ON THE COHOMOLOGY OF OPERATOR ALGEBRAS

A. CONNES*

Institut des Hautes Etudes Scientifiques
Bures-sur-Yvette (France)

Mai 1976

---

* I.H.E.S and PARIS VI

ON THE COHOMOLOGY OF OPERATOR ALGEBRAS

A. CONNES

Abstract. Any amenable C* algebra is nuclear. Injective von Neumann algebras are characterized by the vanishing of their cohomology with coefficients in dual normal Banach bimodules.

Introduction. As in [1] or [3], a Banach algebra A is called amenable when any norm continuous derivation δ of A in a dual Banach A-bimodule is a coboundary. In [5] 7.9, 7.6, B.E. Johnson showed that any C* algebra A which is an inductive limit of type I C^* algebras is amenable as a Banach algebra. He left open the existence of non amenable C^* algebras. We show below that any amenable C^* algebra is also nuclear. It is very likely that any nuclear C^* algebra is amenable but we leave this question open.

In their work on the cohomology of operator algebras [6], [7], [8], B.E. Johnson, R.V. Kadison and J. Ringrose introduced the notion of normal dual Banach M-bimodule, where M is a von Neumann algebra. They showed that any von Neumann algebra M which is generated by an increasing sequence of finite dimensional * algebras has the following property : Any derivation δ of M in a normal dual Banach M-bimodule is a coboundary.

We shall prove the converse : if M is a von Neumann algebra with separable predual and all (continuous) derivations of M in normal dual Banach bimodules are coboundaries, then M is generated by an increasing sequence of finite dimensional * algebras. This class also coincides with the class of injective or of semi discrete von Neumann algebras [3] . It seems that "amenable" is the best terminology to qualify this class.

We now fix the notations for the sequel of the paper. Let A be a Banach algebra. Then a Banach space X which is also an A-bimodule is called a Banach A-bimodule when for some $K > 0$ one has $\|a\xi b\|_X \leq K\|a\| \, \|\xi\|_X \, \|b\|$ for all $a,b \in A$, $\xi \in X$.

If furthermore X is (isometric to) the dual of a Banach space X_* and for each $a \in A$ the operators $\xi \to a\xi$, $\xi \to \xi a$ of X in X are $\sigma(X,X_*)$ continuous, then X is called a <u>dual</u> Banach A-bimodule.

Now if M is a von Neumann algebra and X is a dual Banach M bimodule, one says that X is normal when for any $\xi \in X$ and $\eta \in X_*$ the functionals $x \in M \to \langle \eta, x\xi \rangle$ or $\langle \eta, \xi x \rangle$ are normal functionals on M.

Since von Neumann algebras are in general not norm separable the requirement of normalcy for M-bimodules is a necessity to avoid pathological phenomenas.

Let A be a Banach algebra, X a Banach A-bimodule. Then a derivation δ of A in X is a continuous linear map of A in X such that
$\delta(ab) = \delta(a)b + a\delta(b)$, $\forall a,b \in A$.

We say that a von Neumann algebra is <u>amenable</u> when all derivations of M in <u>normal</u> dual Banach M-bimodules are coboundaries. By [6] thm 5.6, this is the same as asking that all <u>normal</u> derivations (i.e. $\eta \circ \delta$ is normal for any $\eta \in X_*$) be coboundaries.

<u>Theorem 1</u>. Let M be a von Neumann algebra with separable predual. Then M is amenable if and only if it is injective.

Corollary 2. Let A be a separable C^* algebra which is amenable as a Banach algebra, then A is nuclear.

Proof of the corollary. Using [2], to show that A is nuclear, one just needs to show that any representation π of A in a hilbert space $\mathfrak{h}_\pi$ generates an injective von Neumann algebra. By [4] one can assume that $\mathfrak{h}_\pi$ is separable. Let then $M = \pi(A)''$ and X a dual normal Banach M-bimodule. Then any normal derivation δ of M in X defines by composition with π a derivation of A in X. Since A is amenable and X is a dual Banach A-bimodule one gets that $\delta.\pi$ is a coboundary, and δ, being normal, is also a coboundary.

Proof of Injective ⇒ Amenable. By [3] any injective von Neumann algebra M with separable predual is the weak closure of an increasing sequence of finite dimensional * algebras. Hence the result of Johnson, Kadison and Ringrose [6] Corollary 6.4 p.95 gives the implication.

Proof of Amenable ⇒ Injective. In this proof we shall not make use of the hypothesis M_* separable.

We first assume that M is semi-finite, and let τ be a faithful semi-finite normal trace on M. Let M act in a hilbert space $\mathfrak{H}$. We say that an operator $T \in \mathcal{L}(\mathfrak{h})$ is of τ-finite rank when its initial and final supports are majorized by projections $e,f \in M$ with $\tau(e) < \infty$, $\tau(f) < \infty$.

Since the sup of projections $e,f \in M$ with $\tau(e) < \infty$, $\tau(f) < \infty$ also satisfies $\tau(e \vee f) < \infty$, the set $\mathfrak{J}$ of τ-finite rank operators is an M submodule of $\mathcal{L}(\mathfrak{h})$.

For $x \in M$, put $\|x\|_2 = \tau(x^*x)^{1/2}$ - (One can have $\|x\|_2 = +\infty$)- Let Y be the space of all linear functionals φ on $\mathfrak{J}$ with :

$\exists K > 0$, $|\varphi(aTb)| \le K\|a\|_2\|T\|_\infty\|b\|_2$, $\forall a,b \in M \cap \mathfrak{J}$, $T \in \mathfrak{J}$. The smallest possible K is noted $\|\varphi\|_Y$ and defines a norm on Y . The corresponding unit ball of Y is compact for the weak topology $\sigma(Y,\mathfrak{J})$, so Y is a Banach space and is the dual of some Banach space Y_* in which $\mathfrak{J}$ is dense.

For $\varphi \in Y$, $x,y \in M$, define $x\varphi y \in Y$ by :

$$x\varphi y(T) = \varphi(yTx) \quad \forall T \in \mathfrak{J}$$

then, since $\|ya\|_2 \le \|y\|_\infty\|a\|_2$, $\|bx\|_2 \le \|b\|_2\|x\|_\infty$ for $a,b \in M \cap \mathfrak{J}$, it follows that $\|x\varphi y\|_Y \le \|x\|_\infty\|y\|_\infty\|\varphi\|_Y$. Hence Y is a Banach M bimodule. For $x \in M$ the map $\varphi \to x\varphi$ from Y to Y is $\sigma(Y,\mathfrak{J})$ continuous because $\mathfrak{J}$ is an M-bimodule. So Y is a dual Banach M-bimodule. Finally for fixed $T \in \mathfrak{J}$ and $\varphi \in Y$ the map $x \in M \to \varphi(xT)$ is normal, because letting $e \in M$ be a projection, with $\tau(e) < \infty$, majorizing the two supports of T , one has :

$$|\varphi(xT)| = |\varphi(xeTe)| \le \|xe\|_2\|T\|_\infty\|e\|_2 \quad .$$

So Y is a dual normal M-bimodule.

Let X be the submodule of Y defined as follows :

$$X = \{\varphi \in Y \ , \ \varphi(x) = 0 \quad \forall x \in M \cap \mathfrak{J}\}$$

It is a submodule because $M \cap \mathfrak{J}$ is an M-submodule of $\mathfrak{J}$. It is $\sigma(Y,\mathfrak{J})$ closed in Y by construction. Hence it is also a dual normal Banach M-bimodule.

Now since the trace τ on M is normal, there exists in $\mathfrak{h}$ a family $(\xi_\alpha)_{\alpha\in I}$ of unit vectors $\|\xi_\alpha\|=1$, such that, for any $x \in M \cap \mathfrak{J}$: $\tau(x) = \sum_{\alpha\in I} < x\xi_\alpha,\xi_\alpha>$ where $\Sigma|< x\xi_\alpha,\xi_\alpha>| < \infty$.

For any $T \in \mathfrak{J}$ there is a projection $e \in M$, $\tau(e) < \infty$ such that, with $T = T_1 + iT_2$, one has : $-\|T_j\|e \le T_j \le \|T_j\|e$, $j = 1,2$. Hence the sum $\sum_{\alpha \in I} |< T\xi_\alpha, \xi_\alpha >|$ is convergent and

$$\tilde{\tau}(T) = \sum_{\alpha \in I} < T\xi_\alpha, \xi_\alpha >$$

defines a linear functional on $\mathfrak{J}$ satisfying :

a) $\tilde{\tau}(T) \ge 0$, $\forall T \ge 0$, $T \in \mathfrak{J}$

b) $\tilde{\tau}(x) = \tau(x)$, $\forall x \in M \cap \mathfrak{J}$.

As by constuction $\mathfrak{J}$ is a * subalgebra of $\mathcal{L}(h)$ one deduces from a) that $|\tilde{\tau}(T_1^*, T_2)|^2 \le \tilde{\tau}(T_1^* T_1)\, \tilde{\tau}(T_2^* T_2)$, $\forall T_1, T_2 \in \mathfrak{J}$. For $T \in \mathfrak{J}$, $a,b \in M \cap \mathfrak{J}$, let $T_1^* = aT$, $T_2 = b$ then :

$$|\tilde{\tau}(aTb)|^2 \le \tilde{\tau}(aTT^*a^*)\, \tilde{\tau}(b^*b) \le \tilde{\tau}(aa^*)\, \|T\|_\infty^2\, \tilde{\tau}(b^*b)$$

because by construction of $\tilde{\tau}$ one has $\tilde{\tau}(aTT^*a^*) = \Sigma\|T^*a^*\xi_\alpha\|^2 \le \|T\|_\infty^2 \Sigma\|a^*\xi_\alpha\|^2 =$ $= \|T\|_\infty^2 \Sigma\|a^*\xi_\alpha\|^2 = \|T\|_\infty^2\, \tilde{\tau}(aa^*)$. Hence by b) : $|\tilde{\tau}(aTb)| \le \|a\|_2\, \|b\|_2\, \|T\|_\infty$. Thus $\tilde{\tau} \in Y$, $\|\tilde{\tau}\|_Y \le 1$.

Now let δ be the derivation of M in X defined by :

$$\delta(x) = x\tilde{\tau} - \tilde{\tau}x \qquad \forall x \in M .$$

One has $\delta(x) \in X$ because $\tilde{\tau}(xy - yx) = \tau(xy - yx) = 0$ for any $y \in M \cap \mathfrak{J}$.

One can also check directly that δ is normal. If M is amenable, δ must be a coboundary, so there is a $\varphi \in X$ such that $\delta(x) = x\varphi - \varphi x$, $\forall x \in M$.

Now $\tilde{\tau} - \varphi = \psi \in Y$ and one has :

1°) $\tau(x) = \psi(x)$, $\forall x \in M \cap \mathfrak{J}$ (because $\varphi \in X$)

2°) $x\psi = \psi x$, $\forall x \in M$ (because $x\tilde{\tau} - \tilde{\tau}x = x\varphi - \varphi x$) .

Let e be a projection of M , $\tau(e) < \infty$. Let $\mathcal{K} = e\mathcal{H}$ be the range of e . Then M_e is a finite von Neumann algebra acting in the hilbert space $\mathcal{K}$. Since τ is faithful, its restriction τ_1 to M_e is also faithful, and in particular by 1°) the restriction ψ_1 of ψ to $\mathcal{L}(\mathcal{K}) = \{T \in \mathcal{L}(\mathcal{H}) , eT = Te = T\}$ is non zero and satisfies $2)'$: $x\psi_1 = \psi_1 x$ for any x in M_e .

Replacing ψ_1 by $\frac{1}{2}(\psi_1 + \psi_1^*)$ does not affect $2)'$ and $1)'$: $\psi_1(x) = \tau_1(x)$ for any $x \in M_e$. So one can assume that $\psi_1 = \psi_1^*$ as an element of $\mathcal{L}(\mathcal{K})^*$. Writing its unique Jordan decomposition $\psi_1 = \psi_1^+ - \psi_1^-$, with $\|\psi_1^+\| + \|\psi_1^-\| = \|\psi_1\|$ one gets :

α) $\psi_1^+(x) \geq \tau_1(x)$, $\forall x \in M_e$, $x \geq 0$.

β) $\psi_1^+(xT) = \psi_1^+(Tx)$, $\forall T \in \mathcal{L}(\mathcal{K})$, $\forall x \in M_e$.

Now let $K = \|\psi_1^+\|$, we shall prove that for any $a_1,\dots,a_n, b_1,\dots,b_n \in P = M_e$ one has :

$$|\tau_1(\Sigma\, a_j b_j^*)| \leq K \| \Sigma\, a_j \otimes b_j^c \|_{\mathcal{K}\otimes\mathcal{K}^c}$$

where $\mathcal{K}^c$ is the conjugate hilbert space of $\mathcal{K}$.

In fact, let $\varphi_1 = \frac{1}{K}\psi_1^+ \in \mathcal{L}(\mathcal{K})^*$. Then, as φ_1 is a weak limit of normal states on $\mathcal{L}(\mathcal{K})$, there is for any $\epsilon > 0$, a normal state φ on $\mathcal{L}(\mathcal{K})$ such that :

$$\|b_j\varphi - \varphi b_j\| \leq \epsilon \, , \; j = 1,\dots,n \quad , \quad |\varphi(\Sigma\, a_j b_j^*) - \varphi_1(\Sigma\, a_j b_j^*)| \leq \epsilon$$

Hence there exists, for each $\epsilon > 0$, a hilbert schmidt operator ρ in $\mathcal{K}$ with $\|\rho\|_{HS} = 1$, $\|\rho b_j - b_j \rho\|_{HS} \leq \epsilon$ and :

So again for any $\epsilon > 0$, there is a hilbert schmidt operator ρ in $\mathcal{H}$ with $\|\rho\|_{HS} = 1$ and :

$$\|\Sigma\, a_j \rho b_j^*\|_{HS} \geq |\varphi_1(\Sigma\, a_j b_j^*)| - \epsilon$$

But the canonical identification of $\mathcal{H} \otimes \mathcal{H}^c$ with the hilbert schmidt operators of $\mathcal{H}$ intertwines $a \otimes b^c$ with the operator $\rho \to a\rho b^*$ so, we have shown that $|\varphi_1(\Sigma\, a_j b_j^*)| \leq \|\Sigma\, a_j \otimes b_j^c\|$. Now we know that the linear functional ψ_o on $P \odot P^c$, defined by $\psi_o(\Sigma\, a_i \otimes b_i^c) = \tau_1(\Sigma\, a_i b_i^*)$ is continuous for the minimal norm of the tensor product ([2]). Letting P act in the hilbert space $\mathcal{H}_1$ of the Gelfand Segal construction of τ_1 with canonical vector ξ_1 and involution J, one gets :

$$| < \Sigma\, a_i J b_i J \xi_1, \xi_1 > | \leq K \|\Sigma\, a_i \otimes J b_i J\|_{min}$$

It follows then from the cyclicity of ξ_1 for P that the canonical homomorphism η of $P \odot P'$ in $\mathcal{L}(\mathcal{H}_1)$ is bounded. Then by [4] we know that P is semi discrete and hence injective. It follows that M itself is injective.

<u>The general case</u> : Since the finite case is already treated it is easy to see that we can assume M to be properly infinite.

Let then $(N, (\theta_t)_{t \in \mathbb{R}})$ be a continuous decomposition of M, [9], where N is semi finite and M is isomorphic to the cross product of N by the one parameter group $(\theta_t)_{t \in \mathbb{R}}$ of automorphisms of N. We just have to show that if M is amenable then N is amenable. In fact, then N is injective and so is M by [3].

But by construction N is generated by a von Neumann subalgebra P isomorphic to M and a one parameter group of unitaries $(v_s)_{s \in \mathbb{R}}$ such that $v_s P v_s^* = P$, $\forall s \in \mathbb{R}$. Now let X be a dual normal Banach N-bimodule. Then it is also a P bimodule. So given a normal derivation δ of N in X we can assume that $\delta(x) = 0$, $\forall x \in P$, since P is amenable. For each $s \in \mathbb{R}$, let $\xi_s = v_s\delta(v_s^{-1})$. As, for $y \in P$, one has $yv_s\delta(v_s^{-1}) = v_s(v_s^{-1}yv_s)\delta(v_s^{-1}) = v_s\delta(v_s^{-1}y) = v_s\delta(v_s^{-1})y$ we get $y\xi_s = \xi_s y$, $s \in \mathbb{R}$, $y \in P$. Also, for any $t \in \mathbb{R}$ and $s \in \mathbb{R}$: $v_t\xi_s = v_{t+s}\delta(v_{t+s}^{-1}v_t) = (v_{t+s}\delta(v_{t+s}^{-1}))v_t + \delta(v_t)$.

As X is a dual of a Banach space X_* , let ξ be a $\sigma(X,X_*)$ limit point of the $\frac{1}{F^\#}\sum_F \xi_s$, where F runs through a summing family for $\mathbb{R}$ as a discrete amenable group. Then by the above equality one gets $v_t\xi = \xi v_t + \delta(v_t)$, for any $t \in \mathbb{R}$, and $y\xi = \xi y$, $\forall y \in P$. Q.E.D.

Bibliography

[1] F.F. Bonsall and J. Duncan, Complete normed algebras, Ergebnisse der Mathematik und ihrer Grenzgebiete. Band 80.

[2] M.D. Choi, E. Effros, Nuclear C* algebras (preprint).

[3] A. Connes, Classification of injective factors. Annals of Mathematics 103 (1976).

[4] E. Effros and C. Lance, Tensor products of operator algebras (preprint).

[5] B. Johnson, Cohomology in Banach algebras, Memoir A.M.S. 127 (1972)

[6] B. Johnson, R.V. Kadison and J. Ringrose, Cohomology of operator algebras III, Bull. Soc. Math. France 100 (1972), p. 73-96.

[7] R.V. Kadison and J. Ringrose, Cohomology of operator algebras I, type I von Neumann algebra. Acta Math., t. 126 (1971).

[8] R.V. Kadison and J. Ringrose, Cohomology of operator algebras II. Arkiv. for Math., t.9 (1971).

[9] M. Takesaki, Duality in cross products and the structure of von Neumann algebras of type III. Acta Math. 131 (1973).

IHES/P/76/142