, \mathbf{R})$ acts by automorphisms of the Fredholm module $(\mathcal{H}, F)$, generalising the above invariance by translations and homotheties. Indeed, given $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbf{R})$ (so that $a, b, c, d \in \mathbf{R}$. $ad - bc = 1$), we let g^{-1} act in $L^2(\mathbf{R})$ by the unitary operator:

(5) $$(g^{-1}\, \xi)(s) = \xi\left(\frac{as+b}{cs+d}\right)(cs+d)^{-1} \qquad \forall \xi \in L^2(\mathbf{R})\ ,\ s \in \mathbf{R}.$$

One checks that this representation of $SL(2, \mathbf{R})$ commutes with F. Its restriction to $\{\xi\ ,\ F\xi = \pm\xi\}$ are the two mock discrete series. The corresponding automorphisms of the algebra of functions on $\mathbf{R}$ are given by:

(6) $$(g^{-1}\, f)(s) = f\left(\frac{as+b}{cs+d}\right) \qquad \forall f \in L^\infty(\mathbf{R})\ ,\ s \in \mathbf{R}.$$

Using an arbitrary fractional linear transformation from the line $\mathbf{R}$ to the unit circle $S^1 = \{z \in \mathbf{C}\ ,\ |z| = 1\}$, such as

(7) $$s \in \mathbf{R} \to \frac{s-i}{s+i} \in S^1$$

we can transport the above Fredholm module to functions on the circle S^1. It is described as follows:

(8) $\mathcal{H} = L^2(S^1, d\theta)$ with functions on S^1 acting by multiplication (as in (1))

$$F = 2P - 1 \text{ where } P \text{ is the orthogonal projection on}$$
$$H^2(S^1) = \{\xi \in L^2\ ;\ \widehat{\xi}(n) = 0 \quad \forall n < 0\}$$

(where $\widehat{\xi}$ is the Fourier transform of ξ).

The two situations, with $\mathbf{R}$ or S^1, are unitarily equivalent provided we take in both cases the von Neumann algebras of all measurable bounded functions. We shall keep both. Our first and easy task will be to quote a number of well known results of analysis (cf. [Stein] [Pow] [Pel]) allowing to control the order of $df = [F, f]$ in terms of the regularity of the function $f \in L^\infty$.

The *strongest* condition we can ask to df is to belong to the smallest non trivial ideal of operators, namely the ideal R of finite rank operators. The necessary and sufficient condition for this to hold is a result of Kronecker (cf. [Pow]):

(9) Let $f \in L^\infty$, then $df \in R \Leftrightarrow f$ is a rational fraction.

This result holds for both $\mathbf{R}$ and S^1, in both cases the rational fraction $\frac{P(s)}{Q(s)}$ is equal a.e. to f and has no pole on $\mathbf{R}$ (resp. S^1).

The *weakest* condition we can ask to df is to be a compact operator. In fact we should restrict to the subalgebra of L^∞ determined by this condition if we want to comply with condition β) of definition 1).

The answer is known (cf. [Pow]) and easy to formulate for S^1. It involves the *mean oscillation* of the function f. Let us recall that given any interval I of S^1 one lets $I(f)$ be the mean: $\frac{1}{|I|}\int_I f\, dx$ of f on I and one defines for $a > 0$ the mean oscillation of f by:

$$M_a(f) = \sup_{|I| \le a} \frac{1}{|I|} \int_I |f - I(f)|.$$

A function is said to have bounded mean oscillation (BMO) if the $M_a(f)$ are bounded independently of a. This is of course true if $f \in L^\infty(S^1)$. A function f is said to have *vanishing* mean oscillation (VMO) if $M_a(f) \to 0$ when $a \to 0$. Let us then state the result of Fefferman and Sarazon (cf. [Pow]).

(10) Let $f \in L^\infty(S^1)$ then $[F, f] \in k \Leftrightarrow f \in VMO$.

Every continuous function $f \in C(S^1)$ belongs to VMO but the algebra $VMO \cap L^\infty$ is strictly larger than $C(S^1)$. Its elements are called quasi-continuous functions. For instance the boundary values of any bounded univalent holomorphic function $f \in H^\infty(S^1)$ belong to VMO but not necessarily to $C(S^1)$.

The next question is to characterize the functions $f \in L^\infty$ for which

$$[F, f] \in \mathcal{L}^p$$

for a given real number $p \in [1, \infty[$.

Here $\mathcal{L}^p$ is the Schatten ideal.

(11) $$\mathcal{L}^p = \{T \in k \ , \ \Sigma\ \mu_n(T)^p < \infty\}.$$

This question has a remarkably nice answer due to V.V. Peller [Pel] in terms of the Besov spaces $B_p^{1/p}$ of measurable functions.

Definition 1. *Let $p \in [1, \infty[$. Then the Besov space $B_p^{1/p}$ is the space of measurable functions f on S^1 such that*

$$\int\int |f(x+t) - 2f(x) + f(x-t)|^p\ t^{-2}\ dx\ dt < \infty.$$

For $p > 1$ this condition is equivalent to:

$$\int\int |f(x+t) - f(x)|^p\ t^{-2}\ dx\ dt < \infty$$

and the corresponding norms are equivalent. For $p = 2$ one recovers the Sobolev space of Fourier series,

$$f(t) = \sum_{n \in \mathbf{Z}} a_n\ \exp(2\pi i n t)\ ,\ \sum |n|\ |a_n|^2 < \infty.$$

The result of V.V. Peller is then the following:

(12) Let $f \in L^\infty(S^1)$, $p \in [1, \infty[$, then $[F, f] \in \mathcal{L}^p \Leftrightarrow f \in B_p^{1/p}$.

(Note at this point the nuance between $\mathcal{L}^p$ and $\mathcal{L}^{(p,\infty)}$, i.e. for $p > 1$ between $\Sigma\ \mu_n^p < \infty$ and $\mu_n = O(n^{-1/p})$. One has for instance $\mathcal{L}^p \subset \mathcal{L}^{(p,\infty)} \subset \mathcal{L}^q$ for any $q > p$.)

To end this summary of classical results let us note that, for $f \in L^\infty(S^1)$, the operator $df = [F, f]$ anticommutes with F by construction and is hence given by an off diagonal 2×2 matrix in the decomposition of $L^2(S^1)$ as a direct sum of eigenspaces of F. This 2×2 matrix is *lower triangular*, i.e. $(1-P)fP = 0$ with $P = \frac{1+F}{2}$, iff $f \in H^\infty(S^1)$, i.e. f is the boundary value of an holomorphic function in the disk.

(13) Let $f \in L^\infty(S^1)$, then $f \in H^\infty(S^1) \Leftrightarrow (1-P)fP = 0$.

In particular $df_1\ df_2 = 0$ for any $f_1, f_2 \in H^\infty(S^1)$. Moreover, for $f \in H^\infty(S^1)$ there is a very simple criterion for f to belong to the Besov space $B_p^{1/p}$, $p > 1$, which is a straightforward consequence of the definition of these spaces (cf. [Stein]).

(14) Let $f \in H^\infty(S^1)$, $p > 1$, then $f \in B_p^{1/p} \Leftrightarrow \int_{\text{Disk}} |f'(z)|^p\ (1 - |z|)^{p-2}\ dz\ d\bar{z} < \infty$.

(We have used the same letter f for the function f in the unit disk with boundary S^1.)

The right hand side can easily be controlled when the function f is *univalent* in the disk, in terms of the domain $\Omega = f(\text{Disk})$. Indeed by the Koebe $\frac{1}{4}$ theorem (cf. [Ru]) one has, for f univalent:

$$1/4\,(1-|z|^2)\ |f'(z)| \leq \operatorname{dist}(f(z),\partial\Omega) \leq (1-|z|^2)\ |f'(z)|. \tag{15}$$

It is thus straightforward to estimate the size of the quantum differential $df = [F,f]$ of a univalent map f in terms of the geometry of the domain $f(\text{Disk}) = \Omega$.

Theorem 2. ([Co-Su]) *For any $p_0 > 1$, there exists finite constants bounding the ratio of the following two quantities:*

$$\textit{Trace}\,(|[F,f]|^p) \asymp \int_\Omega \operatorname{dist}(z,\partial\Omega)^{p-2}\ dz\ d\bar{z}$$

for any univalent function f and any $p \geq p_0$.

(We use the symbol $\alpha \asymp \beta$ to mean that $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$ are bounded.)

The interval of p's such that the right hand side is finite has a lower bound, known as the Minkowski dimension of the boundary $\partial\Omega$ (cf. [Fe]). It is easy to construct domains with a given Minkowski dimension $p \in\,]1,2[$ for $\partial\Omega$. We want to go further and relate the p-dimensional Hausdorff measure Λ_p with the following formula:

$$\operatorname{Trace}_\omega\,(f(Z)\ |dZ|^p) \qquad \forall f \in C(\partial\Omega) \tag{16}$$

where $\operatorname{Trace}_\omega$ is the Dixmier trace and $dZ = [F,Z]$ the quantum differential. Here Z is the boundary value, $Z \in H^\infty(S^1)$ of a univalent map: $Z : \text{Disk} \to \Omega$. The formula (16) defines a Radon measure provided

$$dZ \in \mathcal{L}^{(p,\infty)} \tag{17}$$

which insures that $|dZ|^p \in \mathcal{L}^{(1,\infty)}$ is in the domain of the Dixmier trace.

Of course Z is in general not of bounded variation and, had we taken dZ as a distribution the symbols $|dZ|$ and $|dZ|^p$ would be meaningless.

To compare (16) with the Hausdorff measure Λ_p, let us first remind the reader of the construction of the latter. We thus open a small parenthesis.

Hausdorff measure and the Caratheodory construction.

Let (X,d) be a metric space. For any function ϕ from $\mathbf{R}^+$ to $\mathbf{R}^+$, (the prototype being $\phi(t) = t^p$ for some positive real number p), one defines a countably additive measure Λ_ϕ on the Borel tribe of X by the following equality:

$$\Lambda_\phi(A) = \lim_{\varepsilon\to 0} \operatorname*{Inf}_{(B_i)} \sum \phi(\text{diam } B_i) \tag{18}$$

where (B_i) runs through all coverings of A by balls B_i of diameter less than ε. Note that as ε decreases the quantity

$$\Lambda_\phi^\varepsilon(A) = \operatorname*{Inf}_{(B_i),\text{diam} B_i \le \varepsilon} \sum \phi(\text{diam } B_i) \tag{19}$$

increases. In particular the limit always exists in $[0, +\infty]$. The fact that Λ_ϕ is always countably additive is non trivial. Nothing insures that $\Lambda_\phi(A)$ is finite and not zero for some subset A. Given (X, d) this requires in general a very careful choice of the gauge function ϕ. The most standard choice is $\phi(t) = t^p$ giving rise to the notion of ***Hausdorff dimension*** of a subset A

$$\text{Hausdorff } \dim(A) = p \text{ iff } \quad \Lambda_{p+\varepsilon}(A) = 0 \quad \forall\varepsilon > 0 \qquad \Lambda_{p+\varepsilon}(A) = \infty \quad \forall\varepsilon < 0. \tag{20}$$

Nothing insures that $\Lambda_p(A)$ is not 0 or ∞.

In full generality the Hausdorff dimension is not the same as the Minkowski dimension (cf. [Fed]) but we shall now specialize to domains Ω whose boundaries $\partial\Omega$ are the Julia sets of iteration theory. More precisely let us consider a complex parameter c, inside the Mandelbrot cardioid, and iterate the mapping:

$$z \to \psi(z) = z^2 + c. \tag{21}$$

The set $\overline{\Omega} = \{z \in \mathbb{C} : \{\psi^{\circ n}(z)\} \text{ bounded}\}$ is the closure of a simply connected domain Ω whose boundary $C = \partial\Omega$ is a Jordan curve, the *Julia set* of ψ, whose Hausdorff dimension, satisfies $1 < p < 2$, by a result of D. Sullivan ($c \neq 0$). Let then Z : Disk $\to \Omega$ be the Riemann mapping.

Theorem 3. ([Co-Su]) *There exists a smallest $p \in]1, 2[$ such that $dZ \in \mathcal{L}^{(p,\infty)}$, p is equal to the Hausdorff dimension of C. There exists a non zero finite constant such that:*

$$\int f \, d\Lambda_p = \lambda \ \operatorname{Tr}_\omega(f(Z) \, |dZ|^p) \qquad \forall f \in C(\partial\Omega).$$

The value of the non zero finite constant λ is related to the rational approximation of Z by rational fractions (cf. [Co-Su]). This theorem could not hold if the Dixmier trace had been a *normal* functional. Indeed the two natural measures carried by the Julia set C are α) The Hausdorff measure Λ_p $\quad\beta$) The Harmonic measure. The latter is defined by the equality:

$$\int f \, d\nu_{z_0} = \tilde{f}(z_0) \qquad \forall f \in C(\partial\Omega) \tag{22}$$

where z_0 is a chosen point in Ω and $\tilde{f}$ is the unique harmonic function in Ω, continuous on $\overline{\Omega}$ and equal to f on $\partial\Omega = C$. The class of the harmonic measures ν_{z_0} is independent of the choice of z_0 and is equal to the image by $Z : S^1 \to C$ of the Lebesgue measure class.

Now, and as soon as $p > 1$ one has

$$\Lambda_p \text{ is } \textit{disjoint} \text{ of } \nu_{z_0} \qquad \forall z_0 \in \Omega. \tag{23}$$

By construction any measure defined by the formula:

$$\mu(f) = L\left(f(Z)\,|dZ|^p\right) \qquad \forall f \in C(\partial\Omega) \tag{24}$$

is automatically absolutely continuous with respect to the image by Z of the Lebesgue measure, (i.e. the harmonic measure), provided that the functional L is *normal*. It is thus a great virtue of the Dixmier trace to allow, by its non normality, to exit from the harmonic measure class.

We shall end the discussion of this example by two important remarks.

Remark 4. Let $C \subset \mathbf{C}$ be a Jordan curve whose 2-dimensional area is *positive*, $\Lambda_2(C) > 0$. (The existence of such curves is an old result of analysis.) Let Ω be a bounded simply connected domain with boundary $\partial\Omega = C$ and let $Z : \text{Disk} \to \Omega$ be the Riemann mapping. Then the finiteness of the area of $Z(\Omega)$ shows that:

$$dZ \in \mathcal{L}^2 \qquad (\text{i.e. } \operatorname{Trace}((dZ)^* \, dZ) < \infty). \tag{25}$$

This provides us with a very interesting example of a space C (or equivalently S^1 with the metric $dZ \, d\overline{Z}$) which has non zero 2-dimensional Lebesgue measure, but whose dimension in our sense is not 2 but rather $2-$, inasmuch as $\operatorname{Trace}((dZ)^* \, dZ)$ is *finite* rather than logarithmically divergent.

Remark 5. The above quantized calculus extends to functions of several real variables, i.e. to functions on manifolds (cf. [Co]). This extension is particularly nice in the context of even dimensional conformal manifolds (cf. [Co] [C-S-T]) and in particular for compact Riemann surfaces, i.e. complex curves. We shall thus briefly explain how to quantize the calculus on a compact Riemann surface Σ. In view of the above theorems 2 and 3, this might be relevant when working with random surfaces in statistical mechanics or in the higher dimensional iteration theory involved in renormalization group techniques. We are given Σ as a complex curve or, equivalently, as an oriented real conformal surface. We assume it is compact. We then define the Fredholm module $(\mathcal{H}, F)$ over the algebra of functions on S^1 as follows:

a) $\mathcal{H}$ is the Hilbert space of 1-forms on Σ with the inner product:

$$\langle \omega_1, \omega_2 \rangle = \int_\Sigma \omega_1 \wedge * \, \omega_2.$$

The action of functions is by multiplication operators,

$$(f\omega)(p) = f(p)\, \omega(p) \qquad \forall p \in \Sigma.$$

b) $F = 2P - 1$ where P is the orthogonal projection on the image of the operator d : $\{df\ ,\ f \text{ a function on } \Sigma\}$.

This construction obviously only depends upon the conformal oriented structure of Σ. With a little more care (cf. [Co] [C-S-T]) one can also insure that F anticommutes with the natural $\mathbb{Z}/2$ grading γ of 1-forms. It is also very important that the notions of Beltrami differentials and of modification of F due to a change of conformal structure fit remarkably well with our framework and are meaningful for arbitrary Fredholm modules, thanks to the Moebius group of von Neumann algebras (cf. [Co]).

Example 2. $\int EdEdE \in \mathbb{Z}$

Let $\mathcal{A}$ be an involutive algebra and $(\mathcal{H}, F)$ be a Fredholm module over $\mathcal{A}$. We shall say that $(\mathcal{H}, F)$ is *even* if the Hilbert space $\mathcal{H}$ is $\mathbb{Z}/2$ graded by a grading γ, $\gamma = \gamma^*$, $\gamma^2 = 1$ such that:

$$(1) \qquad \gamma a = a\gamma \qquad \forall a \in \mathcal{A}\ ,\ \gamma F = -F\gamma.$$

Otherwise we say that $(\mathcal{H}, F)$ is odd.

Let then $n \in \mathbb{N}$ be an integer, assume that $(\mathcal{H}, F)$ has the same parity as n and that it is $n+1$ summable, that is:

$$(2) \qquad [F, a] \in \mathcal{L}^{n+1}(\mathcal{H}) \qquad \forall a \in \mathcal{A}.$$

We shall now imitate the usual construction of differential forms on a manifold to construct an n-dimensional cycle $(\Omega, d, \int)$ on $\mathcal{A}$ in the following sense:

Definition 1. a) *A cycle of dimension n is a triple $(\Omega, d \int)$ where $\Omega = \bigoplus_{j=0}^{n} \Omega^j$ is a graded algebra over $\mathbb{C}$, d is a graded derivation of degree 1 such that $d^2 = 0$, and $\int : \Omega^n \to \mathbb{C}$ is a closed graded trace on Ω.*

b) *Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. Then a cycle over $\mathcal{A}$ is given by a cycle $(\Omega, d, \int)$ and a homomorphism $\rho : \mathcal{A} \to \Omega^0$.*

One can show ([Co]) that a cycle of dimension n over $\mathcal{A}$ is essentially determined by its *character*, the $(n+1)$-linear function τ,

$$\tau(a^0,\ldots,a^n) = \int \rho(a^0)\, d(\rho(a^1))\, d(\rho(a^2))\cdots d(\rho(a^n)) \qquad \forall a^j \in \mathcal{A}$$

and the functionals thus obtained are exactly the cyclic cocycles on $\mathcal{A}$.

Let us now construct the cycle associated to the above Fredholm module. The graded algebra $\Omega^* = \oplus\Omega^k$ is obtained as follows. For $k=0$, $\Omega^0 = \mathcal{A}$, for $k>0$ one lets Ω^k be the linear span of the operators:

$$(3) \qquad \omega = a^0[F,a^1]\cdots[F,a^k] \qquad \forall a^j \in \mathcal{A}.$$

The Hölder inequality shows that $\Omega^k \subset \mathcal{L}^{\frac{n+1}{k}}(\mathcal{H})$. The *product* is the product of operators, one has

$$(4) \qquad \omega\omega' \in \Omega^{k+k'} \ \text{ for any } \ \omega \in \Omega^k\ ,\ \omega' \in \Omega^{k'}$$

as one checks using the equality:

$$(5) \qquad (a^0[F,a^1]\cdots[F,a^k])\ a^{k+1}$$
$$= \sum_{j=1}^{k}(-1)^{k-j}\, a^0[F,a^1]\cdots[F,a^ja^{j+1}]\cdots[F,a^{k+1}] + (-1)^k\, a^0a^1[F,a^2]\cdots[F,a^{k+1}] \quad \forall a^j \in \mathcal{A}.$$

The differential $d:\Omega^* \to \Omega^*$ is defined as follows:

$$(6) \qquad d\omega = F\omega - (-1)^k\,\omega F \qquad \forall \omega \in \Omega^k.$$

Again one checks that ω belongs to Ω^{k+1} using the equality:

$$(7) \qquad F\,(a^0[F,a^1]\cdots[F,a^k]) - (-1)^k\ (a^0[F,a^1]\cdots[F,a^k])\,F = [F,a^0][F,a^1]\cdots[F,a^k] \quad \forall a^j \in \mathcal{A}.$$

(Since $F^2=1$, F anticommutes with $[F,a]$ for any $a \in \mathcal{A}$ and hence $[F,a^1]\cdots[F,a^k]F = (-1)^k\ F\ [F,a^1]\cdots[F,a^k]$.)

By construction d is a graded derivation. i.e.:

$$(8) \qquad d(\omega_1\omega_2) = (d\omega_1)\omega_2 + (-1)^{k_1}\ \omega_1\ d\omega_2 \qquad \forall \omega_j \in \Omega^{k_j}$$

moreover using $F^2 = 1$ it is straightforward that $d^2 = 0$ since for $\omega \in \Omega^k$,

$$F(F\omega - (-1)^k\ \omega F) + (-1)^k\ (F\omega - (-1)^k\ \omega F)F = 0.$$

We thus have a graded differential algebra $(\Omega^\bullet, d)$ and it remains to define a closed graded trace of degree n:

$$\int : \Omega^n \to \mathbf{C}.$$

For this we introduce the following notation. Given an operator T in $\mathcal{H}$ such that $FT + TF \in \mathcal{L}^1(\mathcal{H})$ we set:

$$\mathrm{Tr}'(T) = \frac{1}{2}\ \mathrm{Trace}\ (F(FT + TF)). \tag{9}$$

We then define $\int \omega$ for $\omega \in \Omega^n$ by the formulae:

$$\begin{array}{ll} \int \omega = \mathrm{Tr}'(\omega) & \text{if } n \text{ is } \textit{odd} \\ \int \omega = \mathrm{Tr}'(\gamma\omega) & \text{if } n \text{ is } \textit{even} \end{array} \tag{10}$$

In the last formula γ is the $\mathbf{Z}/2$ grading operator provided by the evenness of the Fredholm module $(\mathcal{H}, F)$.

These formulae make sense. Indeed for n ***odd*** and $\omega \in \Omega^n$, one has $F\omega + \omega F = d\omega \in \Omega^{n+1} \subset \mathcal{L}^{n+1/n+1} = \mathcal{L}^1$, while for n ***even*** one has $F\gamma\omega + \gamma\omega F = \gamma d\omega \in \mathcal{L}^1$ by the same argument.

Proposition 2. [Co] $(\Omega, d, \int)$ *is a cycle of dimension n over $\mathcal{A}$.*

Proof. We just need to check that $\int$ is a closed graded trace. Since $\int \omega$ only involves $d\omega$ and since $d^2 = 0$, it is clearly closed:

$$\int d\omega = 0 \qquad \forall \omega \in \Omega^n.$$

Let then $\omega \in \Omega^k$, $\omega' \in \Omega^{k'}$ with $k + k' = n$. One has, for n odd:

$$\begin{aligned} \int \omega\omega' &= \frac{1}{2}\ \mathrm{Trace}\ (Fd(\omega\omega')) = \frac{1}{2}\ \mathrm{Trace}\ (F(d\omega)\omega' + (-1)^k\ F\omega\ d\omega') \\ &= \frac{1}{2}\ \mathrm{Trace}\ ((-1)^{k+1}\ d\omega\ F\omega' + (-1)^k\ (F\omega)\ d\omega'). \end{aligned}$$

As trace $(F\omega\ d\omega') =$ Trace $(d\omega'\ F\omega)$, using the equality:

$$\mathrm{Trace}\ (T_1T_2) = \mathrm{Trace}\ (T_2T_1)\ \ ,\ \ T_j \in \mathcal{L}^{p_j}\ \ ,\ \ \frac{1}{p_1} + \frac{1}{p_2} = 1$$

we get $\int \omega\omega' = (-1)^{kk'} \int \omega'\omega$.

For n even the computation is similar.

The *character* of the above n-dimensional cycle is the following cyclic cocycle τ_n on $\mathcal{A}$:

$$\tau_n(a^0, \ldots, a^n) = \text{Trace}' \ (a^0[F, a^1] \cdots [F, a^n]) \qquad \forall a^j \in \mathcal{A} \tag{11}$$

with a γa^0 instead of a^0 when n is even.

In the construction of the character τ_n of $(\mathcal{H}, F)$ the parity of n is fixed by the parity of the module but the dimension n is only subject to a lower bound since condition (2) holds for any $n' \geq n$ if it does hold for n. This means that we get in fact a whole sequence of cyclic cocycles, τ_{n+2k}, $k \in \mathbf{N}$ associated to $(\mathcal{H}, F)$, with n the smallest compatible with condition (2).

It is the comparison between these cyclic cocycles which was ([Co]) at the origin of the periodicity operator in cyclic cohomology (section 4):

$$S : HC^n(\mathcal{A}) \to HC^{n+2}(\mathcal{A}). \tag{12}$$

Proposition 3. ([Co]) *Let $(\mathcal{H}, F)$ be an $(n+1)$ summable Fredholm module over $\mathcal{A}$ of the same parity as n. The characters τ_{n+2q} satisfy*

$$\tau_{m+2} = -\tfrac{2}{m+2}\, S\tau_m \ \ in \ \ HC^{m+2}(\mathcal{A}) \ , \ m = n + 2q \ , \ q \geq 0.$$

We refer to [Co] for the proof.

Now the main property of the cyclic cocycles obtained by formula (11), i.e. as characters of Fredholm modules, is their *integrality*, intimately related with the quantization of the calculus. This integrality means that when evaluated on K-theory classes, as in section 4, these cyclic cocycles give *integers*. These integers are indices of Fredholm operators which for instance in the *even* case are constructed as follows. We let the K-theory class $x \in K_0(\mathcal{A})$ be given by an idempotent $e \in M_q(\mathcal{A})$ and we assume for simplicity that $q = 1$ (cf. [Co]).

The matrix of F in the $\mathbf{Z}/2$ decomposition $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ such that $\gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, is of the form $F = \begin{bmatrix} 0 & Q \\ P & 0 \end{bmatrix}$ with $PQ = 1$ in $\mathcal{H}^-$ and $QP = 1$ in $\mathcal{H}^+$. Let $\mathcal{H}_1 = e\mathcal{H}^+$, $\mathcal{H}_2 = e\mathcal{H}^-$ and $P' = eP/\mathcal{H}_1$, $Q' = eQ/\mathcal{H}_2$. Then P' is a Fredholm operator from $\mathcal{H}_1$ to $\mathcal{H}_2$, Q' is a quasi inverse of P' such that $P'Q' - 1$ and $Q'P' - 1$ are both in $\mathcal{L}^{\frac{n+1}{2}}$ if $(\mathcal{H}, F)$ is $n+1$ summable. Indeed these operators are restrictions of

$$e - eFeFe = -e\,[F, e]^2\, e \in \mathcal{L}^{\frac{n+1}{2}}.$$

It then follows that:

$$\text{Index } P' = \text{ Trace } (1 - Q'P')^k - \text{ Trace } (1 - P'Q')^k$$

for any integer $k \geq \frac{n+1}{2}$. thus we get for any $m \geq \frac{n+1}{2}$

$$\text{Index } P' = \text{ Trace } (\gamma(e - eFeFe)^m).$$

Now the pairing of $[e] \in K_0(\mathcal{A})$ with the representative τ_{2m} of the Chern character is given by: (for $m > \frac{n+1}{2}$ so that $\mathrm{Tr}' = \mathrm{Tr}$),

$$(-1)^m \text{ Trace } \left(\gamma e\, [F, e]^{2m}\right)$$

which is precisely the same formula.

We thus have the following fundamental fact:

Theorem 4. *Any representative $\tau \in HC^*(\mathcal{A})$ of the Chern character of a finitely summable Fredholm module pairs* integrally *with K-theory, i.e.*

$$\tau(K) \subset \mathbb{Z}.$$

Before we proceed and apply, with Bellissard, this result to the integrality of the Hall conductivity on the plateaux, we need to make an important point on the nuance between the *ordinary* trace, Tr, used in the definition of the character τ_n of a Fredholm module and the Dixmier trace Tr_ω used for instance in example 1.

The point is that there is an easy to compute formula, using the Dixmier trace, namely:

$$\varphi_n(a^0, \ldots, a^n) = \mathrm{Tr}_\omega \left(a^0[D, a^1] \cdots [D, a^n]\, |D|^{-n}\right) \qquad \forall a^j \in \mathcal{A} \tag{13}$$

(with γa^0 instead of a^0 in the even case) where D is any selfadjoint unbounded opertor such that

$$\text{Sign } D = F \;.\; [D, a] \text{ bounded } \forall a \in \mathcal{A} \;,\; |D|^{-n} \in \mathcal{L}^{(1,\infty)}. \tag{14}$$

The formula (13) defines a *Hochschild cocycle* and this cocycle is equal to the image $I(\tau_n)$ of the character τ_n (cf. [Co], chapter 6). Thus the Dixmier trace computes only the Hochschild class of the character, i.e. the obstruction (cf. section 4) to lower the dimension n. For instance in the case of manifolds (cf. section 4 for the computation of the cyclic cohomology of C^∞ (manifold)), the Dixmier trace formula above, (13), will only compute the top dimensional current of the character, but not the lower dimensional homology classes, (the Pontrjagin classes of the manifold) essential to insure the integrality result of theorem 4. In other words while the Dixmier trace is easily computable and yields classical formulae, the integrality results involve using the ordinary trace instead. The nuance between the two is given by the characteristic classes of the space.

Let us now come back to the quantum Hall effect. We shall now describe the construction by Bellissard of a Fredholm module $(\mathcal{H}, F)$ over the algebra of functions on the non commutative Brillouin zone. We thus use the notations of section 4 and consider the natural representation $(a, \xi) \to a\xi$ of the C^*-algebra A in the one particle Hilbert space $L^2(\mathbf{R}^2) = \mathcal{H}$.

The even Fredholm module $(\mathcal{H}', F, \gamma)$ over A is defined as follows:

$$\mathcal{H}' = L^2(\mathbf{R}^2) \otimes \mathbf{C}^2 = \mathcal{H}^+ \oplus \mathcal{H}^- \tag{15}$$

$$A \text{ acts by } a \to a \otimes 1 = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \quad \forall a \in A$$

$$\gamma = 1 \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$F = \begin{bmatrix} 0 & U \\ U^* & O \end{bmatrix}$ where U is the operator in $L^2(\mathbf{R}^2)$ of multiplication by the function u,

$$u(x_1, x_2) = \frac{x_1 + ix_2}{|x_1 + ix_2|}.$$

It is not difficult to see in the case of periodic crystals that the formula (15) defines an even summable Fredholm module over the C^*-algebra A which quantizes the cyclic 2-cocycle giving the Hall conductivity by the Kubo formula (cf. section 4).

What is quite remarkable is that the same formula (15) continues to work in the presence of disorder and continues to quantize the Hall conductivity in situations where the spectrum of the Hamiltonian no longer has gaps. This is the real situation, in real samples, due to the presence of small amounts of impurities the charge carrier density N and the Fermi level μ are monotonic functions of each other and the spectrum of the Hamiltonian H is equal to $[0, +\infty[$. This shows that the spectral projection E_μ no longer belongs to the C^*-algebra A. Thus E_μ only belongs, a priori, to the von Neumann algebra W weak closure of A in the Hilbert space $\mathcal{H}'$. However J. Bellissard proved that, when μ lies in a region of *localized states*, not contributing to the conductivity, the spectral projection E_μ is ***quasicontinuous*** in the following sense (compare example 1).

Definition 5. *Let A be a C^*-algebra and $(\mathcal{H}, F)$ a Fredholm module over A. Let W be the von Neumann algebra weak closure of A. Then an element $f \in W$ is called* quasicontinuous *iff*

$$[F, f] \in k$$

(i.e. is a compact operator).

Using on the algebra of functions of the Hamiltonian $\{f(H)\ ,\ f \in C_0(\mathbf{R})\}$ the Fredholm module given by formula (15), we can thus state the general result of J. Bellissard:

Lemma 6. *If μ lies in a gap of extended states of H then the characteristic function $E_\mu(\lambda) = \{1$ if $\lambda \leq \mu$; 0 if $\lambda > \mu\}$ is quasicontinuous on* Spectrum H.

It follows then that exactly as in the periodic case, the Fredholm module (15) quantizes the Hall conductivity on the plateaux. We refer to [Bel] and [Co] for a more detailed description of the general case.

Example 3. The trace of the metric.

In this section we shall use the quantized calculus to find the analogue in dimension 4 of the 2-dimensional Polyakov action, namely:

$$(1) \qquad I = \frac{1}{2\pi} \int_\Sigma \eta_{ij}\, dX^i \wedge * \, dX^j$$

for a Riemann surface Σ and a map X from Σ to a d-dimensional space M.

Our first task will be to write the Polyakov action (1) as the Dixmier trace of the operator:

$$(2) \qquad \Sigma\, \eta_{ij}\, dX^i\, dX^j$$

where now $dX = [F, X]$ is the quantum differential of X taken using the canonical Fredholm module $(\mathcal{H}, F)$ of the Riemann surface Σ.

The same expression will then continue to make sense in dimension 4, i.e. with Σ replaced by a 4-dimensional conformal manifold. The action we shall get will be conformally invariant by construction and intimately related to the Einstein action of gravity.

In general, given an even dimensional conformal manifold Σ, $\dim \Sigma \doteq n = 2m$, we let $\mathcal{H} = L^2(\Sigma, \wedge_{\mathbb{C}}^m T^*)$ be the Hilbert space of square integrable forms of middle dimension, in which functions on Σ act as multiplication operators.

We let $F = 2P - 1$ be the operator in $\mathcal{H}$ obtained from the orthogonal projection P on the image of d. It is clear that both $\mathcal{H}$ and F only depend upon the conformal structure of Σ, which we assume to be *compact*.

In terms of an arbitrary Riemannian metric compatible with the conformal structure of Σ one has the formula:

$$(3) \qquad F = (dd^* - d^*d)(dd^* + d^*d)^{-1} \quad \text{on} \quad L^2(\Sigma, \wedge^m T^*)$$

which ignores the finite dimensional subspace of Harmonic forms, irrelevant in our later computations (cf. [Co] for a definition of F taking this in account).

By construction F is a pseudodifferential operator of order 0, whose principal symbol is given by:

Lemma 1. *The principal symbol $\sigma_0(F)$ is given by:*

$$\sigma_0(x,\xi) = (e_\xi\ i_\xi - i_\xi\ e_\xi)\ \|\xi\|^{-2}\ ,\ \forall (x,\xi) \in T^*(\Sigma).$$

We have denoted by e_ξ (resp. i_ξ) the exterior multiplication (resp. interior) by ξ.

When $n = \dim \Sigma = 2$, one has $\wedge^m_{\mathbb{C}}\ T^* = T^*_{\mathbb{C}}$ and σ_0 associates to any $\xi \neq 0$, $\xi \in T^*_x(\Sigma)$, the symmetry with axis ξ. For any function $f \in C^\infty(\Sigma)$, the operator $[F,f]$ is pseudodifferential of order -1. Its principal symbol is the Poisson bracket $\{\sigma_0, f\}$,

$$\{\sigma_0, f\}(x,\xi) = 2\left(e_{df}\ i_\xi + e_\xi\ i_{df} - 2e_\xi\ i_\xi\ \langle \xi, df\rangle\ \|\xi\|^{-2}\right)\ \|\xi\|^{-2}. \tag{4}$$

For $\|\xi\| = 1$, decompose df as $\langle df,\xi\rangle\xi + \eta$ where $\eta \perp \xi$. Then $\{\sigma_0, f\}(x,\xi) = 2(e_\eta\ i_\xi + e_\xi\ i_\eta)$, and its Hilbert Schmidt norm, for $n = 2$, is given by:

$$\operatorname{trace}\left(\{\sigma_0,f\}(x,\xi)^*\ \{\sigma_0,f\}(x,\xi)\right) = 8\|\eta\|^2\ ,\ \eta = df - \langle df,\xi\rangle\xi.$$

The Dixmier trace $\mathrm{Tr}_\omega\left(f_0[F,f_1]^*\ [F,f_2]\right)$ is thus easy to compute for $n = 2$, as the integral on the unit sphere $S^*\Sigma$ of the cotangent bundle of Σ, of the function:

$$\operatorname{trace}\left(f_0\ \{\sigma_0,f_1\}^*\ \{\sigma_0,f_2\}\right) = 8\ f_0(x)\ \langle df_1^\perp, df_2^\perp\rangle$$

where $df^\perp = df - \langle df,\xi\rangle\xi$ by convention. One thus gets:

Proposition 2. *Let Σ be a compact Riemann surface ($n = 2$), then for any smooth map $X = (X^i)$ from Σ to $\mathbb{R}^d$ and metric $\eta_{ij}(x)$ on $\mathbb{R}^d$ one has*

$$\frac{1}{2\pi}\int_\Sigma \eta_{ij}\ dX^i \wedge * \, dX^j = \lambda\ \mathrm{Tr}_\omega\left(\eta_{ij}\ [F,X^i][F,X^j]\right).$$

Both sides of the equality have obvious meaning when the η_{ij} are constants. In general one just views them as functions on Σ namely $\eta_{ij} \circ X$.

Let us now pass to the more involved 4-dimensional case. We want to compute the following action defined on smooth maps $X : \Sigma \to \mathbb{R}^d$ of a 4-dimensional compact conformal manifold Σ to $\mathbb{R}^d$, endowed with the metric $\eta_{ij}\ dx^i\ dx^j$.

$$I = \mathrm{Tr}_\omega\left(\eta_{ij}\ [F,X^i][F,X^j]\right). \tag{5}$$

Here we are beyond the natural domain of the Dixmier trace Tr_ω but we can use the remarkable fact, due to Wodzicki, that it extends uniquely as a trace on the algebra of pseudodifferential operators (cf. [Wo$_2$]). For practical purposes the local formula for this extension, which we still denote by Tr_ω, is given as follows:

$$(6) \qquad \mathrm{Tr}_\omega(P_\sigma) = \int_{S^*\Sigma} \sigma_{-4}(x,\xi)\, d^4x\, d^3\xi$$

where P_σ is a pseudodifferential operator whose total symbol

$$(7) \qquad \sigma(x,\xi) = \sigma_0(x,\xi) + \sigma_{-1}(x,\xi) + \sigma_{-2}(x,\xi) + \cdots$$

has $\sigma_{-4}(x,\xi)$, as the component of order -4.

This formula makes sense for scalar pseudodifferential operators, defined in local coordinates x^j by the usual formula:

$$(8) \qquad (P_\sigma)(x,y) = \int e^{i\langle x-y,\xi\rangle}\, \sigma(x,\xi)\, d^4\xi$$

but, by [Wo$_2$] it is independent of the choice of local coordinates and defines a trace, Tr_ω, on the algebra of scalar pseudodifferential operators.

When we consider a vector bundle E over a manifold Σ, and a pseudodifferential operator P acting on sections of E, we compute $\mathrm{Tr}_\omega(P)$ as follows. Choose local coordinates x^j and local basis of sections α_k for the bundle E. Then P appears as a matrix P_k^ℓ of scalar pseudodifferential operators:

$$P(f^k\, \alpha_k) = (P_k^\ell\, f^k)\, \alpha_\ell.$$

The expression $\mathrm{Tr}_\omega(P) = \mathrm{Tr}_\omega(P_k^k)$ is then independent of the choice of the local basis (α_k) of E and defines a trace.

It is clear that to compute the action I we just need to compute the following trilinear form τ on $C^\infty(\Sigma)$.

$$(9) \qquad \tau(f_0, f_1, f_2) = \mathrm{Tr}_\omega\,(f_0\,[F, f_1][F, f_2]) \qquad \forall f_j \in C^\infty(\Sigma).$$

By construction τ is a Hochschild 2-cocycle on $C^\infty(\Sigma)$. We let $\Omega(f_1, f_2)$ be the 4-dimensional differential form on Σ uniquely determined by the equation:

$$(10) \qquad \tau(f_0, f_1, f_2) = \int_\Sigma f_0\, \Omega(f_1, f_2) \qquad \forall f_0 \in C^\infty(\Sigma).$$

The existence of Ω follows from the general formula for the total symbol of the product of two pseudodifferential operators P_{σ_1}, P_{σ_2}, in terms of σ_1 and σ_2:

$$\sigma(x,\xi) = \sum \frac{1}{\alpha!}\, \partial_\xi^\alpha\, \sigma_1(x,\xi)\; D_x^\alpha\; \sigma_2(x,\xi) \tag{11}$$

where $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a multiindex, $\alpha! = \alpha_1!\ \alpha_2!\ \alpha_3!\ \alpha_4!$ and $D_x^\alpha = (-i)^{|\alpha|}\ \partial_x^\alpha$.

This formula, applied with $P_{\sigma_1} = f_0$, $P_{\sigma_2} = [F, f_1][F, f_2]$ shows the existence of Ω.

Our task is to compute, given $x \in \Sigma$, the value of the differential form $\Omega(f_1, f_2)$ at x, in terms of f_1, f_2 and the conformal structure of Σ.

We shall take local coordinates x^j around x and let $\omega^a = dx^i\ \wedge\ dx^j$ be the corresponding basis for our vector bundle $E = \wedge_{\mathbb{C}}^2\, T^*$ over Σ.

Let $P = [F, f_1][F, f_2]$. It is a pseudodifferential operator of order -2 and in terms of its symbol up to order -4:

$$\sigma = \sigma_{-2} + \sigma_{-3} + \sigma_{-4} \tag{12}$$

where we have omitted the α, β matrix indices. We get the following formula for $\Omega(f_1, f_2)$ at x:

$$\Omega(f_1, f_2) = \left(\int_{S^3} \operatorname{trace}(\sigma_{-4}(x,\xi))\, d^3\xi\right)\ dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4 \tag{13}$$

where S^3 is the unit sphere in the ξ variable and $d^3\xi$ the normalized volume on S^3.

Next the total symbol σ, up to order -4 included, is obtained by formula (11) (and with more matrix indices) from the total symbols $\sigma([F, f_1])$, $\sigma([F, f_2])$ which we only need to know up to order -3 included. To compute them we again use formula (11) for $Ff - fF$ and we thus only need to know the total symbol of F up to order -2.

The computation is done using formula (3) and (11). What matters is the way the variables g_{ij} enter the formula:

Lemma 3. *The total symbol σ^F of F, up to order -2 included, is a 6×6 matrix of the form:*

$$\sigma^F = \sigma_0^F + \sigma_{-1}^F + \sigma_{-2}^F$$

where σ_0^F only invokes $g_{ij}(x)$, σ_{-1}^F is linear in the 1-jet of the metric (at x) with coefficients depending smoothly on $g_{ij}(x)$, σ_{-2}^F is linear in the 2-jet of the metric + quadratic in the 1-jet of the metric, with coefficients depending smoothly on $g_{ij}(x)$.

Proof. Both operators $dd^* - d^*d$ and $\Delta = dd^* + d^*d$ acting on $\wedge_{\mathbb{C}}^2\, T^* = E$ can be expanded in our local basis in the form:

$$\sigma(dd^* - d^*d) = q_2 + q_1 + q_0$$

(14)

$$\sigma(\Delta) = p_2 + p_1 + p_0$$

where p_2, q_2 only invoke the $g_{ij}(x)$, and p_1, q_1, p_0, q_0 have the properties indicated in the lemma for σ_{-1}^F, σ_{-2}^F (cf. for instance [Gi$_1$] lemma 2.4.2 p.118).

Now to compute the total symbol $\sigma(\Delta^{-1})$ up to order -2 let us denote by $\circ$ the product of symbols as defined by formula (11). One has:

$$\sigma(\Delta^{-1}) = p \circ (1 - \varepsilon_{-1} - \varepsilon_{-2} + \varepsilon_{-1}^2) \tag{15}$$

where, with $p(x,\xi) = (p_2(x,\xi))^{-1}$ one lets

$$\Delta \circ p = 1 + \varepsilon_{-1} + \varepsilon_{-2} \tag{16}$$

be the total symbol of $\Delta \circ p$ up to order -2 included. By construction p only depends upon the $g_{ij}(x)$, so that by the formula (11) the symbols ε_{-1}, ε_{-2} satisfy the conditions of lemma 3 (with ε_{-k} linear in the k-jet of the metric + square of 1-jet for $k = 2$).
It thus follows from the formula (15) that $\sigma(\Delta^{-1})$ has a similar expansion:

$$\sigma(\Delta^{-1}) = \sigma_{-2}(\Delta^{-1}) + \sigma_{-3}(\Delta^{-1}) + \sigma_{-4}(\Delta^{-1}) \tag{17}$$

with $\sigma_{-2-k}(\Delta^{-1})$ linear in the k-jet of the metric + eventual quadratic terms for $k = 2$.

Finally when we compute the composition

$$\sigma(dd^* - d^*d) \circ \sigma(\Delta^{-1}) = \sigma_0^F + \sigma_{-1}^F + \sigma_{-2}^F$$

we get, using formulas (14) and (11) the required property.

Now the total symbol of $[F, f]$, up to order -3 included, is of the form:

$$\begin{aligned} &\textstyle\sum_{|\alpha|=1} \partial_\xi^\alpha\, \sigma_0^F\, D^\alpha f \;+ \sum_{|\alpha|=2} \frac{1}{\alpha!}\, \partial_\xi^\alpha\, \sigma_0^F\, D^\alpha f \;+ \sum_{|\alpha|=3} \frac{1}{\alpha!}\, \partial_\xi^\alpha\, \sigma_0^F\, D^\alpha f \\ &\qquad\textstyle+ \sum_{|\alpha|=1} \partial_\xi^\alpha\, \sigma_{-1}^F\, D^\alpha f \;+ \sum_{|\alpha|=2} \frac{1}{\alpha!}\, \partial_\xi^\alpha\, \sigma_{-1}^F\, D^\alpha f \\ &\qquad\qquad\textstyle+ \sum_{|\alpha|=1} \partial_\xi^\alpha\, \sigma_{-2}^F\, D^\alpha f. \end{aligned} \tag{18}$$

Note that the differentiation ∂_ξ^α does not alter the properties of σ_{-k}^F stated in the lemma, so that for instance $\partial_\xi^\alpha\, \sigma_{-1}^F$ is linear in the 1-jet of the metric.

To compute $\Omega(f_1, f_2)$ we need the component of order -4 of the total symbol of $[F, f_1][F, f_2]$. This component σ_{-4} is obtained by composition (i.e. using formula (11)) of the expressions (18) applied to f_1 and f_2. We thus get:

$$\sigma_{-4} = \sum \left(\frac{1}{\alpha!} \, \partial_\xi^\alpha \; \sigma_{-k}^F \; D^\alpha \; f_1 \right) \circ \left(\frac{1}{\beta!} \, \partial_\xi^\beta \; \sigma_{-\ell}^F \; D^\beta \; f_2 \right) \tag{19}$$

where the sum is restricted to $|\alpha| \geq 1$, $|\beta| \geq 1$, $|\alpha| + k + |\beta| + \ell \leq 4$, and one takes in the composition $\circ$ of the symbols, its component of degree -4 only. In other words, using (11) and $\partial_\xi^\alpha (D^\alpha f_1) = 0$ we get:

$$\sigma_{-4} = \sum \frac{1}{\alpha!} \, \frac{1}{\beta!} \, \frac{1}{\gamma!} \, \frac{1}{\delta!} \, (D^\alpha f_1) \left(\partial_\xi^{\alpha+\gamma+\delta} \; \sigma_{-k}^F \right) \left(\partial_\xi^\beta \; D^\gamma \; \sigma_{-\ell}^F \right) (D^{\beta+\delta} f_2) \tag{20}$$

where the sum is restricted to $|\alpha| \geq 1$, $|\beta| \geq 1$, $|\alpha| + |\beta| + |\gamma| + |\delta| + k + \ell = 4$. The inequality $k + |\gamma| + \ell \leq 2$ allows to write σ_{-4} as a sum of 3 terms according to the value of $k + |\gamma| + \ell \in \{0, 1, 2\}$. The term $\sigma_{-4}^{(0)} = \sum\limits_{k=\ell=|\gamma|=0}$ depends only upon the $g_{ij}(x)$. The term $\sigma_{-4}^{(1)} = \sum\limits_{k+\ell+|\gamma|=1}$ is linear in the 1-jet of the metric with coefficients depending smoothly on the g_{ij}. The term $\sigma_{-4}^{(2)} = \sum\limits_{k+\ell+|\gamma|=2}$ is the sum of a linear term in the 2-jet of the metric and of a quadratic term in the 1-jet, both with coefficients depending smoothly on the g_{ij}. Since $|\alpha| + |\beta| + |\delta| = 2$ if $k + \ell + |\gamma| = 2$ we see that $\sigma_{-4}^{(2)}$ only involves the 1-jet of f_1 and f_2 at x.

These properties of $\sigma_{-4}^{(k)}$ obviously persist after integration of the ξ variable on the unit sphere S^3 of $T_x^*(\Sigma)$. Choosing the coordinates x^j to be geodesic normal coordinates at the point x, we can assume that $g_{ij}(x) = \delta_{ij}$, that the 1-jet of g_{ij} at x vanishes and that the 2-jet is expressed in terms of the curvature tensor $R_{ijk\ell}$, at x. We thus get:

Lemma 4. *There exists a universal bilinear expression $B(\nabla^\alpha \, df_1 \, , \, \nabla^\beta \, df_2)$ and a trilinear form $C(R, df_1, df_2)$ such that:*

$$\Omega(f_1, f_2) = \left(B(\nabla^\alpha \; df_1, \nabla^\beta \; df_2) + C(R, df_1, df_2) \right) \; dv$$

where R is the curvature tensor, ∇ the covariant differentiation and dv the volume form of a Riemannian structure compatible with the given conformal structure.

In order to determine the bilinear expression B we just need to perform the computation of $\Omega(f_1, f_2)$ in the flat case. Note that our notation ∇^α is ambiguous for $|\alpha| > 1$ since the covariant derivatives do not commute, but only $|\alpha| \leq 2$ will be involved and the corresponding ambiguity is absorbed by the term $C(R, df_1, df_2)$. We shall determine C using conformal invariance of $\Omega(f_1, f_2)$, but let us begin by the computation of Ω in the flat case.

In the flat case we have:

$$(21) \qquad \sigma_0^F(x,\xi) = (e_\xi \ i_\xi - i_\xi \ e_\xi) \ \|\xi\|^{-2} \ , \ \sigma_{-k}^F = 0 \qquad \forall k > 0.$$

As σ_0^F is independent of x, the formula (20) simplifies to

$$(22) \qquad \sigma_{-4}(x,\xi) = \Sigma \ \frac{1}{\alpha!} \ \frac{1}{\beta!} \ \frac{1}{\delta!} \ \left(\partial_\xi^{\alpha+\delta} \ \sigma_0^F\right) \left(\partial_\xi^\beta \ \sigma_0^F\right) (D^\alpha \ f_1)(x) \left(D^{\beta+\delta} \ f_2\right)(x)$$

where the sum is performed for multiindices α, β, δ such that $|\alpha| + |\beta| + |\delta| = 4$, $|\alpha| \geq 1$, $|\beta| \geq 1$.

Let us consider the function of three vector variables ξ, μ, ν given by

$$(23) \qquad f(\xi,\mu,\nu) = \Sigma \ \frac{1}{\alpha!} \ \frac{1}{\beta!} \ \frac{1}{\delta!} \ \mathrm{trace}\left(\left(\partial_\xi^{\alpha+\delta} \ \sigma_0^F\right) \left(\partial_\xi^\beta \ \sigma_0^F\right)\right) \ \mu^\alpha \ \nu^{\beta+\delta}$$

where the sum is performed with the same conditions as in (22). By construction we thus have in the flat case

$$(24) \qquad \Omega(f_1,f_2) = \left(\Sigma \ A_{\alpha,\beta} \ \ (D^\alpha \ f_1)(x) \left(D^\beta \ f_2\right)(x)\right) \ dx^1 \wedge \ldots \wedge dx^4$$

where $\Sigma \ A_{\alpha,\beta} \ \mu^\alpha \ \nu^\beta = \int_{S^3} \ f(\xi,\mu,\nu) \ d^3\xi$.

To determine the function $f(\xi,\mu,\nu)$ we use the equality:

$$(25) \qquad f(\xi,\mu,\nu) = g(\xi,\mu+\nu,\nu) + \text{ terms not involving } \mu$$

where $g(\xi,\mu,\nu) = \Sigma \ \frac{1}{\alpha!} \ \frac{1}{\beta!} \ \mathrm{trace}\left(\partial_\xi^\alpha \ \sigma_0^F \ \partial_\xi^\beta \ \sigma_0^F\right) \ \mu^\alpha \ \nu^\beta$ with the sum performed for $|\alpha| \geq 1$, $|\beta| \geq 1$, $|\alpha| + |\beta| = 4$. Thus $g(\xi,\mu,\nu)$ reads of from the Taylor expansion of $h(\xi+\mu,\xi+\nu)$ with:

$$(26) \qquad h(\xi,\eta) = \ \mathrm{trace}\left(\sigma_0^F(\xi) \ \sigma_0^F(\eta)\right) = 2\langle\xi,\eta\rangle^2 \ \|\xi\|^{-2} \ \|\eta\|^{-2} + \ \text{constant.}$$

A straightforward calculation of the Taylor expansion of h on the diagonal gives:

$$(27) \qquad \begin{aligned} g(\xi,\mu,\nu) = {} & 2\|\mu\|^2 \ \langle\xi,\mu\rangle \ \langle\xi,\nu\rangle - 4\langle\xi,\mu\rangle^2 \ \langle\mu,\nu\rangle - 4\langle\xi,\mu\rangle^3 \ \langle\xi,\nu\rangle \\ & + \langle\mu,\nu\rangle^2 - \|\nu\|^2 \ \langle\xi,\mu\rangle^2 - \|\mu\|^2 \ \langle\xi,\nu\rangle^2 + \|\mu\|^2 \ \|\nu\|^2 \\ & + 2\|\nu\|^2 \ \langle\xi,\nu\rangle \ \langle\xi,\mu\rangle - 4\langle\xi,\nu\rangle^2 \ \langle\nu,\mu\rangle - 4\langle\xi,\nu\rangle^3 \ \langle\xi,\mu\rangle. \end{aligned}$$

If we use on S^3 the normalized volume element of integral one we have:

$$(28) \qquad \int_{S^3} \langle \xi, \mu \rangle \, \langle \xi, \nu \rangle \, d^3\xi = \frac{1}{4} \, \langle \mu, \nu \rangle$$

$$(29) \qquad \int_{S^3} \langle \xi, \mu \rangle^3 \, \langle \xi, \nu \rangle \, d^3\xi = \frac{1}{8} \, \langle \mu, \nu \rangle \, \|\mu\|^2.$$

We thus get:

$$(30) \qquad \int_{S^3} g(\xi, \mu, \nu) \, d^3\xi = -\|\mu\|^2 \, \langle \mu, \nu \rangle + \langle \mu, \nu \rangle^2 + \frac{1}{2} \, \|\mu\|^2 \, \|\nu\|^2 - \|\nu\|^2 \, \langle \mu, \nu \rangle.$$

Using equality (25) we just need to determine the terms involving both μ and ν in the expression:

$$-\|\mu + \nu\|^2 \, \langle \mu + \nu, \nu \rangle + \langle \mu + \nu, \nu \rangle^2 + \frac{1}{2} \, \|\mu + \nu\|^2 \, \|\nu\|^2 - \|\nu\|^2 \, \langle \mu + \nu, \nu \rangle$$

and we get the desired result:

$$(31) \qquad \Sigma \, A_{\alpha,\beta} \, \mu^\alpha \, \nu^\beta = -\|\mu\|^2 \, \langle \mu, \nu \rangle - \langle \mu, \nu \rangle^2 - \frac{1}{2} \, \|\mu\|^2 \, \|\nu\|^2 - \|\nu\|^2 \, \langle \mu, \nu \rangle.$$

Using equality (24) we get the following formula for $\Omega(f_1, f_2)$ in the flat case:

$$(32) \qquad \Omega(f_1, f_2) = \left(-\Delta \left(\langle df_1, df_2 \rangle \right) + \langle \nabla df_1, \nabla df_2 \rangle - \frac{1}{2} \, \Delta f_1 \, \Delta f_2 \right) \, dx^1 \wedge \ldots \wedge dx^4$$

where $\Delta = \Sigma \, \partial_j^2$ is the Laplacian and ∇ the covariant derivative. We can thus use lemma 4 and summarize what we have found so far in the following:

Lemma 5. *There exists a universal trilinear form $C(R, df_1, df_2)$ in the curvature R and the covectors df_1, df_2 such that, in full generality, one has:*

$$(33) \quad \Omega(f_1, f_2) = \left(-\Delta \left(\langle df_1, df_2 \rangle \right) + \langle \nabla df_1, \nabla df_2 \rangle - \frac{1}{2} \, (\Delta f_1)(\Delta f_2) + C(R, df_1, df_2) \right) \, dv.$$

Our next task is to use conformal invariance of $\Omega(f_1, f_2)$ to determine the term $C(R, df_1, df_2)$. Thus let us replace the metric g_{ij} of Σ by $(1 + \delta) \, g_{ij}$ where δ is a smooth function on Σ and compute, to first order in δ, the variation of the various terms of formula (33).

The perturbation of the Levi Civita connection is given, up to order one in δ, by the following bundle map $T^* \to T^* \otimes T^*$:

$$(\nabla' - \nabla)\omega = -\frac{1}{2}\,(\omega \otimes d\delta + d\delta \otimes \omega - \langle d\delta, \omega\rangle\, g) \in T^* \otimes T^* \tag{34}$$

where we used the symbol g for the metric viewed as an element of $T^* \otimes T^*$.

We can then compute the perturbation, up to order one:

$$\begin{gathered}\langle \nabla' df_1, \nabla' df_2\rangle' - \langle \nabla df_1, \nabla df_2\rangle = \\ \langle (\nabla' - \nabla) df_1, \nabla df_2\rangle + \langle \nabla df_1, (\nabla' - \nabla) df_2\rangle + \langle \nabla df_1, \nabla df_2\rangle' - \langle \nabla df_1, \nabla df_2\rangle.\end{gathered}$$

The first term gives, using (34) and the equality

$$\langle \nabla df, g\rangle = \Delta f \qquad \forall f \in C^\infty(\Sigma). \tag{35}$$

$$\begin{aligned}&-\frac{1}{2}\,\langle (df_1 \otimes d\delta + d\delta \otimes df_1 - \langle d\delta, df_1\rangle\, g)\,, \nabla df_2\rangle \\ &\quad = -\frac{1}{2}\,(2\langle df_1, \nabla_{d\delta}(df_2)\rangle - \langle d\delta, df_1\rangle\, \Delta f_2)\,.\end{aligned}$$

We can thus rewrite the sum of the first two terms as

$$-\langle d\delta, d\langle df_1, df_2\rangle\rangle + \frac{1}{2}\,\langle d\delta, df_1\rangle\, \Delta f_2 + \frac{1}{2}\,\langle d\delta, df_2\rangle\, \Delta f_1. \tag{36}$$

The last two terms just contribute

$$-2\delta\,\langle \nabla df_1, \nabla df_2\rangle. \tag{37}$$

We thus have to add (36) and (37) to get the perturbation of the middle term in (33).

Similarly the general formula to order one in δ:

$$(\Delta' - \Delta)h = \langle dh, d\delta\rangle - \delta\Delta h \qquad \forall h \in C^\infty(\Sigma) \tag{38}$$

shows that the perturbations of the first and third terms of (33) are respectively:

$$-\Delta'\,\langle df_1, df_2\rangle' + \Delta\,\langle df_1, df_2\rangle = -\langle d\delta, d\,\langle df_1, df_2\rangle\rangle + \delta\Delta\,\langle df_1, df_2\rangle + \Delta(\delta\,\langle df_1, df_2\rangle) \tag{39}$$

$$(40) \qquad -\frac{1}{2}\left(\Delta' f_1\ \Delta' f_2 - \Delta f_1\ \Delta f_2\right) = \delta\ \Delta f_1\ \Delta f_2 - \frac{1}{2}\ \langle df_1, d\delta\rangle\ \Delta f_2 - \frac{1}{2}\ \langle df_2, d\delta\rangle\ \Delta f_1.$$

Adding (36), (37), (39) and (40) gives the following expression for the perturbation $T' - T$ of the sum of the first three terms of (33)

$$(41) \qquad T' - T = -2\delta T - 2\,\langle d\delta, d\ \langle df_1, df_2\rangle\rangle + \Delta\,(\delta\langle df_1, df_2\rangle) - \delta\ \Delta\,(\langle df_1, df_2\rangle).$$

The general identity

$$(42) \qquad \Delta(fh) - f\ \Delta h - (\Delta f)h = 2\langle df, dh\rangle \qquad \forall f, h \in C^\infty(\Sigma)$$

applied with $f = \delta$, $h = \langle df_1, df_2\rangle$ thus gives:

$$(43) \qquad T' - T = -2\delta T + (\Delta\delta)\ \langle df_1, df_2\rangle.$$

Thus, as up to order one in δ we have $(dv)' - dv = 2\delta\ dv$, the differential form $T\ dv$ satisfies, to order one in δ:

$$(44) \qquad T'\ (dv)' - T\ dv = \Delta\delta\ \langle df_1, df_2\rangle\ dv.$$

We hence just need to find $C(R, df_1, df_2)$ such that

$$(45) \qquad C'\ dv' - C\ dv = -\Delta\delta\ \langle df_1, df_2\rangle\ dv.$$

The perturbation of the Riemannian curvature R viewed as a linear map $R : \wedge^2 T^* \to \wedge^2 T^*$ is given by:

$$(47) \qquad R' - R = -\delta R + 1/2\ \wedge^2(\nabla d\delta)$$

where $\wedge^2(\nabla d\delta)$ is the natural action of the second derivative $\nabla d\delta$ on $\wedge^2 T^*$, at the Lie algebra level. The curvature scalar, $r = \text{trace}\ R$ thus satisfies:

$$(48) \qquad r' - r = -\delta r + 3/2\ (\Delta\delta).$$

We hence get the following natural solution of (45):

(49) $$C(R, df_1, df_2) = -2/3\ r\ \langle df_1, df_2\rangle.$$

What we know so far is that $-\frac{2}{3}\ r\ \langle df_1, df_2\rangle$ is a possible solution. It is in fact the only one since the only other invariant expression $C(R, df_1, df_2)$ that could be added is a multiple of the Ricci tensor applied to $df_1 \otimes df_2$ and one checks that it fails to give, when multiplied by dv, a conformally invariant answer.

We can thus summarize what we found as follows:

Theorem 6. *Let Σ be a 4-dimensional conformal manifold, $X : \Sigma \to \mathbb{R}^d$ a smooth map, $\eta = \eta_{\mu\nu}\ dx^\mu\ dx^\nu$ a smooth metric on $\mathbb{R}^d$. One has:*

$$\mathrm{Tr}_\omega\ (\eta_{\mu\nu}[F, X^\mu]\ [F, X^\nu]) = \lambda \int_\Sigma \eta_{\mu\nu} \left\{ -\frac{2}{3}\ r\ \langle dX^\mu, dX^\nu\rangle \right.$$
$$\left. -\Delta\ \langle dX^\mu, dX^\nu\rangle + \langle\nabla\ dX^\mu, \nabla\ dX^\nu\rangle - \frac{1}{2}\ (\Delta X^\mu)(\Delta X^\nu)\right\}\ dv$$

where r is the curvature scalar of Σ, dv its volume form, ∇ its covariant derivative, Δ its Laplacian for an arbitrary Riemannian metric compatible with the given conformal structure.

Of course as we saw in the proof the various terms of the formula such as $-\frac{2}{3}\ r\ \langle dX^\mu, dX^\nu\rangle$ are not conformally invariant themselves, only their sum is. It is also important to check that the right hand side of the formula is, like obviously the left hand side, a Hochschild 2-cocycle. This allows to double check the constants in front of the various terms, except for the first one.

Theorem 6 gives a natural 4-dimensional analogue of the Polyakov action, and in particular in the special case when the $\eta_{\mu\nu}$ are constant, a natural conformally invariant action for scalar fields $X : \Sigma \to \mathbb{R}$.

(50) $$I(X) = \mathrm{Tr}_\omega\ ([F, X]^2)$$

which by theorem 6, can be expressed in local terms, and defines an elliptic differential operator P of order 4 on Σ such that:

(51) $$I(X) = \int_\Sigma P(X)\ X\ dv.$$

This operator P is (up to the factor $\frac{1}{2}$) equal to the Paneitz' operator $P = \Delta^2 + d^*\{2\mathrm{Ricci} - 4/3r\}d$ ([]) already known to be the analogue of the scalar Laplacian in 4-dimensional conformal geometry.

Equation 51 uses the volume element dv so that P itself is not conformally invariant, its principal symbol is:

$$\sigma_4(P)\ (x,\xi) = \frac{1}{2}\ \|\xi\|^4 \tag{52}$$

which is *positive.*

The conformal anomaly of the functional integral

$$\int e^{-I(X)}\ \pi\ dX(x)$$

is that of $(\det P)^{-1/2}$ and can be computed (cf. [B-O]). The above discussion gives a very clear indication that the induced gravity theory from the above scalar field theory in dimension 4 should be of great interest, in analogy with the 2-dimensional case.

6. The metric aspect and classical matter fields.

We have seen in section 5 how to develop a calculus of "infinitesimals", given a Fredholm module $(\mathcal{H}, F)$ over the algebra $\mathcal{A}$ of coordinates on a (possibly quantum) space X. For instance when X was an ordinary manifold we saw how to construct canonically a Fredholm module from a conformal structure on X. This shows that the above data does not specify the *metric* structure of X. In fact, the first example of section 5, where $X = S^1$ and $(\mathcal{H}, F)$ is the Hilbert transform, shows that the quantum differential expression:

$$dZ d\overline{Z} = [F, Z]\,[F, \overline{Z}] \quad , \quad Z : S^1 \to \mathbf{C} \tag{1}$$

where Z is the boundary value of a univalent map, yields infinitesimal units of length intimately tied up with the metric on $Z(S^1)$ induced by the usual Riemannian metric $dz\, d\overline{z}$ of $\mathbf{C}$. If we vary Z, even the *dimension* of S^1 for the "metric" (1) will change. More generally, let $\mathcal{A}$ be an involutive algebra and $(\mathcal{H}, F)$ a Fredholm module over $\mathcal{A}$. To define a "unit of length" in the corresponding space X, we shall consider an operator of the form:

$$G = \sum_1^d\, [F, X^\mu]^*\, \eta_{\mu\nu}\, [F, X^\nu] \tag{2}$$

where the X^μ are elements of $\mathcal{A}$ (usually selfadjoint but not necessarily) and where $\eta = (\eta_{\mu\nu})_{\mu,\nu=1,\ldots,d}$ is a positive element of the matrix algebra $M_d(\mathcal{A})$.

We want to think of G as of the ds^2 of Riemannian geometry. It is by construction a positive "infinitesimal", i.e. it is a positive compact operator. The unit of length is its positive square root:

$$ds = G^{1/2}. \tag{3}$$

Now the way we shall measure distances in the (possibly quantum) space X is the following. Given two pure states φ, ψ of the C^*-algebra closure of $\mathcal{A}$, i.e. two points $p, q \in X$ in the commutative case, with:

$$\varphi(f) = f(p)\ ,\ \psi(f) = f(q) \qquad \forall f \in \mathcal{A} \tag{4}$$

we want to use the following formula:

$$\mathrm{dist}(p, q) = \ \mathrm{Sup}\ \{|f(p) - f(q)|\ ;\ f \in \mathcal{A}\ ,\ \|df/ds\| \leq 1\}\,. \tag{5}$$

It is clear that both sides only involve p, q through the associated pure states (by (4)). Moreover since we are in the non commutative case we need to deal with the ambiguity in the order of the terms in an expression such as df/ds which can be either $df(ds)^{-1}$ of $(ds)^{-1}\ df$ or $(ds)^{-\alpha}\ df\ (ds)^{-(1-\alpha)}$ for instance. Instead of handling this problem directly we shall assume that G commutes with F (i.e. that $dG = 0$, a condition similar to the Kaehler metric condition), and introduce the following selfadjoint operator:

$$D = F(ds)^{-1} = F\,G^{-1/2} \tag{6}$$

whose existence assumes that G is non singular.

We shall then formulate equality (5) as follows:

$$\mathrm{dist}(p,q) = \ \mathrm{Sup}\ \{|f(p) - f(q)|\ ;\ f \in \mathcal{A}\ ,\ \|[D,f]\| \leq 1\}\,. \tag{7}$$

Now the operator F is by construction the sign of D, while G is obtained by the formula:

$$G = D^{-2}. \tag{8}$$

Thus it is more economical to give, from the start, the triple $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is an involutive algebra represented in the Hilbert space $\mathcal{H}$, while D is a ***selfadjoint operator*** in $\mathcal{H}$ whose resolvent $(D-\lambda)^{-1}$, $\lambda \notin \mathbf{R}$ is ***compact***. We shall formalise precisely this notion of "unbounded Fredholm module" under the name of K-cycle in definition 2 below. (We refer to [Co] (theorem 4 chapter 4 section 7) for a general construction of D given $\mathcal{A}$ and $(\mathcal{H}, F)$.)

Our task in this section will first be to show that Riemannian spaces are special cases of the above notion of geometric spaces. We shall then see how the construction of the "QED" action involving matter fields and gauge bosons extends to the more general geometric spaces we are dealing with. Finally we shall see that our notion of geometric space treats on equal footing the continuum and the discrete, while the "QED" action for the simplest mixture of continuum and discrete (the product of 4-dim-continuum by a space having two points) gives the Glashow-Weinberg-Salam action for the leptons. With more work the full standard model action can be interpreted as the "QED" action of a space time with a more involved "fine structure", but we postpone this discussion to the final section since it involves a more detailed description of the notion of ***manifold***.

Riemannian manifolds and the Dirac operator.

Let M be a compact Riemannian spin manifold and let $D = \partial_M$ be the corresponding Dirac operator (cf. [Gi_1]). Thus, D is an unbounded self-adjoint operator acting in the Hilbert space $\mathfrak{H}$ of L^2 spinors on the manifold M.

We shall give four formulas below that show how to reconstruct the ***metric space*** (M, d), where d is the geodesic distance, the ***volume measure*** dv on M, the space of ***gauge potentials***, and, finally, the ***Yang–Mills*** action functional, from the purely operator-theoretic data

$$(\mathcal{A}, \mathfrak{H}, D),$$

where D is the Dirac operator in the Hilbert space $\mathfrak{H}$ and were $\mathcal{A}$ is the abelian von Neumann algebra of multiplication by bounded measurable functions on M.

Thus, $\mathcal{A}$ is an abelian von Neumann algebra on $\mathfrak{H}$, and knowing the pair $(\mathfrak{H}, \mathcal{A})$ yields essentially no information (cf. section 2) except for the multiplicity, which is here the constant $2^{d/2}$, where $d = \dim M$. Similarly, the mere knowledge of the operator D in $\mathfrak{H}$ is equivalent to giving its list of eigenvalues $(\lambda_n)_{n\in\mathbf{N}}$, $\lambda_n \in \mathbf{R}$, and is an impractical point of departure for reconstructing M. The growth of these eigenvalues, i.e., the behavior of $|\lambda_n|$ as $n \to \infty$, is again governed by the dimension d of M, namely, $|\lambda_n| \sim Cn^{1/d}$ as $n \to \infty$.

What is relevant is the triple $(\mathcal{A}, \mathfrak{H}, D)$. Elements of $\mathcal{A}$ other than the constants do not commute with D, and the *boundedness* of the commutator $[D, a]$ already implies the following regularity condition on a measurable function a:

Lemma 1. *If a is a bounded measurable function on M, then the densely defined operator $[D, a]$ is bounded if and only if a is almost everywhere equal to a Lipschitz function f, $|f(p) - f(q)| \le Cd(p, q)$ $(\forall p, q \in M)$.*

Here, d is the geodesic distance in M. The operator $[D, a]$ should be viewed in effect as a quadratic form

$$\xi, \eta \to \langle a\xi, D\eta\rangle - \langle D\xi, a^*\eta\rangle,$$

which is well-defined for ξ, η in the domain of D; its boundedness signifies an inequality of the form

$$|\langle a\xi, D\eta\rangle - \langle D\xi, a^*\eta\rangle| \le c\|\xi\|\|\eta\| \qquad (\forall \xi, \eta \in \mathrm{dom} D).$$

The proof of the lemma follows immediately from [Fe].

Now, every Lipschitz function on M is continuous and the algebra of Lipschitz functions is norm-dense in the algebra of continuous functions on M; it follows that the C^*-algebra $C(M)$ of continuous functions on M is identical to the *norm closure* A in $\mathcal{L}(\mathfrak{H})$ of $\mathbf{A} = \{a \in \mathcal{A};\ [D, a] \text{ is bounded}\}$.

By Gelfand's theorem (section 3), we can recover the compact topological space M as the spectrum of A. Thus, a point p of M is a $*$-homomorphism $\rho : A \to \mathbf{C}$,

$$\rho(ab) = \rho(a)\rho(b) \quad (\forall a, b \in A).$$

Any such homomorphism ρ is given by evaluation of a at p for some point $p \in M$,

$$\rho(a) = a(p) \in \mathbf{C}.$$

All of this is still qualitative; we now come to the first interesting formula, giving us a natural distance function, which turns out in this case to be the geodesic distance:

Formula 1. *For any pair of points $p, q \in M$, the geodesic distance between them is given by the formula*

$$d(p,q) = \sup\{|a(p) - a(q)|;\ a \in \mathcal{A},\ \|[D,a]\| \leq 1\}.$$

The proof is straightforward, but it is relevant to go through it to see what is involved. The operator $[D,a]$, which (Lemma 1) is bounded if and only if a is Lipschitz, is given by the Clifford multiplication $i\gamma(\mathrm{d}a)$ by the gradient $\mathrm{d}a$ of a. This gradient is ([Fe]) a bounded measurable section of the cotangent bundle T^*M of M, and we have

$$\|[D,a]\| = \operatorname{ess\,sup} \|\mathrm{d}a\| = \text{the Lipschitz norm of } a.$$

It follows at once that the right-hand side of Formula 1 is less than or equal to the geodesic distance $d(p,q)$. However, fixing the point p and considering the function $a(q) = d(q,p)$, one checks that a is Lipschitz with constant 1, so that $\|[D,a]\| \leq 1$, which yields the desired equality. Note that Formula 1 is in essence dual to the original formula

$$d(p,q) = \text{infimum of the length of paths } \gamma \text{ from } p \text{ to } q, \tag{$*$}$$

in the sense that, instead of involving arcs, namely copies of $\mathbf{R}$ inside the manifold M, it involves *functions* a, that is, mappings from M to $\mathbf{R}$ (or to $\mathbf{C}$).

This is an essential point for us since in the case of discrete spaces or of noncommutative spaces X, there are no interesting arcs in X but there are plenty of *functions*, namely, the elements $a \in \mathcal{A}$ of the defining algebra. We note at once that the right-hand side of Formula 1 is meaningful in that general context and it defines a metric on the space of *states* of the C^*-algebra A, the norm closure of $\mathbf{A} = \{a \in \mathcal{A};\ [D,a] \text{ is bounded}\}$:

$$d(\varphi,\psi) = \sup\{|\varphi(a) - \psi(a)|;\ \|[D,a]\| \leq 1\}.$$

Finally, we also note that, although both Formula 1 and the formula $(*)$ give the same result for Riemannian manifolds, they are of quite different nature if we try to use them in actual measurements of distances. The formula $(*)$ uses the idealized notion of a path, and quantum mechanics teaches us that there is nothing like 'the path followed by a particle'. Thus, for measurements of very small distances, it is more natural to use wave functions and Formula 1.

We have now recovered from our original data $(\mathcal{A}, \mathfrak{H}, D)$ the metric space (M,d), where d is the geodesic distance. However, we still need tools of Riemannian geometry, the first obvious example being the measure given by the volume form

$$f \to \int_M f \,\mathrm{d}v,$$

where, in local coordinates $x^\mu, g_{\mu\nu}$, we have

$$\mathrm{d}v = \left(\det(g_{\mu\nu})\right)^{1/2} |\mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^n|.$$

This brings us to our second formula, which is nothing more than a restatement of H. Weyl's theorem about the asymptotic behavior of elliptic differential operators ([Gi$_1$]). It does, however, involve *the Dixmier trace* Tr_ω, which, unlike asymptotic expansions, makes sense in full generality in our context and is the correct operator-theoretic substitute for integration (cf. section 5):

Formula 2. *For every $f \in A$, we have $\int_M f dv = \mathrm{Tr}_\omega(fD^{-d})$, where $d = \dim M$.*

Now, getting the integral of functions, i.e., the Riemannian volume form, is a good indication but quite far from the full story. In particular, many distinct Riemannian metrics yield the same volume form. Since our aim is to investigate physical space-time at the level of elementary particle physics, we shall now make a deliberate choice: instead of focusing on the intrinsic Riemannian curvature, which would drive us toward general relativity, we shall concentrate on the measurement (using $(\mathfrak{H}, D)$) of the curvature of connections on vector bundles, and on the Yang–Mills functional, which takes us to the theory of matter fields. This line is of course easier since it does not involve derivatives of the $g_{\mu\nu}$.

Let us then clearly state our aim: it is to recover the Yang–Mills functional on connections on vector bundles, making use of only the following data:

Definition 2. *A K-cycle $(\mathfrak{H}, D)$, over an algebra $\mathcal{A}$ with involution $*$, consists of a representation of $\mathcal{A}$ on a Hilbert space $\mathfrak{H}$ together with an unbounded self-adjoint operator D with compact resolvent, such that $[D, a]$ is bounded for every $a \in \mathcal{A}$.*

If the eigenvalues λ_n of $|D|$ are of the order of $n^{1/d}$ as $n \to \infty$, we say that the K-cycle is (d, ∞)-summable (cf. Section 5). On the algebra of functions on a compact Riemannian spin manifold, the Dirac operator determines a K-cycle that is (d, ∞)-summable, where $d = \dim M$. Finer regularity of functions, such as infinite differentiability, is easily expressed using the domain of powers of the derivation δ, $\delta(a) = [|D|, a]$.

We shall not be too specific about the choice of regularity; our discussion applies to any degree of regularity higher than Lipschitz.

The value of the following construction is that it will also apply when the $*$-algebra $\mathcal{A}$ is noncommutative, or when D is no longer the Dirac operator. The reader can have in mind either the Riemannian case or the slightly more involved case in which the algebra $\mathcal{A}$ is the $*$-algebra of matrices of functions on a Riemannian manifold, provided one bears in mind that the notion of exterior product no longer makes sense over such an algebra.

We shall begin with the notion of connection on the trivial bundle, i.e., the case of 'electrodynamics', and we shall first define vector potentials and the Yang–Mills action in that case. We shall then treat the general case of arbitrary hermitian bundles, i.e., in algebraic terms, of arbitrary hermitian, finitely generated projective modules over $\mathcal{A}$.

We wish to define k-forms over $\mathcal{A}$ as operators in $\mathfrak{H}$ of the form

$$\omega = \sum a_0^j [D, a_1^j] \dots [D, a_k^j],$$

where the a_i^j are elements of $\mathcal{A}$ represented as operators on $\mathfrak{H}$. This idea arises because, although the operator D fails to be invariant under the representation on $\mathfrak{H}$ of the unitary group $\mathcal{U}$ of $\mathcal{A}$,

$$\mathcal{U} = \{u \in \mathcal{A};\ u^* u = uu^* = 1\},$$

the following equality shows that the failure of invariance is governed by a 1-form in the above sense: $\omega_u = u[D, u^*]$, that is,

$$uDu^* = D + \omega_u.$$

Note that ω_u is self-adjoint as an operator in $\mathfrak{H}$, thus it is natural to make the following definition:

Definition 3. **A vector potential** V is a self-adjoint element of the space of 1-forms $\sum a_0^j[D, a_1^j]$, where $a_k^j \in \mathcal{A}$.

One can immediately check that in the basic example of the Dirac operator on a spin Riemannian manifold, a vector potential in the above sense is exactly a 1-form v on the manifold M and that this form is imaginary, the corresponding operator in the space of spinors being given by the Clifford multiplication:

$$V = i\gamma(v) \quad (i = \sqrt{-1}).$$

The action of the unitary group $\mathcal{U}$ on vector potentials is such that it replaces the operator $D + V$ by $u(D + V)u^*$, thus it is given by the algebraic formula

$$\gamma_u(V) = u[D, u^*] + uVu^* \quad (u \in \mathcal{U}).$$

We now need only define the curvature or field strength θ for a vector potential, and use the analogue of the above Formula 2 to integrate the square of θ: the formula

$$\mathrm{YM}(V) = \mathrm{Tr}_\omega(\theta^2 |D|^{-d})$$

should give us the Yang–Mills action.

The formula for θ should be of the form $\theta = \mathrm{d}V + V^2$; the only difficulty is in defining properly the 'differential' $\mathrm{d}V$ of a vector potential, as an operator in $\mathfrak{H}$.

Let us examine what happens: the naive formula is:

$$\text{If } V = \sum a_0^j[D, a_1^j] \text{ then } \mathrm{d}V = \sum [D, a_0^j][D, a_1^j].$$

Before we point out what the difficulty is, let us check that if we replace V by $\gamma_u(V)$, where

$$\gamma_u(V) = u[D, u^*] + \sum ua_0^j[D, a_1^j]u^*,$$

then the curvature is transformed covariantly:

$$\mathrm{d}(\gamma_u(V)) + \gamma_u(V)^2 = u(\mathrm{d}V + V^2)u^*.$$

As this computation is instructive, we shall carry it out in detail. First, in order to write $\gamma_u(V)$ in the same form as V, we use the equality

$$[D, a_1^j]u^* = [D, a_1^j u^*] - a_1^j[D, u^*].$$

Thus, $\gamma_u(V) = u[D, u^*] + \sum u a_0^j[D, a_1^j u^*] - \sum u a_0^j a_1^j[D, u^*]$, and we have

$$\mathrm{d}\gamma_u(V) = [D, u][D, u^*] + \sum [D, u a_0^j][D, a_1^j u^*] - \sum [D, u a_0^j a_1^j][D, u^*].$$

We now claim that the following operators in $\mathfrak{H}$ are indeed equal:

α) $\mathrm{d}\gamma_u(V) + \gamma_u(V)^2$,

β) $u(\mathrm{d}V + V^2)u^*$.

For, the operator α) is equal to

$$\begin{aligned}
&\mathrm{d}\gamma_u(V) + (u[D, u^*] + uVu^*)^2 \\
&\quad = \mathrm{d}\gamma_u(V) + u[D, u^*]u[D, u^*] + u[D, u^*]uVu^* + uVu^*u[D, u^*] + uV^2u^* \\
&\quad = \mathrm{d}\gamma_u(V) - [D, u][D, u^*] - [D, u]Vu^* + uV[D, u^*] + uV^2u^* \\
&\quad = \sum [D, u a_0^j][D, a_1^j u^*] - \sum [D, u a_0^j a_1^j][D, u^*] - [D, u]Vu^* + uV[D, u^*] + uV^2u^* \\
&\quad = u\mathrm{d}Vu^* + uV^2u^*,
\end{aligned}$$

where the last equality follows from

$$\begin{aligned}
&\sum [D, u]a_0^j[D, a_1^j u^*] - \sum [D, u]a_0^j a_1^j[D, u^*] = [D, u]Vu^*, \\
&\sum u[D, a_0^j][D, a_1^j u^*] - \sum u[D, a_0^j]a_1^j[D, u^*] = u\mathrm{d}Vu^*, \\
&\sum u a_0^j[D, a_1^j][D, u^*] = uV[D, u^*].
\end{aligned}$$

The difficulty that we overlooked is the following: the same vector potential V might be written in several ways as $V = \sum a_0^j[D, a_1^j]$, so that the definition of $\mathrm{d}V$ as

$$\mathrm{d}V = \sum [D, a_0^j][D, a_1^j]$$

is ambiguous.

To understand the nature of the problem, let us introduce some algebraic notation. We let $\Omega^*\mathcal{A}$ be the universal differential graded algebra over $\mathcal{A}$. It is by definition equal to $\mathcal{A}$ in degree 0 and is generated by symbols $\mathrm{d}a$ ($a \in \mathcal{A}$) of degree 1 with the following presentation:

α) $\mathrm{d}(ab) = (\mathrm{d}a)b + a\mathrm{d}b$ ($\forall a, b \in \mathcal{A}$),

β) $\mathrm{d}1 = 0$.

One can check that $\Omega^1\mathcal{A}$ is isomorphic as an $\mathcal{A}$-bimodule to the kernel $\mathrm{Ker}(m)$ of the multiplication mapping $m : \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$, the isomorphism being given by the mapping

$$\sum a_i \otimes b_i \in \mathrm{Ker}(m) \to \sum a_i \mathrm{d}b_i \in \Omega^1\mathcal{A}.$$

The involution $*$ of $\mathcal{A}$ extends uniquely to an involution on Ω^* with the rule

γ) $(\mathrm{d}a)^* = -\mathrm{d}a^*$.

The differential d on $\Omega^*\mathcal{A}$ is defined *unambiguously* by

$$\mathrm{d}(a^0\mathrm{d}a^1 \ldots \mathrm{d}a^n) = \mathrm{d}a^0\mathrm{d}a^1 \ldots \mathrm{d}a^n \quad (\forall a^j \in \mathcal{A}),$$

and it satisfies the relations

δ) $\mathrm{d}^2\omega = 0 \quad (\forall \omega \in \Omega^*\mathcal{A})$,

ε) $\mathrm{d}(\omega_1\omega_2) = (\mathrm{d}\omega_1)\omega_2 + (-1)^{\partial\omega_1}\omega_1\mathrm{d}\omega_2 \quad (\forall \omega_j \in \Omega^*\mathcal{A})$.

Proposition 4. (1) *The following equality defines a $*$-representation π of the universal algebra $\Omega^*(\mathcal{A})$ on $\mathfrak{H}$:*

$$\pi(a^0\mathrm{d}a^1 \ldots \mathrm{d}a^n) = a^0[D,a^1]\ldots[D,a^n] \quad (\forall a^j \in \mathcal{A}).$$

(2) *Let $J_0 = \mathrm{Ker}\pi \subset \Omega^*$ be the graded two-sided ideal of Ω^* given by $J_0^{(k)} = \{\omega \in \Omega^k,\ \pi(\omega) = 0\}$; then $J = J_0 + \mathrm{d}J_0$ is a graded differential two-sided ideal of $\Omega^*(\mathcal{A})$.*

The first statement is obvious; let us discuss the second. By construction, J_0 is a two-sided ideal but it is not in general a *differential* ideal, i.e., if $\omega \in \Omega^k(\mathcal{A})$ and $\pi(\omega) = 0$, one does not in general have $\pi(\mathrm{d}\omega) = 0$. This is exactly the reason why the above definition of $\sum[D,a_0^j][D,a_1^j]$ as the differential of $\sum a_0^j[D,a_1^j]$ was ambiguous.

Let us show however that $J = J_0 + \mathrm{d}J_0$ is still a two-sided ideal. Since $\mathrm{d}^2 = 0$ it is obvious that J is then a differential ideal. Let $\omega \in J^{(k)}$ be a homogeneous element of J; then ω is of the form $\omega = \omega_1 + \mathrm{d}\omega_2$, where $\omega_1 \in J_0 \cap \Omega^k$, $\omega_2 \in J_0 \cap \Omega^{k-1}$. Let $\omega' \in \Omega^{k'}$ and let us show that $\omega\omega' \in J^{(k+k')}$. We have

$$\begin{aligned}\omega\omega' &= \omega_1\omega' + (\mathrm{d}\omega_2)\omega' = \omega_1\omega' + \mathrm{d}(\omega_2\omega') - (-1)^{k-1}\omega_2\mathrm{d}\omega' \\ &= (\omega_1\omega' + (-1)^k\omega_2\mathrm{d}\omega') + \mathrm{d}(\omega_2\omega').\end{aligned}$$

However, the first term belongs to $J_0 \cap \Omega^{k+k'}$ and $\omega_2\omega' \in J_0 \cap \Omega^{k+k'-1}$.

Using (2) of Proposition 4, we can now introduce the graded differential algebra

$$\Omega_D^* = \Omega^*(\mathcal{A})/J.$$

Let us first investigate Ω_D^0, Ω_D^1 and Ω_D^2.

We have $J \cap \Omega^0 = J_0 \cap \Omega^0 = \{0\}$ provided that we assume, as we shall, that $\mathcal{A}$ is a ***subalgebra*** of $\mathcal{L}(\mathfrak{H})$. Thus $\Omega_D^0 = \mathcal{A}$.

Next, $J \cap \Omega^1 = J_0 \cap \Omega^1 + \mathrm{d}(J_0 \cap \Omega^0) = J_0 \cap \Omega^1$, thus Ω_D^1 is the quotient of Ω^1 by the kernel of π, and it is thus exactly the $\mathcal{A}$-bimodule $\pi(\Omega^1)$ of operators ω of the form

$$\omega = \sum a_j^0[D, a_j^1] \quad (a_j^k \in \mathcal{A}).$$

Finally, $J \cap \Omega^2 = J_0 \cap \Omega^2 + \mathrm{d}(J_0 \cap \Omega^1)$ and the representation π gives an isomorphism

$$\Omega_D^2 \cong \pi(\Omega^2)/\pi\big(\mathrm{d}(J_0 \cap \Omega^1)\big). \qquad (*)$$

More precisely, this means that we can view an element ω of Ω_D^2 as a class of elements ρ of the form

$$\rho = \sum a_j[D, a_j^1][D, a_j^2] \quad (a_j^k \in \mathcal{A})$$

modulo the sub-bimodule of elements of the form

$$\rho_0 = \sum [D, b_j^0][D, b_j^1] \quad (b_j^k \in \mathcal{A},\ \sum b_j^0[D, b_j^1] = 0).$$

It is now clear that since we work modulo this subspace $\pi\big(\mathrm{d}(J_0 \cap \Omega^1)\big)$, the question of ambiguity in the definition of $\mathrm{d}\omega$ for $\omega \in \pi(\Omega^1)$ no longer arises.

The equality $(*)$ makes sense for all k,

$$\Omega_D^k \cong \pi(\Omega^k)/\pi\big(\mathrm{d}(J_0 \cap \Omega^{k-1})\big), \qquad (*)$$

and allows us to define the following inner product on Ω_D^k: for each k let $\mathfrak{H}_k$ be the Hilbert space completion of $\pi(\Omega^k)$ with the inner product

$$\langle T_1, T_2\rangle_k = \mathrm{Tr}_\omega(T_2^* T_1 |D|^{-d}) \quad (\forall T_j \in \pi(\Omega^k)).$$

Let P be the orthogonal projection of $\mathfrak{H}_k$ onto the orthogonal complement of the subspace $\pi\big(\mathrm{d}(J_0 \cap \Omega^{k-1})\big)$. By construction, the inner product $\langle P\omega_1, \omega_2\rangle = \langle P\omega_1, P\omega_2\rangle$ for $\omega_j \in \pi(\Omega^k)$ depends only on their class in Ω_D^k. We denote by Λ^k the Hilbert space completion of Ω_D^k for this inner product; it is, of course, equal to $P\mathfrak{H}_k$.

Proposition 5. (1) *The actions of $\mathcal{A}$ on Λ^k by left and right multiplication define commuting unitary representations of $\mathcal{A}$ on Λ^k.*

(2) *The functional* $\mathrm{YM}(V) = \langle \mathrm{d}V + V^2, \mathrm{d}V + V^2\rangle$ *is positive, quartic and invariant under gauge transformations.*

$$\gamma_u(V) = u\mathrm{d}u^* + uVu^* \quad (\forall u \in \mathcal{U}(\mathcal{A})).$$

(3) *The functional* $I(\alpha) = \mathrm{Tr}_\omega(\theta^2|D|^{-d})$, $\theta = \pi(\mathrm{d}\alpha + \alpha^2)$ *is positive, quartic and gauge invariant on* $\{\alpha \in \Omega^1(\mathcal{A}),\ \alpha = \alpha^*\}$.

(4) *One has* $\mathrm{YM}(V) = \inf\{I(\alpha);\ \pi(\alpha) = V\}$.

Let us say a few words about the easy proof. Since $\pi\big(\mathrm{d}(J_0 \cap \Omega^{k-1})\big) \subset \pi(\Omega^k)$ is a sub-bimodule of $\pi(\Omega^k)$ and since the left and right actions of $\mathcal{A}$ on $\mathfrak{H}_k$ are unitary, it follows that P is a bimodule morphism:

$$P(a\xi b) = aP(\xi)b \quad (\forall a, b \in \mathcal{A},\ \xi \in \mathfrak{H}_k).$$

Thus (1) follows. As for (2), one merely notes that by the above calculation, with $\mathrm{d}V$ now unambiguous, $\theta = \mathrm{d}V + V^2$ is covariant under gauge transformations, whence the result. For (3), one again uses the above calculation to show that $\mathrm{d}\alpha + \alpha^2$ transforms covariantly under gauge transformations.

Finally, (4) follows from the property of the orthogonal projection P: as an element of Λ^2, $\mathrm{d}V + V^2$ is equal to $P\big(\pi(\mathrm{d}\alpha + \alpha^2)\big)$ for any α with $\pi(\alpha) = V$, and since the ambiguity in $\pi(\mathrm{d}\alpha)$ is exactly $\pi\big(\mathrm{d}(J_0 \cap \Omega^1)\big)$ one gets (4).

Stated in simpler terms, the meaning of Proposition 5 is that the ambiguity that we met above in the definition of the operator curvature $\theta = \mathrm{d}V + V^2$ can be ignored by taking the infimum

$$\mathrm{YM}(V) = \inf \mathrm{Tr}_\omega(\theta^2 |D|^{-d})$$

over all possibilities for $\theta = \mathrm{d}V + V^2$, $\mathrm{d}V = \sum [D, a_j^0][D, a_j^1]$ being ambiguous. The action obtained is nevertheless quartic by (2).

We shall now check that in the case of Riemannian manifolds with the Dirac K-cycle, the graded differential algebra Ω_D^* is canonically isomorphic to the *de Rham algebra of ordinary forms on M with their canonical pre-Hilbert space structure*. The whole point is that Proposition 5 gives us these concepts in far greater generality and the formula in (4) will allow extending to this generality, in the case $d = 4$, the inequality between the topological action and the Yang–Mills action YM.

We now specialize to the Riemannian case, where $\mathcal{A}$ is the algebra of functions (with some regularity) on the compact spin manifold M, and $D = \partial_M$ is the Dirac operator in the Hilbert space $L^2(M, S)$ of spinors. We let C be the bundle over M whose fiber at each $p \in M$ is the complexified Clifford algebra $\mathrm{Cliff}_{\mathbf{C}}(T_p^*(M))$ of the cotangent space at $p \in M$. Any bounded measurable section ρ of C defines a bounded operator $\gamma(\rho)$ on $\mathfrak{H} = L^2(M, S)$. For any $f^0, \ldots, f^n \in \mathcal{A}$ one has

$$\pi(f^0 \mathrm{d}f^1 \ldots \mathrm{d}f^n) = i^n \gamma(f^0 \mathrm{d}f^1 \cdot \mathrm{d}f^2 \cdot \ldots \cdot \mathrm{d}f^n),$$

where the usual differential $\mathrm{d}f$ is regarded as a section of C, and $\cdot$ denotes the product in C.

For each $p \in M$, the Clifford algebra C_p has a $\mathbf{Z}/2$ grading given by the parity of the number of terms ξ_j, $\xi_j \in T_p^*(M)$ in a product $\xi_1 \cdot \xi_2 \cdot \ldots \cdot \xi_n$, and a filtration, where $C_p^{(k)}$ is the subspace spanned by products of $n \leq k$ elements of $T_p^*(M)$. The associated graded algebra is canonically isomorphic to the complexified exterior algebra $\bigwedge_{\mathbf{C}}(T_p^*(M))$ and we shall write $\sigma_k : C^{(k)} \to \bigwedge_{\mathbf{C}}^k(T_p^*)$ for the quotient mapping.

Using the canonical inner product on C given by the trace in the spinor representation, one can also identify $\bigwedge_{\mathbf{C}}^k$ with the orthogonal complement of $C^{(k-1)}$ in $C^{(k)}$; equivalently, if we let C^k be the subspace of $C^{(k)}$of elements of the same parity as k, then $\bigwedge_{\mathbf{C}}^k = C^k \ominus C^{k-2}$.

The differential algebra Ω_D^* is determined by the following lemma:

Lemma 6. *Let $(\mathfrak{H}, D)$ be the Dirac K-cycle on the algebra $\mathcal{A}$ of functions on M and let $k \in \mathbf{N}$. Then, a pair T_1, T_2 of operators in $\mathfrak{H}$ is of the form $T_1 = \pi(\omega)$, $T_2 = \pi(\mathrm{d}\omega)$ for some $\omega \in \Omega^k(\mathcal{A})$ if and only if there exist sections ρ_1, ρ_2 of C^k and C^{k+1} such that*

$$T_j = \gamma(\rho_j) \ \ (j = 1,2), \quad \sqrt{-1}\, \mathrm{d}\sigma^k(\rho_1) = \sigma_{k+1}(\rho_2).$$

Here $\sigma_k(\rho_1)$ is an ordinary k-form on M and d is the ordinary differential. Note that for $k > d = \dim M$ one has $\sigma_k(\rho) = 0$. The proof is straightforward.

We can now easily determine the graded differential algebra Ω_D^*. First, let us identify $\pi(\Omega^k)$ with the space of sections of C^k; Lemma 6 then shows that

$$\pi\big(\mathrm{d}(J_0 \cap \Omega^{k-1})\big) = \mathrm{Ker}\sigma_k.$$

(If ρ is a section of C^k with $\sigma_k(\rho) = 0$ then the pair of sections $\rho_1 = 0$, $\rho_2 = \rho$ of C^{k-1} and C^k fulfills the condition of Lemma 6, so that $\rho = \pi(\mathrm{d}\omega)$ for some ω, $\pi(\omega) = 0$.) Thus σ_k is an isomorphism $\Omega_D^k \cong$ sections of $\bigwedge_{\mathbf{C}}^k(T^*)$, which, again by Lemma 6, commutes with the differential. We can then state:

Formula 3. *The mapping $a^0\mathrm{d}a^1 \ldots \mathrm{d}a^n \to i^n a^0 \mathrm{d}_c a^1 \ldots \mathrm{d}_c a^n$ for $a^j \in \mathcal{A}$ extends to a canonical isomorphism of the differential graded algebra Ω_D^* with the de Rham algebra of differential forms on M. Under this isomorphism, the inner product on Ω_D^k is the Riemannian inner product of k-forms:*

$$\langle \omega, \omega' \rangle = \int_M \omega \wedge *\omega'.$$

The last equality follows from the computation of the Dixmier trace for the operator in $\mathfrak{H} = L^2(M, S)$ associated with a section ρ of the bundle C of Clifford algebras:

$$\mathrm{Tr}_\omega(\rho|D|^{-d}) = \int_M \mathrm{trace}(\rho(p))\mathrm{d}v(p).$$

As an immediate corollary of Formula 3, we have

$$\mathrm{YM}(V) = \int \|\mathrm{d}V\|^2 \mathrm{d}v.$$

Let us now extend the definition of the action YM to connections on arbitrary hermitian vector bundles.

First of all, we need to express in algebraic terms—i.e., using only the involutive algebra $\mathcal{A}$—the concept of hermitian vector bundle over M. A vector bundle E is entirely characterized by the right $\mathcal{A}$-module $\mathcal{E}$ of sections of E (having the same regularity as the elements of $\mathcal{A}$); the local triviality of E and the finite-dimensionality of its fibers translate algebraically into the statement that $\mathcal{E}$ is a direct summand of a free module $\mathcal{A}^N$ for some finite N, or, in fancier terms, that $\mathcal{E}$ is a finitely generated projective module over $\mathcal{A}$.

The hermitian structure on E, that is, the inner product $\langle \xi, \eta \rangle_p$ on each fiber E_p, permits constructing a sesquilinear mapping

$$\langle \ , \ \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A},$$

given by $\langle \xi, \eta \rangle(p) = \langle \xi(p), \eta(p) \rangle_p$. The mapping $\langle \ , \ \rangle$ satisfies the following conditions:

1) $\langle \xi a, \eta b \rangle = a^* \langle \xi, \eta \rangle b \quad (\forall \xi, \eta \in \mathcal{E}, \ a, b \in \mathcal{A})$,
2) $\langle \xi, \xi \rangle \geq 0 \quad (\forall \xi \in \mathcal{E})$,
3) $\mathcal{E}$ is self-dual for $\langle \ , \ \rangle$.

Thus, the hermitian vector bundles over M correspond bijectively to the hermitian, finitely generated projective modules over $\mathcal{A}$ in the following sense:

Definition 7. *Let $\mathcal{A}$ be a $*$-algebra with unity and let $\mathcal{E}$ be a finitely generated projective module over $\mathcal{A}$. A* **hermitian structure** *on $\mathcal{E}$ is given by a sesquilinear mapping $\langle \ , \ \rangle : \mathcal{E} \times \mathcal{E} \to \mathcal{A}$ satisfying the above conditions 1), 2) and 3).*

We shall use this concept only in the case where $\mathcal{A}$ is a subalgebra stable under the holomorphic functional calculus in a C^*-algebra, in which case all reasonable notions of positivity coincide in $\mathcal{A}$.

In this case, all hermitian structures on a given finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ are isomorphic to each other and are thus obtained as follows: one writes $\mathcal{E}$ as a direct summand $\mathcal{E} = e\mathcal{A}^N$ of a free module $\mathcal{E}_0 = \mathcal{A}^N$, where the idempotent $e \in M_N(\mathcal{A})$ is *self-adjoint*, and one then restricts to $\mathcal{E}$ the hermitian structure on $\mathcal{A}^N$ given by

$$\langle \xi, \eta \rangle = \sum \xi_i^* \eta_i \in \mathcal{A} \quad (\forall \xi = (\xi_i), \eta = (\eta_i) \in \mathcal{A}^N).$$

The algebra $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ of endomorphisms of a hermitian, finitely generated projective module $\mathcal{E}$ has a natural involution, given by

$$\langle T^* \xi, \eta \rangle = \langle \xi, T\eta \rangle \quad (\forall \xi, \eta \in \mathcal{E}).$$

With this involution, $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$ is isomorphic to the reduced $*$-algebra $eM_N(\mathcal{A})e$.

As above, we now let $(\mathfrak{H}, D)$ be a K-cycle over $\mathcal{A}$, and Ω_D^1 the $\mathcal{A}$-bimodule of operators in $\mathfrak{H}$ of the form $V = \sum a_i[D, b_i]$, $a_i, b_i \in \mathcal{A}$.

Definition 8. *Let $\mathcal{E}$ be a hermitian, finitely generated projective module over $\mathcal{A}$. A* **connection** *on $\mathcal{E}$ is given by a linear mapping $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$ such that*

$$\nabla(\xi a) = (\nabla \xi)a + \xi \otimes da \quad (\forall \xi \in \mathcal{E},\ a \in \mathcal{A}).$$

A connection ∇ is **compatible** *(with the metric) if and only if*

$$\langle \xi, \nabla \eta \rangle - \langle \nabla \xi, \eta \rangle = \mathrm{d}\langle \xi, \eta \rangle \quad (\forall \xi, \eta \in \mathcal{E}).$$

The last equality has a clear meaning in Ω^1_D. In the computations, one should remember that $(\mathrm{d}a)^* = -\mathrm{d}a^*$ $(\forall a \in \mathcal{A})$, and if $\nabla \xi = \sum \xi_i \otimes \omega_i$, $\omega_i \in \Omega^1_D$, then $\langle \nabla \xi, \eta \rangle = \sum \omega_i^* \langle \xi_i, \eta \rangle$.

Such connections always exist; for $\mathcal{E}$ expressed as $e\mathcal{A}^N$ as above, one may take ∇ as follows:

$$(\nabla_0 \xi) = e\eta, \text{ where } \eta_j = \mathrm{d}\xi_j.$$

Two compatible connections ∇ and ∇' on $\mathcal{E}$ differ by an element $\Gamma \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D)$.

As in Proposition 5, we shall now give two equivalent definitions of the action functional $\mathrm{YM}(\nabla)$ on the affine space $C(\mathcal{E})$ of compatible connections.

The group $\mathcal{U}(\mathcal{E})$ of unitary automorphisms of $\mathcal{E}$, $\mathcal{U}(\mathcal{E}) = \{u \in \mathrm{End}_{\mathcal{A}}(\mathcal{E});\ uu^* = u^*u = 1\}$, acts by conjugation $\gamma_u(\nabla) = u\nabla u^*$ on the space $C(\mathcal{E})$. To define the curvature θ of a connection ∇, one first extends ∇ to a unique linear mapping $\tilde{\nabla}$from $\tilde{\mathcal{E}}$ to $\tilde{\mathcal{E}}$, $\tilde{\mathcal{E}} = \mathcal{E} \otimes_{\mathcal{A}} \Omega^*_D$, such that

$$\tilde{\nabla}(\xi \otimes \omega) = (\nabla \xi)\omega + \xi \otimes \mathrm{d}\omega \quad (\forall \xi \in \mathcal{E},\ \omega \in \Omega^*_D),$$

and one checks that this mapping satisfies

$$\tilde{\nabla}(\eta\omega) = (\tilde{\nabla}\eta)\omega + (-1)^{\partial\eta}\eta \mathrm{d}\omega$$

for every homogeneous $\eta \in \tilde{\mathcal{E}}$ and $\omega \in \Omega^*_D$. It then follows that $\theta = \tilde{\nabla}^2$ is an endomorphism of the right Ω^*_D-module $\tilde{\mathcal{E}}$; it is determined by its restriction to $\mathcal{E}$, again denoted θ:

$$\theta \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D).$$

Next, using the inner product on Ω^2_D and the hermitian structure on $\mathcal{E}$, one gets a natural inner product on

$$\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^2_D).$$

Using this, we make the following definition.

Definition 9. $\mathrm{YM}(\nabla) = \langle \theta, \theta \rangle$.

By construction, this action is gauge invariant, positive and quartic. It is moreover obvious from the above Formula 3 in the case of the Dirac K-cycle on a Riemannian spin manifold that one has:

Formula 4. *Let M be a Riemannian spin manifold with its Dirac K-cycle $(\mathfrak{H}, D)$. Then, the notion of connection* (Def. 8) *is the usual one, and*

$$\mathrm{YM}(\nabla) = \int_M \|\theta\|^2_{\mathrm{HS}} \mathrm{d}v,$$

where θ is the usual curvature of ∇.

Thus, we recover in this case the usual Yang–Mills action. For computational purposes, and also to see the curvature as an operator in $\mathfrak{H}$, we shall now mention the easy adaptation of Proposition 5, (4) to the general case.

First of all, any compatible connection in the sense of Definition 8 is the composition with π of a ***universal compatible connection***, i.e., a linear mapping fulfilling precisely the conditions of Definition 8:

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1.$$

To see the surjectivity of the mapping $\pi : CC(\mathcal{E}) \to C(\mathcal{E})$, where $CC(\mathcal{E})$ is the space of universal compatible connections, it is enough to check that the special 'Grassmannian' connection ∇_0 is of this form and that π is a surjection of Ω^1 onto $\pi(\Omega^1)$. Next, a universal compatible connection extends uniquely as a linear mapping

$$\tilde{\nabla} : \mathcal{E} \otimes_{\mathcal{A}} \Omega^* \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^*$$

such that $\tilde{\nabla}$ is equal to ∇ on $\mathcal{E} \otimes 1$ and such that

$$\tilde{\nabla}(\eta\omega) = (\tilde{\nabla}\eta)\omega + (-1)^{\deg \eta}\eta \mathrm{d}\omega$$

for every homogeneous η in $\mathcal{E} \otimes_{\mathcal{A}} \Omega^*$ and $\omega \in \Omega^*$.

The curvature $\theta = \tilde{\nabla}^2$ is then an endomorphism of the induced module $\tilde{\mathcal{E}} = \mathcal{E} \otimes_{\mathcal{A}} \Omega^*$ over Ω^*, and $\pi(\theta)$ makes sense as a bounded operator in the Hilbert space $\mathcal{E} \otimes_{\mathcal{A}} \mathfrak{H}$, as does the following operator D_∇:

$$D_\nabla(\xi \otimes \eta) = \xi \otimes D\eta + \big((1 \otimes \pi)\nabla\xi\big)\eta \qquad (\forall \xi \in \mathcal{E},\ \eta \in \mathfrak{H});$$

the analogue of the action I of Proposition 5, (1) is then given by

$$I(\nabla) = \mathrm{Tr}_\omega(\pi(\theta)^2 |D_\nabla|^{-d}).$$

One then proves in the same way that, for a given compatible connection $\nabla \in C(\mathcal{E})$,

$$\mathrm{YM}(\nabla) = \inf\{I(\nabla_1);\ \pi(\nabla_1) = \nabla\}.$$

Finally let us mention that the basic inequality between the characteristic number $c_1^2 + c_2$ of a hermitian vector bundle and the infimum of the Yang-Mills action does extend to the general non commutative case (cf. [Co] theorem 5 of chapter 6 section 4).

Product of continuum by discrete and the symmetry breaking mechanism.

We have shown how to extend, to our context of finitely summable K-cycles $(\mathfrak{H}, D)$ over an algebra $\mathcal{A}$, the concepts of gauge potentials and Yang–Mills action, as well as the way in which this action is related to a topological action in the case of dimension 4. In this section we shall give several examples of computations of this action. We first recall briefly its definition and use the opportunity to add to it a fermionic part.

We are given a $*$-algebra $\mathcal{A}$ and a (d,∞)-summable K-cycle $(\mathfrak{H}, D)$ over $\mathcal{A}$. This gives us a representation on $\mathfrak{H}$ of the universal differential algebra $\Omega^*\mathcal{A}$:

$$\pi(a^0 \mathrm{d}a^1 \ldots \mathrm{d}a^k) = a^0[D,a^1]\ldots[D,a^k] \quad (\forall a^j \in \mathcal{A}),$$

which defines a quotient differential graded algebra

$$\Omega_D^*(\mathcal{A}) = \Omega^*(\mathcal{A})/J, \quad J = J_0 + \mathrm{d}J_0, \quad J_0^{(k)} = \Omega^k \cap \mathrm{Ker}\pi.$$

A compatible connection ∇ on a hermitian, finitely generated projective module $\mathcal{E}$ over $\mathcal{A}$ is given by a linear mapping

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1$$

that satisfies the Leibniz rule and is compatible with the inner product. The affine space $C(\mathcal{E})$ of such connections is acted on by the unitary group $\mathcal{U}(\mathcal{E})$ of the $*$-algebra of endomorphisms $\mathrm{End}_{\mathcal{A}}(\mathcal{E})$. This action transforms covariantly the curvature $\theta = \nabla^2$ of such connections, and

$$\mathrm{YM}(\nabla) = \mathrm{Tr}_\omega\big(\pi(\theta)^2 |D|^{-d}\big)$$

is a gauge invariant quartic, positive action on $C(\mathcal{E})$.

In the case of the trivial module $\mathcal{E} = \mathcal{A}$ (with the right action of $\mathcal{A}$ on itself), a vector potential is a self-adjoint element V of Ω_D^1, and the following expression is also gauge invariant:

$$\langle \psi, (D + \pi(V))\psi \rangle \quad (\psi \in \mathfrak{H},\ V \in \Omega_D^1),$$

where the unitary group $\mathcal{U} = \mathcal{U}(\mathcal{E}) = \mathcal{U}(\mathcal{A})$ acts on $\mathfrak{H}$ by restriction of the action of $\mathcal{A}$, whereas it acts on vector potentials by gauge transformations:

$$\gamma_u(V) = u\mathrm{d}(u^*) + uVu^* \quad (u \in \mathcal{U},\ V \in \Omega_D^1).$$

This is the fermionic action that we want to add to the action $\mathrm{YM}(\nabla)$; it extends to the case of arbitrary hermitian, finitely generated projective modules $\mathcal{E}$ over $\mathcal{A}$, by means of the next lemma.

Lemma 10. *Let $\mathcal{A}$, $\mathcal{E}$, $(\mathfrak{H}, D)$ be as above. Then:*

(1) *The tensor product $\mathcal{E} \otimes_{\mathcal{A}} \mathfrak{H}$ is a Hilbert space with inner product given by*

$$\langle \xi_1 \otimes \eta_1, \xi_2 \otimes \eta_2 \rangle = \langle \eta_1, (\xi_1, \xi_2)\eta_2 \rangle \quad (\forall \xi_j \in \mathcal{E},\ \eta_j \in \mathfrak{H}).$$

(2) *For any compatible connection ∇, the following equality defines a self-adjoint operator D_∇ in the above Hilbert space:*

$$D_\nabla(\xi \otimes \eta) = \xi \otimes D\eta + ((1 \otimes \pi)\nabla\xi)\eta \quad (\forall \xi \in \mathcal{E},\ \eta \in \mathfrak{H}).$$

Thus, the fermionic action is now given by

$$\langle \psi, D_\nabla \psi \rangle \quad (\psi \in \mathcal{E} \otimes_{\mathcal{A}} \mathfrak{H},\ \nabla \in C(\mathcal{E})),$$

and one checks that it is invariant under gauge transformations by elements of $\mathcal{U}(\mathcal{E})$.

The total action is

$$\mathcal{L}(\nabla, \psi) = \lambda \mathrm{YM}(\nabla) + \langle \psi, D_\nabla \psi \rangle,$$

where λ is a coupling constant.

We shall compute it in two cases: a) the discrete case of a 2-point space; b) the product case of a 4-dimensional manifold by case a). In order to see what the relevant concepts are in the 0-dimensional case a), we first need to discuss product spaces briefly. We are given two triples

$$(\mathcal{A}_1, \mathfrak{H}_1, D_1),\ (\mathcal{A}_2, \mathfrak{H}_2, D_2)$$

and we assume that one of them is *even*, i.e., that we are given a $\mathbb{Z}/2$ grading, say γ_1 on $\mathfrak{H}_1$. The product is then given by the triple $(\mathcal{A}, \mathfrak{H}, D)$, where

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2,\quad \mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2,\quad D = D_1 \otimes 1 + \gamma_1 \otimes D_2.$$

This corresponds to the external product of K-cycles. There is an obvious notion of external product of hermitian finitely generated projective modules over the $\mathcal{A}_j$.

Next, the formula $D^2 = D_1^2 \otimes 1 + 1 \otimes D_2^2$, which follows from the anticommutation of D_1 with γ_1, shows that dimensions add up, that is, if D_j is (p_j, ∞)-summable then D is $(p_1 + p_2, \infty)$-summable; moreover, once the limiting procedure Lim_ω is fixed, one can show that if one of the two terms is convergent then

$$\mathrm{Tr}_\omega((T_1 \otimes T_2)|D|^{-p}) = \mathrm{Tr}_\omega(T_1|D_1|^{-p_1})\mathrm{Tr}_\omega(T_2|D_2|^{-p_2}) \quad (\forall T_j \in \mathcal{L}(\mathfrak{H}_j)).$$

All of this is true provided that $p_j \geq 1$, but in the case we are interested in (Example b)) we have $p_1 = 4$, $p_2 = 0$. The corresponding formula turns out to be

$$\mathrm{Tr}_\omega((T_1 \otimes T_2)|D|^{-p}) = \mathrm{Tr}_\omega(T_1|D_1|^{-p})\mathrm{Trace}(T_2),$$

where Trace is the ordinary trace.

To understand how this occurs, one can use the following general equality assuming that $|D|^{-1} \in \mathcal{L}^{(p,\infty)}$:

$$\lim_{\lambda \to \omega} \left(\frac{1}{\lambda} \mathrm{Trace}(T e^{-\lambda^{-2/p} D^2}) \right) = \mathrm{Tr}_\omega(T|D|^{-p})$$

for all $\in \mathcal{L}(\mathfrak{H})$.

Thus, the 0-dimensional analogue of the action $\mathrm{YM}(\nabla)$ is just given by $\mathrm{Trace}(\pi(\theta)^2)$.

Example a). The space we are dealing with has *two points* a and b. Thus, the algebra $\mathcal{A}$ is just the direct sum $\mathbf{C} \oplus \mathbf{C}$ of two copies of $\mathbf{C}$. An element $f \in \mathcal{A}$ is given by two complex numbers $f(a), f(b) \in \mathbf{C}$. Let $(\mathfrak{H}, D, \gamma)$ be a 0-dimensional K-cycle over $\mathcal{A}$; then $\mathfrak{H}$ is *finite-dimensional* and the representation of $\mathcal{A}$ in $\mathfrak{H}$ corresponds to a decomposition of $\mathfrak{H}$ as a direct sum $\mathfrak{H} = \mathfrak{H}_a + \mathfrak{H}_b$, with the action of $\mathcal{A}$ given by

$$f \in \mathcal{A} \to \begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix}.$$

If we write D as a 2×2 matrix in this decomposition,

$$D = \begin{bmatrix} D_{aa} & D_{ab} \\ D_{ba} & D_{bb} \end{bmatrix},$$

we can ignore the diagonal elements since they commute exactly with the action of $\mathcal{A}$. We shall thus take D to be of the form

$$D = \begin{bmatrix} 0 & D_{ab} \\ D_{ba} & 0 \end{bmatrix},$$

where $D_{ba} = D_{ab}^*$ and D_{ba} is a linear mapping from $\mathfrak{H}_a$ to $\mathfrak{H}_b$. We shall denote this linear mapping by M and take for γ the $\mathbf{Z}/2$ grading given by the matrix $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \gamma$. We thus have

$$\mathcal{A} = \mathbf{C} \oplus \mathbf{C}, \quad \mathfrak{H} = \mathfrak{H}_a \oplus \mathfrak{H}_b, \quad D = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}, \quad \gamma = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Let us first compute the metric on the space $X = \{a, b\}$, given by Formula 1 of the first part of this section. Given $f \in \mathcal{L}$, we have

$$\begin{aligned} [D, f] &= \left[\begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}, \begin{bmatrix} f(a) & 0 \\ 0 & f(b) \end{bmatrix} \right] = \begin{bmatrix} 0 & M^*(f(b) - f(a)) \\ -M(f(b) - f(a)) & 0 \end{bmatrix} \\ &= (f(b) - f(a)) \begin{bmatrix} 0 & M^* \\ -M & 0 \end{bmatrix}. \end{aligned}$$

Thus, the norm of this commutator is $|f(b) - f(a)|\lambda$, where λ is the largest eigenvalue $\|M\|$ of M. Therefore

$$d(a,b) = \sup\{|f(a) - f(b)|,\ \|[D,f]\| \leq 1\} = 1/\lambda.$$

Let us now determine the space of gauge potentials, the curvature and the action in two cases.

case α): $\mathcal{E} = \mathcal{A}$ (i.e., the trivial bundle over X)

First, the space $\Omega^1(\mathcal{A})$ of universal 1-forms over $\mathcal{A}$ is given by the kernel of the multiplication $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, $m(f \otimes g) = fg$. These are functions on $X \times X$ that vanish on the diagonal. Thus, $\Omega^1(\mathcal{A})$ is a 2-dimensional space; if $e \in \mathcal{A}$ is the idempotent $e(a) = 1$, $e(b) = 0$, this space has as basis

$$ede,\ (1-e)\mathrm{d}e,$$

so that every element of $\Omega^1(\mathcal{A})$ is of the form $\lambda ede + \mu(1-e)\mathrm{d}(1-e)$. The differential $\mathrm{d} : \mathcal{A} \to \Omega^1(\mathcal{A})$ is the finite difference

$$\mathrm{d}f = (\Delta f)e\mathrm{d}e - (\Delta f)(1-e)\mathrm{d}(1-e),\quad \Delta f = f(a) - f(b);$$

it is a derivation with values in the bimodule $\Omega^1(\mathcal{A})$ which ***fails to be commutative***: $f\omega \neq \omega f$ for $\omega \in \Omega^1$, $f \in \mathcal{A}$.

Also, if $M \neq 0$ then the representation $\pi : \Omega^*(\mathcal{A}) \to \mathcal{L}(\mathfrak{H})$ is injective on $\Omega^1(\mathcal{A})$, so that $\Omega^1(\mathcal{A}) = \Omega^1_D(\mathcal{A})$. We have

$$\pi\big(\lambda e\mathrm{d}e + \mu(1-e)\mathrm{d}e\big) = \begin{bmatrix} 0 & -\lambda M^* \\ \mu M & 0 \end{bmatrix} \in \mathcal{L}(\mathfrak{H}).$$

A vector potential is given by a self-adjoint element of Ω^1_D, i.e., by a single complex number Φ, with

$$\pi(V) = \begin{bmatrix} 0 & \overline{\Phi} M^* \\ \Phi M & 0 \end{bmatrix}.$$

Since $V = -\overline{\Phi}e\mathrm{d}e + \Phi(1-e)\mathrm{d}e$, its curvature is

$$\theta = \mathrm{d}V + V^2 = -\overline{\Phi}\mathrm{d}e\mathrm{d}e - \Phi\mathrm{d}e\mathrm{d}e + \big(\overline{\Phi}e\mathrm{d}e - \Phi(1-e)\mathrm{d}e\big)^2,$$

and, using the equalities $ede(1-e) = ede$, $e(\mathrm{d}e)e = 0$, $(1-e)\mathrm{d}e(1-e) = 0$, we have

$$\theta = -(\Phi + \overline{\Phi})\mathrm{d}e\mathrm{d}e - (\Phi\overline{\Phi})\mathrm{d}e\mathrm{d}e.$$

Under the representation π, we have $\pi(\mathrm{d}e) = \begin{bmatrix} 0 & -M^* \\ M & 0 \end{bmatrix}$ and $\pi(\mathrm{d}e\mathrm{d}e) = \begin{bmatrix} -M^*M & 0 \\ 0 & -MM^* \end{bmatrix}$. This yields the following formula for the Yang–Mills action:

$$\mathrm{YM}(V) = 2(|\Phi + 1|^2 - 1)^2 \mathrm{Trace}\big((M^*M)^2\big),$$

where Φ is an arbitrary complex number. The action of the gauge group $\mathcal{U} = U(1) \times U(1)$ on the space of vector potentials, i.e., on Φ, is given by

$$\gamma_u(V) = u du^* + uVu^*;$$

for $u = u_a e + u_b(1-e)$, this gives

$$\begin{aligned}\gamma_u(V) =& \big(u_a e + u_b(1-e)\big)(\overline{u}_a de - \overline{u}_b de)\\ &+ \big(u_a e + u_b(1-e)\big)\big(-\overline{\Phi} e de + \Phi(1-e)de\big)\big(\overline{u}_a e + \overline{u}_b(1-e)\big)\\ =& e de + u_b\overline{u}_a(1-e)de - u_a\overline{u}_b e de - (1-e)de\\ &- u_a\overline{u}_b\overline{\Phi} e de + u_b\overline{u}_a\Phi(1-e)de,\end{aligned}$$

which, on the variable $1+\Phi$, just means multiplication by $u_b\overline{u}_a$.

Thus, in this very simple case our action $\mathrm{YM}(V)$ reproduces the usual situation of broken symmetries; it has a non-unique minimum, $|\Phi + 1| = 1$, which is acted upon nontrivially by the gauge group. The fermionic action is in this case given by

$$\langle \psi, (D + \pi(V))\psi \rangle,$$

where the operator $D + \pi(V)$ is equal to

$$\begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix} + \begin{bmatrix} 0 & \overline{\Phi} M^* \\ \Phi M & 0 \end{bmatrix} = \begin{bmatrix} 0 & (1+\overline{\Phi})M^* \\ (1+\Phi)M & 0 \end{bmatrix},$$

which is a term of Yukawa type coupling the fields $(1+\Phi)$ and ψ.

case β): Let us take for $\mathcal{E}$ the nontrivial bundle over $X = \{a, b\}$ with fibers of dimensions n_a and n_b, respectively, over a and b. This bundle is nontrivial if and only if $n_a \neq n_b$; we shall consider the simplest case $n_a = 2$, $n_b = 1$. The finitely generated projective module $\mathcal{E}$ of sections is of the form

$$\mathcal{E} = f\mathcal{A}^2,$$

where the idempotent $f \in M_2(\mathcal{A})$ is given by the formula

$$f = \begin{bmatrix} (1,1) & 0 \\ 0 & (1,0) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}$$

in terms of the notations of α).

To the idempotent f there corresponds a particular compatible connection on $\mathcal{E}$, given by $\nabla_0\xi = f d\xi$ with obvious notations. An arbitrary compatible connection on $\mathcal{E}$ has the form

$$\nabla\xi = \nabla_0\xi + \rho\xi,$$

where $\rho = \rho^*$ is a self-adjoint element of $M_2(\Omega^1_D(\mathcal{A}))$ such that $f\rho = \rho f = \rho$. If we write ρ as a matrix,

$$\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix},$$

these conditions read as follows:

$$e\rho_{21} = \rho_{21}, \; e\rho_{22} = \rho_{22} = \rho_{22}e, \; \rho_{12}e = \rho_{12},$$

thus we get

$$\rho_{11} = -\overline{\Phi}_1 e\mathrm{d}e + \Phi_1(1-e)\mathrm{d}e, \; \rho_{21} = \overline{\Phi}_2 e\mathrm{d}e, \; \rho_{12} = \rho_{21}^*, \; \rho_{22} = 0,$$

where Φ_1, Φ_2 are arbitrary complex numbers.

The curvature θ is given by

$$\begin{aligned} \theta &= f\mathrm{d}f\mathrm{d}f + f\mathrm{d}\rho f + \rho^2 \\ &= \begin{bmatrix} 0 & 0 \\ 0 & e\mathrm{d}e\mathrm{d}e \end{bmatrix} + \begin{bmatrix} \mathrm{d}\rho_{11} & (\mathrm{d}\rho_{12})e \\ e\mathrm{d}\rho_{21} & 0 \end{bmatrix} + \begin{bmatrix} \rho_{11}\rho_{11} + \rho_{12}\rho_{21} & \rho_{11}\rho_{12} \\ \rho_{21}\rho_{11} & \rho_{21}\rho_{12} \end{bmatrix}. \end{aligned}$$

An easy calculation gives the action $\mathrm{YM}(\nabla)$ in terms of the variables Φ_1, Φ_2:

$$\mathrm{YM}(\nabla) = \Big(1 + 2\big(1 - (|\Phi_1 + 1|^2 + |\Phi_2|^2)\big)^2\Big)\mathrm{Tr}\big((M^*M)^2\big).$$

It is by construction invariant under the gauge group $U(1) \times U(2)$. What we learn in example β) rather than α) is that the choice of vacuum corresponds to a choice of connection minimizing the action, and in case β) there is really no preferred choice of ∇_0, the point 0 of the space of vector potentials (case α)) having no intrinsic meaning. In fact, the space of connections realizing the minimum of the action YM is a 3-sphere

$$\{(\Phi_1, \Phi_2) \in \mathbb{C}^2: \; |\Phi_1 + 1|^2 + |\Phi_2|^2 = 1\}$$

whose elements have the following meaning. Let E_a (resp. E_b) be the fiber of our hermitian bundle over the point a (resp. b) of X; then $\dim E_a = 2$, $\dim E_b = 1$. As we saw above, the differential $\mathrm{d} : \mathcal{A} \to \Omega^1(\mathcal{A})$ is the finite difference. One way to extend it to the bundle E is to use an isometry $u : E_b \to E_a$ and the formula

$$(\Delta\xi)_a = \xi_a - u\xi_b, \quad (\Delta\xi)_b = \xi_b - u^*\xi_a.$$

All minimal connections ∇ are of the form

$$\nabla\xi = (\Delta\xi)_a \otimes e\mathrm{d}e + (\Delta\xi)_b \otimes (1-e)\mathrm{d}(1-e).$$

Since the minimum of $\mathrm{YM}(\nabla)$ is > 0 we also see that the bundle E is not flat; it does not admit any compatible connection with vanishing curvature. Since the dimension of the space X is 0, the action YM has of course no topological meaning. However, we shall now return to the 4-dimensional case and work out the case of the product space in detail.

Example b). (4-dimensional Riemannian manifold$M$) $\times$ (2-point space X).

Let us fix the notations: M is a compact Riemannian spin 4-manifold, $\mathcal{A}_1$ the algebra of functions on M and $(\mathfrak{H}_1, D_1, \gamma_1)$ the Dirac K-cycle on $\mathcal{A}_1$; let $\mathcal{A}_2$, $\mathfrak{H}_2$, D_2 be as in Example a) above, that is, $\mathcal{A}_2 = \mathbf{C} \oplus \mathbf{C}$, $\mathfrak{H}_2$ is the direct sum $\mathfrak{H}_2 = \mathfrak{H}_{2,a} \oplus \mathfrak{H}_{2,b}$, and D_2 is given by the matrix

$$D_2 = \begin{bmatrix} 0 & M^* \\ M & 0 \end{bmatrix}.$$

Let $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ and $D = D_1 \otimes 1 + \gamma_1 \otimes D_2$.

The algebra $\mathcal{A}$ is commutative; it is the algebra of complex-valued functions on the space $Y = M \times X$, which is the union of two copies of the manifold M: $Y = M_a \cup M_b$.

Let us first compute compute the metric on Y associated with the K-cycle $(\mathfrak{H}, D)$:

$$d(p,q) = \sup_{f \in \mathcal{A}} \{|f(p) - f(q)|; \quad \|[D,f]\| \leq 1\}.$$

To the decomposition $Y = M_a \cup M_b$there corresponds a decomposition of $\mathcal{A}$ as $\mathcal{A}_a \oplus \mathcal{A}_b$, so that every $f \in \mathcal{A}$ is a pair (f_a, f_b) of functions on M. Also, to the decomposition of $\mathfrak{H}_2$ as

$$\mathfrak{H}_2 = \mathfrak{H}_{2,a} \oplus \mathfrak{H}_{2,b},$$

there corresponds a decomposition $\mathfrak{H} = \mathfrak{H}_a \oplus \mathfrak{H}_b$, in which the action of $f = (f_a, f_b) \in \mathcal{A}$ is diagonal:

$$f \to \begin{bmatrix} f_a & 0 \\ 0 & f_b \end{bmatrix} \in \mathcal{L}(\mathfrak{H}).$$

In this decomposition the operator D becomes

$$D = \begin{bmatrix} \partial_M \otimes 1 & \gamma_5 \otimes M^* \\ \gamma_5 \otimes M & \partial_M \otimes 1 \end{bmatrix},$$

where ∂_M is the Dirac operator on M and γ_5 is the $\mathbf{Z}/2$ grading of its spinor bundle.

This gives us the following formula for the 'differential' of a function $f \in \mathcal{A}$:

$$[D,f] = \begin{bmatrix} i\gamma(\mathrm{d}f_a) \otimes 1 & (f_b - f_a)\gamma_5 \otimes M^* \\ (f_a - f_b)\gamma_5 \otimes M & i\gamma(\mathrm{d}f_b) \otimes 1 \end{bmatrix}.$$

The differential $[D,f]$ thus contains three parts:

α) the usual differential $\mathrm{d}f_a$ of the restriction of f to the copy M_a of M;

β) the usual differential $\mathrm{d}f_b$ of the restriction of f to the copy M_b of M;

γ) the finite difference $\Delta f = f(p_a) - f(p_b)$, where p_a and p_b are the points of M_a, M_b above a given point p of M.

The norm of the operator $[D,f]$ can be computed easily: if λ is the norm of M, i.e., the largest eigenvalue of $|M| = (M^*M)^{1/2}$, then

$$\|[D,f]\| = \operatorname*{ess\,sup}_{p\in M} \begin{bmatrix} \|df_a(p)\| & -i\lambda(\Delta f)(p) \\ i\lambda(\Delta f)(p) & \|df_b(p)\| \end{bmatrix},$$

where $\|df_a(p)\|$ is the length of the gradient of f_a at $p \in M_a$.

We thus obtain easily:

Proposition 11. (1) *The restriction of the metric d on $M_a \cup M_b$ to each copy (M_a or M_b) of M is the Riemannian geodesic distance of M.*

(2) *For each point $p = p_a$ of M_a, the distance $d(p_a, M_b)$ is equal to λ^{-1} and is attained at the unique point p_b.*

Now recall that, given a metric space (Y,d) and two subsets Y_1, Y_2 of Y, their Hausdorff distance $d(Y_1,Y_2)$ is given by

$$d(Y_1,Y_2) = \sup\{d(x,Y_2), x \in Y_1;\ d(x,Y_1), x \in Y_2\}.$$

Thus, the metric d on $M_a \cup M_b = Y$ is clearly related to the following definition of the Gromov distance between two metric spaces (X_1,d_1) and (X_2,d_2):

Definition 12. Let (X_1,d_1) and (X_2,d_2) be two metric spaces. Then, given $\epsilon > 0$, the **Gromov distance** $\delta(X_1,X_2)$ is smaller than ϵ if and only if there exists a metric d on $X = X_1 \cup X_2$ such that a) $d|X_j = d_j$, and b) $d(X_1,X_2) < \epsilon$.

Thus, we see that if we try to take different Riemannian metrics g_a, g_b on the two copies M_a, M_b of M, say by letting

$$D = \begin{bmatrix} D_a \otimes 1 & \gamma \otimes M^* \\ \gamma \otimes M & D_b \otimes 1 \end{bmatrix},$$

then Proposition 11 fails unless the Gromov distance between g_a and g_b is less than $\epsilon = 1/\lambda$, $\lambda = \|M\|$.

Let us now pass to the computation of the $\mathcal{A}$-bimodule Ω_D^1 of 1-forms over the space Y. The above computation of $[D,f] = \pi(df)$ for $f \in \mathcal{A}$ shows that an element α of the $\mathcal{A}$-bimodule $\Omega_D^1 = \pi(\Omega^1)$ is given by:

α) a usual differential form ω_a on M_a;

β) a usual differential form ω_b on M_b;

γ) a pair of complex-valued functions δ_a, δ_b on M.

The corresponding operator in $\mathfrak{H}$ is given by

$$\begin{bmatrix} i\gamma(\omega_a) \otimes 1 & \delta_a\gamma_5 \otimes M^* \\ \delta_b\gamma_5 \otimes M & i\gamma(\omega_b) \otimes 1 \end{bmatrix} = \alpha;$$

the bimodule structure over $\mathcal{A}$ is given, with obvious notations, by

$$(f_a, f_b)(\omega_a, \omega_b, \delta_a, \delta_b) = (f_a\omega_a, f_b\omega_b, f_a\delta_a, f_b\delta_b),$$
$$(\omega_a, \omega_b, \delta_a, \delta_b)(f_a, f_b) = (f_a\omega_a, f_b\omega_b, f_b\delta_a, f_a\delta_b).$$

The involution $*$ is given by $(\omega_a, \omega_b, \delta_a, \delta_b)^* = (-\overline{\omega}_a, -\overline{\omega}_b, \overline{\delta}_b, \overline{\delta}_a)$.

The terms δ_a, δ_b correspond to the bimodule of finite differences on passing from one copy M_a to the other copy M_b of M. Note that even though $\mathcal{A}$ is commutative, this bimodule is *not commutative*; for, if it were commutative then the finite difference would fail to be a derivation. With the above notations, the differential $f \in \mathcal{A} \to \pi(\mathrm{d}f)$ reads as follows:

$$f = (f_a, f_b) \to (\mathrm{d}f_a, \mathrm{d}f_b, f_b - f_a, f_a - f_b) \in \Omega^1_D.$$

When we project on M, the bimodule Ω^1_D can be viewed as a 10-dimensional bundle over M, given by two copies of the complexified cotangent bundle, and a trivial 2-dimensional bundle:

$$T^*_p(M)_{\mathbf{C}} \oplus T^*_p(M)_{\mathbf{C}} \oplus \mathbf{C} \oplus \mathbf{C};$$

however, one must keep in mind the nontrivial bimodule structure in the last two terms.

As in the case of the Dirac operator on Riemannian manifolds (Lemma 6), let us compute the pairs of operators of the form $\pi(\rho) = T_1$, $\pi(\mathrm{d}\rho) = T_2$ for $\rho \in \Omega^1(\mathcal{A})$. Given $\rho = \sum f_j \mathrm{d}g_j \in \Omega^1(\mathcal{A})$, with $f_j, g_j \in \mathcal{A}$, we have

$$\pi(\rho) = \begin{bmatrix} i\gamma(\omega_a) \otimes 1 & \delta_a\gamma_5 \otimes M^* \\ \delta_b\gamma_5 \otimes M & i\gamma(\omega_b) \otimes 1 \end{bmatrix},$$

where $\omega_a = \sum f_{ja}\mathrm{d}g_{ja}$, $\omega_b = \sum f_{jb}\mathrm{d}g_{jb}$ and

$$\delta_a = \sum f_{ja}(g_{jb} - g_{ja}), \quad \delta_b = \sum f_{jb}(g_{ja} - g_{jb}).$$

We have $\pi(\mathrm{d}\rho) = \sum \pi(\mathrm{d}f_j)\pi(\mathrm{d}g_j)$, which gives the 2×2 matrix

$$\pi(\mathrm{d}\rho) = \begin{bmatrix} -\gamma(\xi_a) \otimes 1 + (\delta_a + \delta_b) \otimes M^*M & \gamma_5 i\gamma(\eta_a) \otimes M^* \\ \gamma_5 i\gamma(\eta_b) \otimes M & -\gamma(\xi_b) \otimes 1 + (\delta_a + \delta_b) \otimes MM^* \end{bmatrix},$$

where $\xi_a = \sum \mathrm{d}f_{ja}\mathrm{d}g_{ja}$ and $\xi_b = \sum \mathrm{d}f_{jb}\mathrm{d}g_{jb}$ are sections of the Clifford algebra bundle C^2 over M, whereas

$$\eta_b = \sum \left((f_{ja} - f_{jb})\mathrm{d}g_{ja} - (g_{ja} - g_{jb})\mathrm{d}f_{jb}\right),$$
$$\eta_a = \sum \left((f_{jb} - f_{ja})\mathrm{d}g_{jb} - (g_{jb} - g_{ja})\mathrm{d}f_{ja}\right).$$

Using the equalities

$$
\begin{aligned}
\mathrm{d}\delta_a &= \sum \left(f_{ja}(\mathrm{d}g_{jb} - \mathrm{d}g_{ja}) + (g_{jb} - g_{ja})\mathrm{d}f_{ja}\right),\\
\mathrm{d}\delta_b &= \sum \left(f_{jb}(\mathrm{d}g_{ja} - \mathrm{d}g_{jb}) + (g_{ja} - g_{jb})\mathrm{d}f_{jb}\right),\\
\omega_a &= \sum f_{ja}\mathrm{d}g_{ja}, \quad \omega_b = \sum f_{jb}\mathrm{d}g_{jb},
\end{aligned}
$$

we can rewrite η_a and η_b as follows:

$$
\eta_a = \omega_b - \mathrm{d}\delta_a - \omega_a, \quad \eta_b = \omega_a - \mathrm{d}\delta_b - \omega_b.
$$

As in the Riemannian case, the sections ξ_a, ξ_b of C^2 are arbitrary except for $\sigma_2(\xi_a) = \mathrm{d}\omega_a$ and $\sigma_2(\xi_b) = \mathrm{d}\omega_b$. This shows that the subspace $\pi\big(\mathrm{d}(J_0 \cap \Omega^1)\big)$ of $\pi(\Omega^2)$ is the space of 2×2 matrices of operators of the form

$$
T = \begin{bmatrix} \gamma(\xi_a) \otimes 1 & 0 \\ 0 & \gamma(\xi_b) \otimes 1 \end{bmatrix},
$$

where ξ_a and ξ_b are sections of C^0, i.e., are just arbitrary scalar-valued functions on M, so that $\gamma(\xi_a) = \xi_a$, $\gamma(\xi_b) = \xi_b$.

A general element of $\pi(\Omega^2)$ is a 2×2 matrix of operators of the form

$$
T = \begin{bmatrix} -\gamma(\alpha_a) \otimes 1 + h_a \otimes M^*M & \gamma_5 i\gamma(\beta_a) \otimes M^* \\ \gamma_5 i\gamma(\beta_b) \otimes M & -\gamma(\alpha_b) \otimes 1 + h_b \otimes MM^* \end{bmatrix},
$$

where α_a, α_b are arbitrary sections of C^2, h_a, h_b are arbitrary functions on M and β_a, β_b are arbitrary sections of C^1 (i.e., 1-forms). We thus get:

Lemma 13. *Assume that M^*M is not a scalar multiple of the identity matrix. Then, an element of Ω^2_D is given by:*

1) *a pair of ordinary 2-forms σ_a, σ_b on M;*
2) *a pair of ordinary 1-forms β_a, β_b on M;*
3) *a pair of scalar functions h_a, h_b on M.*

The hypothesis $M^*M \neq \lambda\mathrm{id}$ is important since otherwise the functions h_a, h_b are eliminated by $\pi\big(\mathrm{d}(J_0 \cap \Omega^1)\big)$.

Using the above computation of $\pi(\mathrm{d}\rho)$ we can, moreover, compute $(\sigma_a, \sigma_b, \beta_a, \beta_b, h_a, h_b)$; for the differential $\mathrm{d}\omega$ of an element $\omega = (\omega_a, \omega_b, \delta_a, \delta_b)$ of Ω^1_D, we get:

1) $\sigma_a = \mathrm{d}\omega_a$, $\sigma_b = \mathrm{d}\omega_b$;
2) $\beta_a = \omega_b - \omega_a - \mathrm{d}\delta_a$, $\beta_b = \omega_a - \omega_b - \mathrm{d}\delta_b$;
3) $h_a = \delta_a + \delta_b$, $h_b = \delta_a + \delta_b$.

Thus, we see that the differential $\mathrm{d}\omega \in \Omega^2_D$ involves the differential terms $\mathrm{d}\omega_a$, $\mathrm{d}\omega_b$, $\mathrm{d}\delta_a$, $\mathrm{d}\delta_b$ as well as the finite-difference terms $\omega_a - \omega_b$, $\delta_a + \delta_b$, but in the combinations such as

$\omega_b - \omega_a - d\delta_a$ imposed by $d(df) = 0$. Next, let us compute the product $\omega\omega' \in \Omega_D^2$ of two elements $\omega = (\omega_a, \omega_b, \delta_a, \delta_b)$, $\omega' = (\omega'_a, \omega'_b, \delta'_a, \delta'_b)$ of Ω_D^1; we get:

1) $\sigma_a = \omega_a \wedge \omega'_a$, $\sigma_b = \omega_b \wedge \omega'_b$;
2) $\beta_a = \delta_a \omega'_b - \delta'_a \omega_a$, $\beta_b = \delta_b \omega'_a - \delta'_b \omega_b$;
3) $h_a = \delta_a \delta'_b$, $h_b = \delta_b \delta'_a$.

The next step is to determine the inner product on the space Ω_D^2 of 2-forms. By definition, we take the orthogonal complement of $\pi(d(J_0 \cap \Omega^1))$ in $\pi(\Omega^2)$ equipped with the inner product $\langle T_1, T_2 \rangle = \mathrm{Tr}_\omega(T_1^* T_2 |D|^{-4})$. An easy calculation then gives:

Lemma 14. *Let $\lambda(M^*M)$ be the orthogonal projection of the matrix M^*M onto the scalar matrices $\lambda\mathrm{id}$. Then, the square norm of an element $(\sigma_a, \sigma_b, \beta_a, \beta_b, h_a, h_b)$ of Ω_D^2 is given by*

$$\int_M (N_a \|\sigma_a\|^2 + N_b \|\sigma_b\|^2 dv + \mathrm{tr}(M^*M) \int_M (\|\beta_a\|^2 + \|\beta_b\|^2) dv$$
$$+ \mathrm{tr}\Big(\big(M^*M - \lambda(M^*M)\big)^2\Big) \times \int_M (\|h_a\|^2 + \|h_b\|^2) dv,$$

where $N_a = \dim \mathfrak{H}_a$, $N_b = \dim \mathfrak{H}_b$.

We are now ready to compute the action $\mathrm{YM}(\nabla)$. We shall take the hermitian bundle on $Y = M_a \cup M_b$ that has complex fiber $\mathbf{C}^2$ on the copy M_a of M, and trivial with one-dimensional fiber $\mathbf{C}$ on the copy M_b of M. In other words, we consider the product of the example of Section 1 by the example a), β) on the space X. From the above description of Ω_D^1, we see that if ∇ is a compatible connection on E then it is given by a triple:

α) a usual connection ∇_a on the restriction of E to M_a;
β) a usual connection ∇_b on the restriction of E to M_b;
γ) a section u on M of the bundle $\mathrm{Hom}(E_b, E_a)$ of linear mappings from the fiber $E_{b,p}$ to the fiber $E_{a,p}$.

Both α) and β) have the obvious meaning, while γ) prescribes the value of the finite-difference operation on sections ξ of E. At the point p_a, this finite difference is

$$(\Delta\xi)(p_a) = \xi(p_a) - u_p \xi(p_b) \in E_{p_a} = \mathbf{C}^2,$$

whereas at the point p_b it is given by

$$(\Delta\xi)(p_b) = \xi(p_b) - u_p^* \xi(p_a) \in E_{p_b} = \mathbf{C}.$$

Of course, the choice of u is given by a pair Φ_1, Φ_2 of complex scalar fields on M, namely the two components of $u(1)$ for the basis of $\mathbf{C}^2$ (cf. Example a), β)).

The gauge group $\mathcal{U} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ is the group of unitary endomorphisms of the bundle E over $Y = M_a \cup M_b$, or, equivalently, the group

$$\mathcal{U} = \mathrm{Map}(M, U(1) \times U(2)).$$

Its actions on the $U(2)$ connection ∇_a and on the $U(1)$ gauge connection ∇_b are the obvious ones, and the action on a field $u \in \mathrm{Hom}(E_b, E_a)$ is given by composition.

Let us be more explicit in the description of ∇ as a linear mapping from the space $\mathcal{E}$ of sections of E to $\mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1$. An element of $\mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1$ is given by:

1) a usual differential form $\omega = (\omega_a, \omega_b)$ on $Y = M_a \cup M_b$, with coefficients in $\mathcal{E}$:
2) a section $\delta = (\delta_a, \delta_b)$ on $Y = M_a \cup M_b$ of the bundle E.

The mapping ∇ is then given by

$$\nabla(\xi_a, \xi_b) = (\nabla_a \xi_a, \nabla_b \xi_b), (\xi_a - u\xi_b), (\xi_b - u^* \xi_a)$$

for any section $\xi = (\xi_a, \xi_b)$ of E.

Since the restriction of E to M_a is trivial with fiber $\mathbf{C}^2$, we may as well describe ∇_a by a 2×2 matrix $\begin{bmatrix} \omega_{11}^a & \omega_{12}^a \\ \omega_{21}^a & \omega_{22}^a \end{bmatrix}$ of 1-forms on M that is skew-adjoint. Similarly, ∇_b is given by a single skew-adjoint 1-form $[\omega_{11}^b]$, and u by a pair of complex fields $(1 + \varphi_1, \varphi_2) = u(1)$.

With these notations, the connection ∇ is given by the equality

$$\nabla \xi = f \mathrm{d}\xi + \rho\xi \in \mathcal{E} \otimes_{\mathcal{A}} \Omega_D^1 \quad (\forall \xi \in \mathcal{E}),$$

where $\mathcal{E} = f\mathcal{A}^2$, $f \in M_2(\mathcal{A})$ being the idempotent $f = \begin{bmatrix} 1 & 0 \\ 0 & e \end{bmatrix}$, $e = (0,1) \in \mathcal{A}$, and where $\rho \in M_2(\Omega_D^1)$ is the 2×2 matrix whose entries are the following elements of Ω_D^1:

$$\begin{aligned} \rho_{11} &= (\omega_{11}^a, \omega_{11}^b, \varphi_1, \overline{\varphi}_1), \\ \rho_{12} &= (\omega_{12}^a, 0, 0, \overline{\varphi}_2), \\ \rho_{21} &= (\omega_{21}^a, 0, \varphi_2, 0), \\ \rho_{22} &= (\omega_{22}^a, 0, 0, 0), \end{aligned}$$

or, equivalently,

$$\rho = \left[\begin{bmatrix} \omega_{11}^a & \omega_{12}^a \\ \omega_{21}^a & \omega_{22}^a \end{bmatrix}, \begin{bmatrix} \omega_{11}^b & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{bmatrix}, \begin{bmatrix} \overline{\varphi}_1 & \overline{\varphi}_2 \\ 0 & 0 \end{bmatrix} \right].$$

The curvature θ is then the following element of $fM_2(\Omega_D^2)f$:

$$\theta = f\mathrm{d}f\mathrm{d}f + f\mathrm{d}\rho f + \rho^2,$$

which is easily computed using the above computation of

$$\mathrm{d} : \Omega_D^1 \to \Omega_D^2, \quad \wedge : \Omega_D^1 \times \Omega_D^1 \to \Omega_D^2.$$

As we saw in Lemma 13, elements of Ω_D^2 have a differential degree and a finite-difference degree (α, β) adding up to 2. Let us thus begin with terms in θ of bi-degree $(2,0)$. To compute them we just use the formulas 1) following Lemma 6:

$$\sigma_a = \mathrm{d}\omega_a,\ \sigma_b = \mathrm{d}\omega_b;\quad \sigma_a = \omega_a \wedge \omega'_a,\ \sigma_b = \omega_b \wedge \omega'_b.$$

We thus see that the component $\theta^{(2,0)}$ of bi-degree $(2,0)$ is the following 2×2 matrix of 2-forms on $M_a \cup M_b$:

$$\theta_a^{(2,0)} = \mathrm{d}\omega^a + \omega^a \wedge \omega^a,\quad \theta_b^{(2,0)} = \mathrm{d}\omega^b + \omega^b \wedge \omega^b = \begin{bmatrix} \mathrm{d}\omega_{11}^b & 0 \\ 0 & 0 \end{bmatrix}.$$

Next, we look at the component $\theta^{(1,1)}$ of bi-degree $(1,1)$ and use the formulas 2):

$$\begin{matrix} \beta_a = \omega_b - \omega_a - \mathrm{d}\delta_a \\ \beta_b = \omega_a - \omega_b - \mathrm{d}\delta_b \end{matrix} \quad ; \quad \begin{matrix} \beta_a = \delta_a \omega'_b - \omega_a \delta'_a \\ \beta_b = \delta_b \omega'_a - \omega_b \delta'_b \end{matrix}.$$

Thus, $\theta^{(1,1)}$ is the following 2×2 matrix of 1-forms on $M_a \cup M_b$:

$$\begin{aligned} \theta_a^{(1,1)} &= \left(\begin{bmatrix} \omega_{11}^b & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \omega_{11}^a & \omega_{12}^a \\ \omega_{21}^a & \omega_{22}^a \end{bmatrix} - \begin{bmatrix} \mathrm{d}\varphi_1 & 0 \\ \mathrm{d}\varphi_2 & 0 \end{bmatrix} \right) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ &\quad + \begin{bmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{bmatrix} \begin{bmatrix} \omega_{11}^b & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \omega_{11}^a & \omega_{12}^a \\ \omega_{21}^a & \omega_{22}^a \end{bmatrix} \begin{bmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} -\mathrm{d}\omega_1 - (\omega_{11}^a - \omega_{11}^b)(\varphi_1 + 1) - \omega_{12}^a \varphi_2 & 0 \\ -\mathrm{d}\omega_2 - \omega_{21}^a(\varphi_1 + 1) - (\omega_{22}^a - \omega_{11}^b)\varphi_2 & 0 \end{bmatrix}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} \theta_b^{(1,1)} &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \left(\begin{bmatrix} \omega_{11}^a & \omega_{12}^a \\ \omega_{21}^a & \omega_{22}^a \end{bmatrix} - \begin{bmatrix} \omega_{11}^b & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \mathrm{d}\overline{\varphi}_1 & \mathrm{d}\overline{\varphi}_2 \\ 0 & 0 \end{bmatrix} \right) \\ &\quad + \begin{bmatrix} \overline{\varphi}_1 & \overline{\varphi}_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \omega_{11}^a & \omega_{12}^a \\ \omega_{21}^a & \omega_{22}^a \end{bmatrix} - \begin{bmatrix} \omega_{11}^b & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{\varphi}_1 & \overline{\varphi}_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{d}\overline{\varphi}_1 + (\omega_{11}^a - \omega_{11}^b)(\overline{\varphi}_1 + 1) + \omega_{21}^a \overline{\varphi}_2 & -\mathrm{d}\overline{\varphi}_2 + \omega_{12}^a(\overline{\varphi}_1 + 1) + (\omega_{22}^a - \omega_{11}^b)\varphi_2 \\ 0 & 0 \end{bmatrix}. \end{aligned}$$

Finally, we have to compute the component $\theta^{(0,2)}$; we use the formlas 3):

$$\begin{matrix} h_a = \delta_a + \delta_b \\ h_a = \delta_a + \delta_b \end{matrix} \quad ; \quad \begin{matrix} h_a = \delta_a \delta'_b \\ h_b = \delta_b \delta'_a \end{matrix}.$$

We then have

$$\begin{aligned} \theta_a^{(0,2)} &= \begin{bmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{bmatrix} + \begin{bmatrix} \overline{\varphi}_1 & \overline{\varphi}_2 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{bmatrix} \begin{bmatrix} \overline{\varphi}_1 & \overline{\varphi}_2 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \varphi_1 + \overline{\varphi}_1 + \varphi_1 \overline{\varphi}_1 & \overline{\varphi}_2(1 + \varphi_1) \\ \varphi_2(1 + \overline{\varphi}_1) & 0 \end{bmatrix}, \end{aligned}$$

$$\theta_b^{(0,2)} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 + \overline{\varphi}_1 & \overline{\varphi}_2 \\ \varphi_2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$
$$+ \begin{bmatrix} \overline{\varphi}_1 & \overline{\varphi}_2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \varphi_1 & 0 \\ \varphi_2 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 + \varphi_1 + \overline{\varphi}_1 + \overline{\varphi}_1\overline{\varphi}_1 + \overline{\varphi}_2\overline{\varphi}_2 & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus,

$$\mathrm{YM}(\nabla) = I_2 + I_1 + I_0,$$

where each I_j is the integral over M of a Lagrangian density given by the following formulas.

First,

$$I_2 : \ |\mathrm{d}\omega^a + \omega^a \wedge \omega^a|^2 N_a + |\mathrm{d}\omega^b|^2 N_b,$$

where $N_a = \dim \mathfrak{H}_a$, $N_b = \dim \mathfrak{H}_b$ and the norms are the square norms for the curvatures of the connections ∇^a and ∇^b, respectively.

Next,

$$I_1 : \ 2\left|\nabla \begin{pmatrix} 1 + \varphi_1 \\ \varphi_2 \end{pmatrix}\right|^2 \operatorname{tr}(M^* M),$$

where ∇ is the covariant differentiation of a pair of scalar fields, given by

$$\mathrm{d} + \begin{bmatrix} \omega_{11}^a - \omega_{11}^b & \omega_{12}^a \\ \omega_{21}^a & \omega_{22}^a - \omega_{11}^b \end{bmatrix}.$$

Finally,

$$I_0 : \ \left(1 + 2(1 - (|1 + \varphi_1|^2 + |\varphi_2|^2))^2\right) \operatorname{Tr}\left(\left(\lambda^\perp(M^* M)\right)^2\right),$$

where $\lambda^\perp$ is the orthogonal projection in the Hilbert–Schmidt space of matrices onto the orthogonal complement of the scalar multiples of the identity. These terms are obtained, with the right coefficients, from the computation of the Hilbert space norm on Ω_D^2.

The fermionic action is even easier to compute. We have

$$\langle \psi, D_\nabla \psi \rangle = J_0 + J_1,$$

where $\psi \in \mathcal{E} \otimes_{\mathcal{A}} \mathfrak{H}$, $\gamma\psi = \psi$, is given by a pair of left-handed sections of $S \otimes \mathfrak{H}_a$ denoted by $\begin{bmatrix} \psi_1^a \\ \psi_2^a \end{bmatrix}$, and a right-handed section of $S \otimes \mathfrak{H}_a$ denoted by ψ^b. Both J_0 and J_1 are given by Lagrangian densities:

$$J_0:\ \overline{\psi}^a(\partial + i\gamma(\omega^a))\psi^a + \overline{\psi}^b(\partial + i\gamma(\omega^b))\psi^b,$$
$$J_1:\ \overline{\psi}_b M[(1+\varphi_1),\varphi_2]\psi_a + \text{h.c.}$$

We can now make the point concerning this example b): modulo a few nuances that we shall deal with, the five terms of our action

$$I_0 + I_1 + I_2 + J_0 + J_1$$

are the five terms of the Glashow–Weinberg–Salam unification of electromagnetic and weak forces for N generations of leptons (where $N = N_a = N_b$ is the dimension of $\mathfrak{H}_a$, $\mathfrak{H}_b$).

Let us describe, using conventional notations of physics, what are the five constituents of the G.W.S. Lagrangian, which we write directly in the Euclidean (i.e., imaginary time) framework. For each constituent, we give the corresponding fields and Lagrangian:

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_f + \mathcal{L}_\Phi + \mathcal{L}_\gamma + \mathcal{L}_V,$$

where:

1) $\mathcal{L}_G$: The *pure gauge boson* part is just

$$\mathcal{L}_G = \frac{1}{4}(G_{\mu\nu a}G_a^{\mu\nu}) + \frac{1}{4}(F_{\mu\nu}F^{\mu\nu}),$$

where $G_{\mu\nu} = \partial_\mu W_{\nu a} - \partial_\nu W_{\mu a} + g\varepsilon_{abc}W_{\mu b}W_{\nu c}$ and $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$ are the field strength tensors of an $SU(2)$ gauge field $W_{\mu a}$ and a $U(1)$ gauge field B_μ.

2) $\mathcal{L}_f$: The *fermion kinetic term* has the form

$$\mathcal{L}_f = -\sum\left[\overline{f}_L\gamma^\mu\left(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + ig'\frac{Y_L}{2}B_\mu\right)f_L + \overline{f}_R\gamma^\mu\left(\partial_\mu + ig'\frac{Y_R}{2}B_\mu\right)f_R\right],$$

where the f_L (resp. f_R) are the left-handed (resp. right-handed) fermion fields, which for leptons and for each generation are given by a pair. i.e., an isodoublet, of left-handed spinors (such as $\begin{bmatrix}\nu_L\\ e_L\end{bmatrix}$), and a singlet ($e_R$), i.e., a right-handed spinor.

We shall return later to the hypercharges Y_L, Y_R, which for leptons are given by $Y_L = -1$, $Y_R = -2$.

3) $\mathcal{L}_\Phi$: The *kinetic terms for the Higgs fields* are

$$\mathcal{L}_\Phi = -\left|\left(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + i\frac{g'}{2}B_\mu\right)\Phi\right|^2,$$

where $\Phi = \begin{bmatrix}\Phi_1\\ \Phi_2\end{bmatrix}$ is an $SU(2)$ doublet of complex scalar fields Φ_1 and Φ_2 with hypercharge $Y_\Phi = 1$.

4) $\mathcal{L}_Y$: The ***Yukawa coupling*** of Higgs fields with fermions is

$$\mathcal{L}_Y = -\sum [H_{ff'}(\overline{f}_L \cdot \Phi) f'_R + H^*_{ff'} \overline{f'}_R (\Phi^+ \cdot f_L)],$$

where $H_{ff'}$ is a general coupling matrix in the space of different families.

5) $\mathcal{L}_V$: The ***Higgs self-interaction*** is the potential

$$\mathcal{L}_V = \mu^2(\Phi^+\Phi) - \frac{1}{2}\lambda(\Phi^+\Phi)^2,$$

where $\lambda > 0$ and $\mu^2 > 0$ are scalars.

Let us now spell out the dictionary between our action and the Glashow-Weinberg-Salam action:

Noncommutative geometry	**Classical field theory**
vector $\psi \in \mathcal{E} \otimes_{\mathcal{A}} \mathfrak{H}$, $\gamma\psi = \psi$	chiral fermion f
differential components of connection ω^a, ω^b	pure gauge bosons W, B
finite-difference component of connection $(1 + \delta^a)$, δ^b	Higgs field Φ
I_2	$\mathcal{L}_G$
I_1	$\mathcal{L}_\Phi$
I_0	$\mathcal{L}_V$
J_0	$\mathcal{L}_f$
J_1	$\mathcal{L}_Y$

A lot more work is necessary to interpret the standard model Lagrangian (with its quark sector) in the above way. It will be done (cf. [Co-L]) in the next sections.

7. The notion of manifold in non commutative geometry.

In this section we shall first expound classical results of the theory of manifolds and their characteristic classes, in particular those of D. Sullivan, which exhibit the central role played by K-homology. We shall then explain how to formulate Poincaré duality in K-homology in the non commutative context. Our aim is to free the notion of manifold from the hypothesis of commutativity of the coordinates. We shall then apply the obtained notion to give a geometric interpretation of the standard model. The phenomenology of high energy physics is contained in the Lagrangian of the standard model which in turns determines the fine structure of space time as a manifold in the above sense.

The classical notion of manifold.

A d-dimensional closed topological manifold X is a compact space locally homeomorphic to open sets in Euclidean space of dimension d. Such local homeomorphisms are called charts. If two charts overlap in the manifold one obtains an overlap homeomorphism between open subsets of Euclidean space. A smooth (resp. $PL\ldots$) structure on X is given by a covering by charts so that all overlap homeomorphisms are smooth (resp. $PL\ldots$). We refer to the course by François David in this volume for a more detailed discussion of PL manifolds. By definition a PL homeomorphism is simply a homeomorphism which is piecewise affine.

Smooth manifolds can be triangulated and the resulting PL structure up to equivalence is uniquely determined by the original smooth structure. We can thus write:

$$\text{Smooth} \Rightarrow PL \Rightarrow \text{Top}. \tag{1}$$

The above three notions of smooth, PL and Topological manifolds are compared using the respective notions of tangent bundles. A smooth manifold X possesses a tangent bundle TX which is a *real vector bundle* over X. The stable isomorphism class of TX in the real K-theory of X is classified by the homotopy class of a map:

$$X \to BO. \tag{2}$$

Similarly a PL (resp. Top) manifold possesses a tangent bundle but it is no longer a vector bundle but rather a suitable neighborhood of the diagonal in $X \times X$ for which the projection $(x, y) \to x$ on X defines a PL (resp. Top) bundle. Such bundles are stably classified by the homotopy class of a natural map:

$$X \to BPL \qquad (\text{resp. } B\ \text{Top}). \tag{3}$$

The implication (1) yields natural maps:

$$BO \to BPL \to B\ \text{Top} \tag{4}$$

and the nuance between the three above kinds of manifolds is governed by the ability to lift up to homotopy the classifying maps (3) for the tangent bundles. (In dimension 4 this statement has to be made unstably to go from Top to PL.) It follows for instance that every PL manifold of dimension $d \leq 7$ possesses a compatible smooth structure. Also for $d \geq 5$, a topological manifold X^d admits a PL structure iff a single topological obstruction $\delta \in H^4(X, \mathbb{Z}/2)$ vanishes.

For $d = 4$ one has Smooth $= PL$ but topological manifolds only sometimes possess smooth structure (and when they do they are not unique up to equivalence) as follows from the works of Donaldson and Freedman.

The KO orientation of a manifold.

Any finite simplicial complex can be embedded in Euclidean space and has the homotopy type of a manifold with boundary. The homotopy types of these manifolds with boundary is thus rather arbitrary. For closed manifolds this is no longer true and we shall now discuss this point.

Let X be a closed oriented manifold. Then the orientation class $\mu_X \in H_n(X, \mathbb{Z}) = \mathbb{Z}$ defines a natural isomorphism:

$$(5) \qquad a \in H^i\, X \to a \cap \mu_X \in H_{n-i}\, X$$

which is called the ***Poincaré duality*** isomorphism. This continues to hold for any space Y homotopic to X since homology and cohomology are invariant under homotopy.

Conversely let X be a finite simplicial complex which satisfies Poincaré duality (5) for a suitable class μ_X, then X is called a Poincaré complex. If one assumes that X is simply connected ($\pi_1(X) = \{e\}$), then [Mi-S] there exists a unique up to fibre homotopy equivalence spherical fibration $E \xrightarrow{p} X$ over X (the fibers $p^{-1}(b)$, $b \in X$ have the homotopy type of a sphere) which plays the role of the stable tangent bundle when X is homotopy equivalent to a manifold. Moreover, in the simply connected case and with $d = \dim X \geq 5$, the problem of finding a PL manifold in the homotopy type of X is the same as that of promoting the Spivak normal bundle to a PL bundle. There are, in general, obstructions for doing that, but a key result of D. Sullivan [ICM. Nice 1970] asserts that after tensoring the relevant abelian obstruction groups by $\mathbb{Z}\left[\frac{1}{2}\right]$, a PL bundle is the same thing as a spherical fibration together with a KO orientation. This shows first that the characteristic feature of the homotopy type of a PL manifold is to possess a KO orientation

$$(6) \qquad \nu_X \in KO_*(X)$$

which defines a Poincaré duality isomorphism in real K theory, after tensoring by $\mathbb{Z}[1/2]$:

$$(7) \qquad a \in KO^*(X)_{1/2} \to a \cap \nu_X \in KO_*(X)_{1/2}.$$

Moreover it was shown that this element $\nu_X \in KO_*(X)$ describes all the invariants of the PL manifolds in a given homotopy type, provided the latter is simply connected and all relevant abelian obstruction groups are tensored by $\mathbf{Z}\left[\frac{1}{2}\right]$. Among these invariants are the rational Pontrjagin classes of the manifold. For smooth manifolds they are the Pontrjagin classes of the tangent vector bundle, but in general they are obtained from the Chern character of the KO orientation class ν_X. These classes continue to make sense for topological manifolds and are homeomorphism invariants thanks to the work of S. Novikov.

We can thus assert that, in the simply connected case, a closed manifold is in a rather deep sense more or less the same thing as a homotopy type X satisfying Poincaré duality in ordinary homology together with a preferred element $\nu_X \in KO_*(X)$ which induces Poincaré duality in KO theory tensored by $\mathbf{Z}[1/2]$. In the non simply connected case one has to take in account the equivariance with respect to the fundamental group $\pi_1(X) = \Gamma$ acting on the universal cover $\widetilde{X}$.

Fredholm modules and K-homology.

In the above discussion we used, without defining it properly, the KO homology $KO_*(X)$. By definition this is the generalized homology theory dual to real K-theory. The latter classifies real vector bundles over X exactly as $K(X)$ classifies complex vector bundles. M.F. Atiyah gave in [At$_2$] a natural operator theoretic interpretation of the cycles in $KO_*(X)$ (in fact in $K_*(X)$ the K-homology dual to (complex) K-theory). This interpretation was fully developed thanks to the work of Brown-Douglas-Fillmore in operator theory and of Miscenko and Kasparov on the Novikov conjecture.

The basic cycles in the K-homology $K_*(X)$, X a compact space, are given by Fredholm modules $(\mathcal{H}, F)$ (definition 1 of section 5) on the commutative C^*-algebra $C(X)$. (We refer to [Co] for the small nuance between Fredholm modules and pre Fredholm modules for which the condition $F^2 = 1$ is relaxed to $F^2 - 1$ compact.) Using even (resp. odd) Fredholm modules up to homotopy (cf. [Kas]) and the obvious operation of direct sum of such modules, one obtains an abelian group $K_0(X)$ (resp. $K_1(X)$). The generalized homology theory thus obtained on the category of metrisable compact spaces is the Steenrod homology dual to K-theory [Kam-S]. The description of KO homology $KO_*(X)$ is entirely similar but requires the discussion of real C^*-algebras and a more systematic use of Clifford algebras. In [Sin$_2$], I. Singer conjectured that the Sullivan KO orientation of PL or of topological manifolds could be directly constructed from elliptic theory on the manifold, as a Fredholm module $(\mathcal{H}, F)$ over $C(X)$, thus providing a direct analytical proof of Novikov's theorem on rational Pontrjagin classes. This conjecture was proven by Sullivan and Teleman [S-T]. First, D. Sullivan showed that any topological manifold X of dimension different from 4 admits an essentially unique Lipschitz (resp. quasi conformal) structure. To define these notions we just need to give the definitions of Lipschitz (resp. quasi conformal) homeomorphisms between open subsets of Euclidean space. They are:

(8) *Lipschitz* $\exists \lambda > 0$ such that

$$\lambda\, d(x,y) \leq d(\varphi(x), \varphi(y)) \leq \lambda^{-1}\, d(x,y) \qquad \forall x, y$$

where $d(a,b) = \|b - a\|$ is the Euclidean distance.

(9) ***Quasi conformal*** $\exists K < \infty$ such that

$$\overline{\lim_{r\to 0}} \; \frac{\max\{d(\varphi(x),\varphi(y))\;,\; d(x,y)=r\}}{\inf\{d(\varphi(x),\varphi(y))\;,\; d(x,y)=r\}} \leq K \qquad \forall x,y.$$

Oriented Lipschitz manifolds admit a natural signature operator which defines an unbounded Fredholm module $(\mathcal{H}, D)$ over the C^*-algebra $C(X)$.

Quasi conformal manifolds admit bounded measurable conformal structures and the work of Donaldson and Sullivan [D-S] provides exactly the ingredients necessary and sufficient to construct the even Fredholm module $(\mathcal{H}, F)$ that we considered in section 5 for even dimensional manifolds (cf. [C-S-T] [Co]). This leads ([C-S-T]) to local formulae for the rational Pontrjagin classes of topological manifolds of dimension different from 4. In dimension 4 the work of Donaldson and Sullivan [D-S] shows that the Yang-Mills theory can still be done for quasi conformal manifolds and thus that many 4-dimensional topological manifolds do not admit quasi conformal structures. In some sense a quasi conformal structure is the minimal amount of structure required to develop the Yang-Mills theory and to construct the Donaldson invariants. As is clear from section 5 it is also the minimal amount of structure required to develop the quantized calculus.

Poincaré duality in K-homology and non commutative C^*-algebras.

The notions of Fredholm module and of homotopy between them make sense over any, not necessarily commutative C^*-algebra A, thus defining the analogue of K-homology in the non commutative case. Both groups $K_*(A)$ of K-theory and $K^*(A)$ of K-homology of a C^*-algebra A are special cases of the bivariant functor KK of Kasparov (cf. [Kas]) and we shall use the corresponding notation:

$$KK(\mathbb{C}, A) = K_*(A) \tag{10}$$

is the K-theory of the C^*-algebra A (cf. section 4)

$$KK(A, \mathbb{C}) = K^*(A) \tag{11}$$

is the K-homology of the C^*-algebra A.

Thus for $A = C(X)$ one has $KK(\mathbb{C}, A) = K^*(X)$ and $KK(A, \mathbb{C}) = K_*(X)$. (The functor $X \to C(X)$ is contravariant.) An even (resp. odd) Fredholm module $(\mathcal{H}, F)$ over a C^*-algebra A yields an element in $KK(A, \mathbb{C})$. Exactly as in the commutative case one has, in the general case, a cup product operation in the Kasparov bivariant theory.

A bilinear, associative ***intersection product*** is defined, given C^*-algebras A_1, A_2, B_1, B_2 and D:

$$KK(A_1, B_1 \otimes D) \otimes_D KK(D \otimes A_2, B_2) \longrightarrow KK(A_1 \otimes A_2, B_1 \otimes B_2). \tag{12}$$

We refer to [Kas] for the properties of the intersection product (cf. also [Co]). One way to remember these properties is to think of the elements of $KK(A, B)$ as (homotopy classes of) generalized morphisms from A to B, the intersection product then being composition. For any C^*-algebra the intersection product provides the abelian group $KK(A, A)$ with a ring structure, whose unit element will be denoted 1_A.

Now let A and B be C^*-algebras and let us assume that we have two elements

$$\alpha \in KK(A \otimes B, \mathbf{C}), \quad \beta \in KK(\mathbf{C}, A \otimes B) \tag{13}$$

such that

$$\begin{aligned} \beta \otimes_A \alpha &= 1_B \in KK(B, B), \\ \beta \otimes_B \alpha &= 1_A \in KK(A, A). \end{aligned} \tag{14}$$

It then follows from the general properties of the intersection product that there are canonical isomorphisms

$$\begin{aligned} K_*(A) = KK(\mathbf{C}, A) &\cong KK(B, \mathbf{C}) = K^*(B), \\ K_*(B) = KK(\mathbf{C}, B) &\cong KK(A, \mathbf{C}) = K^*(A) \end{aligned} \tag{15}$$

that exchange the K-theory of A with the K-homology of B.

More explicitly, the mapping from $K_*(A)$ to $K^*(B)$ is given by the intersection product with α:

$$x \in K_*(A) = KK(\mathbf{C}, A) \to x \otimes_A \alpha \in KK(B, \mathbf{C}) = K^*(B). \tag{16}$$

The inverse mapping from $K^*(B) = KK(B, \mathbf{C})$ to $K_*(A)$ is given by the intersection product with β:

$$y \in K^*(B) = KK(B, \mathbf{C}) \to \beta \otimes_B y \in KK(\mathbf{C}, A) = K_*(A). \tag{17}$$

More generally, for any pair of C^*-algebras C and D, we have canonical isomorphisms

$$\begin{aligned} KK(C, A \otimes D) &\cong KK(C \otimes B, D), \\ KK(C, B \otimes D) &\cong KK(C \otimes A, D), \end{aligned} \tag{18}$$

which show that the above pair α, β establishes a duality between A and B with arbitrary coefficients.

In ([Kas$_7$],[Co-S]) an example of this duality was worked out with $A = C(V)$, $B = C_0(TV)$, where V is a compact smooth manifold, A is the C^*-algebra of continuous functions on V, and TV is the total space of the tangent bundle of V. The differentiable structure of V then provides, through the pseudo-differential calculus, the desired elements $\alpha \in KK(A \otimes B, \mathbf{C})$, $\beta \in KK(\mathbf{C}, A \otimes B)$ fulfilling the above condition (14).

The Thom isomorphism for vector bundles ([Kas$_1$]) provides a natural KK-equivalence (i.e., an isomorphism in the category of C^*-algebras with $KK(A, B)$ as the morphisms from A to B)

$$C_0(TV) \cong C_V, \tag{19}$$

where C_V is the C^*-algebra of continuous sections of the bundle over V of Clifford algebras $C_p = \mathrm{Cliff}_{\mathbf{C}}\,(T_p(V))$. We thus get, using the KK-equivalence, a natural duality between $A = C(V)$ and $B = C_V$. We shall now describe in greater detail the corresponding elements

$$\alpha \in KK\big(C(V) \otimes C_V, \mathbf{C}\big), \quad \beta \in KK\big(\mathbf{C}, C(V) \otimes C_V\big). \tag{20}$$

Since, as a rule, K-homology is always more difficult than K-theory, we shall concentrate on the description of α. The description of β is much simpler.

We shall describe α as a very specific K-cycle on $C(V) \otimes C_V$: we let $\mathfrak{H}$ be the Hilbert space of square-integrable differential forms on V, which is equipped with a Riemannian metric g:

$$\mathfrak{H} = L^2(V, \textstyle\bigwedge_{\mathbf{C}}^* T^*). \tag{21}$$

We let $D = \mathrm{d} + \mathrm{d}^*$ be the self-adjoint operator in $\mathfrak{H}$ given by the sum of the exterior differential d with its adjoint d^*. The action of $C(V)$ on $\mathfrak{H}$ is the obvious one, by multiplication. For the action of C_V we have the following:

Lemma. *If $(\mathfrak{H}, D)$ is the K-cycle over $C(V)$ given above, then the commutant on $\mathfrak{H}$ of the algebra generated by $C(V)$ and the $[D, f]$ ($f \in C(V)$, $\|[D, f]\| < \infty$) is canonically isomorphic to the algebra of bounded measurable sections of the bundle C of Clifford algebras.*

Indeed, the commutant of $C(V)$ on $\mathfrak{H}$ is the algebra of bounded measurable sections of the bundle $\mathrm{End}(\bigwedge)$ of endomorphisms of $\bigwedge_{\mathbf{C}} T^*$, so that it is enough to compute for each $p \in V$ the commutant of the algebra generated by the operators $\gamma(\xi)$, $\xi \in T_p^*(V)$, where

$$\gamma(\xi)\eta = \xi \wedge \eta + i_\xi \eta \quad \big(\forall \eta \in \textstyle\bigwedge_{\mathbf{C}} T_p^*(V)\big). \tag{22}$$

However, γ defines a representation of the Clifford algebra C_p on the Hilbert space $\bigwedge_{\mathbf{C}} T_p^*$ with $1 \in \bigwedge_{\mathbf{C}} T_p^*$ as cyclic and separating vector, so that its commutant is also given by a canonical representation of C_p, given explicitly by the formula

$$(23) \qquad \gamma_p'(\xi)\eta = (-1)^{\partial\eta}(\xi \wedge \eta - i_\xi \eta) \quad (\forall \eta \in \wedge_{\mathbb{C}} T_p^*(V)).$$

This K-cycle $(\mathfrak{H}, D)$ over $C(V) \otimes C_V$ defines the fundamental class of V in K-homology, $\alpha \in KK(C(V) \otimes C_V, \mathbb{C}))$, and yields for instance the construction of the Dirac operator with coefficients in a Clifford bundle ([Gi$_1$]) as the natural mapping

$$(24) \qquad K_*(C_V) \to K^*(C(V)).$$

The K-theory class $\beta \in KK(\mathbb{C}, C(V) \otimes C_V)$ is easier to describe; it is just the family, parametrized by $p \in V$, of Bott elements $\beta_p \in K_*(C_V)$ obtained from the Bott periodicity applied to a small disk centered at $p \in V$.

In general, C_V is not Morita equivalent to $C(V)$. Giving a Spin^c structure on V determines such a Morita equivalence and thus permits replacing α and β by equivalent elements

$$(25) \qquad \alpha \in KK(C(V) \otimes C(V), \mathbb{C}), \quad \beta \in KK(\mathbb{C}, C(V) \otimes C(V)).$$

This time α is given by the Dirac K-cycle on V and the two representations of $C(V)$ on $\mathfrak{H}$ are identical, thus yielding the diagonal representation of $C(V) \otimes C(V)$ on $\mathfrak{H}$. This is a very special feature of the *commutative* case: if A is abelian then every A-module is in a trivial way an A-bimodule, since one then has the diagonal homomorphism $A \otimes A \to A$. In general, as we saw above, the fundamental class in K-homology involves an algebra A, its Poincaré dual B and (A, B)-bimodules.

The lesson that we want to draw from this discussion is that the K-homology fundamental class of a quantum space is given by a *bimodule*, not just a module, $(\mathcal{H}, D)$.

We should of course stress that we are interested in the actual K-cycle $(\mathcal{H}, D)$ over $\mathcal{A} \otimes \mathcal{B}$ and not only in its stable homotopy class.

We also need to discuss briefly the non simply connected case. For a non simply connected manifold X the above construction of the Poincaré duality isomorphism should be done using the universal cover $\widetilde{X}$ instead of X, and Γ-equivariantly where $\Gamma = \pi_1(X)$ is the fundamental group of X (cf. [Kas]). One thus obtains elements α, β of the Γ-equivariant groups KK_Γ (cf. [Kas]).

Non commutative manifolds.

Without giving a final definition of a non commutative Riemannian manifold X we shall combine our original discussion of the *metric* aspect in non commutative geometry (section 6) together with the above discussion of *ordinary manifolds* to describe the natural ingredient of such a notion. We let $\mathcal{A}$ be the involutive algebra of coordinates on the quantum space X. By reference to KO (versus K) theory we no longer assume $\mathcal{A}$ to be an algebra over $\mathbb{C}$ but just over $\mathbb{R}$. The full structure (both manifold and metric) should be given by a K-cycle $(\mathcal{H}, D)$ over the algebra $\mathcal{A}$. (Note that it makes sense to say that the representation of $\mathcal{A}$ in $\mathcal{H}$ is involutive, we take $\mathcal{H}$ to be a complex Hilbert space.)

Following the above lemma, the Poincaré dual algebra $\mathcal{B}$ turning $(\mathcal{H}, D)$ into an $\mathcal{A} \otimes \mathcal{B}^0$ K-cycle, should be defined by:

$$\mathcal{B} = \{b \in \mathcal{A}' \ ; \ [[D,a],b] = 0 \quad \forall a \in \mathcal{A} \ , \ [D,b] \text{ bounded}\} \ . \tag{26}$$

Of course one should assume that $\mathcal{B}$ is large enough to insure that the class α of $(\mathcal{H}, D)$ in $KK(\mathcal{A} \otimes \mathcal{B}^0, \mathrm{C})$ defines a Poincaré duality isomorphism by the formula (15) (at least rationally).

Finally the discussion should involve the "fundamental group" Γ of X, which in the non commutative case has no reason to be a usual group but rather a "discrete quantum group". This "group" should act on $\mathcal{H}$ making the whole picture Γ-equivariant.

We shall not rush to write down final axioms but rather present what should be the prototype of such a generalized Riemannian manifold, namely space time (in its Euclidean version $t \to it$) as revealed by the Lagrangian of high energy physics: the standard model.

In [Co] we give other non commutative examples such as the non commutative torus and we also show how the cohomology of differential forms in $\Omega_D^*(\mathcal{A})$ is related to the cyclic cohomology $HC^*(\mathcal{B})$, in the above general situation.

8. Geometric interpretation of the standard model.

In this section we shall describe the (non commutative) geometric structure of space time which is given by the phenomenological Lagrangian of high energy physics, namely the standard model. This "example" should be analyzed further with great care since it is the basic example of geometry provided by nature.

We shall first describe the Lagrangian itself from the notes by J. Ellis (Les Houches 1981). We shall then describe the geometric structure of the finite space F, it will be a "non commutative Riemannian manifold" in the sense of section 7. After giving the computation of the corresponding Lagrangian on (4 dim continuum) $\times F$ we shall discuss a general unimodularity condition which will give here the known values to the hypercharges and give the equality between the Lagrangian of our model and the standard model Lagrangian.

The standard model

Just as for the Glashow–Weinberg–Salam model for leptons, the Lagrangian of the standard model contains five different terms,

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_f + \mathcal{L}_\varphi + \mathcal{L}_Y + \mathcal{L}_V,$$

which we now recall together with the field content of the theory.

1) The pure gauge boson part $\mathcal{L}_G$.

$$\mathcal{L}_G = \frac{1}{4}(G_{\mu\nu a}G_a^{\mu\nu}) + \frac{1}{4}(F_{\mu\nu}F^{\mu\nu}) + \frac{1}{4}(H_{\mu\nu b}H_b^{\mu\nu}),$$

where $G_{\mu\nu a}$ is the field strength tensor of an $SU(2)$ gauge field $W_{\mu a}$, F_μ is the field strength tensor of a $U(1)$ gauge field B_μ and $H_{\mu\nu b}$ the field strength tensor of an $SU(3)$ gauge field $V_{\mu b}$. This last gauge field, the ***gluon field***, is the carrier of the strong force; the gauge group $SU(3)$ is the ***color group***, and is thus the essential new ingredient. The respective coupling constants for the fields W, B, V fields will be denoted g, g', g'', consistent with the previous notations.

2) The fermion kinetic term $\mathcal{L}_f$.

To the leptonic terms

$$-\sum_f [\overline{f}_L\gamma^\mu(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + ig'\frac{Y_L}{2}B_\mu)f_L + \overline{f}_R\gamma^\mu(\partial_\mu + ig'\frac{Y_R}{2}B_\mu)f_R],$$

one adds the following similar terms involving the quarks:

$$-\sum_f [f_L\gamma^\mu(\partial_\mu + ig\frac{\tau_a}{2}W_{\mu a} + ig'\frac{Y_L}{2}B_\mu + ig''\lambda_b V_{\mu b})f_L + \overline{f}_R\gamma^\mu(\partial_\mu + ig'\frac{Y_R}{2}B_\mu + ig''\lambda_b V_{\mu b})f_R].$$

For each of the three generations of quarks $\begin{bmatrix} u \\ d \end{bmatrix}$, $\begin{bmatrix} c \\ s \end{bmatrix}$, $\begin{bmatrix} t \\ b \end{bmatrix}$ one has a left-handed isodoublet (such as $\begin{bmatrix} u_L \\ d_L \end{bmatrix}$), two right-handed $SU(2)$ singlets (such as $\begin{bmatrix} u_R \\ d_R \end{bmatrix}$), and each quark field appears in 3 colors so that for instance there are three u_R fields: u_R^r, u_R^y, u_R^b. All of these quark fields are thus in the fundamental representation 3 of $SU(3)$.

The hypercharges Y_L, Y_R are identical for different generations and are given by the following table:

	e, μ, τ	$\nu_e \nu_\mu \nu_\tau$	u, c, t	d, s, b
Y_L	-1	-1	$1/3$	$1/3$
Y_R	-2		$4/3$	$-2/3$

These numbers are not explained but are set by hand so as to get the correct electromagnetic charges Q_{em} from the formulas

$$2Q_{\mathrm{em}} = Y_L + 2I_3, \quad 2Q_{\mathrm{em}} = Y_R,$$

where I_3 is the 3rd generator of the weak isospin group $SU(2)$.

3) The kinetic terms for the Higgs fields:

$$\mathcal{L}_\varphi = -\left| \left(\partial_\mu + ig\frac{\tau_a}{2} W_{\mu a} + i\frac{g'}{2} B_\mu \right) \varphi \right|^2,$$

where $\varphi = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ is an $SU(2)$ doublet of complex scalar fields with hypercharge $Y_\varphi = 1$. This term is *exactly the same* as in the G.W.S. model for leptons.

4) The Yukawa coupling of Higgs fields with fermions:

$$\mathcal{L}_Y = -\sum_{f,f'} [H_{ff'} (\overline{f}_L H \cdot \varphi) f'_R + H^*_{ff'} \overline{f'_R} (\varphi^* \cdot f_L)],$$

where $H_{ff'}$ is a general coupling matrix in the space of different fermions, about which we must now be more explicit. First, there is no $H_{ff'} \neq 0$ between leptons and quarks, so that $\mathcal{L}_Y$ is a sum of a leptonic and a quark part. Since there is no right-handed neutrino, the leptonic part can always be put into the form

$$\mathcal{L}_{Y,\mathrm{lepton}} = -G_e(\overline{L}_e \cdot \varphi) e_R - G_\mu(\overline{L}_\mu \cdot \varphi)\mu_R - G_\tau(\overline{L}_\tau \cdot \varphi)\tau_R + \mathrm{h.c.},$$

where L_e is the isodoublet $\begin{bmatrix} \nu_{e,L} \\ e_L \end{bmatrix}$ and similarly for the other generations. The coupling constants G_e, G_μ, G_τ provide the lepton masses through the Higgs vacuum contribution.

The quark Yukawa coupling is more complicated owing to new terms which provide the masses of the up particles, and to the mixing angles. The new terms are of the form

$$G\overline{L}u_R\tilde{\varphi}, \tag{$*$}$$

where the isodoublet $L = \begin{bmatrix} u_L \\ q_L \end{bmatrix}$ is obtained from a left-handed up quark and a mixing q_L of left-handed down quarks (taken from the three families). Also, $\tilde{\varphi}$ needs to have the same isospin but opposite hypercharge to the Higgs doublet φ and is given by

$$\tilde{\varphi} = J\varphi^*, \quad J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \tag{$**$}$$

We refer to [Ell] for more details on this point, to which we shall return later.

5) The Higgs self-interaction:

$$\mathcal{L}_V = \mu^2\varphi^+\varphi - \frac{1}{2}\lambda(\varphi^+\varphi)^2$$

has exactly the same form as in the previous case.

Thus, we see that there are essentially three novel features of the complete standard model with respect to the leptonic case:

A) The new gauge symmetry: *color*, with gluons responsible for the strong interaction.

B) The new values $\frac{1}{3}$, $\frac{4}{3}$, $\frac{-2}{3}$ of the hypercharge for quarks.

C) The new Yukawa coupling terms $(*)$.

We shall now briefly explain how these new features motivate a corresponding modification of the geometric model of section 6, which led us above to the G.W.S. model for leptons. First, our model will still be a *product* of an ordinary Euclidean continuum by a finite space.

In section 6, for the algebra $\mathcal{A}$ of functions on the finite space, we took the algebra $\mathbb{C}_a \oplus \mathbb{C}_b$. But since we then considered a bundle on $\{a, b\}$ with fiber $\mathbb{C}^2$ over a and $\mathbb{C}$ over b, we could have in an equivalent fashion taken $\mathcal{A} = M_2(\mathbb{C}) \oplus \mathbb{C}$ and then dealt with vector potentials, instead of connections on vector bundles. Let us see how C) leads to replacing $\mathcal{A} = M_2(\mathbb{C}) \oplus \mathbb{C}$ by $\mathcal{A} = \mathbb{H} \oplus \mathbb{C}$, where $\mathbb{H}$ is the Hamilton algebra of quaternions. The point is simply that the equation $(**)$ which relates φ and $\tilde{\varphi}$ is the same as the unitary equivalence $2 \sim \overline{2}$ of the fundamental representation 2 of $SU(2)$ with the complex-conjugate or contragradient representation, i.e., we have

$$g \in U(2), \quad JgJ^{-1} = \overline{g} \Leftrightarrow g \in SU(2).$$

Let us simply remark that $x \in M_2(\mathbb{C})$, $JxJ^{-1} = \overline{x}$ defines an algebra, the quaternion algebra $\mathbb{H}$.

Next, let us see how A) leads us to the formalism of bimodules and Poincaré duality of Section 7. Indeed, let us look at any isodoublet of the form $\begin{bmatrix} u_L \\ d_L \end{bmatrix}$ of left-handed quarks. It appears in 3 colors,

$$\begin{matrix} u_L^r & u_L^y & u_L^b \\ d_L^r & d_L^y & d_L^b \end{matrix},$$

which makes it clear that the corresponding representation of $SU(2)\times SU(3)$ is the external tensor product $2_{SU(2)}\otimes 3_{SU(3)}$ of their fundamental representations. It is easy to convince oneself that even if one neglects the nuance between $U(n)$ and $SU(n)$ in general, there is no way to obtain such groups and representations from a single algebra and its unitary group. The solution that we found, namely to take $(\mathcal{A},\mathcal{B})$-bimodules, with $\mathcal{B} = \mathbf{C}\oplus M_3(\mathbf{C})$ (and $\mathcal{A} = \mathbf{C}\oplus\mathbf{H}$ as above) is in fact already suggested by the following picture in the paper [Ell] of J. Ellis, very close to that of the diagonal $\Delta\subset X\times X$ in Poincaré duality:

[[[reproduction of Fig. 6.1 of cited reference goes here if possible]]

We just refine it by taking algebras—$\mathbf{C}\oplus\mathbf{H}$ for the y-axis, $\mathbf{C}\oplus M_3(\mathbf{C})$ for the x-axis—instead of groups, which allows us to better account for the leptons (by the $\mathbf{C}$ of $\mathbf{C}\oplus M_3(\mathbf{C})$).

Finally, we shall get a conceptual understanding of the numbers B) from a general unimodularity condition that makes sense in noncommutative geometry, but we need not anticipate on that point.

We are now ready to describe in detail the geometric structure of the finite space F which, once crossed by $\mathbf{R}^4$, gives the standard model.

Geometric structure of the finite space F

This structure is given by an $(\mathcal{A},\mathcal{B})$-module $(\mathfrak{H}, D, \gamma)$, where $\mathcal{A}$ is the $*$-algebra $\mathbf{C}\oplus\mathbf{H}$ while $\mathcal{B}$ is the $*$-algebra $\mathbf{C}\oplus M_3(\mathbf{C})$. Unlike $\mathcal{B}$, the algebra $\mathcal{A}$ is only an algebra over $\mathbf{R}$. The $*$-representations π of $\mathcal{A}$ on a finite-dimensional Hilbert space are characterized (up to unitary equivalence) by three multiplicities: n_+, n_-, m, where $\mathfrak{H}_\pi = \mathbf{C}^{n_+}\oplus\mathbf{C}^{n_-}\oplus\mathbf{C}^{2m}$; if $a = (\lambda, q)\in\mathcal{A} = \mathbf{C}\oplus\mathbf{H}$, then $\pi(a)$ is the block diagonal matrix

$$\pi(a) = (\lambda\otimes\mathrm{id}_{n_+})\oplus(\bar\lambda\otimes\mathrm{id}_{n_-})\oplus\left(\begin{bmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{bmatrix}\otimes\mathrm{id}_m\right),$$

where the quaternion q is $q = \alpha+\beta j$ with $\alpha,\beta\in\mathbf{C}\subset\mathbf{H}$. The representation of the complex $*$-algebra $\mathcal{B}$ on $\mathfrak{H}$ gives a decomposition

$$\mathfrak{H} = \mathfrak{H}_0\oplus(\mathfrak{H}_1\otimes\mathbf{C}^3)$$

in which $\mathcal{B}$ acts by $\pi(b) = b_0\oplus(1\otimes b_1)$ for $b = (b_0, b_1)\in\mathbf{C}\oplus M_3(\mathbf{C})$, thus the commuting representation of $\mathcal{A}$ is given by *a pair* π_0, π_1 of representations on $\mathfrak{H}_0$ and $\mathfrak{H}_1$. The $(\mathcal{A},\mathcal{B})$-bimodule $\mathfrak{H}$ is thus completely described by the six multiplicities: (n_+^0, n_-^0, m^0) for π_0 and (n_+^1, n_-^1, m^1) for π_1. We shall take these to be of the form

$$(n_+^0, n_-^0, m^0) = N(1,0,1), \quad (n_+^1, n_-^1, m^1) = N(1,1,1)$$

(where N will eventually be the number of generations $N = 3$). We shall take the $\mathbf{Z}/2$ grading γ in $\mathfrak{H}$ to be given), by the element $\gamma = (1,-1)$ of the center of $\mathcal{A}$. Finally, we shall take for D the most general self-adjoint operator in $\mathfrak{H}$ that anticommutes with γ ($D\gamma = -\gamma D$) and commutes with $\mathbf{C} \otimes \mathcal{B}$, where $\mathbf{C} \subset \mathcal{A}$ is the diagonal subalgebra $\{(\lambda, \lambda),\ \lambda \in \mathbf{C}\}$. (As we shall see, D encodes both the masses of the fermions and the Kobayashi–Maskawa mixing parameters.) It follows that the action of $\mathcal{A}$ and the operator D in $\mathfrak{H}_0$ (resp. $\mathfrak{H}_1$) have the following general form (with $q = \alpha + \beta \mathrm{j} \in \mathbf{H}$):

$$\pi_0(f,q) = \begin{bmatrix} f & 0 & 0 \\ 0 & \alpha & \beta \\ 0 & -\overline{\beta} & \overline{\alpha} \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0 & M_e^* & 0 \\ M_e & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\pi_1(f,q) = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & \overline{f} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\overline{\beta} & \overline{\alpha} \end{bmatrix}, \quad D_1 = \begin{bmatrix} & 0 & M_d^* & 0 \\ & & 0 & M_u^* \\ M_d & 0 & & \\ 0 & M_u & 0 & \end{bmatrix},$$

where M_e, M_u, M_d are arbitrary $N \times N$ complex matrices.

Since π_0 is a degenerate case ($M_u = 0$) of π_1, we just restrict to π_1 in order to determine $\Omega_D^1(\mathcal{A})$.

A straightforward computation gives $\pi_1(\sum a_j \mathrm{d}a_j')$ with $a_j, a_j' \in \mathcal{A}$, $a_j = (\lambda_j, q_j)$, $q_j = \alpha_j + \beta_j \mathrm{j}$, $q_j' = \alpha_j' + \beta_j' \mathrm{j}$; we have

$$\pi_1(\sum a_j \mathrm{d}a_j') = \begin{bmatrix} 0 & X \\ Y & 0 \end{bmatrix},$$

where X, Y are the matrices

$$X = \begin{bmatrix} M_d^* \varphi_1 & M_d^* \varphi_2 \\ -M_u^* \overline{\varphi}_2 & M_u^* \overline{\varphi}_1 \end{bmatrix}, \quad Y = \begin{bmatrix} M_d \varphi_1' & M_u \varphi_2' \\ -M_d \overline{\varphi_2'} & M_u \overline{\varphi_1'} \end{bmatrix},$$

with

$$\varphi_1 = \sum \lambda_i (\alpha_i' - \lambda_i'), \quad \varphi_2 = \sum \lambda_i \beta_i',$$
$$\varphi_1' = \sum [\alpha(\lambda_i' - \alpha_i') + \beta_i \overline{\beta_1'}], \quad \varphi_2' = \sum [-\alpha_i \beta_i' + \beta_i (\overline{\lambda_i'} - \overline{\alpha_i'})].$$

It follows that $\Omega_D^1(\mathcal{A}) = \mathbf{H} \oplus \mathbf{H}$ with the $\mathcal{A}$-bimodule structure given by

$$(\lambda, q)(q_1, q_2) = (\lambda q_1, q q_2) \quad (\forall q_1, q_2 \in \mathbf{H}),$$
$$(q_1, q_2)(\lambda, q) = (q_1 q, q_2 \lambda) \quad (\forall \lambda \in \mathbf{C},\ q \in \mathbf{H}),$$

and the differential d being again the *finite difference*:

$$\mathrm{d}(\lambda, q) = (q - \lambda, \lambda - q) \in \mathsf{H} \oplus \mathsf{H}$$

(just set $q_1 = \varphi_1 + \varphi_2 \mathrm{j}$, $q_2 = \varphi_1' + \varphi_2' \mathrm{j}$ with the above φ's).

Finally, the involution $*$ on $\Omega_D^1(\mathcal{A})$ is given by

$$(q_1, q_2)^* = (q_2^*, q_1^*) \quad (\forall q_j \in \mathsf{H}).$$

The space $\mathcal{U}$ of vector potentials is thus naturally isomorphic to H, and a similar computation shows that $\Omega_D^2(\mathcal{A}) = \mathsf{H} \oplus \mathsf{H}$ with the $\mathcal{A}$-bimodule structure

$$(\lambda, q)(q_1, q_2)(\lambda', q') = (\lambda q_1 \lambda', q q_2 q') \qquad (\forall \lambda, \lambda' \in \mathsf{C},\ q, q' \in \mathsf{H});$$

the product $\Omega_D^1 \times \Omega_D^1 \to \Omega_D^2$ is given by

$$(q_1, q_2) \wedge (q_1', q_2') = (q_1 q_2', q_2 q_1'),$$

and the differential $\mathrm{d} : \Omega_D^1 \to \Omega_D^2$ by

$$\mathrm{d}(q_1, q_2) = (q_1 + q_2, q_1 + q_2).$$

Thus, the curvature θ of a vector potential $V = (q, q^*)$ is

$$\theta = \mathrm{d}V + V^2 = (q + q^* + q q^*, q + q^* + q^* q) = (|1 + q|^2 - 1)(1, 1),$$

where $q \to |q|$ denotes the norm of quaternions.

We thus see that the action $\mathrm{YM}(V) = \mathrm{Trace}(\pi(\theta)^2)$ (we are in the 0-dimensional case) is the same symmetry-breaking quartic potential for a pair of complex numbers as in section 6.

The detailed expression for the Hilbert space norm on $\Omega_D^2(\mathcal{A}) = \mathsf{H} \oplus \mathsf{H}$ is given, for $\omega = (q_1, q_2)$, $q_j = \alpha_j + \beta_j \mathrm{j}$, by

$$\|\omega\|^2 = \lambda_1 |\alpha_1|^2 + \mu_1 |\beta_1|^2 + \lambda_2(|q_2|^2),$$

where

$$\begin{aligned}
\lambda_1 &= \mathrm{Trace}(|M_e)|^4) + 3\mathrm{Trace}(|M_d|^4 + |M_u|^4), \\
\mu_1 &= 6\mathrm{Trace}(|M_d|^2 |M_u|^2), \\
\lambda_2 &= \frac{1}{2}\mathrm{Trace}(|M_e|^4 + 3(|M_d|^4 + |M_u|^4 + 2|M_d|^2 |M_u|^2)).
\end{aligned}$$

Finally, we shall investigate what freedom we have in the choice of the self-adjoint operators D_0, D_1 in $\mathfrak{H}_0$, $\mathfrak{H}_1$ in the above example. Two pairs $(\mathfrak{H}_j, D_j)$ and $(\mathfrak{H}_j', D_j')$ give identical results if there exist unitaries $U_j : \mathfrak{H}_j \to \mathfrak{H}_j'$ such that:

α) $U_j D_j U_j^* = D_j$ $(j = 1, 2)$,

β) $U_j \pi_j(a) U_j^* = \pi_j'(a)$ $(\forall a \in \mathcal{A},\ j = 1, 2)$.

Making use of this freedom, we can assume that D_0 is diagonal in $\mathfrak{H}_0$ and has positive eigenvalues e_1, e_2, e_3. Thus, the situation for D_0 is described by these 3 positive numbers.

For $\mathfrak{H}_1$, a general element of the commutant of $\pi_1(\mathcal{A})$ is of the form

$$U_1 = \begin{bmatrix} V_1 & 0 & 0 & 0 \\ 0 & V_2 & 0 & 0 \\ 0 & 0 & V_3 & 0 \\ 0 & 0 & 0 & V_3 \end{bmatrix},$$

where the V_j are unitary operators when U_1 is unitary.

Conjugating D_1 by U_1 replaces M_d and M_u, respectively, by

$$M_d' = V_1 M_d V_3^*, \quad M_u' = V_2 M_u V_3^*.$$

We thus see that we can assume that both M_u and M_d are positive matrices and that one of them, say M_u, is diagonal.

The invariants are thus the eigenvalues of M_u and M_d, i.e., a total of 6 positive numbers, and the pair of maximal abelian subalgebras generated by M_u and M_d. Since any pair $\mathcal{A}_1$, $\mathcal{A}_2$ of maximal abelian subalgebras of $M_3(\mathbb{C})$ are conjugate by a unitary W, $W\mathcal{A}_1 W^* = \mathcal{A}_2$, which is given modulo the unitary groups $\mathcal{U}(\mathcal{A}_j)$, there remain 4 parameters with which to specify W so that $W M_d W^*$ is also diagonal. Such a W corresponds exactly to the Kobayashi–Maskawa mixing matrix of the standard model.

Geometric structure of the standard model

We shall show in this section how the standard model is obtained from the product geometry of the usual 4-dimensional continuum by the above finite geometry F. Thus, we let M be a 4-dimensional spin manifold and $(L^2, \partial_M, \gamma_5)$ its Dirac K-cycle. The product geometry is, according to the general rule for forming products, described by the algebras

$$\mathcal{A} = C^\infty(M) \otimes (\mathbb{C} \oplus \mathbb{H}), \quad \mathcal{B} = C^\infty(M) \otimes \big(\mathbb{C} \oplus M_3(\mathbb{C})\big).$$

The Hilbert space $H = L^2(M, S) \otimes \mathfrak{H}_F$, where $\mathfrak{H}_F$ is described in c) above, i.e., $\mathfrak{H}_F = \mathfrak{H}_0 \oplus (\mathfrak{H}_1 \otimes \mathbb{C}^3)$. There is a corresponding decomposition $H = H_0 \oplus (H_1 \otimes \mathbb{C}^3)$, with corresponding representations π_j of $\mathcal{A}$ on H_j.

Then $D = \partial_M \otimes 1 + \gamma_5 \otimes D_F$, where D_F is as above. This gives a decomposition $D = D_0 \oplus (D_1 \otimes 1)$, where, according to c), we take M_e, M_u and M_d to be *positive* matrices:

$$D_0 = \begin{bmatrix} \partial_M \otimes 1 & \gamma_5 \otimes M_e & 0 \\ \gamma_5 \otimes M_e & \partial_M \otimes 1 & 0 \\ 0 & 0 & \partial_M \otimes 1 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} \partial_M \otimes 1 & 0 & \gamma_5 \otimes M_d & 0 \\ 0 & \partial_M \otimes 1 & 0 & \gamma_5 \otimes M_u \\ \gamma_5 \otimes M_d & 0 & \partial_M \otimes 1 & 0 \\ 0 & \gamma_5 \otimes M_u & 0 & \partial_M \otimes 1 \end{bmatrix}.$$

We shall first restrict attention to the algebra $\mathcal{A}$, the case of $\mathcal{B}$ being easier. Note that $\mathcal{A} = \mathcal{C}^\infty(M, \mathsf{C}) \oplus \mathcal{C}^\infty(M, \mathsf{H})$, so that every $a \in \mathcal{A}$ is given by a pair (f, q) consisting of a C-valued function f on M and an H-valued function q on M.

Let us first compute $\Omega^1_D(\mathcal{A})$. Given $\rho = \sum a_s \mathrm{d}a'_s \in \Omega^1(\mathcal{A})$, with $a_s, a'_s \in \mathcal{A}$, we have $a_s = (f_s, q_s)$, $a'_s = (f'_s, q'_s)$, where f_s, f'_s are complex-valued functions on M and q_s, q'_s are H-valued functions on M, of the form

$$q_s = \alpha_s + \beta_s \mathrm{j}, \ \ q'_s = \alpha'_s + \beta'_s \mathrm{j}.$$

Then

$$\pi_1(\rho) = \begin{bmatrix} i\gamma(A) \otimes 1 & 0 & \varphi_1 \gamma_5 \otimes M_d & \varphi_2 \gamma_5 \otimes M_d \\ 0 & i\gamma(\overline{A}) \otimes 1 & -\overline{\varphi}_2 \gamma_5 \otimes M_u & \overline{\varphi}_1 \gamma_5 \otimes M_u \\ \varphi'_1 \gamma_5 \otimes M_d & \varphi'_2 \gamma_5 \otimes M_u & i\gamma(W_1) \otimes 1 & i\gamma(W_2) \otimes 1 \\ -\overline{\varphi'_2} \gamma_5 \otimes M_d & \overline{\varphi'_1} \gamma_5 \otimes M_u & -i\gamma(\overline{W}_2) \otimes 1 & i\gamma(\overline{W}_1) \otimes 1 \end{bmatrix},$$

where $A = \sum f_s \mathrm{d}f'_s$ is a C-valued 1-form on M, and $W_1 + W_2 \mathrm{j} = W = \sum q_s \mathrm{d}q'_s$ is an H-valued 1-form on M (cf. [ ]).

Also, φ_j, φ'_j are complex-valued functions on M given by the same formulas as above for the finite geometry, namely,

$$\varphi_1 = \sum f_s(\alpha'_s - f'_s), \ \ \varphi_2 = \sum f_s \beta'_s,$$
$$\varphi'_1 = \sum \left(\alpha_s(f'_s - \alpha'_s) + \beta_s \overline{\beta'_s}\right), \ \ \varphi'_2 = \sum \left(\beta_s(\overline{f'_s} - \overline{\alpha'_s}) - \alpha_s \beta'_s\right).$$

This means that the pair (q, q') of H-valued functions, given by $q = \varphi_1 + \varphi_2 \mathrm{j}$, $q' = \varphi'_1 + \varphi'_2 \mathrm{j}$, satisfies

$$(q, q') = \sum a_s \Delta a'_s,$$

where, using the notations of b), $\Delta a'_s = (q'_s - \lambda'_s, \lambda'_s - q'_s)$ is the finite-difference operation, while the $\mathcal{A}$-bimodule structure on the space of $\mathbb{H} \oplus \mathbb{H}$-valued functions (q_1, q_2) is given by

$$(f, q)(q_1, q_2) = (fq_1, qq_2), \quad (q_1, q_2)(f, q) = (q_1 q, q_2 f) \quad \text{for all } (f, q) \in \mathcal{A}.$$

This shows that $\Omega^1_D(\mathcal{A})$ is the direct sum of two subspaces:

$$\Omega^1_D(\mathcal{A}) = \Omega^{(1,0)}_D \oplus \Omega^{(0,1)}_D,$$

where:

$\Omega^{(1,0)}_D$, the subspace of elements of ***differential*** type, is the space of pairs (A, W) consisting of a $\mathbb{C}$-valued 1-form A on M and an $\mathbb{H}$-valued 1-form W on M;

$\Omega^{(0,1)}_D$, the subspace of elements of ***finite-difference*** type, is the space of pairs (q_1, q_2) of $\mathbb{H}$-valued functions on M, with the above $\mathcal{A}$-bimodule structure.

The geometric picture is that of two copies M_R and M_L of M, with $\mathbb{C}$-valued functions on M_R, and $\mathbb{H}$-valued functions on M_L. More generally, the differential forms on M_R are $\mathbb{C}$-valued, whereas on M_L they are $\mathbb{H}$-valued, exactly as in Atiyah's book [At$_4$]. Of course the finite difference mixes both sides, so they are not independent.

Given an element $a = (f, q)$ of $\mathcal{A}$, the element $\mathrm{d}a$ of Ω^1_D has a ***differential*** component $(\mathrm{d}f, \mathrm{d}q)$, given by the $\mathbb{C}$-valued 1-form $\mathrm{d}f$ and the $\mathbb{H}$-valued 1-form $\mathrm{d}q$; and a ***finite-difference*** component $(q - f, f - q)$.

The involution $*$ on Ω^1_D is given by

$$((A, W), (q_1, q_2))^* = ((-\overline{A}, -\overline{W}), (\overline{q}_2, \overline{q}_1)),$$

so that a vector potential V is given by:

α) an ordinary $U(1)$ vector potential on M;

β) an $SU(2)$ vector potential on M (cf. [At$_4$]);

γ) a pair $q = \alpha + \beta \mathrm{j}$ of complex scalar fields on M.

The next step is to compute $\Omega^2_D(\mathcal{A})$ as well as the product

$$\Omega^1_D \times \Omega^1_D \to \Omega^2_D$$

and the differential $\mathrm{d} : \Omega^1_D \to \Omega^2_D$.

We shall first state the result. It holds provided a certain nondegeneracy condition is satisfied, namely, that the following matrices are not scalar multiples of the identity matrix:

$$M^2_d \text{ or } \frac{1}{2}(M^2_d + M^2_u), \quad M^2_d \text{ or } M^2_e \text{ or } M^2_u.$$

The result then is that an element of $\Omega^2_D(\mathcal{A})$ has:

1) a component of type $(2, 0)$ given by a pair (F, G) consisting of a $\mathbb{C}$-valued 2-form F and an $\mathbb{H}$-valued 2-form G on M;

2) a component of type $(1,1)$ given by a pair (ω_1, ω_2) of quaternionic 1-forms ω_1, ω_2 on M;

3) a component of type $(0,2)$ given by a pair (q_1, q_2) of quaternionic functions on M.

Moreover, with the obvious notations, the following formulas hold:

$$
\begin{aligned}
&\mathrm{d}\big((A,W),(q_1,q_2)\big) \\
&\qquad = \{(\mathrm{d}A, \mathrm{d}W), (\mathrm{d}q_1 + A - W, \mathrm{d}q_2 + W - A), (q_1 + q_2, q_1 + q_2)\}, \\
&\big((A,W),(q_1,q_2)\cdot\big((A',W'),(q_1',q_2')\big) \\
&\qquad = \{(A \wedge A', W \wedge W'), (Aq_1' - q_1W', Wq_2' - q_2A', (q_1q_2', q_2, q_1')\};
\end{aligned}
$$

they show that we are dealing with the graded tensor product of the differential algebra of M(the de Rham algebra) by the differential algebra Ω_D of the finite space F.

In order to compute the Hilbert space norm on $\Omega_D^2(\mathcal{A})$, we have to explicitly write down the class in $\pi(\Omega^2(\mathcal{A}))/\pi(J)$ associated with an element $\big((F,G), (\omega_1,\omega_2), (q_1,q_2)\big)$ of $\Omega_D^2(\mathcal{A})$. We shall first write what happens for π_1; the case of π_0 is a degenerate case obtained by taking $M_u = 0$. The subspace $\pi(J)$ only interferes with the elements of degree 0 in the Clifford algebra, and any element of $\pi(J)$ is given by 5 complex-valued functions α, β, γ, Y, Z on M, whose representation in $\mathfrak{H}_1$ is given by:

$$
\begin{bmatrix}
\alpha \otimes 1 & 0 & 0 & 0 \\
0 & \overline{\alpha} \otimes 1 & 0 & 0 \\
0 & 0 & T_{11} & T_{12} \\
0 & 0 & T_{21} & T_{22}
\end{bmatrix},
$$

where

$$
R = \begin{bmatrix} \beta & \gamma \\ -\overline{\gamma} & \overline{\beta} \end{bmatrix} \otimes 1 + \begin{bmatrix} Y & Z \\ \overline{Z} & -\overline{Y} \end{bmatrix} \otimes \frac{1}{2}(M_d^2 - M_u^2).
$$

In the Hilbert space $\mathfrak{H}_0$, the same element is represented by

$$
\begin{bmatrix}
\alpha \otimes 1 & 0 & 0 \\
0 & T_{11}' & T_{12}' \\
0 & T_{21}' & T_{22}'
\end{bmatrix},
$$

with

$$
T' = \begin{bmatrix} \beta & \gamma \\ -\overline{\gamma} & \overline{\beta} \end{bmatrix} \otimes 1 + \begin{bmatrix} Y & Z \\ \overline{Z} & -\overline{Y} \end{bmatrix} \otimes \frac{1}{2}M_e^2.
$$

The elements (F,G) and (ω_1,ω_2) of degree $(2,0)$ and $(1,1)$ have canonical representatives given by the following expressions, where $\omega_k = \alpha_k + \beta_k \mathrm{j}$ and α_k, β_k are complex-valued 1-forms on M:

$$
\begin{bmatrix}
i^2\gamma(F)\otimes 1 & 0 & 0 & 0 \\
0 & i^2\gamma(\overline{F})\otimes 1 & 0 & 0 \\
0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$i^2\gamma(G)\otimes 1$}} \\
0 & 0 & &
\end{bmatrix} \text{ for } \mathfrak{H}_1,
$$

$$
\begin{bmatrix}
i^2\gamma(F) & 0 & 0 \\
0 & \multicolumn{2}{c}{\multirow{2}{*}{$i^2\gamma(G)\otimes 1$}} \\
0 & &
\end{bmatrix} \text{ for } \mathfrak{H}_0,
$$

$$
\begin{bmatrix}
0 & 0 & i\gamma(\alpha_1)\gamma_5\otimes M_d & i\gamma(\beta_1)\gamma_5\otimes M_d \\
0 & 0 & -i\gamma(\overline{\beta}_1)\gamma_5\otimes M_u & i\gamma(\overline{\alpha}_1)\gamma_5\otimes M_u \\
i\gamma(\alpha_2)\gamma_5\otimes M_d & i\gamma(\beta_2)\gamma_5\otimes M_u & 0 & 0 \\
-i\gamma(\overline{\beta}_2)\gamma_5\otimes M_d & i\gamma(\overline{\alpha}_2)\gamma_5\otimes M_u & 0 & 0
\end{bmatrix} \text{ for } \mathfrak{H}_1,
$$

$$
\begin{bmatrix}
0 & i\gamma(\alpha_1)\gamma_5\otimes M_e & i\gamma(\beta_1)\gamma_5\otimes M_e \\
i\gamma(\alpha_2)\gamma_5\otimes M_d & 0 & 0 \\
-i\gamma(\overline{\beta}_2)\gamma_5\otimes M_d & 0 & 0
\end{bmatrix} \text{ in } \mathfrak{H}_0.
$$

The component (q_1, q_2) of degree $(0,2)$, $q_i = \alpha_i + \beta_i \mathrm{j}$, has the following representative modulo $\pi(J)$:

$$
\begin{bmatrix}
\alpha_1 M_d^2 & \beta_1 M_d M_u & 0 & 0 \\
-\overline{\beta}_1 M_u M_d & \overline{\alpha}_1 M_u^2 & 0 & 0 \\
0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$q_2\otimes \frac{1}{2}(M_d^2+M_u^2)$}} \\
0 & 0 & &
\end{bmatrix},
$$

$$
\begin{bmatrix}
\alpha_1 M_e^2 & 0 & 0 \\
0 & \multicolumn{2}{c}{\multirow{2}{*}{$q_2\otimes \frac{1}{2}M_e^2$}} \\
0 & &
\end{bmatrix}.
$$

The nondegeneracy condition ensures that the various terms such as $\alpha_1 M_d^2$ do not disappear modulo $\pi(J)$. It is now straightforward, as in Section 6, to compute the action. One gets the five terms of the $U(1) \times SU(2)$ part of the standard model, with the new Yukawa coupling terms C), but before we discuss the coefficients with which they arise we need to show how to reduce the gauge group $\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$ of the theory to the global gauge group $U(1) \times SU(2) \times SU(3)$ and obtain the intricate table of hypercharges of the standard model. Note that we did not make the straightforward calculation of vector potentials and action for the algebra $\mathcal{B}$, which yield a pure gauge action with group $U(1) \times U(3)$.

Unimodularity condition and hypercharges

We shall now see how, from a general condition of unimodularity valid in the general context of noncommutative geometry, one obtains the intricate table of hypercharges of elementary particles:

	e, μ, τ	$\nu_e \nu_\mu \nu_\tau$	u, c, t	d, s, b
Y_L	-1	-1	$1/3$	$1/3$
Y_R	-2		$4/3$	$-2/3$

(Note that since we are dealing with the Lie algebra of $U(1)$, this table is only determined up to a common scale.)

To obtain these values and at the same time obtain the global gauge group $U(1) \times SU(2) \times SU(3)$, we shall simply replace the local gauge group $\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$ by its unimodular subgroup $S\mathcal{U}$ ***relative to*** $\mathcal{A}$.

In our context, the notion of ***determinant*** of a unitary makes sense provided that a ***trace*** τ is given. More precisely by [Harp-S], given a C^*-algebra C and a ***self-adjoint*** trace τ on C (i.e., $\tau(x^*) = \overline{\tau(x)}$ for all $x \in C$), one obtains the phase of the determinant of a unitary u as follows:

$$\mathrm{Phase}_\tau(u) = \frac{1}{2\pi i} \int_0^1 \tau\left(u(t)' u(t)^{-1}\right) \mathrm{d}t,$$

where $u(t)$ is a smooth path of unitaries joining u to 1. Thus, this phase is only defined in the connected component $\mathcal{U}_0(C)$ of the identity, and it is ambiguous, by the image $\langle \tau, K_0(C) \rangle$ of $K_0(C)$ under the trace τ, which is a countable subgroup of $\mathbf{R}$.

The condition $\mathrm{Phase}_\tau(u) = 0$ is well-defined and gives a normal subgroup of $\mathcal{U}(C)$. We let $S_\tau \mathcal{U}(C)$ be the connected component of its identity element.

Now let $\mathcal{A}$ and $\mathcal{B}$ be $*$-algebras and let $(\mathfrak{H}, D)$ be a (d, ∞)-summable bimodule over $\mathcal{A}, \mathcal{B}$. We shall apply the above considerations to the C^*-algebra C on $\mathfrak{H}$ generated by $\mathcal{A}$ and $\mathcal{B}$ and to the family of traces τ on C given by the self-adjoint elements $\rho = \rho^*$ of the ***center*** of $\mathcal{A}$:

$$\tau_\rho(x) = \mathrm{Tr}_\omega(\rho x D^{-d}) \quad (\forall x \in C).$$

We thus get a normal subgroup $S_{\mathcal{A}}(C)$ of the unitary group of C by intersecting all of the S_τ, $\tau = \tau_\rho$ as above. Since $\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$ is a subgroup of $\mathcal{U}(C)$, its intersection with $S_{\mathcal{A}}(C)$ gives a normal subgroup $S(\mathcal{A},\mathcal{B})$ of $\mathcal{U}(\mathcal{A}) \times \mathcal{U}(\mathcal{B})$. We shall see what $S(\mathcal{A},\mathcal{B})$ is, in simple examples, but our main point now is:

Theorem 1. *Let $(\mathcal{A},\mathcal{B},\mathcal{H},D)$ be the product geometry of a 4-dimensional Riemannian spin manifold by the finite geometry F. Then, the group $S(\mathcal{A},\mathcal{B})$ is equal to $\mathrm{Map}(M, U(1) \times SU(2) \times SU(3))$ and its representation in $\mathcal{H}$ is, for the $U(1)$ factor, given by the above table of hypercharges.*

By construction, $\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F$, $\mathcal{B} = C^\infty(M) \otimes \mathcal{B}_F$, $\mathcal{H} = L^2(M,S) \otimes \mathfrak{H}_F$, $D = \partial_M \otimes 1 + \gamma_5 \otimes D_F$ and it follows by a straightforward argument that

$$S(\mathcal{A},\mathcal{B}) = \mathrm{Map}(M, S(\mathcal{A}_F,\mathcal{B}_F)),$$

where $S(\mathcal{A}_F,\mathcal{B}_F)$ is defined as above, but using the ordinary trace in the finite-dimensional space $\mathfrak{H}_F$ instead of the Dixmier trace. Thus, we need only compute the group $S(\mathcal{A}_F,\mathcal{B}_F)$ over a point, and its representation in $\mathfrak{H}_F$. Now, every self-adjoint element ρ of the center $Z(\mathcal{A}_F)$ is of the form $\rho = \lambda_1 e + \lambda_2(1-e)$, where the λ_j are real numbers, $e = (0,1) \in \mathbf{C} \oplus \mathbf{H}$ and $1 - e = (1,0)$. It follows easily that

$$S(\mathcal{A}_F,\mathcal{B}_F) = \big(\mathcal{U}(\mathcal{A}_F) \times \mathcal{U}(\mathcal{B}_F)\big) \cap \big(SU(e\mathfrak{H}_F) \times SU((1-e)\mathfrak{H}_F)\big).$$

In other words, the unimodularity condition means that the action is unimodular on both $e\mathfrak{H}_F$ and $(1-e)\mathfrak{H}_F$. Let, then, U be an element of $\mathcal{U}(\mathcal{A}_F) \times \mathcal{U}(\mathcal{B}_F)$. It is given by a quadruple

$$U = \big((\lambda,q),(u,v)\big); \quad \lambda \in U(1), q \in SU(2), u \in U(1), v \in U(3).$$

We have $\mathfrak{H}_F = \mathfrak{H}_0 \oplus (\mathfrak{H}_1 \otimes \mathbf{C}^3)$ and, with the notations of b), the action of U on $\mathfrak{H}_F$ is given by

$$\pi_0(\lambda,q)u \oplus \big(\pi_1(\lambda,q) \otimes v\big).$$

This operator restricts to both $e\mathfrak{H}_F$ and $(1-e)\mathfrak{H}_F$ and we have to compute the determinants of these restrictions. With $N = 3$ generations, we get

$$\det(U_e) = u^{2\times 3} \times \big(\det(v)\big)^{2\times 3}, \quad \det(U_{1-e}) = (\lambda u)^3 \times (\det v)^{2\times 3},$$

hence the unimodularity condition means exactly that

$$\lambda = u, \quad \det v = u^{-1}. \tag{$*$}$$

It follows that $S(\mathcal{A}_F,\mathcal{B}_F) = U(1) \times SU(2) \times SU(3)$. Let us compute the table of hypercharges, say, for example, by taking u as generator, as it is represented. Since $\lambda = u$, for $\mathfrak{H}_0$ we get

$$\pi_0(\lambda, q)u = \begin{bmatrix} \lambda u & 0 \\ 0 & qu \end{bmatrix} = \begin{bmatrix} u^2 & 0 \\ 0 & qu \end{bmatrix},$$

which corresponds to hypercharge 2 for e_R, μ_R, τ_R and 1 for e_L, μ_L, τ_L and ν_e, ν_μ, ν_τ. For $\mathfrak{H}_1$ we get $v = v_0 u^{-1/3}$ with $v_0 \in SU(3)$ and $u^{1/3}$ a cube root of u, so that

$$\pi_1(\lambda, q)u^{-1/3} = \begin{bmatrix} \lambda u^{-1/3} & 0 & 0 & 0 \\ 0 & \lambda^{-1} u^{-1/3} & 0 & 0 \\ 0 & 0 & \multicolumn{2}{c}{\multirow{2}{*}{$qu^{-1/3}$}} \\ 0 & 0 & & \end{bmatrix},$$

which corresponds to hypercharge $\frac{2}{3}$ for d_R, s_R, b_R; $\frac{-4}{3}$ for u_R, c_R, t_R; and $-\frac{1}{3}$ for the left-handed quarks. An overall sign is of course irrelevant (change u to u^{-1}), so we get the desired table.

Remarks. 1) *Let $\mathcal{A}$ be the algebra of functions on a 4-dimensional spin manifold M, let $(\mathfrak{H}, D)$ be the sum of N copies of the Dirac K-cycle (L^2, ∂_M) on M, and $\mathcal{B}$ the commutant of $\mathcal{A} \cup [D, \mathcal{A}]$ on $\mathfrak{H}$. Then, the group $S(\mathcal{A}, \mathcal{B})$ is the group $\mathrm{Map}(M, SU(N))$ of local gauge transformations associated with the global gauge group $SU(N)$.*

2) *To the above reduction of the gauge group there corresponds a similar reduction of bivector potentials V, which in the case of Theorem 1 means that V is traceless in the spaces $e\mathfrak{H}_F$ and $(1-e)\mathfrak{H}_F$, and in the case of Remark 1 yields the corresponding $SU(N)$ pure gauge theory.*

We are now ready to compare our theory with the standard model of electro-weak and strong interactions. We first note that the bimodule $(\mathfrak{H}, D)$ over $\mathcal{A}, \mathcal{B}$ of Theorem 1 is ***not*** irreducible, i.e., the commutant of the algebra generated by $\mathcal{A}$, $\mathcal{B}$ and D does not reduce to $\mathbb{C}$. With generic M_e, M_u, M_d, one can check that this commutant is $\mathbb{C}^4$ so that $(\mathfrak{H}, D)$ splits as a direct sum of 4 irreducible bimodules, three for the three lepton generations, i.e., for $\mathfrak{H}_0$, and one for $\mathfrak{H}_1$ which is irreducible.

Theorem 2. *Let $(\mathcal{A}, \mathcal{B}, \mathfrak{H}, D)$ be the product geometry of a 4-dimensional spin Riemannian manifold by the finite geometry F. Let $S(\mathcal{A}, \mathcal{B})$ be the reduced gauge group and $\mathcal{V}$ the corresponding space of bivector potentials. Then, the following action gives the standard model with its* 18 *free parameters:*

$$\mathrm{Tr}_\omega\big((\lambda_{\mathcal{A}}\theta_{\mathcal{A}}^2 + \lambda_{\mathcal{B}}\theta_{\mathcal{B}}^2)D^{-4}\big) + \langle \psi, D_V \psi \rangle,$$

where $\lambda_{\mathcal{A}}$, $\lambda_{\mathcal{B}}$ belong to the commutant of the bimodule and $\theta_{\mathcal{A}}$, $\theta_{\mathcal{B}}$ are the respective curvatures.

The proof is straightforward, given Theorem 1 and the above computations.

In order to transform our interpretation of the standard model into a predictive theory it is important to solve the following problems:

1) Find a non trivial finite quantum group of symmetries of the finite space F.

2) Determine the structure of the Clifford algebra of the finite space F, given by the linear map from the space $\mathsf{H} \oplus \mathsf{H}$ of 1-forms in the algebra of endomorphisms of $\mathcal{H}_F$ which to a 1-form $\Sigma a_i \, db_i$ associates $\Sigma a_i[D, b_i]$.

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