A SURVEY OF FOLIATIONS AND OPERATOR ALGEBRAS

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INTRODUCTION.

Let V be a smooth manifold and TV its tangent bundle, so that for each $x \in V$, $T_x V$ is the tangent space of V at x. A smooth subbundle F of TV is called integrable iff one of the following equivalent conditions is satisfied :

a) Every $x \in V$ is contained in a submanifold W of V such that

$$T_y(W) = F_y \qquad \forall y \in W \quad .$$

b) Every $x \in V$ is in the domain $U \subset V$ of a submersion $p : U \to \mathbb{R}^q$ (q = Codim F) with $F_y = \mathrm{Ker}(p_*)_y \quad \forall y \in U$.

c) $C^\infty(F) = \{ X \in C^\infty(TV),\ X_x \in F_x \ \forall x \in V \}$ is a Lie algebra.

d) The ideal $J(F)$ of smooth exterior differential forms which vanish on is stable by exterior differentiation.

Any 1-dimensional subbundle F of TV is integrable, but for $\dim F \geq 2$ the condition is non trivial, for instance if $P \xrightarrow{p} B$ is a principal H-bundle (with compact structure group H) the bundle of horizontal vectors for a given connection is integrable iff this connection is flat.

A foliation of V is given by an integrable subbundle F of TV The leaves of the foliation (V, F) are the maximal connected submanifolds L of V with $T_x(L) = F_x \quad \forall x \in L$, and the partition of V in leaves $V = \cup L_\alpha$, $\alpha \in A$ is characterized geometrically by its "local triviality" : Every point $x \in V$ has a neighborhood U and a system of local coordinates (1) $(x^j)_{j=1,\dots,\dim V}$ so that the partition of U in connected components of leaves (2) corresponds to the partition of $\mathbb{R}^{\dim V} = \mathbb{R}^{\dim F} \times \mathbb{R}^{\mathrm{Codim}\, F}$ in the parallel affine subspaces $\mathbb{R}^{\dim F} \times pt$.

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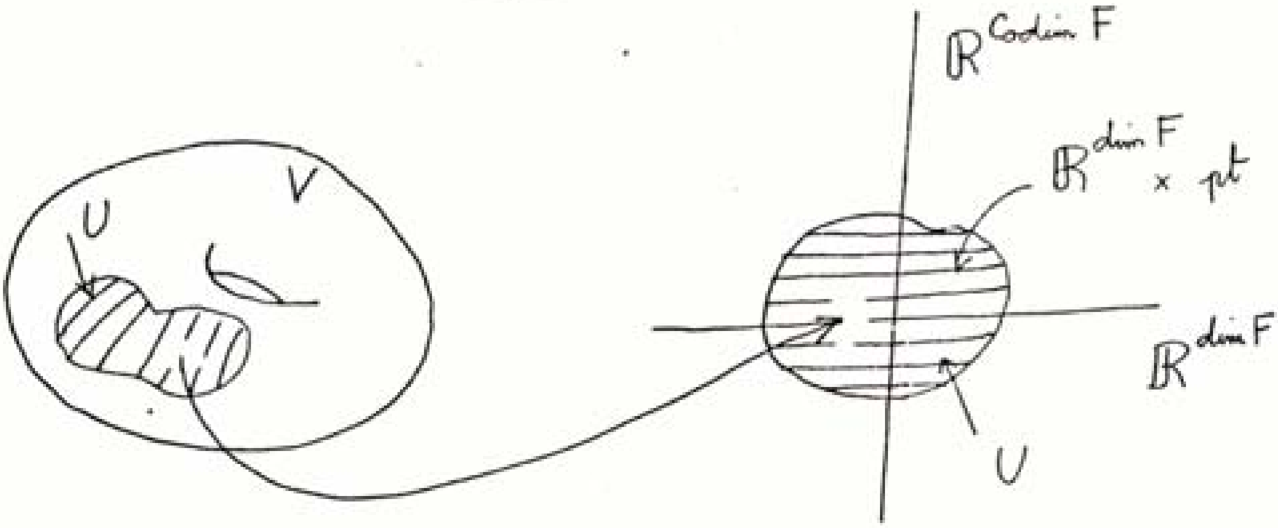
(1) Such charts are called foliation charts.

(2) Called plaques, they are the leaves of the restriction of F.



In the simplest examples, such as the Kronecker foliation of the 2-torus $V = \mathbb{R}^2/\mathbb{Z}^2$ given by the differential equation $dx = \theta\, dy$ where $\theta \notin \mathbb{Q}$, one sees that :

1) Though V is compact, the leaves L_α, $\alpha \in A$ can fail to be compact.

2) The space A of leaves L_α, $\alpha \in A$ can fail to be hausdorff and in fact the quotient topology can be trivial (with no non trivial open subset).

We shall study 1) the generic leaf L of the foliation 2) the parameter space A, i.e. the space of leaves. In 1) the problem is to be able to say something about the global properties of the non compact manifold L using only the local information given by the bundle $F \subset TV$ (If F is given it is in general not so easy to integrate it explicitly), the main difficulty lies in the non compactness of L, since otherwise many classical results of differential geometry allow to pass say from the knowledge of the tangent bundle to global invariants. The first part of these lectures is occupied with problem 1), where the word "generic" is taken from the point of view of probability theory, i.e. a suitable "measure" is put on the space A and a property of leaves L_α is generic if it holds for almost all $\alpha \in A$. Thus, in this, the space is only considered from the point of view of <u>measure theory</u>. It is well known that the measure theory of usual spaces gives no information about the space : all Lebesgue measure spaces are isomorphic. However for spaces like the space of leaves A , the usual measure theory (measurable sets, measurable functions, positive measures, L^p spaces etc...) is totally inappropriate, for instance in the above Kronecker foliation (V,F) of the 2-torus, any measurable function f on A is, (by ergodicity of the irrational rotation of angle $2\pi\theta$) almost everywhere equal to a constant, thus $L^p(A) = \mathbb{C}$ for any $p \in [1,+\infty]$ is about as informative as the quotient topology on A .

The proper notion of measure on the set V/F of leaves of (V,F) is exactly the notion of <u>transverse measure</u> for the foliation (V,F) . This is a well established notion, defined in purely geometric terms as a measure Λ on transversals which is invariant under the holonomy pseudogroup or in a more canonical manner as a current, of the same dimension as F , on the manifold V which is a) positive in the leaf direction b) closed. We shall carefully explain this classical notion in the first two sections, beginning with the very simple one dimensional case (i.e. flows provided is oriented). We give there a proof of the result of Ruelle and Sullivan characterizing transverse measures as closed currents. The main point of these two sections is to show that, given a transverse measure Λ for (V,F) the measure $\Lambda(N)$ of a transversal N only depends on the "function" on the leaf space V/F which associates to each $L \in V/F$ the number of points of intersection of the leaf L with N .

We then sketch in a remark, how the classical measure theory has to be modified to treat spaces like V/F .

In section 3 we first give two geometric interpretations of the scalar $\chi(F,\Lambda)$ defined for a measured foliation as the Euler class $e(F)$ of the bundle F, evaluated on the Ruelle Sullivan cycle $[C]$, $[C] \in H_*(V,\mathbb{R})$. We then define analytically the Betti numbers β_i as the Murray-von Neumann dimension of the space of L^2 harmonic forms and give an application of the equality

$$\chi(F,\Lambda) = \sum (-1)^i \beta_i$$

In section 4 we state the index theorem for differential operators elliptic along the leaves, as the equality :

$$\dim_\Lambda(\operatorname{Ker} D) - \dim_\Lambda(\operatorname{Ker} D^*) = \varepsilon \operatorname{ch} D \operatorname{Td} V\,[C]$$

and we apply it in the simplest case to construct certain meromorphic functions on $\mathbb{C}$.

From section 5 we begin the discussion of the topology of the "space of leaves". In sections 1-2-3-4 there is no mention of operator algebras, they disappear from the statements but are however crucial in the proof of the theorems of sections 3 and 4. From section 5 the basic tool is the C^* algebra $C^*(V,F)$ canonically associated to the foliation (V,F). We define analytically the K-theory of the space of leaves of (V,F) as the algebraic K-theory $K_*(C^*(V,F))$. We then define geometrically (1) the K-theory of the space of leaves of (V,F) as a certain twisted K-homology, $K_{*,\tau}(BG)$ of the classifying space BG of the graph G of the foliation. This geometric group is rather easy to compute, and for instance we prove that in the set up of the Thom isomorphism (cf. [10]), i.e. for foliations coming

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(1) This definition is part of joint work with P. Baum .

from actions of solvable simply connected Lie groups, the geometric group is equal to $K^*(V)$ (up to a parity shift). In general, for foliations with contractible K-oriented leaves the same result holds.

Before connecting the geometric group with the analytical one, we discuss the functoriality of the analytical group $K^*(V/F)$. In algebraic topology, one can under certain circumstances define, given a K-oriented map $\varphi : X \to Y$, a push forward map $\varphi!$ in K-theory. These $\varphi!$ maps play a crucial role in the proof of the index theorem of [1] in the spirit of the Grothendieck Riemann Roch theorem. The most remarquable property of the analytical group $K^*(V/F)$ is its <u>wrong way functoriality</u> which to each K-oriented map φ of leaf spaces associates an element $\varphi!$ of the Kasparov group $KK(C^*(V_1,F_1), C^*(V_2,F_2))$. In section 10 we discuss this functoriality in simple cases and then in section 11 we construct $\varphi!$ given a smooth K-oriented map φ of a manifold M in the leaf space of (V, F). This enables us to define a natural map from the geometric theory to the analytical theory.

It is then easy to define geometrically the topological index $\mathrm{Ind}_t(D)$ for differential operators elliptic along the leaves, the equality of this topological index with the analytical index $\mathrm{Ind}_a(D) \in K^*(V/F)$ extends the index theorem for measured foliations to foliations without transverse measures. We present it here as a problem, whose solution would follow from a general proof of the functoriality : $(\varphi \circ \psi)! = \psi! \otimes \varphi!$ for smooth K-oriented maps of leaf spaces (cf. Section 10).

In section 12 we discuss when the natural map μ : $\mu : K_{*,\tau}(BG) \to K^*(V/F)$ is bijective. All evidence is that it is in full generality an isomorphism.

Finally in section 13 we discuss, by an example, how classical differential geometry extends naturally to C^* algebras such as $C^*(V,F)$ where F is a smooth foliation of V. We introduce the notion of curvature by studying a purely C^* algebraic problem. We end by discussing an integrality result, and stating an index theorem for difference differential operators on the real line.

We have kept these lectures at a very informal level so most proofs are only sketched, a more refined and complete version will appear later.

1. TRANSVERSE MEASURE FOR FLOWS.

To get acquainted with the notion of transverse measure for a foliation, we shall describe it in the simplest case : $\dim F = 1$, i.e. when the leaves are one dimensional. The foliation is then given by an arbitrary smooth 1-dimensional subbundle F of TV, the integrability condition being automatically satisfied. To simplify even further, we assume that F is oriented, so that the complement of the zero section has two components F^+ and $F^- = -F^+$. Then using partitions of unity, one gets the existence of smooth sections of F^+, and any two such vector fields $X, X' \in C^\infty(F^+)$ are related by $X' = \varphi X$ $\varphi \in C^\infty(V)$. The leaves are then the orbits of the flow $\exp tX$ so the flows $H_t = \exp tX$, $H'_t = \exp tX'$ have the same orbits and differ by a time change :

$$H'_t(p) = H_{T(t,p)}(p) \qquad \forall t \in \mathbb{R},\ p \in V$$

The dependence in p of this time change $T(t,p)$ makes it clear that a measure μ on V which is invariant under the flow H (i.e. $H_t \mu = \mu$, $\forall t \in \mathbb{R}$), is not in general invariant under H' (take the simplest case $X = \frac{\partial}{\partial\theta}$ on S^1).

To be more precise let us first translate the invariance $H_t \mu = \mu$ by a condition involving the vector field X rather than the flow $H_t = \exp tX$. Recall that a current C on V is a continuous linear form on the complex topological vector space $C^\infty(\wedge T^*_{\mathbb{C}})$ of smooth complex valued differential forms on V, and that C is of dimension q iff $\langle C, \omega \rangle = 0$ whenever ω is of degree $q' \neq q$.

In particular a measure μ on V defines a 0-dimensional current by the equality $\langle \mu, \omega \rangle = \int \omega \, d\mu$, $\forall \omega \in C^\infty(V)$. All the usual operations, the Lie derivative ∂_X with respect to a vector field, the boundary d, the contraction i_X with a vector field, are extended to currents by duality, and the equality $\partial_X = d i_X + i_X d$ remains of course true. Now the condition $H_t \mu = \mu$, $\forall t \in \mathbb{R}$ is equivalent to $\partial_X \mu = 0$ and since the boundary $d\mu$ is always 0 it is equivalent to $d(i_X \mu) = 0$. This condition is obviously not invariant by changing X in $X' = \varphi X$, $\varphi \in C^\infty(V)$. However if we replace X by $X' = \varphi X$ and change μ into $\mu' = \varphi^{-1}\mu$, the current $i_{X'}\mu'$ is equal to $i_X \mu$ and hence is closed, so that μ' is now invariant for $H'_t = \exp(tX')$.

So while we do not have a single measure μ on V invariant under all possible flows defining the foliation we can trace the invariant measures for each of these flows as 1-dimensional currents $i_X \mu = C$. To reconstruct μ from the current C and the vector field X, define :

$$<\mu, f> = <C, \omega>, \forall \omega \in C^\infty(\Lambda^1 T_{\mathbb{C}}^*), \omega(X) = f .$$

Given a 1-dimensional current C on V and a vector field $X \in C^\infty(F^+)$ the above formula will define a positive invariant measure for $H_t = \exp tX$ iff C satisfies the following conditions :

1) C is closed, i.e. $dC = 0$

2) C is positive in the leaf direction, i.e. if ω is a smooth 1-form whose restriction to leaves is positive then $<C, \omega> \geqq 0$.

We could also replace condition 1) by any of the following :

1)' There exists a vector field $X \in C^\infty(F^+)$ such that $\exp tX$ leaves C invariant.

1)" Same as 1)', but for all $X \in C^\infty(F^+)$.

In fact if C satisfies 2) then $\langle C, \omega \rangle = 0$ for any ω whose restriction to F is 0, thus $i_X C = 0$, $\forall X \in C^\infty(F^+)$, and

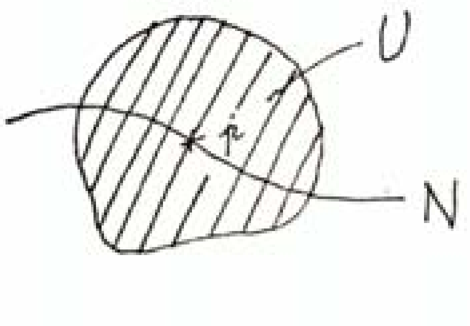
$\partial_X C = d i_X C + i_X dC = i_X dC$ is zero iff $dC = 0$.

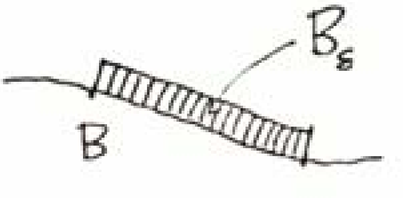
From the above discussion we get two equivalent points of view on what the notion of an invariant measure should be for the one dimensional foliation F:

α) An equivalence class of pairs (X, ν), where $X \in C^\infty(F^+)$, ν is an $\exp tX$ invariant measure on V, and $(X, \nu) \sim (X', \nu')$ when $X' = \varphi X$, $\nu = \varphi \nu'$ for $\varphi \in C^\infty(V)$.

β) A 1-dimensional current C, positive in the leaf direction (cf. 2)) and invariant under all (or equivalently some) flows $\exp tX$, $X \in C^\infty(F^+)$.

Before we proceed to describe α) and β) for arbitrary foliations we relate them to a third point of view γ) that of holonomy invariant transverse measure. A submanifold N of V is called a transversal if at each $p \in N$, $T_p(V)$ splits as the direct sum of the subspaces $T_p(N)$ and F_p. Thus the dimension of N is equal to the codimension of F. Let then $p \in N$, and U, $p \in U$, the domain of a foliation chart, one can choose U small enough so that the plaques of U correspond bijectively to points of $N \cap U$, each plaque of U meeting N in one and only one point.





Starting from a pair (X,ν) as in α), one defines on each transversal N a positive measure by observing that the conditional measures of ν (restricted to U) on the plaques, are, since ν is invariant by H_t $H_t = \exp tX$, proportional to the obvious Lebesgue measures determined by X, so the formula $\Lambda(B) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\nu(B_\varepsilon)$, $B_\varepsilon = \bigcup_{t\in[0,\varepsilon]} H_t(B)$ makes sense.

If one replaces (X,ν) by an equivalent (X',ν') it is obvious that Λ does not change, since $\nu' = \varphi^{-1}\nu$ while $X' = \varphi X$. By construction Λ is invariant under any of the flows $H_t = \exp tX$ i.e. $\Lambda(H_t B) = \Lambda(B)$, $\forall t\in\mathbb{R}$, and any borel subset B of a transversal. In fact much more is true :

LEMMA. Let N_1, N_2 be two transversals, B_i a borel subset of N_i, $i=1,2$ and $\Psi : B_1 \to B_2$, a borel bijection such that for each $x\in B_1$, $\Psi(x)$ is on the leaf of x, then we have $\Lambda(B_1) = \Lambda(B_2)$.

To prove this, note that if $p_1 \in N_1$ and $H_t(p_1) = p_2 \in N_2$ for some $t\in\mathbb{R}$, then there exists a smooth function φ defined in a neighborhood of p_1 and such that $\varphi(p_1) = t$, $H_{\varphi(p)}(p) \in N_2$. Thus there exists a sequence φ_n of smooth functions defined on open sets of N_1 and such that

$$\{(p,t)\in N_1\times\mathbb{R},\ H_t(p)\in N_2\} = \bigcup \mathrm{Graph}\ \varphi_n$$

Let then (P_n) be a borel partition of B_1 such that for $p\in P_n$ one has $\Psi(p) = H_{\varphi_n(p)}(p)$, it is enough to show that $\Lambda(\Psi(P_n)) = \Lambda(P_n)$ for all n, but since on P_n, Ψ coincides with $p \mapsto H_{\varphi_n(p)}(p)$ it follows from the invariance of Λ under all flows $\exp tY$, $Y\in C^\infty(F^1)$.

Thus the transverse measure $\Lambda(B)$ depends in a certain sense only of the intersection number of the leaves L, of the foliation, with the borel set B. For instance if the current C is carried by a single closed leaf L the transverse measure $\Lambda(B)$ only depends on the number of points of intersection of L and B and hence is proportional to $B \mapsto (B \cap L)^{\#}$.

By a borel transversal B to (V, F) we mean a borel subset B of V such that $B \cap L$ <u>is countable for any leaf</u> L. If there exists a borel injection Ψ of B in a transversal N with $\Psi(x) \in \mathrm{Leaf}(x)$ $\forall x \in B$, define $\Lambda(B)$ as $\Lambda(\Psi(B))$ (which by the lemma, is independent of the choices of N and Ψ). Then extend Λ to arbitrary borel transversals by σ-additivity, (remarking that any borel transversal is a countable union of the previous ones).

We thus obtain a transverse measure Λ for (V, F) in the following sense :

DEFINITION γ). A transverse measure Λ for the foliation (V, F) is a σ - additive map $B \to \Lambda(B)$ from borel transversals (i.e. borel sets in V with $V \cap L$ countable for any leaf L) to $[0, +\infty]$ such that
1°) If $\Psi : B_1 \to B_2$ is a borel bijection and $\Psi(x)$ is on the leaf of x for any $x \in B_1$, then $\Lambda(B_1) = \Lambda(B_2)$.
2°) $\Lambda(K) < \infty$ if K is a compact subset of a smooth transversal.

We have seen that the points of view α) and β) are equivalent and how to pass from α) to γ). Given a transverse measure Λ as in γ), we get for any distinguished open set U (1), a measure μ_U on the set of

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(1) By this we mean that U is the domain of a foliation chart.

plaques π of U , such that for any borel transversal $B \subset U$ one has :

$$\Lambda(B) = \int \operatorname{Card}(B \cap \pi) \, d\mu_U(\pi)$$

Put $\langle C_U , \omega \rangle = \int \left(\int_\pi \omega \right) d\mu_U(\pi)$, where ω is a differential form on U (2) and $\int_\pi \omega$ is its integral on the plaque π of U . Then on $U \cap U'$ the currents C_U and $C_{U'}$ agree so that one gets a current C on V which obviously satisfies the conditions β), 1) and 2). One thus gets the equivalence between the three points of view α) β) and γ) .

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(2) With compact support in U .

2. TRANSVERSE MEASURE FOR FOLIATIONS

We first state how to modify α) and β) for arbitrary foliations $(\dim F \neq 1)$. To simplify we assume that the bundle F is oriented. For α) we considered in the case $\dim F = 1$, pairs (X, μ) up to the very simple equivalence relation saying that only $X \otimes_{C^\infty(V)} \mu$ matters. In the general case, since F is oriented we can talk of the positive part $(\Lambda^{\dim F} F)^+$ of $\Lambda^{\dim F} F$, and, using partitions of unity, construct sections X of this bundle. These will play the role of the vector field X. Given a smooth section $X \in C^\infty(\Lambda^{\dim F} F)^+$, we have on each leaf L of the foliation a corresponding volume element, it corresponds to the unique k-form ω on L such that $\omega(X) = 1$. In the case $\dim F = 1$, the measure μ had to be invariant under the flow $H_t = \exp(tX)$. This occurs iff <u>in each domain of foliation chart</u> U, <u>the conditional measures of</u> μ <u>on the plaques of</u> U <u>are proportional to the measures determined by the volume element</u> X. Thus we shall define the X-invariance of μ in general by this condition. So α) becomes classes of pairs (X, μ) where $X \in C^\infty(\Lambda^{\dim F} F)^+$ and μ is an X-invariant measure, while $(X, \mu) \sim (X', \mu')$ iff $X' = \varphi X$ and $\mu = \varphi \mu'$ for some $\varphi \in C^\infty(V)$. The condition of invariance of μ is of course local, if it is satisfied and $Y \in C^\infty(F)$ is a vector field leaving the volume element X invariant, i.e. $\partial_Y(X) = 0$, then Y also leaves μ invariant. Moreover since Lebesgue measure in $\mathbb{R}^k$ is characterized by its invariance under translation, one checks that if $\partial_Y \mu = 0$ $\forall Y \in C^\infty(F)$ with $\partial_Y(X) = 0$, then μ is X-invariant. To see this one can choose local coordinates in U transforming X in the k-vector field $X = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^k} \in C^\infty(\Lambda F)$. With the above trivialization of X it is clear that the current $i_X \mu$ one gets by contracting X with the 0-dimensional current μ is a closed current, which is locally of the form :

$$<C,\omega> = \int \left(\int_\pi \omega \right) d\nu_U(\pi)$$

where ω is a k-form with support in U, $\int_\pi \omega$ is its integral on a plaque π of U and ν_U is the measure on the set of plaques coming from the desintegration of ν restricted to U with respect to the conditional measures associated to X.

Clearly C satisfies conditions 1) and 2) of β) and we may also check that $\partial_Y C = 0$, $\forall Y \in C^\infty(F)$, thus for β) in general we take the same as in the case $k=1$, namely a <u>closed current positive in the leaf direction</u>, the condition "closed" being equivalent to

$$\partial_Y C = 0 \qquad \forall Y \in C^\infty(F)$$

To recover ν, given C and X, one considers an arbitrary k-form ω on V such that $\omega(X) = 1$ and puts $<f,\nu> = <C, f\omega>$ $\forall f \in C^\infty(V)$. One checks in this way that α) and β) are equivalent points of view. For γ) one takes exactly the same definition as for $k=1$. Given a transverse measure Λ satisfying γ) one constructs a current exactly as for $k=1$. Conversely given a current C satisfying β), one gets for each domain U of foliation chart a measure ν_U, on the set of plaques of U, such that the restriction of C to U is given by :

$$<C,\omega> = \int \left(\int_\pi \omega \right) d\nu_U(\pi)$$

Thus one can easily define $\Lambda(B)$ for each borel transversal B : if B is contained in U then one has :

$$\Lambda(B) = \int \mathrm{Card}(B \cap \pi)\, d\nu_U(\pi)$$

Now we have to check that Λ satisfies condition δ), i.e. that if B_1, B_2 are two borel transversals, Ψ a borel bijection of B_1 on B_2 such that $\Psi(x)$ is on the leaf of x for all x, then $\Lambda(B_1) = \Lambda(B_2)$. Let ν_j be the restriction of Λ to B_j, then to show that $\Psi(\nu_1) = \nu_2$ we can assume that both B_1, B_2 are contained in domains U_1, U_2 of foliation charts. We let P_i be the set of plaques of U_i, we can assume that each plaque $\pi \in P_i$ contains at most one point of B_i so the obvious projection p_i of B_i in P_i (to each point $x \in B_i$ is associated its plaque) sends ν_i to the restriction of ν_{U_i} to $p_i(B_i)$. We want to show that any borel bijection Ψ of a borel subset D_1 of P_1 on a borel subset D_2 of P_2 such that the plaque $\Psi(\pi)$ is on the same leaf as the plaque π, $\forall \pi \in D_1$, transforms ν_{U_1} in ν_{U_2}. We shall do this with more care than required, to get acquainted with the notion of holonomy, to be used later. For each $\pi \in P_1$, let $I(\pi)$ be the set of plaques of P_2 which are on the same leaf as π, this is a countable set and we just have to construct, as in the case $k = 1$, a sequence φ_n of smooth maps from open sets of P_1 to open sets of P_2 and such that :

1) Each φ_n transforms ν_{U_1} in ν_{U_2}.

2) For all $\pi \in P_1$, $I(\pi) = \{\varphi_n(\pi), n \in \mathbb{N}\}$.

Then the borel map Ψ coincides on borel subsets D_n of D_1 with φ_n and hence maps ν_{U_1} to ν_{U_2}.

Now the current C is invariant under all flows $H_t = \exp tY$ where $Y \in C^\infty(F)$, since $\partial_Y C = 0$ (cf. β)). For $\pi_2 \in I(\pi_1)$ we can construct a simple arc $\gamma(t)$, $\gamma(0) \in \pi_1$, $\gamma(1) \in \pi_2$ on the leaf of π_1, i.e. with $\dot{\gamma}(t) \in F_{\gamma(t)}$, $\forall t \in [0, 1]$. If

$Y \in C^\infty(F)$ is a smooth vector field with $Y_{\gamma(t)} = \dot{\gamma}(t)$, $\forall t \in [0,1]$ and $H_t = \exp tY$ the corresponding flow, one has a smooth map φ from an open subset of the transversal N_1 of U_1 parametrizing the plaques, to an open set in P_2, and such that :

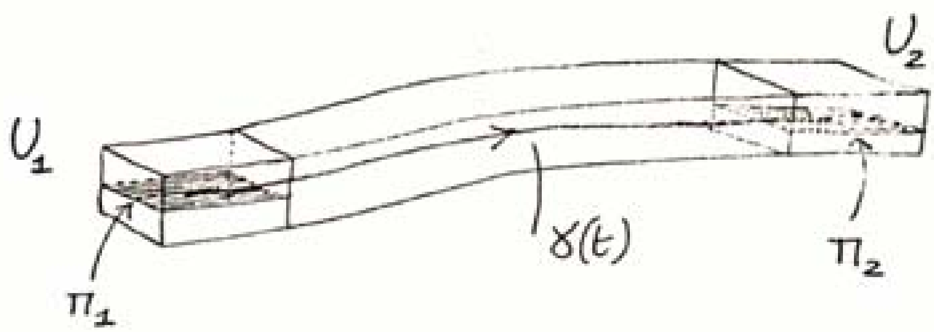
1) φ transforms $\mathcal{N}_{U_1}$ in $\mathcal{N}_{U_2}$

2) φ maps π_1 to π_2

3) $\varphi(\pi)$ is on the leaf of π for all π.

This allows to define a sheaf of sets on P_1, given an open set $\Omega \subset P_1$ we let $\Xi(\Omega)$ be the space of smooth maps from Ω to P_2 which are locally of the above form. Letting E be the covering of P_1 associated to the above sheaf (so that the fiber over each $\pi \in P_1$ is the set of germs of maps φ as above at π), it is enough, to get the existence of the sequence φ_n, to show that the (not necessarily hausdorff) space E is a countable union of compact subsets.

The main point about holonomy is that in the above construction of φ (from the path γ, the vector field Y and the section N_1) the germ of φ at π_1 only depends on the homotopy class of γ, not of the other datas. To see this, note that, with Ω a small enough neighborhood of π_1 and $W = \bigcup_{t\in[0,1]} H_t\Omega$ the restriction of the foliation to W will have its leaves parametrized by Ω, two distinct plaques of Ω being in different leaves of the restriction of F to W.



Thus in the neighborhood W of γ the only plaque $\pi_2 \in P_2$ which is on the leaf of π_1 is $\varphi(\pi_1)$.

In fact this picture shows that one can construct the map φ in other ways, for instance if $\| \ \|$ is a smooth Euclidean structure on the bundle F and $\gamma(t)$ is a geodesic in the corresponding Riemannian structure on leaves, with $(\gamma(0), \dot\gamma(0)) \in F$, it follows that for all Z in a suitable neighborhood of $(\gamma(0), \dot\gamma(0))$ in F , the geodesic $\exp(tZ)$, $t \in [0,1]$, lies entirely in W and hence connects a plaque $\pi \in P_1$ with $\varphi(\pi) \in P_2$. To each $Z \in F$ such that $p_F(Z) \in N_1$ and $p_F(G_1(Z)) \in U_2$, where G_t is the geodesic flow on F , one can associate the element of E given by $p_F(Z)$ and the path $\gamma(t) = p_F(G_t Z)$. The above discussion shows that we get a continuous surjection to E . Thus E is a countable union of compact subsets.

Remark : Abstract measure theory.

Let X be a standard borel space. Then given a standard borel space Y and a borel map $Y \xrightarrow{p} X$ with countable fibers ($p^{-1}\{x\}$ countable for any $x \in X$) , there exists [14] a borel bijection Ψ of Y on the subgraph $\{(x,n) , 0 \leq n < F(x)\}$ of the integer valued function F
$F(x) = \mathrm{Card}\,(p^{-1}\{x\})$ on X . In particular if $(Y_1, p_1),(Y_2,p_2)$ are as above, then the two integer valued functions $F_i(x) = \mathrm{Card}\,(p_i^{-1}\{x\})$ coincide iff there exists a borel bijection $\Psi : Y_1 \to Y_2$ with $p_2 \circ \Psi = p_1$. Let X be a set, and Y_1, Y_2 be standard borel spaces with maps $p_i : Y_i \to X$ with countable fibers and such that $\{(y_i, y_j) \in Y_i \times Y_j , p_i(y_i) = p_j(y_j)\}$ is borel in $Y_i \times Y_j$, $i,j = 1,2$. Then we say that the "functions" from X to the integers defined by (Y_1, p_1) and (Y_2, p_2) are the same, iff there exists a borel bijection

$\Psi : Y_1 \to Y_2$ with $p_2 \circ \Psi = p_1$. By the above, if : $\forall x \in X$ $\mathrm{Card}\,(p_1^{-1}\{x\}) = \mathrm{Card}\,(p_2^{-1}\{x\})$ and <u>if the quotient borel structure on</u> X <u>is standard</u>, then the two "functions" are the same. However, in general we obtain a more refined notion of integer valued function and of measure space. Let (V, F) be a foliation with transverse measure Λ, we shall now define, using Λ, such a generalized measure on the set X of leaves of (V, F). Each borel transversal B is a standard borel space gifted with a projection $p : x \in B \mapsto$ (leaf of $x) \in X$ with countable fiber. Clearly for any two transversals we check the compatibility condition that in $B_1 \times B_2$, the set $\{(b_1, b_2)$, b_1 on the leaf of $b_2\}$ is borel.

DEFINITION. Let p be a map with countable fibers from a standard borel space Y to the space X of leaves, we say that p is borel iff $\{(y, x),\ y \in Y,\ x \in \text{leaf } p(y)\}$ is borel in $Y \times V$.

Equivalently, one could say that the pair (Y, p) is compatible with the pairs associated to any borel transversals. Given a borel pair (Y, p) we shall define its transverse measure $\Lambda(Y, p)$ by observing that if (B, q) is a borel transversal with $q(B) = X$, we can define on $B \cup Y$ the equivalence relation coming from the projection to X and then find a borel partition $Y = \bigcup_1^\infty Y_n$ and borel maps $\Psi_n : Y_n \to B$ with $q \circ \Psi_n = p$, Ψ_n injective. Thus $\sum_1^\infty \Lambda(\Psi_n(Y_n))$ is an unambiguous definition of $\Lambda(Y, p)$.

We then obtain probably the most interesting example of the following abstract measure theory :

<u>A measure space</u> $(X, \mathcal{B})$ is a set X together with a collection $\mathcal{B}$

of pairs (Y, p), Y standard borel space, p map with countable fibers, from Y to X with the only axiom :

<u>A pair</u> (Y, p) <u>belongs to</u> $\mathcal{B}$ <u>iff it is compatible with all other pairs of</u> $\mathcal{B}$ (i.e. iff for any (Z, q) in $\mathcal{B}$ one has $\{(y, z)\ ,\ p(y) = q(z)\}$ borel in $Y \times Z$).

<u>A measure</u> Λ on $(X, \mathcal{B})$ is a map which assigns a real number : $\Lambda(Y, p) \in [0, +\infty]$ to any pair (Y, p) in $\mathcal{B}$ with the following axioms :

<u>σ-additivity</u> $\Lambda(\Sigma(Y_n, p_n)) = \Sigma\, \Lambda(Y_n, p_n)$ (Here $\Sigma(Y_n, p_n)$ is the disjoint union with the obvious projection).

<u>Invariance</u> If $\psi : Y_1 \to Y_2$ is a borel bijection with $p_2 \circ \psi = p_1$ then :

$$\Lambda(Y_1, p_1) = \Lambda(Y_2, p_2)$$

Of course the obtained measure theory contains as a special case the usual measure theory on standard borel spaces, it is however much more suitable for spaces like the space of leaves of a foliation, since, giving a transverse measure for the foliation (V, F) is the same as giving a measure (in the above sense) on the space of leaves, which satisfies the following finiteness condition :

$$\Lambda(K, p) < \infty \quad \text{for any compact subset } K \text{ of a smooth transversal.}$$

The role of the abstract theory of transverse measures ([7]) thus obtained, is made clear by its <u>functorial property</u> : if h is a borel map of the leaf space of (V_1, F_1) to the leaf space of (V_2, F_2) then

$h_*(\Lambda)$ is a "measure" on V_2/F_2 for any "measure" Λ on V_1/F_1 . (V/F is the space of leaves of (V, F) .

3. THE RUELLE-SULLIVAN CYCLE AND THE EULER NUMBER OF A MEASURED FOLIATION.

By a measured foliation we mean a foliation (V,F) equipped with a transverse measure Λ. We assume that F is oriented and we let C be the current defining Λ in the point of view β). As constructed C is an <u>odd</u> current, i.e. it is an <u>odd</u> form with distributions as local coefficients. As it is closed : $dC = 0$, it defines a cyle $[C] \in H_k(V, \mathbb{R})$, by looking at its de Rham homology class ([26]). The distinction here between cycles and cocycles is only a question of orientability, if one assumes that F is transversally oriented then the current becomes even and it defines a cohomology class (cf. [23]).

Let now $e(F) \in H^k(V, \mathbb{R})$ be the Euler class of the oriented real vector bundle F on V (cf. [21]). Using the pairing which makes $H^k(V, \mathbb{R})$ the dual of the finite dimensional vector space $H_k(V, \mathbb{R})$, we get a scalar $\chi(F,\Lambda) = \langle e(F), [C] \rangle \in \mathbb{R}$ which we shall first interpret in two ways as the average Euler characteristic of the leaves of the measured foliation. First recall that for an oriented compact manifold M the Euler characteristic $\chi(M)$ is given by the well known theorem of Poincaré and H. Hopf :

$$\chi(M) = \sum_{p \in \mathrm{Zero}\, X} \omega(X, p)$$

where X is a smooth vector field on M with only finitely many zeros, while $\omega(X,p)$ is the local degree of X around p, generically p is a non degenerate zero, i.e. in local coordinates, $X = \sum a^i \frac{\partial}{\partial x^i}$, the matrix $\frac{\partial a^i}{\partial x^j}(p)$ is non degenerate and the local degree is the sign of its determinant.

Also, choosing arbitrarily on M a Riemannian metric, one has the generalized Gauss Bonnet theorem which expresses the Euler characteristic as the integral over M of a form Ω on M, equal to the Pfaffian of $(2\pi)^{-1}K$ where K is the curvature form. These two interpretations of the Euler characteristic extend immediately to the case of measured foliations (V, F, Λ).

First if $X \in C^\infty(F)$ is a generic vector field tangent to the foliation, its zeros will form on each leaf L a locally finite subset on which the local degree $\omega(X, p) = \pm 1$ is well defined. Using the transverse measure we can thus form $\int_{p \in \mathrm{Zero}\, X} \omega(X, p)\, d\Lambda(p)$ (using γ). This scalar is again independent of the choice of X and equals $\chi(F, \Lambda) = \langle e(F), [C] \rangle$ as is easily seen from the geometric definition of the Euler class by taking the odd cycle associated to the zeros of a generic section.

Second, let $\| \ \|$ be a Euclidean structure on the bundle F, (cf. [21]), then each leaf L is equipped with a Euclidean structure on its tangent bundle, i.e. with a Riemannian metric. So the curvature form of each leaf L allows to define a form Ω_L, of maximal degree, on L, by taking as above the Pfaffian of the $(2\pi)^{-1}\times$ curvature form. Now Ω_L is only defined on L, but one can define easily its integral, using the transverse measure Λ, which is formally :

$$\int \left(\int_L \Omega_L \right) d\Lambda(L)$$

(In point of view α), this integral is $\nu(\varphi)$ where φ is the function $\varphi(p) = \Omega_p(X_p)$, in β) it is $\langle \Omega', C \rangle$ where Ω' is any k-form on V whose restriction to each leaf L is Ω_L, and in γ) it is the above integral computed using a covering by domains U_i of foliation charts and partition of unity φ_i , the value of $\int\left(\int_L \varphi_i \Omega\right) d\Lambda(L)$ being given by $\int\left(\int_\pi \varphi_i \Omega\right) d\Lambda(\pi)$ where π varies on the set of plaques of U_i).

To show that the above integral is equal to $\chi(F,\Lambda)$, choose a connection ∇ on F compatible with the metric, and from the curvature form K of ∇ , take $\Omega' = Pf(K/2\pi)$, this gives a closed k-form on V and by [21] p. 311, the Euler class of F is represented by Ω' in $H^k(V, \mathbb{R})$. Now the restriction of ∇ to leaves is not necessarily equal to the Riemannian connection, however both are compatible with the metric, it follows then, that on each leaf L there is a canonical $k-1$ form ω_L with $\Omega_L - \Omega'_L = d\omega_L$ where Ω'_L is the restriction of Ω' to L . Now since ω_L is canonical it is the restriction to L of a $k-1$ form ω on V and hence :

$$\int\left(\int_L \Omega_L\right) d\Lambda(L) = \langle \Omega' + d\omega , C \rangle = \langle e(F), [C] \rangle$$

So in fact both interpretations of $\chi(F,\Lambda)$ follow from the general theory of characteristic classes. There is however missing the third interpretation of the Euler number $\chi(M)$ of a compact manifold :

$\chi(M) = \sum (-1)^i \beta_i$, where the β_i are the Betti numbers :

$\beta_i = \dim(H^i(M, \mathbb{R}))$. The first idea to define the β_i in the foliation case is to consider the transverse measure Λ as a way of defining the density of discrete subsets of the generic leaf and then take β_i as the density of holes of dimension i . However the simplest examples show that one may very well have a foliation with all leaves diffeomorphic to $\mathbb{R}$ while $\chi(F,\Lambda) < 0$, so that $\beta_1 > 0$ cannot be defined in the above naive sense. Specifically, let π be a faithful orthogonal representation of the fundamental group Γ of a Riemann surface S of genus 2 , then the corresponding principal bundle V on S has a natural foliation : V is the quotient of $\tilde{S} \times SO(n)$ (*) by the action

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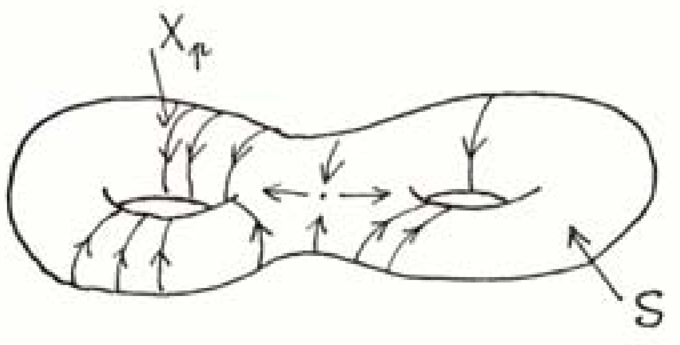
(*) $\tilde{S}$ is the universal covering of S .

of Γ and the foliation with leaves $\tilde{S} \times pts$ is globally invariant under Γ and hence drops down as a foliation on V. The bundle F is the bundle of horizontal vectors for a flat connection on V, each fiber $p^{-1}\{x\}$ is a closed transversal which intersects each leaf in exactly one orbit of Γ. So the Haar measure on $SO(n)$ defines a transverse measure Λ (using point of view γ)). One has $\chi(F,\Lambda) = -2$ as is easily seen by taking a generic vector field X on S and pulling it back to an horizontal vector field $\tilde{X} \in C^\infty(F)$ on V, since the transverse measure of each fiber is one it is clear that $\chi(F,\Lambda) = -2$



Moreover, since the representation π of Γ in $SO(n)$ is faithful, each leaf L of the foliation is equal to the covering space $\tilde{S}$ i.e. is conformal to the unit disk in $\mathbb{C}$. So each leaf is simply connected while "$\beta_0 - \beta_1 + \beta_2$" < 0. However though the Poincaré disk (i.e. the unit disk of $\mathbb{C}$ as a complex curve) is simply connected it has plenty of non zero harmonic 1-forms. For instance if f is an holomorphic function in the disk D, then the form $\omega = f(z)dz$ is harmonic, and its L^2 norm $\int \omega \wedge *\omega$ is finite when $f \in L^2(D)$, in particular if f is bounded. Thus the space $H^1(D, \mathbb{C})$ thus defined, of square integrable harmonic 1-forms, is infinite dimensional. Note that the definition depended on the conformal structure of D, not only on its smooth structure, since on $\mathbb{C}$ all square integrable harmonic 1-forms are 0 (end of the example) . Given a compact foliated manifold we can, in many ways, choose a Euclidean metric $\|\ \|$ on F , however since V is compact, two such metrics $\|\ \|, \|\ \|'$ always satisfy an inequality of the form $C^{-1}\|\ \|' \leq \|\ \| \leq C\|\ \|'$. So for each leaf L the two riemannian structures defined by $\|\ \|$ and $\|\ \|'$ will be well related : the identity map from (L) to $(L)'$ being a quasi isometry. Letting then $\mathfrak{h}$ be the hilbert space of square integrable 1-forms on L with respect to $\|\ \|$ (resp. $\mathfrak{h}'$ with respect to $\|\ \|'$) the above quasi isometry determines a bounded invertible operator T from $\mathfrak{h}$ to $\mathfrak{h}'$. We let P (resp. P') be the orthogonal projection of $\mathfrak{h}$ (resp. $\mathfrak{h}'$) on the subspace of harmonic 1-forms. Then $P'T$ (resp. PT^{-1}) is a bounded operator from $H^1(L,\mathbb{C})$ to $H^1((L)',\mathbb{C})$ (resp. from $H^1((L)',\mathbb{C})$ to $H^1(L,\mathbb{C})$. These operators are inverse of each other since for instance the form $PT^{-1}P'T\omega - \omega$ is harmonic on L , and is in the closure of the range of the boundary operator, and hence is 0 . (At this point, of course, one has to know precisely the domains of the unbounded operators used, the compactness of the ambient manifold V shows that each leaf with its quasi isometric structure is complete, in particular there is no boundary condition needed to define the Laplacian, its minimal and maximal domains coincide. To see that $P'T\omega - T\omega$ is in the closure of the image of d , note that $T\omega = \omega'$ is closed, so that in the decomposition $\omega' - P'\omega' = \omega'_1 + \omega'_2$ with $\omega'_1 \in \overline{\mathrm{Im}\, d}$, $\omega'_2 \in \overline{\mathrm{Im}(d)^{*'}}$, one has $\omega'_2 = 0$).

From the above discussion we get that the hilbert space $H^j(L,\mathbb{C})$ of square integrable harmonic j-forms on the leaf L is well defined up to a quasi isometry. Of course this fixes only its dimension,

$\dim H^j(L,\mathbb{C}) \in \{0,1,\dots,+\infty\}$. In the above example all leaves were equal to the Poincaré disk D and hence for each L :

$\dim H^1(L,\mathbb{C}) = +\infty$. However in this example $\chi(F,\Lambda) = -2$ was finite, which is not compatible with a definition of β_1 as the constant value $+\infty$ of $\dim H^1(L,\mathbb{C})$. In this example one can easily see that $H^0(L,\mathbb{C}) = 0$ and $H^2(L,\mathbb{C}) = 0$, thus we should have $\beta_1 = 2$.

Now note that the quasi isometry defined above from $H^j(L,\mathbb{C})$ to $H^j((L)',\mathbb{C})$ $(j=1)$ was canonical, this means that, on the space of leaves V/F of (V,F) , the two "bundles" of hilbert spaces are isomorphic as "bundles" and not only fiberwise. The point that there is more information in the "bundle" than in the invidual fibers is well known, however it is also well known that in measure theory all bundles are trivial. If (X,μ) is a Lebesgue measure space and $(H_x)_{x\in X}$, $(H'_x)_{x\in X}$ are two measurable fields of hilbert spaces (cf. [12]) with isomorphic fibers (i.e. $\dim H_x = \dim H'_x$ a.e.) then they are isomorphic as bundles.

Since in our example : $\dim H^1(L,\mathbb{C}) = +\infty \quad \forall L \in V/F$ one could think that this bundle of hilbert spaces is measurably trivial, let us show in fact that it has no measurable cross section of norm one. To define measurability we pull the section back to the ambient manifold using the projection ℓ of V on the space V/F of leaves (through each point $x \in V$ passes one and only one leaf $\ell(x) = L$, the pull back of the bundle $H^1(L,\mathbb{C})$ having an obvious measurable structure. Now since $H^1(L,\mathbb{C})$ is a subbundle of the bundle of L^2 1-forms on leaves, which is measurably isomorphic to the sum of two copies of the bundle of L^2 functions on leaves, we just have to show that the latter does not have sections of norm one which are measurable. So let for each leaf L, $\varphi_L \in L^2(L)$ be an L^2 function. We can then associate to each L the measure β_L on V which is carried by the leaf L and is the product of $|\varphi_L|^2$ by the volume element on the leaf. If $\|\varphi_L\| = 1$ then the measure β_L is a probability measure : $\beta_L \in P(V) = \{$ probability measures on $V\}$. If Ψ is measurable the map $x \mapsto \beta_{\ell(x)}$ from V to $P(V)$ which we obtain is now measurable (when $P(V)$ is considered as a compact space with the $\sigma(P(V), C(V))$ topology). But the map Ψ, $\Psi(x) = \beta_{\ell(x)}$ is constant on each leaf and hence due to the ergodicity of Λ, which holds if $\pi(\Gamma)$ is dense in $SO(n)$, Ψ is almost everywhere constant, which contradicts the disjointness of the measures $\beta_{\ell(x)}$ for disjoint leaves. (To get the ergodicity of Λ note that the transversal $N \simeq SO(n)$ meets ecah leaf of the foliation, in one orbit of $\pi(\Gamma)$ in $SO(n)$, while the action by translation of $\pi(\Gamma)$ on $SO(n)$ is ergodic).

Thus we see that the measurable bundle $H^1(L, \mathbb{C})$ is not trivial, it is however isomorphic to a much simpler measurable bundle, which we now describe : We let B be a borel transversal, then to each leaf L of the foliation we associate the hilbert space $H_L = \ell^2(L \cap B)$ with orthonormal basis (e_y) canonically parametrized by the discrete countable subset $B \cap L$ of the leaf L. (To define the measurable structure of this bundle, note that its pull back to V assigns to each $x \in V$, the space $\ell^2(L_x \cap B)$ where L_x is the leaf through x, so given a section $(s(x))_{x \in V}$ of this pull back, we shall say that it is measurable iff the function $(x,y) \in V \times B \mapsto \langle s(x), e_y \rangle$ is measurable).

LEMMA. Let B, B' be borel transversals, then if the two bundles of Hilbert spaces $(H_L)_{L\in V/F}$, $H_L = \ell^2(L\cap B)$; $H'_L = \ell^2(L\cap B')$ are measurably isomorphic, one has $\Lambda(B) = \Lambda(B')$.

Proof. Let $(U_L)_{L\in V/F}$ be a measurable family of unitaries from H_L to H'_L . For each $x \in B$ let λ_x be the probability measure on B' given by $\lambda_x(\{y\}) = |\langle U_{\ell(x)}\, e_x, e_y\rangle|^2$. By construction λ_x is carried by the intersection of the leaf of x with B' . From the existence of this map $x \mapsto \lambda_x$ which is obviously measurable one concludes as in the proof of lemma 11 that $\Lambda(B) \leqq \Lambda(B')$. Q.E.D.

From this lemma we get a non ambiguous definition of the dimension for measurable bundles of hilbert spaces of the form $H_L = \ell^2(L\cap B)$ by taking :

$$\dim_\Lambda(H) = \Lambda(B)$$

We can now state the main result of this section. We shall assume that the set of leaves of (V,F) with non trivial holonomy is Λ-negligible. This is not always true and we shall then explain how the statement has to be modified for the general case.

THEOREM 1. a) For each $j = 0, 1, 2, \dots, \dim F$, there exists a borel transversal B_j such that the bundle $(H^j(L,\mathbb{C}))_{L\in V/F}$ of jth square integrable harmonic forms on L is measurably isomorphic to $(\ell^2(L\cap B))_{L\in V/F}$.

b) The scalar $\beta_j = \Lambda(B_j)$ is finite, independent of the choice of B_j, of the choice of the Euclidean structure on F .

c) One has $\sum (-1)^j \beta_j = \chi(F,\Lambda)$.

Of course if $F = TV$ so that there is only one leaf, a borel transversal B is a finite subset of V, $\Lambda(B)$ is its cardinality so that one recovers the usual interpretation of the Euler characteristic and Betti numbers. Les us specialise now to 2-dimensional leaves, i.e. $\dim F = 2$. Then we get $\beta_0 - \beta_1 + \beta_2 = \frac{1}{2\pi} \int K \, d\Lambda$ where K is the intrinsic Gaussian curvature of the leaves. Now β_0 is the dimension of the measurable bundle $H_L = \{$square integrable harmonic 0-forms on $L\}$, thus, as harmonic 0-forms are constant, there are two cases :

If L is not compact, one has $H_L = \{0\}$.

If L is compact, one has $H_L = \mathbb{C}$.

Using the $*$ operation as an isomorphism of $H^0(L, \mathbb{C})$ with $H^2(L, \mathbb{C})$ one gets the same result for $H^2(L, \mathbb{C})$ and hence :

COROLLARY. If the set of compact leaves of (V, F) is Λ-negligible then the integral $\int K \, d\Lambda$ of the intrinsic Gaussian curvature of the leaves is $\leqq 0$.

Proof. $\frac{1}{2\pi} \int K \, d\Lambda = \beta_0 - \beta_1 + \beta_2 = -\beta_1 \leqq 0$. Q.E.D.

Remark. The above theorem was proven in the hypothesis : "The set of leaves with non trivial holonomy is negligible". To state it in general, one has to replace whereever it appears, the generic leaf L by its holonomy covering $\tilde{L}$. Thus for instance the measurable bundle $H^j(L, \mathbb{C})$ is replaced by $H^j(\tilde{L}, \mathbb{C})$, and $\ell^2(L \cap B)$ is replaced by $\ell^2(L \tilde{\cap} B)$ where $L \tilde{\cap} B$ is a shorthand notation for the inverse image of $L \cap B$ in the covering space $\tilde{L}$. One has to be careful at one point, namely the holonomy group of L acts naturally on both : $H^j(\tilde{L}, \mathbb{C})$. $\ell^2(L \tilde{\cap} B)$ and the unitary equivalence

$U_L : H^j(\tilde{L}, \mathbb{C}) \mapsto \ell^2(\tilde{B \cap L})$ is supposed to commute with the action of the holonomy group. Then with these precautions the above theorem holds in full generality. Now unless $\tilde{L}$ is compact one has $H^0(\tilde{L}, \mathbb{C}) = 0$ thus unless L is compact with finite holonomy (which by the Reeb stability theorem implies that nearby leaves are also compact) one has $\beta_0 = \beta_2 = 0$, this of course strengthens the above corollary; to get $\int K d\Lambda \leq 0$ it is enough that the set of leaves isomorphic to S^2 be Λ-negligible. Of course one may have $\int K d\Lambda > 0$ as occurs for foliations with S^2-leaves.

The statement of the corollary is fairly intuitive, if there is enough positive curvature in the generic leaf, this leaf is forced to be closed and hence a sphere. This fact should have a simple geometric proof.

4. THE INDEX THEOREM FOR MEASURED FOLIATIONS.

The above theorem 1, defining the Betti numbers of a measured foliation and proving the formula $\sum (-1)^j \beta_j = \chi(F,\Lambda)$, is a special case of the following index theorem for measured foliations. The main value of this index theorem is that it extends to non compact manifolds results computing the dimension of spaces of solutions of elliptic equations, since even though the ambient manifold V is compact the leaves L of the foliation will in general fail to be compact. The solutions of interest will be the L^2 solutions and though the ordinary dimension of the solution space over each leaf L will be in general 0 or $+\infty$, its average with respect to the transverse measure Λ will be finite, non zero unless the solution space is trivial for almost all leaves. So it is very much in the spirit of the theorems of Atiyah [3] and Singer [28] for covering spaces.

One starts with a pair of smooth vector bundles E_1, E_2 on V together with a differential operator D on V from sections of E_1 to sections of E_2 such that :

1°) D restricts to leaves, i.e. $(D\xi)_x$ only depends on the restriction of ξ to a neighborhood of x in the leaf of x (i.e. D only uses partial differentiation in the leaf direction).

2°) D is elliptic when restricted to any leaf.

For each leaf L, let D_L be the restriction of D to L (replace L by the holonomy covering $\tilde{L}$ if L has holonomy), then D_L is an ordinary elliptic operator on this manifold, and its L^2-kernel : $\{\xi \in L^2(L,E_1),\ D_L(\xi) = 0\}$ is formed of smooth sections of E_1 on L. As in [3] one does not have problems of domains for the definition of D_L as an unbounded operator in L^2 since (cf. [7]) its minimal and maximal domains coincide. In this discussion we fix once and for all a 1-density $\alpha \in C^\infty(\Omega)^+$ which is strictly positive. This choice determines $L^2(L, E_i)$, $i = 1, 2$ as well as the formal adjoint D_L^* of D_L which coincides with its hilbert space adjoint.

The principal symbol $\sigma_D(x,\xi) \in \mathrm{Hom}(E_{1,x}, E_{2,x})$ is defined for any $x \in V$, $\xi \in F_x^*$ and being invertible for $\xi \neq 0$, it gives an element of $K^*(F^*)$ (described as K-theory with compact supports as in [1]). Using the Thom isomorphism as in [1] one defines the chern character $ch(\sigma_D) \in H^*(V, \mathbb{Q})$.

THEOREM 2. a) There exists a borel transversal B (resp. B') such that the bundle $(\ell^2(L \cap B))_{L \in V/F}$ is measurably isomorphic to the bundle $(\mathrm{Ker}\, D_L)_{L \in V/F}$ (resp. to $(\mathrm{Ker}\, D_L^*)_{L \in V/F}$).

b) The scalar $\Lambda(B) < \infty$ is independent of the choice of and noted $\dim_\Lambda(\mathrm{Ker}(D))$

c) $\dim_\Lambda(\mathrm{Ker}(D)) - \dim_\Lambda(\mathrm{Ker}(D^*)) = \varepsilon \langle ch\, \sigma_D\, Td(F_{\mathbb{C}}), [C] \rangle$
$\varepsilon = (-1)^{\frac{k(k+1)}{2}}$, $k = \dim F$.

In this formula $[C]$ is the Ruelle-Sullivan cycle, $Td(F_{\mathbb{C}})$ is the todd genus of the complexification $F_{\mathbb{C}}$ of F, which due to the Bott vanishing theorem (i.e. $F^\perp$ is flat in the leaf direction) together with the orthogonality of C with the ideal of forms vanishing on the leaves, can be replaced by the Todd genus of $T_{\mathbb{C}}V$.

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(1) Assume for simplicity that F is oriented.

Before we illustrate this theorem by an example we note that in c) the two sides of the formula are of a very different nature. The left side gives a global information about the leaves by measuring the dimension of the space of global L^2 solutions. It depends obviously on the transverse measure Λ. The right side only depends on the homology class $[C] \in H^k(V,\mathbb{R})$ (a finite dimensional vector space) of the Ruelle-Sullivan cycle, of the subbundle F of TV which defined the foliation, and on the symbol of D which is also a local data. In particular to compute the right hand side it is not necessary to integrate the bundle F. This makes sense since the conditions on a current C to correspond to a transverse measure on (V,F) are meaningful without integrating the foliation (C should be closed and positive on k-forms ω whose restriction to F is positive). Let us now give an example. We take two lattices Γ_1, Γ_2 in $\mathbb{C}$ and in $V = \mathbb{C}/_{\Gamma_1} \times \mathbb{C}/_{\Gamma_2}$ which is a compact torus we consider the foliation F, which is the projection on V of the foliations of $\mathbb{C}^2$ by affine lines parallel to the diagonal. So F is 2-dimensional and each leaf has a natural complex structure. In fact, if the intersection of the diagonal with $\Gamma_1 \times \Gamma_2$ is $\{0\}$, i.e. if $\Gamma_1 \cap \Gamma_2 = \{0\}$ then each leaf of (V,F) is an affine line $\mathbb{C}$.

On V the normalised haar measure of this compact group is invariant by the translations by the subgroup $\mathbb{C}$ (we assume $\Gamma_1 \cap \Gamma_2 = \{0\}$) and defines a transverse measure Λ for (V,F), taking on $\mathbb{C}$ the natural haar measure. We shall assume that $\Gamma_1^{\perp} \cap \Gamma_2^{\perp} = \{0\}$ where $\Gamma_j^{\perp}$ is the dual lattice of Γ_j.

Then Λ is ergodic and by the ergodic theorem (for actions of $\mathbb{C}$), given any borel transversal B, the <u>density</u> of $B \cap L$ on L exists for almost all leaves L and equals $\Lambda(B)$. (Each leaf L is here an affine line $\mathbb{C}$, $B\cap L$ is a discrete subset and its density exists when for any $x \in B\cap L$ the limit of the number of points of $B\cap L$ in a disk with center x and radius R divided by the area of this disk exists (for $R\to\infty$) and does not depend on the choice of x).

Let p_j be a point of $\mathbb{C}/\Gamma_j$, E_1 (resp. E_2) the holomorphic line bundle on $\mathbb{C}/\Gamma_1$ (resp. $\mathbb{C}/\Gamma_2$) associated to the divisor $-p_1$ (resp. $+p_2$) . Consider the holomorphic line bundle $E_1 \otimes E_2$ on $V = \mathbb{C}/\Gamma_1 \times \mathbb{C}/\Gamma_2$. The restriction of E to each leaf : $L = \{(z_1+z, z_2+z), z\in\mathbb{C}\}$ is a holomorphic line bundle. The constant function 1 is a holomorphic section of E_1 on $\mathbb{C}/\Gamma_1$ and is a meromorphic section of E_2 on $\mathbb{C}/\Gamma_2$, using it, one identifies holomorphic sections of E on L with meromorphic functions φ on $\mathbb{C}$ which have poles only at the points of $P_1 = p_1 - z_1 + \Gamma_1$, these poles being simple, and are equal to zero at all points of $P_2 = p_2 - z_2 + \Gamma_2$.

For such a holomorphic section, to be L^2 (which is well defined since V is compact) means that the meromorphic function φ satisfies :

$$\int_{\mathbb{C}} \left(d(z,P_1)/d(z,P_2)\right)^2 |\varphi(z)|^2 |dz\wedge d\bar{z}| < \infty$$

where $d(z,P_i)$ is the Euclidean distance of z with the set P_i . If we let D be the differential operator on V , acting on sections of E as the $\bar\partial$ operator in the leaf direction, (i.e. $(D\xi)(z_1,z_2) = \bar\partial_{z_1}\xi(z_1,z_2) + \bar\partial_{z_2}\xi(z_1,z_2)$, then D satisfies the hypothesis of the theorem and we can easily compute the right hand side $\varepsilon < ch\,\sigma_D\, Td(F_{\mathbb{C}}), [C]>$ of the index formula, since F is trivial we have $Td(F_{\mathbb{C}}) = 1$ and $ch\,\sigma_D = ch\,E = (1 + c_1(E_1))(1 + c_1(E_2))$, whose component of degree 2 is $c_1(E_1)\otimes 1 + 1\otimes c_1(E_2)$, i.e. is equal to $1\otimes v_2 - v_1\otimes 1$ where v_i is the volume form on $\mathbb{C}/\Gamma_i$. From the construction of the current C we thus get :

$$\varepsilon < ch\, \sigma_D\, Td(F_{\mathbb{C}}), [C]> \quad = \text{density } \Gamma_1 - \text{density } \Gamma_2$$

where the density of Γ_i has the obvious meaning and equals the inverse of its covolume.

Thus, since if the index of D is >0, $\operatorname{Ker} D_L$ is necessarily non zero for almost all leaves we get :

COROLLARY. Let Γ_1, Γ_2 be two lattices in $\mathbb{C}$ with density $\Gamma_1 >$ density Γ_2, then for almost all $p\in\mathbb{C}$ there exists a meromorphic function $\varphi\neq 0$ on $\mathbb{C}$ with :

a) The only poles of φ are simple and are on $p+\Gamma_1$

b) $\varphi(z)=0 \quad \forall z\in\Gamma_2$

c) $$\int_{\mathbb{C}} \left(d(z-p,\Gamma_1)/d(z,\Gamma_2)\right)^2 |\varphi(z)|^2 |dz\wedge d\bar z| < \infty$$

Of course the non trivial part of the statement is the growth condition c), otherwise one could take an entire function φ on $\mathbb{C}$ vanishing on Γ_2, but here one can show that if density $\Gamma_1 \leq$ density Γ_2 then there is no meromorphic function φ on $\mathbb{C}$ satisfying a) b) c). Thus indeed the Murray and von Neumann dimension of the space of φ's is equal to density Γ_1 - density Γ_2 since the kernel of D corresponds to exchanging the roles of Γ_1 and Γ_2 and hence vanishes for density $\Gamma_1 >$ density Γ_2.

5. ANALYTICAL K THEORY OF FOLIATIONS.

The index theorem for measured foliations has, as its main feature, the replacement of the hypothesis of compactness of a manifold M by the hypothesis : M is a leaf of a foliation of a compact manifold. So the emphasis is on the properties of the generic leaf as a non compact manifold, and it uses only measure theory in the transverse direction.

If the foliation (V, F) comes from a submersion $p : V \to B$ so that the leaves are the fibers : $p^{-1}\{x\}$, $x \in B$, the ergodic transverse measures correspond exactly to points of B. Let $x \in B$ and Λ_x the corresponding transverse measure, the index theorem for (V, F, Λ_x) is simply the ordinary index theorem for the compact manifold $p^{-1}\{x\}$.

In this situation, the index theorem for families [2] gives a much better result than the index of the restriction of the operator to each fiber, by showing the equality of the analytical index with the topological index of the family D_x, $x \in B$, both elements of the K-theory group $K^*(B)$. In general, the foliation does not come from a submersion and the space of leaves, with the quotient topology, is of very little value since if, for instance, the foliation is minimal (all leaves are dense) then the only open sets are the empty set and the whole space of leaves.

The main point is to replace this quotient topological space by a canonically defined C^* algebra, which will play the role of the algebra of "continuous functions, vanishing at infinity, on the leaf space".

Thus the first task will be the construction of a canonical C^* algebra $C^*(V, F)$ associated to the foliation (V, F) and which, for foliations coming from a submersion $p : V \to B$ will be stably isomorphic to $C_0(B)$ (the algebra of continuous functions vanishing at ∞ on B).

The local triviality of the foliation (V, F) will be visible on $C^*(V, F)$ from the first important property of the construction : if W is an open subset of V and F_W is the restriction of F to W, then one has a canonical isometric homomorphism of $C^*(W, F_W)$ in $C^*(V, F)$. So, covering V by domains W_i of submersions $p_i : W_i \to B_i$ one generates $C^*(V, F)$ from the subalgebras $C^*(W_i, F_{W_i}) \simeq C_0(B_i) \otimes \{\text{compact operators}\}$. Of course, the way these subalgebras fit together can be very complicated and is related to the global nature of the foliation.

We shall define the analytical K theory $K^*(V/F)$ of the space of leaves as the K theory group $K_*(C^*(V, F))$ (with its natural $\mathbb{Z}_2$ grading). Now among the existing homology or cohomology theories for topological spaces, K theory is particularly well adapted to the replacement of the space X by the algebra $C_0(X)$ of continuous functions. Thus, for instance the basic cocycles of K theory are matrix valued continuous functions (i.e. matrices of continuous functions) and a natural frame for the proof of Bott periodicity is non commutative Banach algebras. When extended to arbitrary non commutative C^* algebras, K theory retains the main features, such as Bott periodicity, it had for locally compact spaces. Moreover, it becomes an extremely flexible tool if one uses the bivariant groups introduced by Kasparov.

In fact we shall devote a whole section to determine the C^* modules over $C^*(V, F)$, making clear that this notion and hence

$K^*(V/F)$ only depend on the "space of leaves" V/F (of course we do not mean here the set V/F with the quotient topology, but the meaning of the equality of the spaces of leaves V_1/F_1, V_2/F_2 of two foliations should be clear). Our nextpoint is that the analytical index of a differential operator on V elliptic along the leaves is, in a natural manner, an element of $K^*(V/F)$. When the foliation comes from a submersion $p: V \to B$, $K^*(V/F)$ is equal to $K^*(B)$ and the analytical index for families is indeed an element of $K^*(B)$. Moreover a transverse measure Λ on (V, F) corresponds to a trace on $C^*(V,F)$ and thus defines a canonical morphism : $\dim_\Lambda : K^*(V/F) \to \mathbb{R}$, the index theorem for measured foliations only determined the composition $\dim_\Lambda \circ \mathrm{Ind}_a$ not the more primitive analytical index, which makes sense without any choice of transverse measure.

Our next two problems will be 1) to relate $K^*(V/F)$ with a geometrically defined group, 2) to compute Ind_a in purely geometric terms.

<u>Construction of</u> $C^*(V,F)$.

Let (V, F) be a foliated manifold. The first step in the construction of the C^* algebra $C^*(V, F)$ is the construction of a manifold G, $\dim G = \dim V + \dim F$, called the graph, (or holonomy groupoid), of the foliation, which is made by Winkelnkemper in [30].

An element γ of G is given by two points $x = s(\gamma)$, $y = r(\gamma)$ of V together with an equivalence class of smooth paths : $\gamma(t)$, $t \in [0,1]$; $\gamma(0) = x$, $\gamma(1) = y$, tangent to the bundle F (i.e. with $\dot{\gamma}(t) \in F_{\gamma(t)}$, $\forall t \in \mathbb{R}$) up to the following equivalence : γ_1 and γ_2 are equivalent iff the holonomy of the path $\gamma_2 . \gamma_1^{-1}$ at the point x is the identity. The graph G has an obvious composition law, for $\gamma, \gamma' \in G$, the composition $\gamma \circ \gamma'$ makes sense if $s(\gamma) = r(\gamma')$.

If the leaf L which contains both x and y has no holonomy (this is generic in the topological sense of dense G_δ's) then the class in G of the path $\gamma(t)$ only depends on the pair (y, x) . In general, if one fixes $x = s(\gamma)$, the map from $G_x = \{\gamma, s(\gamma) = x\}$ to the leaf L through x , given by $\gamma \in G_x \mapsto y = r(\gamma)$ is the holonomy covering of L .

Both maps r and s from the manifold G to V are smooth submersions and the map (s, r) to $V \times V$ is an immersion whose image in $V \times V$ is the (often singular) subset $\{(y, x) \in V \times V$, y and x are on the same leaf $\}$. In first approximation one can think of elements of $C^*(V, F)$ as continuous matrices $k(y, x)$, where (y, x) varies in the above set, but we shall now describe this C^* algebra in all details. We shall assume for notational convenience that the manifold G is hausdorff, but as this fails to be the case in very interesting examples we shall then devote a whole section to remove this hypothesis.

The basic elements of $C^*(V, F)$ are smooth half densities with compact support on G , $f \in C_c^\infty(G, \Omega^{1/2})$ where $\Omega_\gamma^{1/2}$ for $\gamma \in G$ is the one dimensional complex vector space $\Omega_x^{1/2} \otimes \Omega_y^{1/2}$ where $\gamma : x \to y$ and $\Omega_x^{1/2}$ is the one dimensional complex vector space of maps from the exterior power $\wedge^k F_x$, $k = \dim F$, to $\mathbb{C}$

such that :

$$\rho(\lambda v) = |\lambda|^{1/2}\rho(v) \qquad \forall v \in \Lambda^k F_x\ ,\ \forall \lambda \in \mathbb{R}$$

Of course the bundle $(\Omega_x^{1/2})_{x\in V}$ is trivial, and we could Choose once and for all (as in [7]) a trivialisation ν making elements of $C_c^\infty(G,\Omega^{1/2})$ into functions, let us however stress how canonical the algebraic operation is. For $f, g \in C_c^\infty(G,\Omega^{1/2})$, the convolution product $f * g$ is defined by the equality :

$$(f*g)(\gamma) = \int_{\gamma_1\circ\gamma_2=\gamma} f(\gamma_1)\, g(\gamma_2)$$

This makes sense because for fixed $\gamma : x \to y$, fixing $v_x \in \Lambda^k F_x$, $v_y \in \Lambda^k F_y$, the product $f(\gamma_1)\, g(\gamma_1^{-1}\gamma)$ defines a 1-density on $G^y = \{\gamma_1 \in G,\ r(\gamma_1) = y\}$, which is smooth with compact support (it vanishes if $\gamma_1 \notin$ support f), and hence can be integrated over G^y to give a scalar : $f*g\,(\gamma)$ evaluated on v_x, v_y. One has to check that $f*g$ is still smooth with compact support, we shall check that in detail when G is not Hausdorff later.

The $*$ operation is defined by $f^*(\gamma) = \overline{f(\gamma^{-1})}$, i.e. if $\gamma : x \to y$ and $v_x \in \Lambda^k F_x$, $v_y \in \Lambda^k F_y$ then $f^*(\gamma)$ evaluated on v_x, v_y is equal to $\overline{f(\gamma^{-1})}$ evaluated on v_y, v_x. We thus get a $*$ algebra $C_c^\infty(G,\Omega^{1/2})$. For each leaf L of (V,F) one has a natural representation of this $*$ algebra on the L^2 space of the holonomy covering $\tilde{L}$ of L. Fixing a base point $x \in L$, one identifies $\tilde{L}$ with $G_x = \{\gamma\ ,\ s(\gamma) = x\}$ and defines :

$$(\pi_x(f)\,\xi)(\gamma) = \int_{\gamma_1\circ\gamma_2=\gamma} f(\gamma_1)\,\xi(\gamma_2) \qquad \forall \xi \in L^2(G_x)$$

where ξ is a square integrable half density on G_x. Given $\gamma : x \to y$ one has a natural isometry of $L^2(G_x)$ on $L^2(G_y)$ which transforms the representation π_x in π_y.

DEFINITION. $C^*(V,F)$ is the C^* algebra, completion of $C_c^\infty(G,\Omega^{1/2})$ with the norm $\| f \| = \sup_{x\in V} \| \pi_x(f) \|$.

(cf. Section 6)

If the leaf L has trivial holonomy then the representation π_x, $x\in L$, is irreducible, in general its commutant is generated by the action of the (discrete) holonomy group G_x^x in $L^2(G_x)$. If the foliation comes from a submersion $p : V \to B$, then its graph G is $\{ (x,y) \in V\times V , p(x) = p(y)\}$ which is a submanifold of $V\times V$, and $C^*(V,F)$ is identical with the algebra of the continuous field of hilbert spaces $L^2(p^{-1}\{x\})_{x\in B}$. Thus (unless $\dim F = 0$) it is isomorphic to the tensor product of $C_0(B)$ with the elementary C^* algebra of compact operators. If the foliation comes from an action of a Lie group H in such a way that the graph is identical with $V\times H$ (this is not always true even for flows, cf. Section 6), then $C^*(V,F)$ is identical with the reduced crossed product of $C_0(V)$ by H. Moreover the construction of $C^*(V,F)$ is local in the following sense :

LEMMA. If $V'\subset V$ is an open set and F' is the restriction of F to V', then the graph G' of (V',F') is an open set in the graph G of (V,F), and the inclusion $C_c^\infty(G',\Omega^{1/2})\subset C_c^\infty(G,\Omega^{1/2})$ extends to isometric $*$ homomorphism of $C^*(V',F')$ in $C^*(V,F)$.

The proof is straightforward (cf. [7]) and also applies in the case of non hausdorff graph .

We now define the analytical K groups :

DEFINITION. Let (V_1, F_1) , (V_2, F_2) be two foliations, then we put $KK(V_1/F_1 , V_2/F_2) = KK(C^*(V_1, F_1), C^*(V_2, F_2))$, the Kasparov group of this pair of C^* algebras.

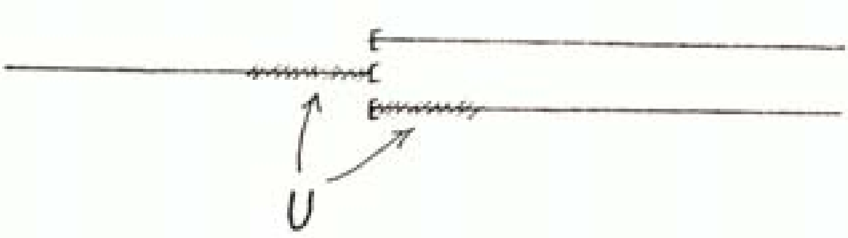
Note that $C^*(V,F)$ is always norm separable. Of course $K^*(V/F) = KK(pt, V/F)$ coincides with the denumerable abelian group obtained by the stable comparison of projections in $M_n(C^*(\tilde{V},F))$ (its odd component involving homotopy classes of unitaries), however it would be narrow minded not to introduce the other groups such as : $K_*(V/F) = KK(V/F , pt)$ which, through the existence of pairings with $K^*(V/F)$ will be useful tools in proving the non triviality of elements of $K^*(V/F)$.

As a minor point we want to relax a little bit the definition of the basic objects "abstract elliptic operators" of the Kasparov group in allowing the operator (F, in the notation of Kasparov ([18])) to be unbounded, then the F of Kasparov is related to D by a formula like $F = D(1+D^*D)^{-1/2}$.

6. $C^*(V,F)$ WHEN THE GRAPH IS NOT HAUSDORFF.

In this section we shall show how the construction of the C^* algebra $C^*(V,F)$ has to be done in the case when the graph of the foliation is not hausdorff. This case is rather rare, since it never occurs if the foliation is real analytic. However, it does occur in cases which are topologically of interest, for foliations, such as the Reeb foliation of the 3 sphere, which are constructed by patching together foliations of manifolds with boundaries (V_i, F_i) where the boundary ∂V_i is a leaf of F_i. In fact most of the constructions done in geometry to produce smooth foliations of given codimension on a given manifold give a non hausdorff graph. We shall then see in a simple example how the $C^*(V_i, F_i)$ are glued together to give $C^*(V,F)$.

Now, let G be the graph of (V,F), being non hausdorff it may have only very few continuous fonctions with compact support, however being a manifold we can give a local chart $U \xrightarrow{\chi} \mathbb{R}^{\dim G}$ take a smooth function $\varphi \in C_c^\infty(\mathbb{R}^{\dim G})$, $\mathrm{Supp}\,\varphi \subset \chi(U)$ and consider the function on G equal to $\varphi \circ \chi$ on U and to 0 outside U. If G was hausdorff this would generate all of $C_c^\infty(G)$ by taking linear combinations, and in general we take this linear span as the <u>definition</u> of $C_c^\infty(G)$. Note that we do not get <u>continuous</u> functions, since there may well be a sequence $\gamma_n \in U$ with two limits, one in $\mathrm{Supp}\,\varphi \circ \chi$ one in the complement of U, as is illustrated in the simplest case :

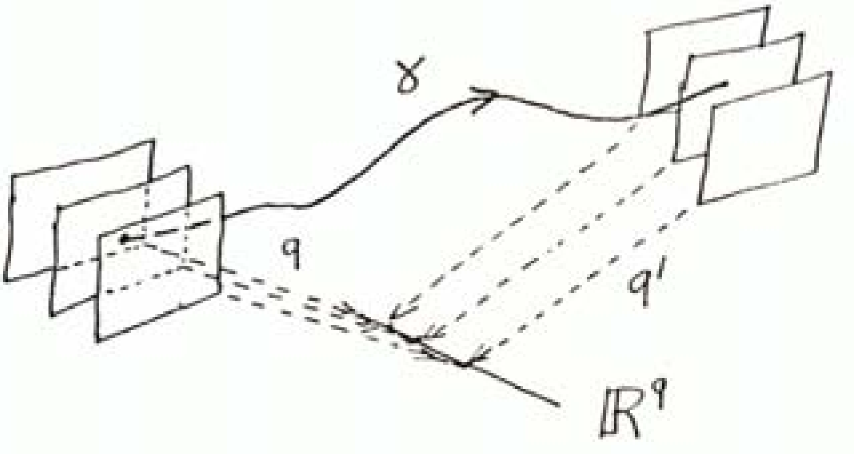


Note also that with the above notations, $\chi^{-1}(\mathrm{Supp}\,\varphi)$ is a compact subset of G but is not necessarily closed.

The above definition of $C_c^\infty(G)$ obviously extends to get $C_c^\infty(G, \Omega^{1/2})$ the space of smooth $\frac{1}{2}$ functions on G . It is not true that the pointwise product of two elements of $C_c^\infty(G)$ is in $C_c^\infty(G)$, it is true however that if φ is a smooth function on V and $\Psi \in C_c^\infty(G, \Omega^{1/2})$ then $(\varphi \circ s)\Psi \in C_c^\infty(G, \Omega^{1/2})$, as is easily seen by writing $\Psi = \sum \Psi_j$, Ψ_j of the form $\varphi_j \circ \chi_j$.

LEMMA. The convolution $\Psi_1 * \Psi_2$ of $\Psi_1, \Psi_2 \in C_c^\infty(G, \Omega^{1/2})$ is in $C_c^\infty(G, \Omega^{1/2})$.

Proof. Let $\Psi \in C_c^\infty(G, \Omega^{1/2})$ and $(W_i)_{i=1,\dots,n}$ an open covering of its support by domains of local charts, then we can find Ψ_i, $i = 1,\dots,n$, $\Psi_i \in C_c^\infty(G, \Omega^{1/2})$ of the form $\varphi_i \circ \chi_i$ so that $\Psi = \sum \Psi_i$. Thus here we can assume that for $j = 1,2$, Ψ_j is of the form $\varphi_j \circ \chi_j$ where χ_j is a local chart in G of the following form : we are given submersions q_j , q'_j from open sets $\mathrm{Dom}\, q_j$, $\mathrm{Dom}\, q'_j$ in V to $\mathbb{R}^q$ which define the restriction of F to their domains (cf. condition b) defining a foliation) and $W(q'_j, q_j)$ is the set of all paths $\gamma \in G$ with $s(\gamma) \in \mathrm{Dom}\, q_j$ $r(\gamma) \in \mathrm{Dom}\, q'_j$,



whose holonomy is the identity with the identification of transversals around $s(\gamma)$, $r(\gamma)$ with an open set of $\mathbb{R}^q$ by q_j, q'_j. Let $Q_1 = s$ (support Ψ_1), $Q_2 = s$ (Support Ψ_2), then for each $x \in Q_1 \cap Q_2$ there is an open set $\Omega \subset$ Domain $q_1 \cap$ Domain q'_2 and a local diffeomorphism θ of $\mathbb{R}^q$ such that, on Ω one has $q'_2 = \theta \circ q_1$. Thus using a smooth partition of unity on $Q_1 \cap Q_2$ in V, and replacing Ψ_1 by $\Psi_1\ \varphi \circ s$, $\varphi \in C_c^\infty(V)$ we may assume that $Q_1 \cap Q_2 \subset \Omega$ and that $q'_2 = q_1$ on Ω. Then taking obvious local coordinates we get for $\Psi_1 * \Psi_2(\gamma)$ the expression :

$$(\Psi_1 * \Psi_2)(t'', t, s) = \int \Psi_1(t'', t', s)\, \Psi_2(t', t, s)\, dt'$$

which obviously defines a smooth function with compact support in the variables (t'', t, s).

Then we proceed exactly as in the hausdorff case, and construct the representation π_x of the $*$ algebra $C_c^\infty(G, \Omega^{1/2})$ in the hilbert space $L^2(G_x)$. We note that though G is not hausdorff, each G_x, being the holonomy covering of the leaf through x is <u>hausdorff</u>.

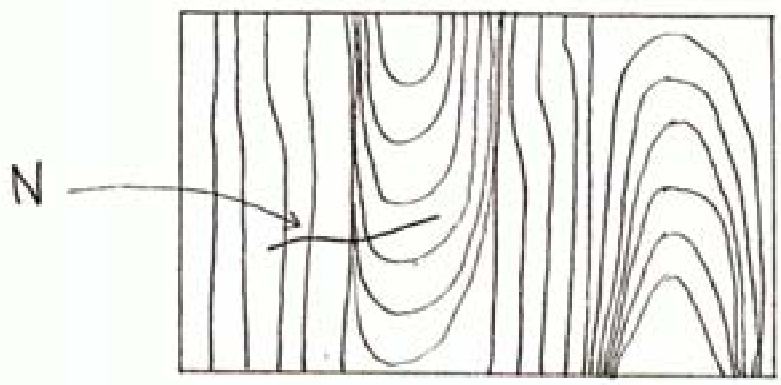
LEMMA. For each $\Psi \in C_c^\infty(G, \Omega^{1/2})$ and $x \in V$, $\pi_x(\Psi)$ is an ordinary smoothing operator, bounded in $L^2(G_x)$.

<u>Proof</u>. Again we can assume that Ψ is of the form $\varphi \circ \chi$ where χ is a local chart in a $W(q', q)$ as above. Now $(\pi_x(\Psi)\xi)(\gamma) = \int_{\gamma_1 \circ \gamma_2 = \gamma} \Psi(\gamma_1)\xi(\gamma_2)$ $\forall \xi \in L^2(G_x)$, thus $\pi_x(\Psi)$ is defined by the kernel $k(\gamma_1, \gamma_2) = \Psi(\gamma_1\gamma_2^{-1})$; $\gamma_1, \gamma_2 \in G_x$. This kernel is invariant by the action of the holonomy group G_x^x (at x) on the holonomy covering and it is equal to 0 unless γ_1, γ_2 are in associated connected components of $r^{-1}(\mathrm{Dom}\, q')$, $r^{-1}(\mathrm{Dom}\, q)$.

so that it is a sum $k = \Sigma k_\alpha$ where the k_α are smooth kernels with compact supports $\mathrm{Supp}\, k_\alpha \subset K'_\alpha \times K_\alpha$ where the K_α (resp. K'_α) are pairwise disjoint. Thus k is a smooth kernel and its L^2 norm is the supremum of the L^2 norms of the k_α which is finite say by the usual hilbert-schmidt estimate. Q.E.D.

Of course $C^*(V,F)$ is defined as the C^* completion of $C_c^\infty(G, \Omega^{1/2})$ with norm : $\sup_{x \in V} \| \pi_x(\Psi) \|$.

To get some feeling of what is different in the non hausdorff situation, we shall discuss the simplest example of a foliation with non hausdorff graph and see what $C^*(V,F)$ looks like. We take $V = \mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ the two torus, together with the following foliation :

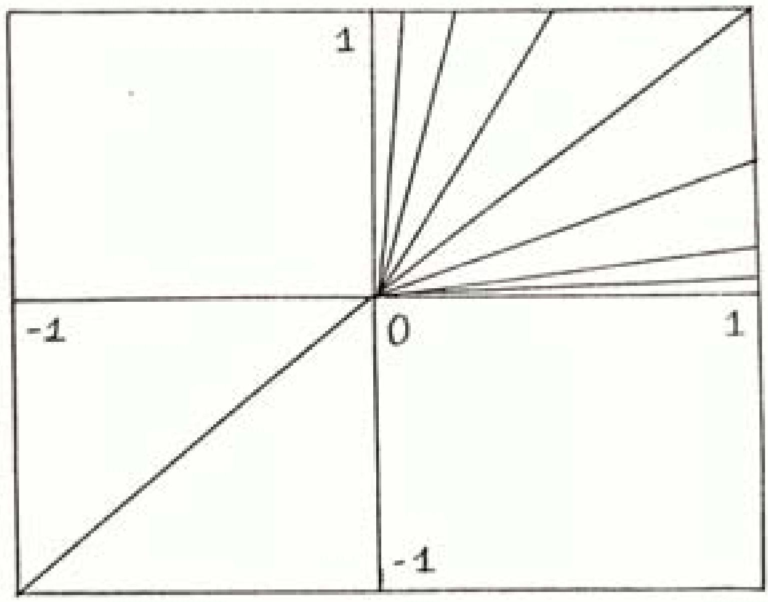


To simplify even further we take the open transversal N (which we identify with $]-1,1[$) as indicated on the figure. We choose it so that 0 is on the leaf with holonomy and so that on N two points a,b are on the same leaf of the foliation iff :

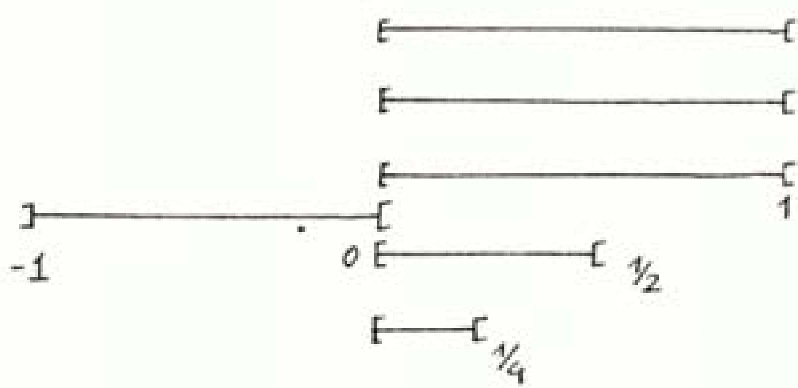
1) If $a \leq 0$ or $b \leq 0$ then $a = b$

2) If $a > 0$, $b > 0$ then $a/b = 2^n$ for some $n \in \mathbb{Z}$.

Thus this equivalence relation is the following subset of the square $(]-1,1[)^2$.



It is not a manifold because of the singularity at 0 , and the corresponding graph G is easily identified as the quotient of : $\{(t,k) \in]-1,1[\times \mathbb{Z} , t<2^k\}$ by the relation which identifies all points (t,n) , $n \in \mathbb{Z}$, if $t<0$. It is obviously a non hausdorff 1-manifold :



All the points $(0,k)$, $k \in \mathbb{Z}$ are in the closure of $]-1,0[$. Since we have cut down to a transversal, we do no longer have to talk about $\frac{1}{2}$ functions but of functions on G , belonging to $C_c^\infty(G)$, i.e. which are finite sums of $\varphi \circ \chi'$s where $\varphi \in C_c^\infty(\mathrm{Im}\,\chi)$ and χ is a local chart. We have :

LEMMA. A function Ψ on G belongs to $C_c^\infty(G)$ iff it is zero on all but a finite number of the intervals J_k, is smooth on each of them and the function $\tilde{\Psi}(t) = \sum_{k\in\mathbb{Z}} \Psi(t,k)$ is smooth on $]-1,1[$.

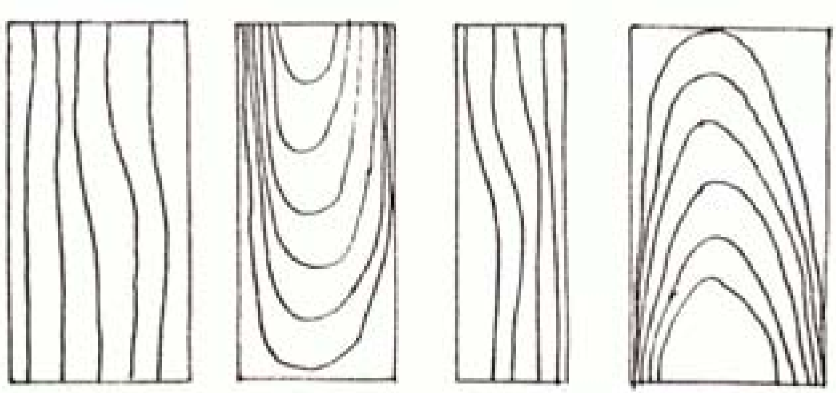
<u>Proof</u>. The condition is satisfied by any $\varphi\circ\chi$ and is stable under linear combinations, also if Ψ satisfies it, then we can extend Ψ_k, $\Psi_k(t) = \Psi(t,k)$, $k\in\mathbb{Z}$, $t\geq 0$ to $]-1,1[$ and express Ψ as a finite linear combination of $\varphi\circ\chi$'s. Q.E.D.

It is obvious that this condition is not stable under pointwise product, and however that it is stable under the convolution product :

$$\Psi_1 * \Psi_2(\gamma) = \sum_{\gamma_1\circ\gamma_2=\gamma} \Psi_1(\gamma_1)\,\Psi_2(\gamma_2)$$

For instance $\Psi_1 * \Psi_2(0,n) = \sum \Psi_1(0,j)\,\Psi_2(0,n-j)$ and hence $(\Psi_1 * \Psi_2)^\sim(0) = \tilde{\Psi}_1(0)\,\tilde{\Psi}_2(0)$, which shows why $(\Psi_1 * \Psi_2)^\sim$ is still continuous at 0.

One can reconstruct $C^*(V,F)$ in our example as a fibered product $\prod C^*(V_i,F_i)$ of the C^* algebras of the foliations (V_i,F_i) of the manifolds with boundaries the compact leaves with holonomy:



V_1 $\quad$ V_2 $\quad$ V_3 $\quad$ V_4

We end up this section with two interesting questions :

1) Compute the K theory $K^*(V/F)$ in the above example as a function of the integer n which is the number of Reeb components separated by stable compact leaves, show in particular that while the foliation is defined by a flow and the rank of $K^*(V)$ is 2+2 that of $K^*(V/F)$ can be very large (thus there is no Thom isomorphism in this case).

2) Compute the extension of C^* algebras obtained for a foliated manifold with boundary (V,F) with ∂V a leaf of F, by taking the ideal in $C^*(V,F)$ of those elements which vanish on the boundary. (This is very interesting even in the simplest case of the full 2 torus in $\mathbb{R}^3$ with the Reeb foliation).

7. C^* MODULES OVER $C^*(V,F)$ AND CONTINUOUS FIELDS OF HILBERT SPACES ON V/F.

Let X be a locally compact space, the notion of continuous field of hilbert spaces on X extends that of hermitian complex vector bundle in a way which not only allows the fibers to be infinite dimensional but also allows the fibers to vary in a less trivial manner. For instance if Y is an open set in X and E_Y is a complex hermitian vector bundle on Y it is not true in general that E_Y is the restriction to Y of a vector bundle on X, however, viewed as a continuous field of hilbert spaces on Y, E_Y extends canonically to a continuous field of hilbert spaces on X, simply by taking all fibers to be 0 in the complement of Y. So if one starts with the usual definition ([1]) of K theory with compact supports, from complexes with compact supports the above pushforward operation from $K^*(Y)$ to $K^*(X)$ though well defined by using trivialisation at infinity in Y is not easy to use since the complex first has to be deformed. In [18] Kasparov has given for arbitrary non unital C^* algebras B (in the above context $B = C_0(X)$ is the C^* algebra of continuous functions vanishing at ∞) a construction of the group $K_*(B) = K^*(X)$ which is well adapted to the pushforward operation. For $B = C_0(X)$ the basic triples (E_1, E_2, T) are given by two continuous fields of hilbert spaces E_1, E_2 on X together with $T \in \mathrm{Hom}(E_1, E_2)$ (i.e. for each $x \in X$, $T_x \in \mathrm{Hom}(E_{1,x}, E_{2,x})$, and if ξ is a continuous section of E_1 tending to 0 at ∞ then $T\xi : (T\xi)_x = T_x \xi_x$ has the same property in E_2) which is invertible modulo the "compact" elements of End (E_i) $i=1,2$ ([18]). It follows then that T_x is an isomorphism outside a compact subset of X and is everywhere a Fredholm operator. The equivalence relation one has to put on such triples to obtain $K^*(X)$ is exactly analogue to the equivalence used for theory with compact supports, namely the degenerate triples, with everywhere an isomorphism, are considered as equivalent to 0, and two homotopic triples are considered as equivalent.

Now let (V, F) be a compact foliated manifold, then $B = C^*(V, F)$ is not unital (unless $\dim F = 0$) and in describing the denumerable abelian group $K^*(V/F) = K_*(C^*(V,F))$ we shall use the point of view of Kasparov, i.e. his refined version of K-theory with compact supports. (It should be stressed at this point that, of course, the definition obtained by adjoining a unit to B and then taking the kernel of the augmentation $\varepsilon_B : K_*(\tilde{B}) \mapsto \mathbb{Z}$ gives the same result, but will not be as useful).

The basic triples are now of the form $(\mathcal{E}_1, \mathcal{E}_2, T)$ where $\mathcal{E}_1, \mathcal{E}_2$ are C^* modules over B, while $T \in \mathrm{Hom}_B(\mathcal{E}_1, \mathcal{E}_2)$ is invertible modulo the "compact" elements of $\mathrm{End}_B(\mathcal{E}_i), i = 1,2$. in the case $B = C_o(X)$ there was an easy correspondence between continuous fields of hilbert spaces on X and C^* modules over $C_o(X)$, which to each field E associates the module $\mathcal{E} = C_o(E)$ of continuous sections of E vanishing at ∞ .

In our case : $B = C^*(V, F)$, we should successively describe the C^* modules over B, the homomorphisms and "compact" endomorphisms of such modules. With the material of [7] Section IV, this will be mainly a matter of translation.

In the case $B = C_o(X)$, the basic data for a continuous field of hilbert spaces on X is a (measurable) field $(H_x)_{x \in X}$ of hilbert spaces on X, together with a linear space Γ of sections $(\xi_x)_{x \in X}$ whose main property is :

(*) $\forall \xi, \eta \in \Gamma$ the function $x \to \langle \xi_x, \eta_x \rangle$ is continuous.

Giving, in a measurable way, one hilbert space per leaf means giving a measurable field $(H_x)_{x \in V}$ of hilbert spaces on V, and for each element γ of the graph G of (V, F), $\gamma : x \to y$ a unitary operator which identifies H_x with H_y and defines a representation of G in the obvious sense. It will be convenient to consider G as acting on the right on H, and to denote the above identification of H_x with H_y by : $\xi \in H_y \mapsto \xi.\gamma \in H_x$.

By a measurable $\frac{1}{2}$ section of H we mean a measurable section of the measurable field $(\Omega_x^{1/2} \otimes H_x)_{x \in V}$, where $\Omega_x^{1/2}$ is the bundle of half densities used already in the construction of $C^*(V,F)$.

Let ξ, η be measurable $\frac{1}{2}$ sections of H, then the coefficient (ξ, η) is the $\frac{1}{2}$ fonction on G defined by

$$(\xi,\eta)(\gamma) = \langle \xi_y . \gamma , \eta_x \rangle$$

(We took the scalar product in H_x antilinear in the first variable and linear in the second).

Given a measurable $\frac{1}{2}$ section ξ of H one defines $\| \xi \|_\infty \in [0, +\infty]$ as the smallest $c \in [0, +\infty]$ with :

$$\int_{G_x} |\langle \xi_y . \gamma , \eta \rangle|^2 \leq c^2 \|\eta\|^2 \quad \forall \eta \in H_x , x \in V.$$

If $\| \xi \|_\infty < \infty$, then ξ defines an intertwining operator

$$T(\xi) \in \mathrm{Hom}_G (H_x , L^2(G_x))$$

by the formula :

$$(T(\xi)\eta)(\gamma) = \langle \xi\circ\gamma , \eta \rangle \quad \forall \eta \in H_x \quad \forall \gamma \in G_x .$$

The representation H of G is square integrable iff the subspace $\{\xi , \|\xi\|_\infty < \infty\}$ is large enough to contain a countable total subset (i.e. total in each H_x).

The basic data to construct a C^* module over $C^*(V,F)$, is the following : a (square integrable) representation H of G and a linear space Γ of $\frac{1}{2}$ sections of H such that :

(*) $\forall \xi, \eta \in \Gamma$, the coefficient (ξ,η) belongs to $C^*(V,F)$.

(The coefficient (ξ,η) is given as a $\frac{1}{2}$ function on G and it makes sense to say that it belongs to the norm closure $C^*(V,F)$ of $C_c^\infty(G,\Omega^{1/2})$.

LEMMA. One has, for $\xi \in \Gamma$, $\|\xi\|_\infty = \|(\xi,\xi)\|^{1/2}$.

<u>Proof</u>. $\|(\xi,\xi)\| = \mathrm{Sup}^{\cdot}\, \pi_x(\xi,\xi)$ and a straightforward computation ([7] Prop. 8 p. 77) shows that :

$$\pi_x(\xi,\xi) = T(\xi)T(\xi)^* \qquad \text{Q.E.D.}$$

Now given $\xi \in \Gamma$ and $f \in C^*(V,F)$ we would like to define a $\frac{1}{2}$ section $\xi * f$ of H by the formula :

$$(\xi * f)(x) = \int_{G_x} \xi\circ\gamma \; f(\gamma)$$

However, for $f \in C^*(V,F)$, $f(\gamma)$ does not make sense in general, so we let $C^*(V,F)_2$ be the left ideal of those f for which the restriction ξ_x to G_x makes sense and is in $L^2(G_x)$ (it is a left ideal since $(g*\xi)_x = \pi_x(g)\xi_x$, $\forall \xi, g$.

We then rewrite the above formula as follows :

$$(\xi * \xi)(x) = T(\xi)^* \xi_x \qquad \forall x \in V$$

So $\xi * \xi$ is a $\frac{1}{2}$ section of H for any $\xi \in C^*(V,F)_2$, and we construct the C^* module $\mathcal{E}$ as the completion of the linear span of the $\xi * \xi$, $\xi \in \Gamma$, $\xi \in C^*(V,F)_2$, with the norm $\| \ \|_\infty$.

LEMMA. $\mathcal{E}$ is a C^* module over $C^*(V,F)$.

<u>Proof</u>. The equality $(\xi * \xi, \eta * g) = \xi^* * (\xi, \eta) * g$ shows that for ξ, η in the linear span Γ' of the $\xi * \xi$ one has $(\xi, \eta) \in C^*(V,F)$ and $\| \xi \|_\infty < \infty$. The map $\xi, \eta \to \xi \times \eta$ extends to $\mathcal{E} \times \mathcal{E}$ by continuity and satisfies conditions 1°) 3°) 4°) of [19] Def. 1. The map $\xi, \xi \mapsto \xi * \xi$ extends to $\mathcal{E} \times C^*(V,F)$ by continuity and 2° of [19] def.1 holds. Q.E.D.

We are only interested in C^* modules which are separable, as Banach spaces, or equivalently (since $B = C^*(V,F)$ is norm separable) which are countably generated in the sense of [19] . The above construction yields such a C^* module when Γ has a dense countable subset (in the $\| \ \|_\infty$ norm).

Exactly as in the case of continuous field of hilbert spaces on a locally compact space, we now define arbitrary continuous $\frac{1}{2}$ sections of H as those measurable $\frac{1}{2}$ sections which belong to $\mathcal{E}$. It is not true in general that all elements of $\mathcal{E}$ define $\frac{1}{2}$ sections of H , this fails in the simplest example, taking $H_x = L^2(G_x), \forall x \in V$, $(\xi \circ \gamma)(\gamma_1) = \xi(\gamma_1 \gamma^{-1}) \quad \forall \gamma_1 \in G_x$, one gets $\mathcal{E} = C^*(V,F)$ and not all its elements define $\frac{1}{2}$ sections of H .

The next lemma shows however how to pass from elements of $\mathcal{E}$ to $\frac{1}{2}$ sections of H :

LEMMA. For any $\xi \in \mathcal{E}$ and $g \in C^*(V,F)_2$, the element $\xi * g$ of $\mathcal{E}$ defines a $\frac{1}{2}$ section of H .

Proof. Let η be a continuous $\frac{1}{2}$ section of H (in the above sense) then $(\eta * g)_x = T(\eta)^* g_x$, $\|(\eta * g)_x\| \leq \|\eta\|_\infty \|g_x\|$. (this makes sense by evaluating both sides on a volume element v, $v \in \Lambda^{\dim F} F_x$). By construction of $\mathcal{E}$ we can approximate $\xi \in \mathcal{E}$ by continuous $\frac{1}{2}$ sections η_m of H in the $\| \quad \|_\infty$ norm and it follows that the $(\eta_m * g)_x$ will converge in norm. Q.E.D.

Definition. A continuous field of hilbert spaces $(H_L)_{L \in V/F}$ on the space of leaves of the foliation (V,F) is given by a representation : $\xi \in H_y \mapsto \xi \circ \gamma \in H_x$ of the graph of (V,F) and a linear space Γ of $\frac{1}{2}$ sections of H such that :

1°) Γ has a countable total subset, $\| \quad \|_\infty$ dense.

2°) $\forall \xi, \eta \in \Gamma$ one has $(\xi, \eta) \in C^*(V,F)$.

3°) Γ is closed in the $\| \quad \|_\infty$ norm.

4°) For $g \in C^*(V,F)_2$, $\xi \in \Gamma$ one has $\xi * g \in \Gamma$.

In condition 4) one can replace $C^*(V,F)_2$ by any subset dense in the C^* norm, since by hypothesis Γ is closed in the $\| \quad \|_\infty$ norm.

Thus it is enough to check 4°) for $f \in C_c^\infty(G, \Omega^{1/2})$. Conditions 1) 2) show that the representation of G in H is square integrable (cf. [7] part IV).

We now pass to the description of the endomorphisms of the module $\mathcal{E}$ associated to a continuous field of hilbert spaces on the space of leaves.

Our aim to describe $\mathrm{End}(\mathcal{E})$ (or more generally $\mathrm{Hom}(\mathcal{E}_1, \mathcal{E}_2)$) as the random operators $T \in \mathrm{End}_G(H)$ (i.e. measurable bounded families $(T_x)_{x \in V}$, with T_x acting in H_x and $T_x(\xi \circ \gamma) = (T_y \xi) \circ \gamma$, $\forall\, \xi \in H_y$ (cf [7])) which are continuous in the sense that : <u>If ξ is a continuous $\frac{1}{2}$ section of H then $T\xi$ and $T^*\xi$ also.</u>

Now, in [7] we introduced the random operator $\theta(\xi, \eta)$ associated to a pair of sections, this with our change of notations translates as follows : let $\xi, \eta \in \Gamma$ be continuous $\frac{1}{2}$ sections of H, then $\theta(\xi, \eta) = T(\xi)^* T(\eta)$ defines a continuous random operator, and the obvious extension of this operator to $\mathcal{E}$ is given by :

$$\alpha \mapsto \eta(\xi, \alpha) \qquad \forall \alpha \in \mathcal{E}$$

(If T is a continuous random operator, it extends from continuous $\frac{1}{2}$ sections of H to arbitrary elements of $\mathcal{E}$ by the inequality $\| T\xi \|_\infty \leq \sup_V \| T_x \| \, \| \xi \|_\infty$ ([7] Prof. 8c)) and the existence of T^* shows that $\xi \mapsto T\xi$ is an endomorphism of $\mathcal{E}$ in the sense of [19]).

By construction the endomorphism of $\mathcal{E}$ defined by $\theta(\xi, \eta)$ agrees with the $\theta_{\xi,\eta}$ of Kasparov ([19]), when we close in norm the linear span of the $\theta(\xi,\eta)$, $\xi,\eta\in\Gamma$ we get the same result as when we close in norm the $\theta_{\xi,\eta}$, $\xi,\eta\in\mathcal{E}$, since by construction Γ is norm dense in $\mathcal{E}$. (Note that if T is a continuous random operator, its norm as an endomorphism of $\mathcal{E}$ is the same as $\sup_{x\in V}\|T_x\|$, since both norms are C^* norms).

The norm closure of the linear span of the $\theta(\xi,\eta)$ will be denoted by $\mathrm{End}_o(\mathcal{E})$. By [19] thm 1, $\mathrm{End}(\mathcal{E})$ is the algebra of multipliers of $\mathrm{End}_o(\mathcal{E})$.

PROPOSITION. $\mathrm{End}\,\mathcal{E}$ is canonically isomorphic to the C^* algebra of continuous random operators $\mathrm{End}_G(H)$.

Proof. We have already seen that every continuous random operator : $T\in\mathrm{End}_G(H)$ defines an endomorphism of $\mathcal{E}$ with the same norm. Conversely, given an endomorphism T of $\mathcal{E}$ we want to find the family $(T_x)_{x\in V}$, $T_x\in\mathrm{End}\,H_x$ with $(T\xi)_x = T_x\xi_x$, $\forall\xi\in\Gamma, x\in V$ But by [19] thm. 1, T is a limit in the multiplier topology of a sequence $T_n\in\mathrm{End}_o(\mathcal{E})$, with $\|T_n\|\leqq\|T\|$. For each T_n the existence of the $T_{n,x}$ is proven by construction, thus it remains only to show that for fixed $x\in V$, the sequence $(T_{n,x})_{n\in\mathbb{N}}$ converges strongly in H_x to an operator T_x (It is then obvious that the family $(T_x)_{x\in V}$ is measurable and that the associated endomorphism is equal to T). So the next lemma ends the proof :

LEMMA. If (T_n) is a sequence of continuous random operators which converges in the multiplier topology of $\mathrm{End}(\mathcal{E})$, then $T_{n,x}$ converges strongly for all $x\in V$.

<u>Proof.</u> Let $S \in \mathrm{End}_o(\mathcal{E})$, $S = (S_x)_{x \in V}$, then by hypothesis $T_n S \to TS$ in norm, thus $T_{n,x} S_x \to T_x S_x$ in norm. As $\|T_{n,x}\|$ is bounded, it remains only to show that the union of the ranges of the S_x, $S \in \mathrm{End}_o(\mathcal{E})$ is total in H_x, but this follows from condition 1°) on H. Q.E.D.

THEOREM. Let $\mathcal{E}$ be a C^* module over $C^*(V,F)$, then there exists a canonical continuous field of hilbert spaces $(H_L)_{L \in V/F}$ on the space of leaves and a canonical isomorphism of $\mathcal{E}$ with the C^* module associated to H.

<u>Proof.</u> We shall first define $(H_x)_{x \in V}$ in a canonical way from $\mathcal{E}$. We consider in $\mathcal{E}$ the linear span $\mathcal{E}_o$ of the $\xi * f$, $\xi \in \mathcal{E}$, $f \in C^*(V,F)_2$ and on $\mathcal{E}_o$, given $x \in V$ we define a scalar product by :

$$<\xi * f, \eta * g>_x = < f_x, \pi_x(\xi,\eta) g_x> \qquad (*)$$

(Thus it is antilinear in the first variable).

Since $\pi_x(\xi,\eta)(h.\gamma) = (\pi_y(\xi,\eta)h).\gamma \quad \forall h \in L^2(G_y)$ we see that there exists, given $\gamma \in G$, a unique isometry $\xi \to \xi \circ \gamma$ of H_y on H_x (the completions of $\mathcal{E}_o$ with the $< \; >_x$ scalar product) such that, for any $\xi \in \mathcal{E}$ and $f_1, f_2 \in C^*(V,F)_2$ with $f_{2,y} \circ \gamma = f_{1,x}$, one has :

$$(\xi * f_2)_y \circ \gamma = (\xi * f_1)_x$$

(Here $\xi \to \xi_x \in H_x$ is the canonical (*) map of $\mathcal{E}_o$ in H_x). With $\xi * f$ and $\eta * g$ as above, we have for $\gamma \in G$, $\gamma : x \mapsto y$,

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(*) Note that, to do that, one needs to choose a volume element $v_x \in \Lambda^{\dim F} F_x$.

$<(\xi * f)_y \cdot \gamma , (\eta * g)_x> = < f_y \cdot \gamma , \pi_x(\xi,\eta) g_x>$
$= (f^* * (\xi,\eta) * g)\ (\gamma)$. So the space of $\frac{1}{2}$ sections of H given by $(\xi_x)_{x \in V}$ for $\xi \in \mathcal{E}_0$ satisfies condition 2°) of the definition of continuous fields of hilbert spaces on V/F . Taking Γ as the closure of the above space in the $\| \ \|_\infty$ norm, we check directly 1) and 3). To check condition 4) one has to show that for $f, g \in C^*(V,F)_2$ $x \in V$, and $\xi \in \mathcal{E}$ one has :

$$\int_{G_x} (\xi * f)_y \cdot \gamma \ g(\gamma) = (\xi * (f * g))_x$$

which easily follows from the definition of the action of G on H . Finally for f, g, ξ, η as above we got that the coefficient $(\xi * f, \eta * g)$ is equal to the $C^*(V,F)$ valued scalar product of these elements, this is enough to show that the C^* module associated with H is canonically isomorphic to $\mathcal{E}$. Q.E.D

In conclusion, we adopt the point of view that giving a C^*module over $C^*(V,F)$ (or equivalently a continuous field of hilbert spaces on V/F) is giving in a continuous manner one hilbert space H_L per leaf of the foliation, to simplify the notation we shall only rarely mention the space Γ of $\frac{1}{2}$ sections, for instance a typical such field is : $(L^2(\tilde{L}))_{L \in V/F}$ where $\tilde{L}$ is the holonomy covering of L .

Given continuous fields of hilbert spaces $(H_L)_{L \in V/F}$, $(H'_L)_{L \in V/F}$ on the space of leaves, and a continuous random operator $(T_L)_{L \in V/F}$, we shall say that T is <u>compact</u> iff the associated morphism of C^* modules is compact in the sense of Kasparov ([19]) i.e. belongs to the norm closure of the $\theta_{\xi,\eta}$.

We shall say that T is a <u>quasi isomorphism</u> iff it is invertible modulo compact random operators. From the definition of $K^*(V/F)$ we get a natural map Ind from quasi isomorphisms T to the group $K^*(V/F)$, which satisfies :

a) $\mathrm{Ind}(T) = 0$ if T is an isomorphism

b) $\mathrm{Ind}(T_1 \oplus T_2) = \mathrm{Ind}(T_1) + \mathrm{Ind}(T_2)$

c) $\mathrm{Ind}(T_1 \circ T_2) = \mathrm{Ind}(T_1) + \mathrm{Ind}(T_2)$

d) If T_1 and T_2 are homotopic then $\mathrm{Ind}(T_1) = \mathrm{Ind}(T_2)$.

e) Any element of $K^*(V/F)$ is of the form $\mathrm{Ind}\,T$ for some quasi isomorphism T.

To end this section we shall just translate the results of the pseudodifferential calculus of [7] to give the definition of the analytical index $\mathrm{Ind}_a(D) \in K^*(V/F)$ of a differential operator on V elliptic along the leaves of the foliation (cf. section 4). First, using an auxiliary Euclidean structure $X \to \|X\|$ on the bundle F, one gets the Laplacian in the leaf direction : $\Delta : C^\infty(V) \mapsto C^\infty(V)$ which is by construction elliptic along the leaves, with principal symbol $\sigma_\Delta(x,\xi) = \|\xi/F\|^2$ (ξ/F is the restriction of ξ to F). The restriction Δ_L of Δ to each leaf L (to the holonomy covering $\tilde{L}$ of L if L has holonomy) defines ([7] Part VI) a positive selfadjoint operator in the hilbert space $L^2(\tilde{L})$. The Sobolev space $W^s(\tilde{L})$ is defined, for $s \geq 0$ as the domain of the operator $(1+\Delta_L)^{s/2}$ anf for $s \leq 0$ as the dual of W^{-s}. We thus get for each $L \in V/F$ a chain $W^s(\tilde{L})$ of hilbert spaces, and for $s' > s$ the random operator $(i_L^{s',s})_{L\in V/F}$ of inclusion of $W^{s'}$ in W^s is compact ([7]). While the hilbert space structure of $W^s(\tilde{L})$ depends on the chosen Euclidean structure, the underlying topological vector space is independent of this choice, moreover if $X \mapsto \|X\|'$ is another choice of Euclidean structure, the natural identification of $W^s(\tilde{L})$ with $W'^s(\tilde{L})$ defines a continuous random operator. All these definitions and properties extend to the sections of a hermitian vector bundle E on V. We use the notation $W^s(\tilde{L},E)$ for the corresponding Sobolev spaces.

PROPOSITION. Let $D : C^\infty(E_1) \mapsto C^\infty(E_2)$ (*) be a differential operator on V elliptic along the leaves of (V,F).

a) For each $s \in \mathbb{R}$, the family $D_L : W^{s+n}(\tilde{L},E_1) \mapsto W^s(\tilde{L},E_2)$ (n = order of D) is a quasi isomorphism.

b) The index $\mathrm{Ind}_a(D)$ of the above quasi isomorphism, is independent of s and only depends on the K-theory class $[\sigma_D] \in K^*(F)$ of the principal symbol of D.

The proof follows from the pseudo differential calculus for families when used in domains of foliation charts (cf. [7]). The analytical index map thus obtained from $K^*(F)$ to $K^*(V/F)$ is of course the same as the one coming from the exact sequence of C^* algebras described in [7] :

$$0 \mapsto C^*(V,F) \to \Psi^* \to C(F_1) \to 0$$

where Ψ^* is the C^* algebra of pseudo differential operators of order 0 along the leaves.

(*) E_1, E_2 are smooth vector bundles on V.

8. THE CANONICAL MAP OF $K^*(N)$ IN $K^*(V/F)$ ASSOCIATED TO A TRANSVERSAL.

Let (V, F) be a foliated manifold, and N a submanifold of V, with $\dim N = \operatorname{Codim} F$, which is transverse to F, i.e. the inclusion map $i : N \mapsto V$ satisfies :

$$i_*(T_x(N)) + F_{i(x)} = T_{i(x)}(V)$$

Our aim in this section is to construct a canonical map $i!$ of $K^*(N)$ in $K^*(V/F)$. This will be done in constructing for each continuous field of hilbert spaces E on N a pushforward $i!\,E$ which is a continuous field of hilbert spaces on the space of leaves. This construction will be functorial (each $T \in \operatorname{Hom}(E_1, E_2)$ yields an $i!(T)$, $i!(T) \in \operatorname{Hom}(i!\,E_1, i!\,E_2)$) compatible with homotopies and will map compact homomorphisms to compact ones, thus yielding a K theory map. Letting L be a leaf of the foliation, and E the given continuous field of hilbert spaces on N, we define the fiber of $i!\,E$ at L as the hilbert space $\ell^2(\tilde{L} \cap N, E)$ where $\tilde{L}$ is the holonomy covering of L. If $T \in \operatorname{Hom}(E_1, E_2)$, then the action of $i!\,T$ on $\xi \in \ell^2(\tilde{L} \cap N, E_1)$ is given by :

$$((i!T)\,\xi)(x) = T_x\,\xi_x \in E_{2,x} \qquad \forall x \in \tilde{L} \cap N$$

This makes $i!$ in a functor and the only thing one has to check is that $i!T$ is compact if T is compact. The details are as follows.

Given E as above, $i!\,E$ corresponds to the representation by right translation of the graph G of (V,F) in the borel field of hilbert spaces $(i!\,E)_x = \ell^2(G_x \cap r^{-1}(N), r^*(E))$ (where $r : G_x \to L$ is the range map, realising the holonomy covering of L). Any ξ,

$\xi \in C_c(G_x \cap r^{-1}(N), r^*(E) \otimes s^*(\Omega^{1/2}))$ (*) defines a $\frac{1}{2}$ section of $i!E$ by restriction to each $G_x \cap r^{-1}(N)$ (Note that r being a submersion $r^{-1}(N)$ is a submanifold of G, if G is not Hausdorff, C_c is taken in the sense of section 6). For ξ as above, the coefficient $(\xi, \xi)(\gamma)$ of the corresponding $\frac{1}{2}$ section of $i!(E)$ is given by :

$$(\xi,\xi)(\gamma) = \sum_{r(\gamma')\in N} < \xi(\gamma' \cdot \gamma^{-1}), \xi(\gamma') >$$

Thus by the same argument as in section 6 Lemma, one checks that $(\xi,\xi) \in C^*(V, F)$. Thus the $\| \ \|_\infty$ closure of the above set of $\frac{1}{2}$ sections of $i!(E)$ satisfies conditions 1) 2) 3) 4) defining a continuous field of hilbert spaces on the space of leaves. To check 4) one notes that if $f \in C_c(G, \Omega^{1/2})$, $\xi \in C_c(G \cap r^{-1}(N), r^*(E) \otimes s^*(\Omega^{1/2}))$ then $\xi * f \in C_c(G \cap r^{-1}(N), r^*(E) \otimes s^*(\Omega^{1/2}))$ where

$$(\xi * f)(\gamma) = \int f(\gamma')\, \xi(\gamma\gamma')$$

Next, if $T \in \mathrm{Hom}(E_1, E_2)$, $i!T$ is given by :

$$((i!T)_x \xi)(\gamma) = T_y\, \xi(\gamma) \quad , \quad \forall \gamma : x \to y \ , \ \xi \in (i!E_1)_x$$

One checks that it is a continuous random operator, in fact it maps C_1 $C_c(G \cap r^{-1}(N), r^*(E) \otimes s^*(\Omega^{1/2}))$ in C_2.

We now check the important point :

LEMMA. If T ($T \in \mathrm{Hom}(E_1, E_2)$) is compact, then $i!T$ also.

<u>Proof</u>. We can assume that $E_1 = E_2$ and that $T = \theta_{\xi,\xi}$ with the

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(*) $\Omega^{1/2}$ is the bundle on V of half densities on F.

notations of Kasparov [19], where ξ is a continuous section with compact support of E on N. Thus $(T_x \xi') = \xi_x < \xi_x, \xi'>$ and for any $\xi' \in (i!\,E)_x$ one has :

$$((i!T)_x \xi')(\gamma) = \xi_y < \xi_y, \xi'(\gamma)> \quad \forall \gamma : x \to y.$$

Now any $\eta \in C_c(G \cap r^{-1}(N), r^*(E) \otimes s^*(\Omega^{1/2}))$ defines a compact random operator on $i!(E)$ by the equality

$$(\theta(\eta,\eta)_x \xi')(\gamma) = \int_{\gamma_1} \sum_{\gamma_2} \eta(\gamma_1) < \eta(\gamma_2 . \gamma^{-1} . \gamma_1), \xi'(\gamma_2)>$$

Let K be the compact support of ξ, $\mathcal{V}$ a neighborhood of $\{(x,x), x \in K\}$ in $r^{-1}(N) \subset G$ so that if $\gamma \in \mathcal{V} . \mathcal{V}^{-1}$ and $s(\gamma) \in N$ then $\gamma \in G^{(0)} = V$. To get such a $\mathcal{V}$ one uses the transversality of $i : N \to V$, with the foliation. Then if η is 0 outside $\mathcal{V}$, we see that, given $y \in N$, $\gamma, \gamma_1 \in G^y$, $\gamma_2 \in r^{-1}(N)$, $s(\gamma_2) = s(\gamma)$ $\eta(\gamma_1) \neq 0$, $\eta(\gamma_2 . \gamma^{-1} . \gamma_1) \neq 0$, $r(\gamma_2 . \gamma^{-1}) \in N$, while $\gamma_2 . \gamma^{-1}$ $= (\gamma_2 . \gamma^{-1} . \gamma_1) . \gamma_1^{-1} \in \mathcal{V} . \mathcal{V}^{-1}$, and $s(\gamma_2 . \gamma^{-1}) = y \in N$ so that $\gamma = \gamma_2$. Thus for such a η the above formula simplifies to :

$$(\theta(\eta,\eta)_x \xi')(\gamma) = \int_{G^y} \eta(\gamma_1) < \eta(\gamma_1), \xi'(\gamma)>$$

Choosing η of the form $\eta(\gamma) = f(\gamma) \xi_{r(\gamma)}$, $r(\gamma) \in N$ where $f \in C_c(G \cap r^{-1}(N), s^*(\Omega^{1/2}))$ one gets the equality $\theta(\eta,\eta) = i!(T)$ provided that f is 0 outside $\mathcal{V}$ and satisfies :

$$\int_{G^y} |f(\gamma_1)|^2 = 1 \qquad \forall y \in K .$$

One constructs such an f using a partition of unity on K.

Q.E.D.

Thus if $T \in \mathrm{Hom}(E_1, E_2)$ is a quasiisomorphism, the same is true for $i!(T)$, and we thus get a well defined map from $K^*(N)$ to $K^*(V/_F)$, the class of T being mapped to the class of $i!(T)$. (It is clear that homotopic T's map to homotopic ones).

To give some reality to this map, we shall take some special cases :

1) Closed transversals and idempotents of $C^*(V, F)$.

Assume that N is a compact transversal to (V, F), then to the trivial bundle with fibers $\mathbb{C}$ on N is associated a K theory element $1_N \in K^*(N)$, thus $i!(1_N)$ is a well defined element of $K^*(V/_F)$. We shall show now that it corresponds to an idempotent $e \in C^*(V, F)$, $[e] = i!(1_N)$. In fact, choose $\xi \in C_c(G \cap r^{-1}(N), s^*(\Omega^{1/2}))$ so that with the notations of the proof of lemma p. 64, ξ is 0 outside $\mathcal{V}$ and $\int_{G^y} |\xi(\gamma_1)|^2 = 1$ $\forall y \in N$.

Then ξ defines a continuous $\frac{1}{2}$ section of $i!(1_N)$ and since $\theta(\xi, \xi) = i!$ (identity) is an idempotent, the same is true for $T(\xi)^* T(\xi) = (\xi, \xi) \in C^*(V, F)$, so that the equality $e(\gamma) = (\xi, \xi)(\gamma) = \sum \bar{\xi}(\gamma' \cdot \gamma^{-1})\, \xi(\gamma')$ determines an idempotent $e \in C^*(V, F)$ with $[e] = i!(1_N)$. Of course foliations can fail to have such a closed transversal N, and we shall show in an example that even $C^*(V, F)$ can fail to have any non zero idempotent. We let Γ be a discrete cocompact subgroup of $SL(2, \mathbb{R})$ $V = SL(2, \mathbb{R})/\Gamma$ has a natural flow H_t, the Horocyle flow defined by the action, by left translations, of the subgroup : $\left\{ \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}, t \in \mathbb{R} \right\}$ of $SL(2, \mathbb{R})$.

We let F be the foliation of V in orbits of the horocycle flow. First the flow is minimal, so, [15][27] $C^*(V,F)$ is a simple C^* algebra, then letting μ be the measure on V associated to the haar measure of $SL(2,\mathbb{R})$; we can associate to μ , which is H_t invariant for all $t \in \mathbb{R}$, a transverse measure Λ for (V,F) and hence a trace τ on $C^*(V,F)$. By simplicity of $C^*(V,F)$ this trace τ is faithful, for any idempotent $e \in C^*(V,F)$ one has :

$$0 < \tau(e) < \infty \qquad \text{if } e \neq 0 \quad .$$

Now let G_s , $s \in \mathbb{R}$, be the geodesic flow on V , defined by the action by left translations of the subgroup $\left\{ \begin{bmatrix} e^s & 0 \\ 0 & e^{-s} \end{bmatrix}, s \in \mathbb{R} \right\}$. For every s , G_s is an automorphism of (V,F) since the equality $G_s H_t G_s^{-1} = H_{e^{-2s}t}$ (which follows from : $\begin{bmatrix} 1 & 0 \\ t e^{-2s} & 1 \end{bmatrix}$ $= \begin{bmatrix} e^s & 0 \\ 0 & e^{s} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} e^{-s} & 0 \\ 0 & e^{s} \end{bmatrix}$) shows that for $y = G_s(x)$, $F_y = (G_s)_* F_x$. Let θ_s be the corresponding automorphism of $C^*(V,F)$. For every $x \in C^*(V,F)$ the map $s \mapsto \theta_s(x)$ from $\mathbb{R}$ to $C^*(V,F)$ is norm continuous, which shows that if e is a self adjoint idempotent, then $\theta_s(e)$ is equivalent to e for all $s \in \mathbb{R}$ and hence $\tau(\theta_s(e)) = \tau(e), \forall s \in \mathbb{R}$.

But though μ is invariant obviously by the geodesic flow the transverse measure Λ is <u>not</u> invariant by G_s , indeed the equality $G_s H_t G_s^{-1} = H_{e^{-2s}t}$ shows that $G_s(\Lambda) = e^{-2s}\Lambda$ for all $s \in \mathbb{R}$. Thus $\tau \circ \theta_s = e^{-2s}\tau$ and $\tau(e) = 0$ for any self adjoint idempotent e . So $C^*(V,F)$ does not have any non zero idempotent though it is simple (cf. [5] for an earlier example of such a C^* algebra). We shall describe in section 12 an other example with a unital C^* algebra.

2) The fundamental class.

Assume that the foliation (V, F) is transversally oriented, so that the real bundle $\tau_x = T_x(V)/F_x$ on V is oriented. We also assume of course that V is connected.

We shall now define a canonical element $[V/F] \in K^*(V/F)$. Let q be the codimension of F, and at fixed $x \in V$, choose a transversal embedding $\varphi : D^q \hookrightarrow V$ (where D^q is the q-dimensional disk) which preserves the orientation. Then letting $\beta_q \in K^*(D^q)$ be the canonical generator of $K^*(D^q) = \mathbb{Z}$, we get an element $\varphi!(\beta_q) \in K^*(V/F)$.

Now note that the choices of $x \in V$ and of φ do not affect $\varphi!(\beta_q)$. (Construct for instance a homotopy (using a path $x(t) \in V$, $\varphi_t : D^q \to V$) from $\varphi_0!(\beta_q)$ to $\varphi_1!(\beta_q)$). Thus we get a well defined class $[V/F]^* \in K^*(V/F)$. The obvious question is whether one can show that this class $[V/F]^*$ is non zero. To do that one should of course use the natural pairing of $K^*(V/F)$ with $K_*(V/F)$ and construct an element $[V/F]_*$ of $K_*(V/F)$ such that $< [V/F]_* , [V/F]^* > = 1$.

Problem. Assuming that (V, F) is transversally K-oriented, i.e. that the transverse bundle τ is gifted with a holonomy invariant spin^c structure, construct the K homology class $[V/F]_*$ using a suitable Dirac type operator on the graph G of (V, F) .

9. THE GEOMETRIC GROUP $K_{*,\tau}(BG)$

In that section we shall describe in purely geometric terms a group $K_{*,\tau}(BG)$ (with a $\mathbb{Z}_2$ grading) associated in a purely geometric way to any foliated manifold (V, F). This construction is the result of a collaboration with P. Baum (cf. his lecture in these proceedings). The basic cycles are triples (M, E, g) where :

1) M is a smooth compact manifold.

2) E is a complex vector bundle on M.

3) g is a K-oriented smooth map from M to the space of leaves of (V, F).

A very convenient way to describe a smooth map from M to the space of leaves of (V, F) is to lift it on small enough open sets Ω_α of M in a smooth map $g_\alpha : \Omega_\alpha \mapsto V$, then, with $\cup \Omega_\alpha = M$, if $x \in M$ belongs to both Ω_α and Ω_β, the points $g_\alpha(x)$ and $g_\beta(x)$ should be on the same leaf. Due to the possible presence of holonomy one thus requires a smooth map, $g_{\beta,\alpha} : \Omega_\alpha \cap \Omega_\beta \mapsto G = \mathrm{Graph}(V, F)$ where $g_{\beta,\alpha}(x)$ goes from $g_\alpha(x)$ to $g_\beta(x)$ and the obvious 1- cocycle condition holds. Thus we get a 1-cocycle with values in the graph G of (V, F), and we can define a smooth map from M to the space of leaves of (V, F) as an equivalence class of such cocycles (two cocycles $(\Omega_\alpha, g_{\beta,\alpha}), (\Omega'_\alpha, g'_{\beta,\alpha})$ being equivalent if they extend to a cocycle on the disjoint union of the two open coverings).

If one wants, one can characterize g by its graph G_g, constructed as the principal G-bundle associated to a cocycle. Thus G_g is a smooth manifold, of dimension : $\dim G_g = \dim M + \dim F$, one has a submersion p of G_g on M and an action of G on G_g (i.e. a smooth map q from G_g to $G^{(0)} = V$ and a smooth map $z \mapsto z \cdot \gamma$ of

$q^{-1}(x)$ to $q^{-1}(y)$, $\forall \gamma : x \mapsto y$ with $z(\gamma_1 \circ \gamma_2) = (z\gamma_1)\gamma_2$ whenever it makes sense) such that the usual conditions for a principal bundle hold when suitably translated. In particular the action of G on each fiber $p^{-1}(x)$, $x \in M$ (note that $\dim p^{-1}(x) = \dim M$) should be free and transitive (given $x \in M$, $z_1, z_2 \in p^{-1}(x)$ there is a unique $\gamma \in G$ with $z_1 \gamma = z_2$, and in particular all $q(z)$, $z \in p^{-1}(x)$ belong to the same leaf).

Given a cocycle, as above, from M to Graph $(V, F) = G$ one constructs G_g as the quotient of $\{(x, \alpha, \gamma), x \in \Omega_\alpha, r(\gamma) = g(x)\}$ by the equivalence relation which identifies two such triples if, with the obvious notations, $x = x'$ and $g_{\alpha', \alpha}(x) = \gamma' \gamma^{-1}$. Conversely, given G_g, since p is a submersion one can cover M by domains Ω_α of local cross sections $c_\alpha : \Omega_\alpha \mapsto G_g$ and then define $g_\alpha(x) = q(c_\alpha(x))$ while $g_{\beta,\alpha}(x)$ is the only element of G which moves $c_\alpha(x)$ to $c_\beta(x)$.

To get the condition of K-orientability of g note that along each leaf L of (V, F) the differential of the holonomy gives a natural trivialisation of the transverse bundle lifted to the holonomy covering $\tilde{L}$ of L. Thus given a 1-cocycle $(\Omega_\alpha, g_\alpha, g_{\beta,\alpha})$ with values in G = Graph (V, F) we can give a meaning of the pull back of τ to M, since the differential of the holonomy of $g_{\beta,\alpha}$ can be used to glue together the bundles $g_\alpha^*(\tau)$. We shall denote this bundle by $g^*(\tau)$ and the K orientation of g is a spin^c structure on $T(M) \oplus g^*(\tau)$

Having described the basic cycles (M, E, g), one constructs the group $K_{*,\tau}(BG)$ in exactly the same way as P. Baum defines the K-homology of topological spaces, (except that the map g is oriented and smooth instead of M being oriented and g continuous). Thus there are three steps defining the equivalence of cycles : 1) direct sum, disjoint union 2) bordism 3) vector bundle modification.

Since in defining these steps all the modifications occur at the source M of the map g , it is obvious how they should be phrased in our context (cf. Loc.cit.). Instead of listing carefully these axioms, we shall express the result in terms of the classifying space of the topological groupoid G . On BG one has a natural principal G-bundle EG and hence one deduces from it, as above, a real q -dimensional vector bundle τ_{BG} . Now given a topological space X with a real vector bundle τ one can slightly modify the definition of the K-homology $K_*(X)$ to take in account the absence of a spin^c structure on τ . This means that in defining the K-cycles (M, E, g) , one will require the existence of a spin^c structure on $TM \oplus g^*\tau$ rather then on TM . We let $K_{*,\tau}(BG)$ be the corresponding group, it agrees by construction with the above group of geometric cycles for the foliation.

If the foliation comes from an action of a Lie group H in such a way that the graph G is identical with $V \times H$ then BG is the quotient $(V \times EH)/_H$ of the product of V by a contractible space EH on which H acts freely, with quotient $BH = EH/_H$, the classifying space of H . Thus BG is quite computable, if, say $H = \mathbb{R}^n$ it is homotopic to V . In general, if the leaves are contractible one has the same equality.

Finally note the strong connection of BG with the classifying space $B\Gamma$ defined by Haefliger ([17]). On V the foliation determines a Γ structure, which the holonomy trivializes along the holonomy covering of each leaf. One then easily gets a continuous map, well defined up to homotopy, from BG to $B\Gamma$, and the above cycles in $K_{*,\tau}(BG)$ map to elements of $K_{*,\tau}(B\Gamma)$.

10. FUNCTORIALITY OF K THEORY FOR FOLIATIONS.

The preceeding construction of the map $i! : K^*(N) \mapsto K^*(V/F)$ shows that in general the interesting functoriality will be the covariant one rather than the obvious contravariant one. The functor $X \mapsto K^*(X)$ for X locally compact is contravariant for proper maps but for smooth manifolds it becomes a covariant functor for not necessarily proper K-oriented smooth maps.(*) In our context, if (V_i, F_i), $i = 1,2$ are foliated manifolds, the notion of proper map from V_1/F_1 to V_2/F_2 is very restrictive and excludes the following very interesting maps :

1°) The projection p from V to V/F .

2°) Given a leaf $L \in V/F$ the map $pt \mapsto L$ from a one point space to V/F .

3°) The map from V/F to a point.

So instead of discussing proper maps and contravariant functoriality, we shall discuss smooth K-oriented maps and covariant functoriality. In the next section we shall construct $f!$ for any smooth K-oriented map f from a manifold M to the leaf space V/F , here we shall discuss the case when $f : V_1/F_1 \to V_2/F_2$ is an etale map and several important special cases. The general problem being :

a) Given a smooth K oriented map $f : V_1/F_1 \to V_2/F_2$ define a natural element $f! \in KK(V_1/F_1, V_2/F_2)$.

b) Show that $(f \circ g)! = g! \boxtimes f!$ (where $\boxtimes$ is the Kasparov cup

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(*) A smooth map $\varphi : X \mapsto Y$ of manifolds, is called K-oriented when the bundle $TX + \varphi^* TY$ on X is gifted with a Spin^c structure.

product).

We shall construct a) in the following examples :

(1) Let $N \overset{i}{\mapsto} V$ be a (not necessarily closed) transversal to (V,F). Then the map $i!$ of K groups constructed in section 8 is given by $i![E] = i! \otimes [E]$, $\forall [E] \in K^*(N)$, where $i!$ is the element of the Kasparov group $KK(C_0(N), C^*(V,F))$ described as follows. Let $\mathcal{E}$ be the C^* module on $C^*(V,F)$ associated to the continuous field of hilbert spaces $i!(1_N)$ on V/F. The fiber of $i!(1_N)$ at the leaf $L \in V/F$ is $\ell^2(L \tilde{\cap} N)$ and one has an obvious homomorphism of $C_0(N)$ in $\mathrm{End}\,\mathcal{E}$, to each $h \in C_0(N)$ corresponds $i!(h) \in \mathrm{End}\,\mathcal{E}$ given by a multiplication operator in each $\ell^2(L \tilde{\cap} N)$. In this way $\mathcal{E}$ becomes a bimodule and since $i!(h)$, $h \in C_0(N)$ is a <u>compact</u> endomorphism of $\mathcal{E}$, we see that the module map $F = 0$ of $\mathcal{E}$ in the trivial bimodule 0 is a quasiisomorphism, hence defining an element $i!$ of $KK(N, V/F)$.

The construction (1) extends to the case of 1-cocycle $(N_\alpha, g_{\beta\alpha})$ on N with coefficients in the graph G of (V,F), and where each of the smooth maps $g_\alpha : N_\alpha \to V$ is etale from N_α to V/F (i.e. $k \circ g_\alpha$ is etale where k is a submersion $\mathrm{Dom}\, k \mapsto \mathbb{R}^q$, defining the restriction of F to the image of g_α).

(2) Let $f : W \mapsto V$ be a smooth map of the manifold W to V, then f is called transverse to the foliation F of V if for any $y \in W$, $x = f(y)$, one has $T_x(V) = F_x + f_* T_y(W)$ ([20]). It follows that on W there is a natural pull back H, of the foliation F, which has the same codimension as F, and whose leaves are the connected components of the inverse images of the leaves of (V,F). We thus have a natural map f from (W/H) to (V/F).

which is <u>etale</u> in an obvious sense. We now describe :

$$g! \in KK(W/H, V/F)$$

Let $\Gamma = \{(y,\gamma) \in W \times G,\ g(y) = r(\gamma)\}$, where G is the graph of (V,F), and for each $x \in V$ let $\Gamma_x = \{(y,\gamma) \in \Gamma,\ s(\gamma) = x\}$. For each $x \in V$, let $H_x = L^2(\Gamma_x)$ be the space of square integrable $\frac{1}{2}$ functions on the manifold Γ_x (which can fail to be connected). The map $(y,\gamma) \mapsto (y, \gamma \circ \gamma_1)$ of Γ_{y_1} in Γ_{x_1} for $\gamma_1 \in G$, $\gamma_1 : x_1 \to y_1$ defines a natural representation of G in $(H_x)_{x \in V}$. To make it into a continuous field of hilbert spaces on V/F, one lets $\Omega_\Gamma^{1/2}$ be the bundle on Γ whose fiber at (y,γ) is $\Omega_{H,y}^{1/2} \otimes \Omega_{F,x}^{1/2}$ $(x = s(\gamma))$. Then any element ξ of $C_c(\Gamma, \Omega_\Gamma^{1/2})$ defines by restriction to the Γ_x, a $\frac{1}{2}$ section of H, and one checks that the closure of this space of $\frac{1}{2}$ sections satisfies conditions 1) 2) 3) 4) defining a continuous field of hilbert spaces on V/F.
For a leaf L of (V,F) without holonomy the fiber of H over L is easy to describe, as $H_L = \oplus L^2(L')$ where L' varies through all leaves of (W,H) which map to L by g. If L has holonomy, then H_L is the L^2 space of the manifold $\widetilde{g^{-1}}(L) = \{(y,x) \in W \times \tilde{L},\ g(y) = r(x) \in V\}$ (where here r stands for the holonomy covering : $r : \tilde{L} \mapsto L$). Thus in general we note $H_L = L^2(\widetilde{g^{-1}}(L))$.

Next, we shall represent $C^*(W,H)$ as endomorphisms of the above continuous field of hilbert spaces $(H_L)_{L \in V/F}$. In fact, any $k \in C^*(W,H)$ defines a compact endomorphism of the continuous field of hilbert spaces $L^2(\tilde{L}')$ on W/H and we shall associate to any continuous endomorphism $(T_{L'})_{L' \in W/H}$ of this field, a continuous endomorphism of $(H_L)_{L \in V/F}$. If L has no holonomy,

$H_L = \oplus_{g(L')=L} L^2(L')$ thus one takes $(g!(T))_L$, as the diagonal operator with $T_{L'}$, as diagonal entries. In general, the connected components of $g^{-1}(L)$ are each of the form $\tilde{L}'$ where L' is a leaf of (W,H) and one defines $(g!(T))_L$ in the same way. We leave it as an exercice to check that if T is compact, then $g!(T)$ is also compact, so that by taking the triple $(H\ ,\, o,\, o)$ one gets an element $g!$ of $KK(W/H\ ,\ V/F)$.

(3) In the above example the map from the leaf space W/H to V/F was etale, so in particular it was obviously K-oriented. In the next example we consider the natural projection ℓ of V on V/F which to each $x \in V$ assigns the leaf $L = \ell(x)$ through x . To K-orient this map means to choose a spin^c structure on the real oriented vector bundle F . We let then $S = S(F)$ be the corresponding complex vector bundle of spinors, while $\mathrm{Cliff}(F)$ is the bundle of Clifford algebras associated to F (*) .

The associated element $\ell! \in KK(V, V/F)$ has a mod 2 degree equal to 0 or 1 according to the parity of the dimension of F . For instance, we shall assume that $\dim F$ is even, then using clifford multiplication by the orientation ε of F one splits S as a direct sum of two bundles $S = S^+ \oplus S^-$, and given $X \in F_x$, the clifford multiplication $c(X)$ by X maps S_x^+ to S_x^- . Let then P be a pseudodifferential operator along the leaves, elliptic of order 0 , with principal symbol σ_P given by :

$$\sigma_P(x, \xi) = \|\xi\|^{-1} c(\xi) \quad \forall \xi \neq 0 .$$

(*) We choose a Euclidean structure $X \mapsto \|X\|$ on the bundle F .

Then P defines a quasi-isomorphism $(P_L)_{L \in V/F}$ of $L^2(\tilde{L}, S^+)$ with $L^2(\tilde{L}, S^-)$. Moreover, one has a natural representation of $C(V)$ as multiplication operators in each of the $L^2(\tilde{L}, S^{\pm})$, and since the commutator of P with multiplication operators is pseudo-differential with negative order, we see that the quasi isomorphism P commutes modulo compact endomorphisms with the obvious action of $C(V)$, we thus have a very natural element $\ell!$ of $KK(V, V/F)$. Note also that to determine the corresponding index map : $K^*(V) \mapsto K^*(V/F)$ is exactly equivalent to the determination of the analytical index : $\mathrm{Ind}_a : K^*(F) \mapsto K^*(V/F)$ for arbitrary elliptic operators along the leaves (since, as F has a spin^c structure, one has a natural Thom isomorphism $K^*(F) \simeq K^*(V)$). In section 11 we shall give a construction of $f!$ for any K-oriented map of a manifold M in the leaf space V/F. This construction of $f!$ is based on the existence of a factorization of f as $f = p \circ q \circ i$ where α) i is an imbedding of M in a manifold N, β) q is an etale map from N to the product $V/F \times \mathbb{R}^n$, γ) p is the projection of $V/F \times \mathbb{R}^n$ to V/F. Applying the construction of section 11 to $\ell : V \mapsto V/F$ thus yields an element of $KK(V, V/F)$ and the longitudinal index problem for foliations is :

<u>Problem.</u> <u>Show that the two definitions of $\ell!$ coincide.</u>

The solution of that problem contains as a special case the equality of the index maps : $K^*(V) \mapsto K^*(V/F)$ associated to the two constructions of $\ell!$ and hence the computation of the analytical index $\mathrm{Ind}_a(D) \in K^*(V/F)$ of arbitrary elliptic pseudo differential operators D along the leaves of (V, F), in terms of a purely geometric construction. This construction amounts essentially to find an open transversal N of the foliation of $V \times \mathbb{R}^n$ by $F \oplus \{0\}$ which is naturally homeomorphic to the total space of a Spin^c vector bundle over V .

The index theorem for measured foliations, of section 4 follows also from the solution of the above problem, one would then obtain a proof very different from the heat equation method used in [4][7]. Finally the above problem is a special case of the problems a) b) stated at the beginning of this section, since it would be settled by the equality :

$$f! = i! \otimes g! \otimes p!$$

To end this section we shall take the simplest case ($F_1 = 0$ and $F_2 = 0$) and, given a smooth K oriented map $f : V_1 \mapsto V_2$ we shall sketch an easy construction of $f!$ (here the two foliations are trivial) and show how the index theorem follows from the functoriality. Using the factorization of f as $f = f_2 \circ f_1$ where : $f_1(x) = (x, f(x)) \in V_1 \times V_2$ and $f_2(x_1, x_2) = x_2$, we shall specialize to the case of f_1 and f_2 and then get $f!$ as the product $f_1! \otimes f_2!$.

(4) Let $f_1 : V_1 \mapsto V_2$ be an imbedding. The obvious thing to do to get $f_1!$ is to take a tubular neighborhood U of the image of f_1 in V_2 and use the element of $KK(V_1, U)$ associated in to the classical Thom isomorphism (note that since f_1 is K-oriented the vector bundle U is a spin^c vector bundle). Then one has to replace the open set U of V_2 by V_2 . To fix the notations let us assume that the codimension of V_1 in V_2 is even. Let then E be the continuous field of hilbert spaces on V_2 , whose fibers are 0 outside U , obtained by pulling back to U the bundle $S(U)$ of spinors associated to the Spin^c structure of U . So, formally E is the pushforward by the inclusion

of ν in V_2 of the pull back $p_\nu^*(S(\nu))$.

Given $h \in C(V_1)$, the composition $h \circ p_\nu$ of h by the projection of ν on V_1 , acts by multiplication as an endomorphism of E , we thus have a natural homomorphism of $C(V_1)$ in $\mathrm{End}(E)$. Finally the orientation ε of ν splits E in E^+ and E^- as in (3) and for $\xi \in \nu$ the clifford multiplication $c(\xi')$ by $\xi' = (1+\|\xi\|)^{-1}\xi$ defines an element F_ξ of $\mathrm{Hom}(E_\xi^+, E_\xi^-)$. The corresponding triple (E^+, E^-, F) defines $g_1! \in KK(V_1, V_2)$.

(5) Let $g_2 : V_1 \times V_2 \mapsto V_2$ be the second projection. Assuming g_2 is K-oriented means that V_1 is a Spin^c manifold, then the construction of $g_2!$ is a very special case of the construction (3) or of the analytical index for V_2-families of elliptic operators on V_1 . One considers the bundle S_1 of Spinors on V_1 associated to the Spin^c structure and the two constant field of hilbert spaces on V_2 with fibers $L^2(V_1, S_1^\pm)$ (*). Then $C(V_1 \times V_2)$ acts, by multiplication operators in each fiber, as endomorphisms of the two above continuous field of hilbert spaces and letting F be a pseudo-differential operator of order 0 on V_1 , from S_1^+ to S_1^- , with principal symbol given by Clifford multiplication by $\xi' = \xi/\|\xi\|$ one gets the desired triple.

(6) Let $g : V_1 \mapsto V_2$ be an arbitrary smooth K-oriented map we have $g = g_2 \circ g_1$ where g_1 is the imbedding $g_1(x) = (x, g(x))$ while g_2 is the projection of $V_1 \times V_2$ on V_2 . If both g_1 , g_2 were K-oriented, we could define $g!$ as $g_1! \otimes g_2!$ using (4) (5). However, this product is easy to compute and makes sense without

(*) We assume again for convenience that $\dim V_1$ is even.

orienting g_1 and g_2. It is given by a family F_{x_2}, $x_2 \in V_2$ of pseudodifferential operators of order 0 on V_1, which we shall describe by giving for each $x_2 \in V_2$ a pair of continuous field of hilbert spaces $(E^{\pm}_{x_1,x_2})_{x_1 \in V_1}$ on V_1 and a symbol of order 0 : $\sigma(x_1,x_2,\xi) \in \mathrm{Hom}\,(E^{+}_{x_1,x_2}, E^{-}_{x_1,x_2})$ for any $\xi \neq 0$, $\xi \in T_{x_1}(V_1)$.

As g is K-oriented we let S be the bundle of Spinors on V_1 associated to the Spin^c structure on $TV_1 \oplus g^* TV_2$, and assuming the even dimensionality of this last bundle, we split S in $S^+ \oplus S^-$ using its orientation.
We let $X \in T(V_2) \mapsto \|X\| \in \mathbb{R}^+$, be a Riemannian metric on V_2 such that, given $x, y \in V_2$, $d(x,y) < 1$ there exists a unique $X \in T_x(V_2)$, $\|X\| < 1$ with exponential equal to y, this unique X is denoted by $X(x,y)$.

We put $E^{\pm}_{x_1,x_2} = 0$ if $d(g(x_1), x_2) \geq 1$, and otherwise $E^{\pm}_{x_1,x_2} = S^{\pm}_{x_1}$. The principal symbol of F_{x_2} is given by $\sigma(x_1, x_2, \xi) = c((\xi, X(g(x_1), x_2))')$ where c means Clifford multiplication, $\xi \in T_{x_1}(V_1)$ is a tangent vector to V_1 at x_1, $X(g(x_1), x_2)$ is the element of :
$g^*(TV_2)_{x_1} = (TV_2)_{g(x_1)}$ with exponential equal to x_2, and the $(\;)'$ means that the vector $Y \in T_{x_1}(V_1) \oplus g^*(TV_2)_{x_1}$ $Y = (\xi, X(g(x_1), x_2))$ is replaced by $Y' = \|Y\|^{-1}\, Y$ if $\|Y\| > \frac{1}{2}$.

One then has to check that this definition of $g!$ is compatible with (4) and (5). Then the equality $(g \cdot q)! = q! \otimes g!$ applied to the factorization of the map of a Spin^c manifold V to a point through an imbedding of V in $\mathbb{R}^n$ gives in particular the K-theory formulation of the Atiyah Singer index theorem.

11. CONSTRUCTION OF $f!$ FOR ANY SMOOTH K-ORIENTED MAP FROM M TO THE SPACE OF LEAVES OF (V,F).

Let M be a smooth manifold, f a K-oriented smooth map from M to the leaf space of (V,F) (cf. Section 9). To construct $f!$, $f! \in KK(M, V/F)$, we shall show how to come back to the special situation of section 8 where the map f was etale.

For each n there is an obvious foliated manifold (V^n, F^n) with space of leaves V^n/F^n identical with the product of V/F by $\mathbb{R}^n$. One takes $V^n = V \times \mathbb{R}^n$ and for each $(x,t) \in V^n$ the subspace $F^n_{(x,t)}$ of the tangent space $T_x(V) \oplus \mathbb{R}^n$ is $F_x \oplus \{0\}$. Each leaf L^n of (V^n, F^n) is of the form $L^n = \{(x,t),\ x \in L\}$ where L is a leaf of (V,F) and $t \in \mathbb{R}^n$. The C^* algebra $C^*(V^n, F^n)$ is naturally isomorphic to: $C^*(V,F) \otimes C_0(\mathbb{R}^n)$ and thus by Bott periodicity one has a natural isomorphism of $K^*(V^n/F^n)$ with $K^*(V/F)$. Letting j be a smooth imbedding of M in $\mathbb{R}^n$, we shall replace the original map: $f : M \mapsto V/F$ by the map $f^n : M \mapsto V^n/F^n$ given by $x \mapsto (f(x), j(x)) \in V/F \times \mathbb{R}^n$. Once $(f^n)!$ is constructed one gets $f!$ by composing with the canonical element of $KK(V^n/F^n, V/F)$ given by Bott periodicity.

By replacing V/F by $V/F \times \mathbb{R}^n$ we made f into an immersion, to make it etale we now have to modify the original manifold M. We let ν be the normal bundle of the imbedding j : $\nu_x = j_*(T_x(M))^\perp$ $\forall x \in M$.

We let h be the Haefliger Γ_q structure on M ([17]) which is the pull back by the map f of the natural Haefliger structure on V/F. To be specific, let (V_i) be an open covering of V by domains of submersions $k_i : V_i \mapsto \mathbb{R}^q$ defining the restriction of the foliation F to V_i . Let then $(\Omega_\alpha, g_{\beta,\alpha})$ be a 1-cocycle on M (with values in the graph G of (V,F)) defining the map g ; we can assume that each $g_{\alpha,\alpha}(\Omega_\alpha)$ is contained in some $V_{i(\alpha)}$. Then $h_\alpha = k_{i(\alpha)} \circ g_{\alpha,\alpha}$ is a smooth map from Ω_α to $\mathbb{R}^q$ and for $x \in \Omega_\alpha \cap \Omega_\beta$, the holonomy $h_{\beta,\alpha}(x)$ of the element $g_{\beta,\alpha}(x)$ of the graph G of (V,F) gives a germ of diffeomorhism of a neighborhood of $h_\alpha(x)$ to a neighborhood of $h_\beta(x)$. The cocycle $(\Omega_\alpha, h_{\beta\alpha})$ defines the Γ_q structure h on M .

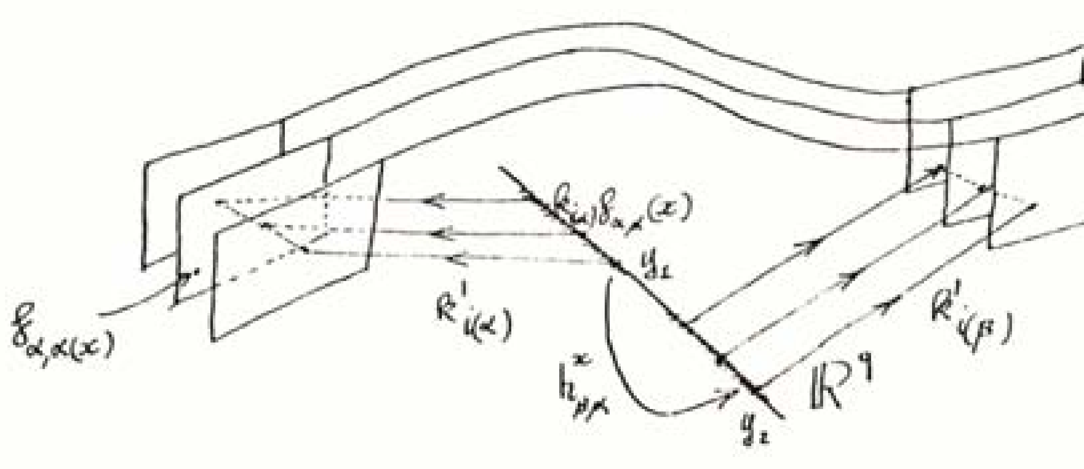
Let $M \xrightarrow{i} H \xrightarrow{p} M$ be the foliated microbundle of rank q on M associated to the Γ_q structure h (cf. [16] p. 188).

With the above notations, one can take for H the quotient of a neighborhood of $\{(x,\alpha,h_\alpha(x)),\ x\in\Omega_\alpha\}$ in $(\amalg\,\Omega_\alpha)\times\mathbb{R}^q$ by the equivalence relation which identifies (x,α,y) with $(x,\beta,\ h_{\beta\alpha}(y))$ for $x\in\Omega_\alpha\cap\Omega_\beta$ and $y\in$ Domain $h_{\beta\alpha}$. The projection $p : H \mapsto M$ is given by $p(x,\alpha,y) = x$, the section $i : M \to H$ by $i(x) = (x,\alpha,h_\alpha(x))$ for $x\in\Omega_\alpha$, and the foliation of H is given by $y = \text{cst.}$.

We let $p^*\nu$ be the pull back to H of the normal bundle ν of the imbedding j of M in $\mathbb{R}^n$, and we imbed M in $p^*\nu$ by the map $\tilde{i} : x \mapsto (i(x),0)$ (composition of the 0-section $H \mapsto p^*\nu$ and the section $i : M \mapsto H$). By construction of $i : M \mapsto H$ the normal bundle of i is equal to $g^*\tau$, the pull back by g of the normal bundle of the foliation (V,F) , thus the normal bundle of the imbedding $\tilde{i} : M \to p^*\nu$ is equal to $\nu \oplus g^*\tau$ and hence, inherits a natural Spin^c structure from the K-orientation of δ .

Thus we have, from the usual Thom isomorphism theorem for Spin^c vector bundles, a natural element of $KK(M,N)$, where N is a neighborhood of M in $p^*\nu$. The whole problem now is to define a natural etale map g of N in the space of leaves V^n/F^n .

For each i choose a smooth cross section k'_i of the submersion $k_i : V_i \mapsto \mathbb{R}^q$. We shall construct g by giving a cocycle (on a neighborhood N of M in $p^*(\nu)$) $(N_\alpha , g_{\beta,\alpha})$ with values in the graph G of (V,F) . For each α we let $\tilde{\Omega}_\alpha$ be the open set in $p^*(\nu)$ of elements of the form $z = (x,\alpha,y,\eta)$, $x \in \Omega_\alpha$, $y \in \mathbb{R}^q$ $y \in$ Image of $k_{i(\alpha)}$. Then we let $g_\alpha(z) \in V^n$ be given by $(k'_{i(\alpha)}(y) , j(x)+\eta) \in V \times \mathbb{R}^n$. For $z \in \tilde{\Omega}_\alpha \cap \tilde{\Omega}_\beta$, so that z is represented by (x,α,y_1,η) and (x,β,y_2,η) with $y_2 = h^x_{\beta,\alpha}(y_1)$. It is clear that $g_\alpha(z)$ and $g_\beta(z)$ have the same component in $\mathbb{R}^n$, namely $j(x)+\eta$, while the leaf L of (V,F) through $k'_{i(\alpha)}(y_1)$ is the same as the leaf of (V,F) through $k'_{i(\beta)}(y_2)$. Indeed the local diffeomorphism $h^x_{\beta\alpha}$ of $\mathbb{R}^q$, from $k_{i(\alpha)}\,\delta_{\alpha\alpha}(x)$ to $k_{i(\beta)}\,\delta_{\beta\beta}(x)$, is given by the holonomy of $\delta_{\beta,\alpha}(x) \in G$, so that corresponding points are on the same leaf.



In fact, we can specify uniquely an element $k'_{\beta,\alpha}(z)$ of the graph $\mathcal{G}$ of (V, F) with source $k'_{i(\alpha)}(y_1)$ and range $k'_{i(\beta)}(y_2)$. We put $g_{\beta,\alpha}(z) = (k'_{\beta,\alpha}(z), j(x)+\eta) \in \mathcal{G}^n = \text{Graph}(V^n, F^n) = \text{Graph}(V, F) \times \mathbb{R}^n$. We choose a neighborhood N of M in $p^*\mathcal{U}$ so that each of the maps $\tilde{g}_\alpha : N_\alpha = \tilde{\Sigma}_\alpha \cap N \mapsto V^n/F^n$ is etale; this is possible since locally $\tilde{g}_\alpha$ is given by :

$$\tilde{g}_\alpha(x, \alpha, y, \eta) = (y, j(x)+\eta)$$

We now have a cocycle $(N_\alpha, g_{\beta,\alpha})$ where each map g_α defines an etale map to V^n/F^n and it remains to show how to adapt the construction of $i!$, where $N \xrightarrow{i} V$ is a transversal to (V, F), done in section 10, to the case of a cocycle. The only important step is the construction of the continuous field of hilbert spaces $i!(1_N)$ on V/F. Let Γ_g be the graph of g, i.e. the principal $\mathcal{G}^n$-bundle on N associated to the above 1-cocycle, we have a natural action of $\mathcal{G}^n$ on Γ_g on the right (cf. section 9) and hence a representation of $\mathcal{G}^n$ on the borel field of hilbert spaces $H_x = \ell^2(\Gamma_{g,x})$, $x \in V^n$.

Note that since g is etale, the set $\Gamma_{g,x}$ is countable. Using the natural structure of manifold of Γ_g (of dimension equal to $\dim V^n$), one makes the above field in a continuous field of hilbert spaces on V^n/F^n. The natural projection of Γ_g on N defines a homomorphism of $C_0(N)$ in End $\mathcal{E}$, where $\mathcal{E}$ is the C^* module on $C^*(V^n, F^n)$ attached to the above continuous field. We define $g!$ by the triple $(\mathcal{E}, 0, 0)$ as in section 10. Now the element $g!$, $g! \in KK(M, V/F)$ is obtained by composition of three terms 1) the element $p_1!$, $p_1! \in KK(V^n/F^n, V/F)$ where p_1 is the projection $V/F \times \mathbb{R}^n \to V/F$, 2) the above $g! \in KK(N, V^n/F^n)$

3) the Thom isomorphism $\tilde{i}! \in KK(M,N)$ where $\tilde{i} : M \mapsto N$ is the inclusion.

We of course have to check that this definition of $g!$ is independent of all the choices, such as the imbedding $j : M \mapsto \mathbb{R}^n$ made along the way. This follows from the homotopy invariance of [18] together with the proof of the invariance of the topological index in [1].

THEOREM. Given an oriented K cycle (M, E, g) the class : $g!([E]) \in K^*(V/F)$ only depends upon the homology class $[(M, E, g)] \in K_{*,\tau}(BG)$ of the K cycle.

We shall sketch the proof assuming that the functoriality $(g_1 \circ g_2)! = g_2! \otimes g_1!$ has been checked in certain special cases.

The compatibility with vector bundle modification (i.e. replacing M by the total space of a suitable sphere bundle $M_1 \xrightarrow{p} M$ and E by a bundle E_1 on M_1) follows from the equality $g_1 = g \circ p$, $p!([E_1]) = [E]$. The compatibility with bordism goes as follows, if $M = \partial M_1$ is a boundary, the inclusion $i : M \mapsto D$ of M in the double D of M_1 is a K-oriented map, since it has a trivial one dimensional normal bundle. So if g is a smooth K oriented map of M_1 in the leaf space V/F, its restriction to M is equal to $\tilde{g} \circ i$ where $\tilde{g}$ is the extension of g to the double of M_1 (this extension can be assumed to be smooth). Thus if the bundle E on M is the restriction $i^*(E_1)$ of a bundle E_1 on M_1, one has $(\tilde{g} \circ i)!\, [E] = \tilde{g}!\,(i!\, i^*(E_1)) = 0$ since $i!\, i^* = 0$. Indeed to check this, it is enough to show that $i!\,(1_M) = 0$, but $i!(1_M)$ is the element of $K^1(D)$ defined by a smooth map $\varphi : D \mapsto S^1$ such that the 1-form on D, $\omega = \varphi^*(d\theta)$, pull back of the fundamental class of S^1, is equal to the element of $H^1(D, \mathbb{Z})$ which is dual to the cycle M and hence is equal to 0.

<u>Remark</u>. Using the above construction one gets a map from the analytical group $K_*(V/F)$ (the K-<u>homology</u> of V/F in the sense of extension theory) to a natural geometric group $K^{*,\tau}(BG)$ of cocycles on BG.

12. COMPARISON OF $K_{*,\tau}(BG)$ AND $K^*(V/F)$.

The most important problem in the K-theory of foliations is to be able to compute the analytical group $K^*(V/F)$ in purely geometric terms. The first non trivial example is the Kronecker foliation F_θ of the two torus $V = \mathbb{R}^2/\mathbb{Z}^2$ by lines of irrational slope θ. This case was settled by the remarkable breakthrough of Pimsner and Voiculescu ([22]) who exhibited an exact sequence for the K group of crossed products of a C^* algebra A by an automorphism $\alpha \in \mathrm{Aut}(A)$. Their result was then extended in [10], [13] to the case of crossed products by the real line $\mathbb{R}$ where it takes an extremely simple form, extending the classical Thom isomorphism. This step allows to compute $K^*(V/F)$ whenever the foliation F comes from a free action of a solvable simply connected Lie group H by diffeomorphisms of V.

In section 9 we introduced the geometric group $K_{*,\tau}(BG)$ which is quite easy to compute in concrete examples, moreover in section 11 we defined a natural map μ from the geometric group to the analytical group. The main problem can now be stated in precise terms :

<u>Problem</u>. Show that the map $\mu : K_{*,\tau}(BG) \to K^*(V/F)$ is always an isomorphism.

The easiest case is when the foliation (V, F) comes from a submersion $p : V \to B$, then the classifying space BG with bundle τ is homotopic to B with its tangent bundle TB, so that the oriented K-homology group $K_{*,\tau}(BG)$ is naturally isomorphic to $K^*(B)$, which in turn is obviously equal to the analytical group $K^*(V/F)$.

The next important case is when the foliation (V, F) comes from a free action by diffeomorphisms of the simply connected solvable Lie group H . Then the classifying space BG is homotopic to the ambient manifold V , so that (up to a shift of mod 2 degree depending on the parity of dim H) the geometric group $K_{*,\tau}(BG)$ is naturally isomorphic to $K^*(V)$. Now by [10] , exactly the same result holds for the analytical group $K^*(V/F)$ and one has :

THEOREM. Let (V, F) be the foliation coming from a free action (*) on V of the solvable simply connected Lie group H then the map $\mu: \quad K_{*,\tau}(BG) \mapsto K^*(V/F)$ is an isomorphism.

Moreover the three following maps from $K^*(V)$ to $K^*(V/F)$ coincide and are isomorphisms :

a) The Thom isomorphism of [10] .

b) The analytical index map $e!$ of section 10.

c) The topological index map of section 11.

Taking for instance the Horocycle foliation discussed at the end of section 8, i.e. the foliation of $V = SL(2,\mathbb{R})/\Gamma$ where Γ is a discrete cocompact subgroup, by the action on the left of the subgroup of lower triangular matrices of the form $\begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}, t \in \mathbb{R}$, one sees that the analytical index gives a (degree 1) isomorphism of $K^*(V)$ on $K^*(V/F)$, while for the only transverse measure Λ for (V, F) (the Horocycle flow being strongly ergodic), the composition of the above

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(*) The important point is that the graph G of (V, F) is equal to $V \times H$, this may occur even if the action is not free.

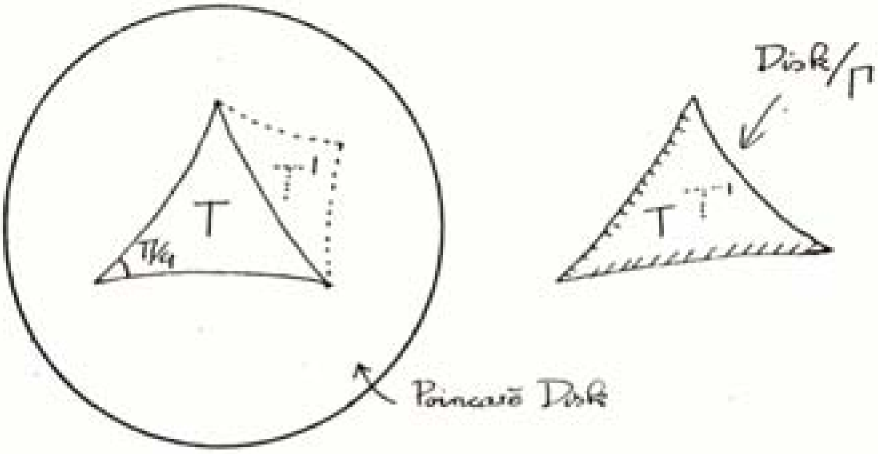
analytical index with $\dim_\Lambda : K^*(V/F) \mapsto \mathbb{R}$ is equal to 0, so in particular $\dim_\Lambda$ is identically 0 (cf. section 8). This example shows that even when the foliation (V,F) does have non trivial transverse measure, the K-theoretic formulation of the index theorem (i.e. here the equality of b) and c)) gives much more information than the index theorem for the measured foliation (F,Λ). As a corollary of the above we get :

COROLLARY. Let (V,F) be as in the above theorem, and let Λ be a transverse measure for (V,F) (a non trivial one exists iff H is unimodular then the image of $K^*(V/F)$ by $\dim_\Lambda : K^*(V/F) \mapsto \mathbb{R}$ is equal to $\{\langle ch(E), [C]\rangle, [E] \in K^*(V)\}$.

Here ch is the usual chern character, mapping $K^*(V)$ to $H^*(V,\mathbb{Q})$ and $[C]$ is the Ruelle-Sullivan cycle of section 3.

COROLLARY. Let V be a compact smooth manifold, φ a minimal diffeomorphism of V. Assume that the first cohomology group $H^1(V,\mathbb{Z})$ is equal to 0, then the crossed product $A = C(V) \rtimes_\varphi \mathbb{Z}$, is a simple unital C^* algebra without any non trivial idempotent.

As a very nice example where this corollary applies one can take the diffeomorphism φ given by left translation by $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in SL(2,\mathbb{R})$ of the manifold $V = SL(2,\mathbb{R})/\Gamma$, where the group Γ, discrete and cocompact is chosen in such a way that V is a homology 3-sphere. For instance one can take in the Poincaré disk a regular triangle T with its three angles equal to $\pi/4$ and as Γ the group formed by products of an even number of hyperbolic reflections along the sides of T.



We now pass to the foliation we used in section 3 to show that the Betti number β_1 of a measured foliation could fail to be 0 while leaves were contractible. We shall deal more abstractly with foliations (V,F) whose leaves L are isometric to the Poincaré disk (with its metric of constant negative curvature -1), when the bundle F is gifted with a suitable Euclidean structure. With this hypothesis it is clear that the classifying space BG of the graph G of (V,F) is homotopic to V. More concretely any smooth map $g : M \mapsto V/F$ from a manifold to the space of leaves V/F can be lifted to V using a partition of unity on M and the hyperbolic barycenter for n-uples of points on the same leaf.

Thus the geometric group $K_{*,\tau}(BG)$ coincides with $K^*(V)$ and a special case of the general problem is the following :

PROBLEM. With the above hypothesis, show that the analytical index map $\ell! : K^*(V) \mapsto K^*(V/F)$ is an isomorphism.

What we shall do here is to construct an element δ of $KK(V/F, V) = KK(C^*(V,F), C(V))$ which satisfies $\ell! \otimes \delta = id_V$ where $id_V \in KK(V, V)$ is the element associated to the identity map. Thus the index map associated to δ, which goes from $K^*(V/F)$ to $K^*(V)$, is a right inverse of $\ell!$, so that we know that $\ell!$ is injective. The next step would be to show that with the other construction of $\ell!$ made in section 11, one has $\delta \boxtimes \ell! = id_{C^*(V,F)}$. This would show at the same time that the two constructions of $\ell!$ do coincide and that $\ell!$ is an isomorphism. We have not however yet checked carefully this important step.

Construction of δ.

In our context, F is a 2-dimensional oriented real vector bundle on V, it thus has a natural Spin^c structure, and we let $S = S(F)$ be the corresponding complex vector bundle of spinors on V with its natural splitting as $S = S^+ \oplus S^-$. For each $X \in F_x$, $\xi \in S_x^+$ we let $c(X)\,\xi \in S_x^-$ be the Clifford product of ξ by X.

Since $\delta \in KK(C^*(V,F), C(V))$, it is described by a C^* module over $C(V)$, i.e. a continuous field $(H_x)_{x \in V}$ of hilbert spaces over V. We put $H_x = L^2(G_x) \otimes S_x$, where $G_x = \{\gamma \in G, s(\gamma) = x\}$ (G is the graph of (V,F)). As each leaf L of (V,F) is simply connected, there is no holonomy and the graph G is hausdorff, which shows that the field $(L^2(G_x))_{x \in V}$ of hilbert spaces is continuous. Now for each x the representation π_x of $C^*(V,F)$ in $L^2(G_x)$ (cf. Section 5) defines uniquely a representation $\pi_x^\pm$ of $C^*(V,F)$ in $H_x^\pm = L^2(G_x) \otimes S_x^\pm$ such that :

$$\pi_x^\pm(f)\,(\xi_1 \otimes \xi_2) = \pi_x(f)\,\xi_1 \otimes \xi_2 \qquad \forall f \in C^*(V,F)$$

In this way we get a homomorphism $\pi^\pm$ of $C^*(V,F)$ in the algebra of endomorphisms of $H^\pm$. We now define an homomorphism Δ of H^+ to H^-, for each $x \in V$, $\Delta_x : H_x^+ \mapsto H_x^-$ is given by the equality :

$$(\Delta_x \xi)(\gamma) = c(\varphi(\gamma)^1)\,\xi(\gamma) \qquad \forall \xi \in L^2(G_x, S_x)$$

Here $\varphi(\gamma)$ for $\gamma \in G_x$ stands for the vector in F_x whose exponential is equal to $y = r(\gamma)$, and the $'$ means that for $\|\varphi(\gamma)\| > 1$ it has been replaced by $\|\varphi(\gamma)\|^{-1}\varphi(\gamma)$.

For $f \in C_c^\infty(G, \Omega^{1/2})$ and each $x \in V$, $\pi_x^\pm(f)$ is a smoothing operator in $L^2(G_x) \otimes S_x$, and since $\gamma \mapsto \|\varphi(\gamma)\|$ is a proper map from G_x to $\mathbb{R}_+$ it follows that with f as above the following operators are compact :

$$\pi_x^+(f)(\Delta_x^*\Delta_x - 1) \qquad , \ \pi_x^-(f)(\Delta_x\Delta_x^* - 1)$$

Also, as, given $\varepsilon > 0$, $C > 0$, there exists a compact subset K of G_x such that for $\gamma_1, \gamma_2 \in G_x \setminus K$, $d(\gamma_1, \gamma_2) \leq C$ one has $\|\varphi'(\gamma_1) - \varphi'(\gamma_2)\| \leq \varepsilon$, we get that Δ_x intertwines, modulo compact operators, the representations $\pi_x^\pm$ of $C^*(V,F)$.

One checks in this way that the triple (H^+, H^-, Δ) defines an element of the Kasparov group $KK(V/F, V)$.

<u>Proof of</u> $\ell! \boxtimes \delta = id_V$.

We shall reduce this check to the classical Thom isomorphism for the vector bundle F over V using the homotopy invariance of KK as in [13]. If one tries to compute directly $\ell! \boxtimes \delta$ the difficulty is the following : one gets as a continuous field of hilbert spaces on V: the field $L^2(G_x, r^*(S)) \otimes S_x$, but the action of $C(V)$ on each fiber is highly non trivial, it is given by multiplication by $h \circ r$ for any $h \in C(V)$. The aim of the homotopy we are going to use is to transform this action of $C(V)$ on the fiber in the multiplication by $h(x)$, $\forall\, h \in C(V)$. The following construction extends the idea of [13], our first version of this homotopy was using a parameter $\hbar \to 0$ in the pseudodifferential calculus of [7]. The present form is due to G. Skandalis.

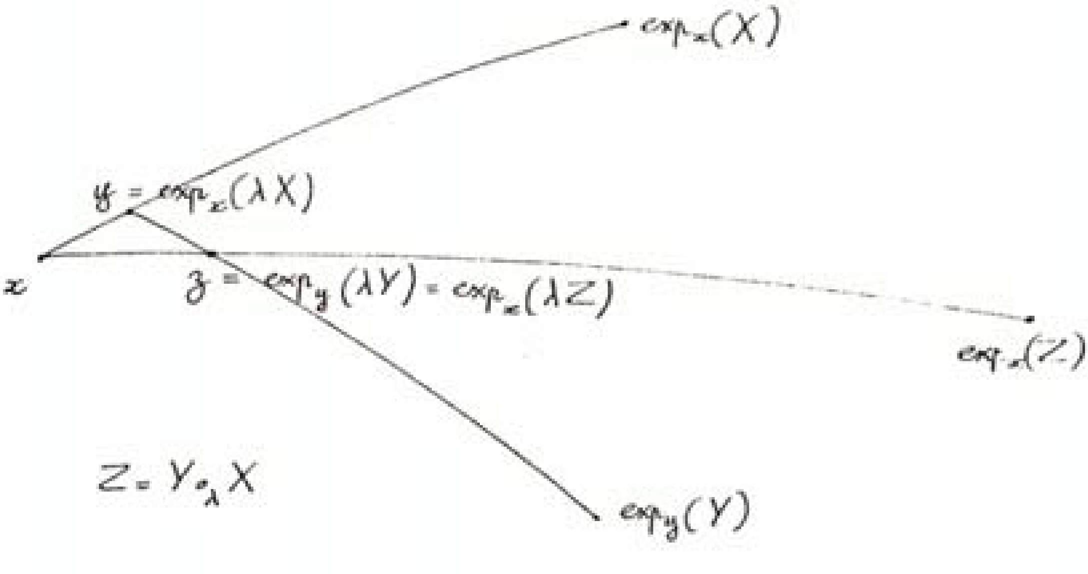
We shall construct a continuous field $(A_\lambda)_{\lambda\in[0,1]}$ of C^* algebras and continuous families $\alpha_\lambda, \beta_\lambda$ of elements, $\alpha_\lambda \in KK(A_\lambda, C(V))$, $\beta_\lambda \in KK(C(V), A_\lambda)$ such that for $\lambda = 1$, $A_1 = C^*(V,F)$, $\alpha_1 = \delta$, $\beta_1 = \ell!$ while for $\lambda = 0$, $A_0 = C_0(F^*)$ (where F^* is the total space of the dual bundle of F) and $\alpha_0 \in KK(F^*, V)$, $\beta_0 \in KK(V, F^*)$ are given by the classical Thom isomorphism for the bundle F^*. The most convenient description of A_λ is as the C^* algebra of a groupoid G_λ and we shall just give the construction of the continuous family G_λ, the construction of the corresponding C^* algebra A_λ can be done as in section 5 (see also [24] for the general theory). For each λ one has $G_\lambda^{(0)} = V$ and the underlying topological space is the total space of the bundle F. For $\lambda = 0$, the source and range maps s_0, r_0 are equal to the projection of F on its base and the groupoid structure of F is the obvious group structure on each fiber F_x. For $\lambda \neq 0$, the groupoid structure of G_λ is determined by the source and range maps s_λ, r_λ given by :

$$s_\lambda(X) = x \quad \forall X \in F_x$$

$$r_\lambda(X) = \exp_x(\lambda X) \in \text{ leaf of } x \text{ in } V$$

where the exponential is taken in the leaf of x.

It is clear that for $\lambda \neq 0$, G_λ is isomorphic to G. The continuity of this family at $\lambda = 0$ should be clear from the following picture :



To describe $\beta_\lambda \in KK(C(V), A_\lambda)$, we consider as in section 10 3), the representation of $\mathcal{G}_\lambda$ in the borel field of hilbert spaces $L^2(\mathcal{G}_{\lambda,x}, r_\lambda^*(S))$ given by right translations. This representation defines a C^* module $\mathcal{E}_\lambda$ over A_λ, which for $\lambda = 0$, corresponds to the action by translations of F_x on each $L^2(F_x) \otimes S_x$. Each $h \in C(V)$ defines an endomorphism of $\mathcal{E}_\lambda$, as a multiplication operator by the function $h \circ r_\lambda$ on each $L^2(\mathcal{G}_{\lambda,x}, r_\lambda^*(S))$. Finally exactly as in section 10 3), we define a right invariant family D_x^λ of pseudo-differential operators of order 0, D_x^λ acting on $\mathcal{G}_{\lambda,x}$ from the bundle $r_\lambda^*(S^+)$ to $r_\lambda^*(S^-)$ and with principal symbol given by Clifford multiplication. For $\lambda = 0$ we obtain a translation invariant operator, which is most easily described, using Fourier transform, as the operator from $L^2(F_x^*) \otimes S_x^+$ to $L^2(F_x^*) \otimes S_x^-$ given by Clifford multiplication by ξ', $\xi \in F_x^*$. To describe $\alpha_\lambda \in KK(A_\lambda, C(V))$, we just have to introduce the parameter λ in the construction of δ given above. By homotopy invariance the check that $\ell! \boxtimes \delta = id_V$ is reduced to $\beta_0 \boxtimes \alpha_0 = id_V$ which is handled by [18].

13. NON COMMUTATIVE DIFFERENTIAL GEOMETRY AND TRANSVERSAL ELLIPTIC THEORY.

In the construction of the Thom isomorphism $K_0(A) \simeq K_1(A \rtimes_\alpha \mathbb{R})$ for C^* dynamical $\mathbb{R}$-systems $(A, \mathbb{R}, \alpha)$ (*), the crucial lemma is the following :

LEMMA. Given any orthoprojection $e \in \mathrm{Proj}(A)$, there exists an equivalent orthoprojection $e' \in \mathrm{Proj}(A)$ and an outer equivalent action α' of $\mathbb{R}$ on A, such that $\alpha'_t(e') = e' \quad \forall t \in \mathbb{R}$.

Though the proof of this lemma is rather trivial, it does not work for actions of $\mathbb{R}^2$, and what we want to explain first is how trying the proof for actions of $\mathbb{R}^2$ one is led to the notion of curvature, and one can then exhibit a non trivial obstruction.

In the case of $\mathbb{R}$ the proof goes as follows : 1) One replaces e by an equivalent $e' \in \mathrm{Proj}(A)$, where e' is now in the domain of the derivation δ, $\delta(x) = \lim_{t \to 0} \frac{1}{t}(\alpha_t(x) - x)$ which defines the action of $\mathbb{R}$. One does this by first smoothing e (i.e. replacing e by $\alpha_f(e) = \int f(t)\alpha_t(e)\,dt$ where $f \in C_c^\infty(\mathbb{R})$ and then replacing $\alpha_f(e)$ by its support. Of course all this works equally well for actions of $\mathbb{R}^2$. 2) Letting H be the unbounded multiplier of the crossed product $A \rtimes_\alpha \mathbb{R}$ such that $\delta(x) = Hx - xH \quad \forall x \in A$ one replaces the off diagonal terms $e'H(1-e')$ and $(1-e')He'$ of H by 0. In other words one defines :

$$H' = \begin{bmatrix} H_{11} & 0 \\ 0 & H_{22} \end{bmatrix} = e'He' + (1-e')H(1-e')$$

(*) A is a C^* algebra on which the real line $\mathbb{R}$ acts by automorphisms $(\alpha_t)_{t \in \mathbb{R}}$.

Let us pass to the case of $\mathbb{R}^2$. We let δ_1, δ_2 be the unbounded derivations of A associated to the generators X_1, X_2 of the Lie algebra of $\mathbb{R}^2$ and H_1, H_2 be the corresponding unbounded multipliers of the crossed product $A \rtimes_\alpha \mathbb{R}^2$.

We may from step 1) above assume that e is smooth, i.e. that the map $t \mapsto \alpha_t(e)$ is C^∞. What we want is to modify H_1, H_2 to make them commute with e. As above, the obvious thing to do is to replace H_j by $H'_j = eHe + (1-e)H(1-e)$ for $j = 1,2$. The problem however, is that now H'_1 and H'_2 no longer commute : blocks of commuting matrices do notnecessarily commute. Thus H'_1, H'_2 no longer define an action of $\mathbb{R}^2$. So this method fails, and we shall now analyse why, by interpreting the commutator of eH_1e and eH_2e as the curvature of a connection.

Recalling the correspondence between projections and finite projective modules we put :

DEFINITION. Let $\mathcal{E}^\infty$ be a finite projective module over $A^\infty = \{x \in A, t \mapsto \alpha_t(x)$ is of class $C^\infty\}$. Then a connection ∇ on $\mathcal{E}^\infty$ is a pair of linear maps $\nabla_1, \nabla_2 : \mathcal{E}^\infty \mapsto \mathcal{E}^\infty$ such that :

$$\nabla_j(\xi x) = (\nabla_j \xi)x + \xi\,\delta_j(x) \qquad \forall \xi \in \mathcal{E}^\infty, x \in A^\infty$$

In the above situation the equality $\mathcal{E}^\infty = eA^\infty$ defines a (right) finite projective module over A^∞, and the following equalities define a special connection : the Grassmannian connection, on $\mathcal{E}^\infty$:

$$\nabla_j \xi = e\,\delta_j(\xi) \qquad \forall \xi \in eA^\infty \quad j = 1,2 .$$

Since $\delta_j(\xi) = H_j\xi - \xi H_j$ with the above notations, we see that $\nabla_j(\xi) = (eH_je)\xi - \xi H_j$ and that :

$$(\nabla_1\nabla_2 - \nabla_2\nabla_1)\xi = [eH_1e, eH_2e]\xi$$

Now quite generally we have :

LEMMA. a) For any connection ∇ on $\mathcal{E}^\infty$, the map $\theta = \nabla_1\nabla_2 - \nabla_2\nabla_1$ of $\mathcal{E}^\infty$ in itself, is an endomorphism : $\theta(\xi x) = \theta(\xi)x \quad \forall \xi \in \mathcal{E}^\infty, x \in A^\infty$ so $\theta \in \mathrm{End}(\mathcal{E}^\infty)$.

b) Let $\nabla'_j = \nabla_j + \Gamma_j$ be another connection, $\Gamma_j \in \mathrm{End}(\mathcal{E}^\infty)$, then one has the equality :

$$\theta' - \theta = \partial_1\Gamma_2 - \partial_2\Gamma_1 + [\Gamma_1, \Gamma_2] \quad \text{where :}$$

$$(\partial_j X)\xi = \nabla_j(X\xi) - X(\nabla_j\xi)$$

for any $X \in \mathrm{End}\,\mathcal{E}^\infty$.

<u>Proof</u>. a) $(\nabla_1\nabla_2 - \nabla_2\nabla_1)(\xi x) = \nabla_1((\nabla_2\xi)x + \xi\delta_2(x)) - \nabla_2((\nabla_1\xi)x + \xi\delta_1(x)) = (\nabla_1\nabla_2\xi)x - (\nabla_2\nabla_1\xi)x$.

b) $(\nabla'_1\nabla'_2 - \nabla'_2\nabla'_1)\xi = \nabla'_1(\nabla_2\xi + \Gamma_2\xi) - \nabla'_2(\nabla_1\xi + \Gamma_1\xi)$

$= \nabla_1(\nabla_2\xi + \Gamma_2\xi) + \Gamma_1(\nabla_2\xi + \Gamma_2\xi) - \nabla_2(\nabla_1\xi + \Gamma_1\xi)$
$- \Gamma_2(\nabla_1\xi + \Gamma_1\xi) =$

$$(\nabla_1\nabla_2 - \nabla_2\nabla_1)\xi + \nabla_1(\Gamma_2\xi) - \Gamma_2(\nabla_1\xi) - \nabla_2(\Gamma_1\xi) + \Gamma_1\nabla_2\xi + [\Gamma_1, \Gamma_2]\xi.$$

Now one easily checks that the maps ∂_j of $\mathrm{End}\,\mathcal{E}^\infty$ in itself are derivations, the commutator $\partial_1\partial_2 - \partial_2\partial_1$ being the inner derivation implemented by θ.

To extract from the above lemma an actual obstruction to our initial problem we shall assume that the C^* algebra A is gifted with a trace τ, $\tau(1) = 1$ which is invariant under the automorphisms α_t. The invariant will be the trace of the curvature θ (up to a normalization factor $2\pi i$), as the trace is defined on A not on $\mathrm{End}\, A^\infty$, one first has to extend it but this is easy. For instance if $\mathcal{E}^\infty = eA^\infty$ then $\mathrm{End}\ \mathcal{E}^\infty = eA^\infty e$ and one just takes the restriction of τ. We let $\tau_{\mathcal{E}}$ be the extension of τ to $\mathcal{E}$.

THEOREM. The scalar $c_1(\mathcal{E}) = (2\pi i)^{-1}\tau_{\mathcal{E}}(\theta)$ is independent of the choice of connection, it depends only on the class $[\mathcal{E}] \in K_0(A)$ and on the outer equivalence class of the action α.

Proof. Since τ is α-invariant one easily checks that $\tau_{\mathcal{E}}(\partial_j X) = 0$ $j = 1,2$, $X \in \mathrm{End}\ \mathcal{E}^\infty$. Thus the above lemma makes clear that $(2\pi i)^{-1}$ $\tau_{\mathcal{E}}(\nabla_1\nabla_2 - \nabla_2\nabla_1)$ is independent of the choice of the connection ∇. Next, if two finite projective modules $\mathcal{E}_1^\infty, \mathcal{E}_2^\infty$ on A^∞ have isomorphic images $\mathcal{E}_1, \mathcal{E}_2$ as modules over A : $\mathcal{E}_j = \mathcal{E}_j^\infty \otimes_{A^\infty} A$, they are isomorphic. As obviously : $c_1(\mathcal{E} \oplus \mathcal{E}') = c_1(\mathcal{E}) + c_1(\mathcal{E}')$ for any pair $\mathcal{E}, \mathcal{E}'$, it is clear that c_1 is a map from $K_0(A)$ to $\mathbb{R}$. Using the two by two matrix trick one gets easily that $c_1(\mathcal{E})$ depends only upon the outer equivalence class of the action α. Q.E.D.

Coming back to the case $\mathcal{E}^\infty = eA^\infty$ with $e \in \mathrm{Proj}(A^\infty)$, one gets the formula :

$$c_1(e) = (2\pi i)^{-1}\tau(e[\delta_1(e), \delta_2(e)])$$

So, in fact, if one can find a projection $e' \sim e$ and an outer equivalent action α' such that just one of the $\delta_j'(e')$ is 0, one gets the vanishing of $c_1(e)$. Thus it remains, to show that this obstruction is non trivial, to exhibit an example where $c_1(e) \neq 0$.

Our example will be the irrational rotation algebra A_θ, generated by two unitaries U, V satisfying the relation $VU = e^{i2\pi\theta}UV$ where $\theta \notin \mathbb{Q}$.

We shall consider A_θ as the crossed product of $C(S^1)$ by the irrational rotation of angle θ and write its generic element as a sum $\sum a_{n,m} U^n V^m = \sum f_m V^m$ where $f_m \in C(S^1)$ admits the $a_{n,m} \in \mathbb{C}$ as Fourier coefficients. There are two commuting derivations δ_1, δ_2 of A_θ which define an action α of $\mathbb{R}^2$ on A_θ and are given by $\delta_1(\sum f_m V^m) = \sum f'_m V^m$ (where f'_m is the ordinary derivative of the periodic function f_m), while $\delta_2(\sum f_m V^m) = \sum (2\pi i m) f_m V^m$. In the U, V notation the two formulas are of course similar.

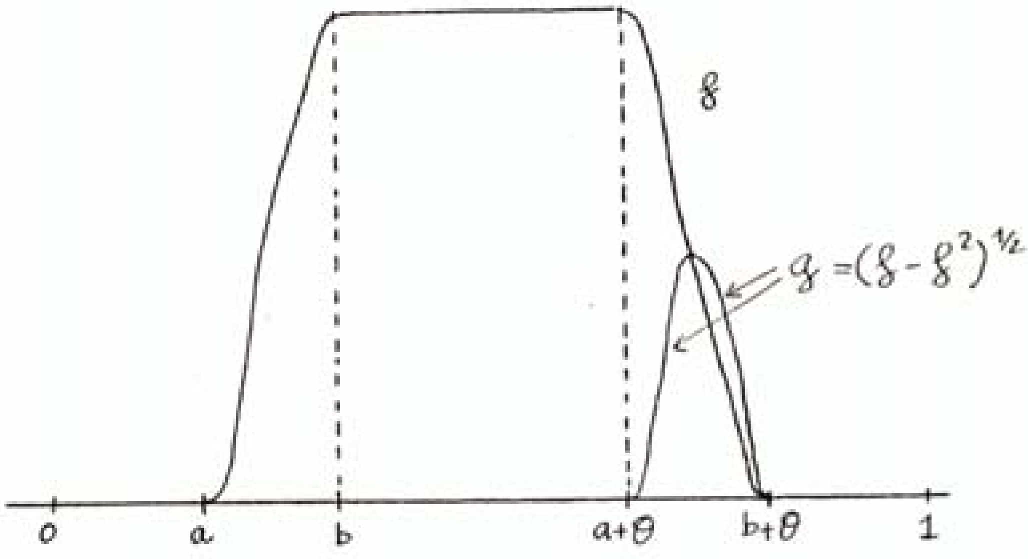
We note that though it is very difficult to give the conditions on the numerical coefficients $a_{n,m} \in \mathbb{C}$ which make $\sum a_{n,m} U^n V^m$ an element of A_θ (it is already difficult to decide when $\sum a_n U^n$ defines an element of $C(S^1)$) it is very easy to characterize elements of $A_\theta^\infty = \{ x \in A_\theta \, , t \mapsto \alpha_t(x) \text{ is } C^\infty \} = \bigcap \text{Domain } \delta_1^{k_1} \delta_2^{k_2}$.

LEMMA. $A_\theta^\infty = \{ \sum a_{n,m} U^n V^m, (a_{n,m})$ is a sequence with rapid decay$\}$.

<u>Proof</u>. The equality $\tau(\sum a_{n,m} U^n V^n) = a_{0,0}$ defines a trace on A_θ, the finiteness of $\tau(x^* x)$ shows that for $\sum a_{n,m} U^n V^m \in A_\theta$ the sequence $(a_{n,m})$ is ℓ^2. This makes it clear that if $\sum a_{n,m} U^n V^m \in A_\theta^\infty$ one has $|n^k m^{k'} a_{n,m}| \mapsto 0$ as $n, m \mapsto \infty$ for any k, k'. Conversely, if $(a_{n,m})$ is of rapid decay, one has $\sum a_{n,m} U^n V^m$ because the sequence is ℓ^1, so obviously then, one has $\sum a_{n,m} U^n V^m \in A_\theta^\infty$. Q.E.D.

Of course to have a chance to get $c_1(e) \neq 0$, we need a non trivial projection $e \in \text{Proj } A_\theta$. We shall use the construction due to R. Powers and M. Rieffel : the projection e is of the form ([25])

$e = V^* g + f + gV$ where $f, g \in C^\infty(S^1)$ have graphs as given in the following picture :



So f is equal to 0 on $[0,a]$ and $[b+\theta, 1]$, to 1 on $[b, a+\theta]$ and for $s \in [a,b]$ one has $f(s) + f(s+\theta) = 1$. While g is equal to 0 except on $[a+\theta, b+\theta]$ where $g = (f - f^2)^{1/2}$.
We then have to compute $C_1(e) = (2\pi i)^{-1} \tau(e[\delta_1(e), \delta_2(e)])$. Here $\tau(\sum f_m V^m) = \int f_0(t)\, dt$ and δ_1, δ_2 are as given above, so the computation is duable and gives : $\frac{1}{2\pi i} [\delta_1(e), \delta_2(e)]$
$= V^*(g f' - g f'^\theta) + 2((gg')^{-\theta} - gg') + (g f' - g f'^\theta) V$
where for $h \in C(S^1)$, h^θ is the translate of h by θ . In the product $\frac{1}{2i\pi} e[\delta_1(e), \delta_2(e)]$ the term which is independent of V is given by :

$$g(f' - f'^\theta) + 2f((gg')^{-\theta} - gg') + (g^2(f' - f'^\theta))^{-\theta}$$

When we integrate this function on S^1 we can delete the last $(\quad)^{-\theta}$ and with $u = g^2$, $v = f - f^\theta$ we get the integral of $2uv' - u'v$, i.e. $3\int uv' = -6\int g^2 f'$. Next since $g = 0$ outside the interval $[a+\theta, b+\theta]$, where f varies from 1 to 0 , and $g^2 = f - f^2$ we get :

$$C_1(e) = 6\int_0^1 (\lambda - \lambda^2)\, d\lambda = 1$$

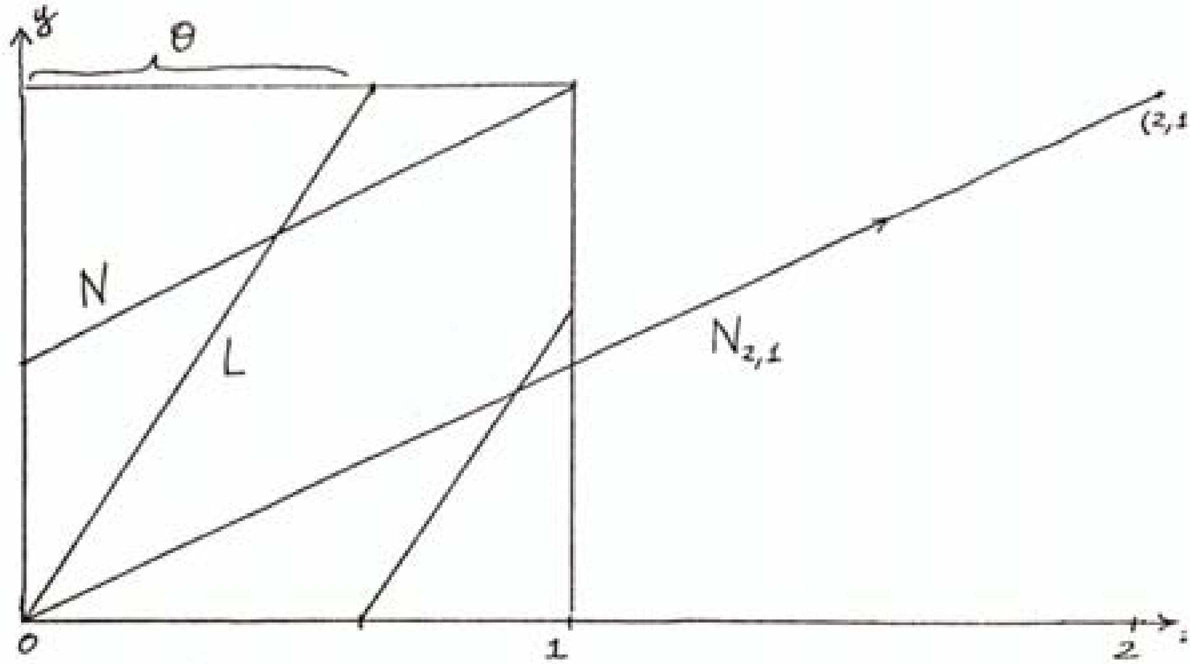
Thus $C_1(e) \neq 0$ and we have shown that the obstruction to our problem can indeed be non trivial. There is however a very interesting observation :

COROLLARY. With A_θ as above, the map $C_1 : K_0(A_\theta) \to \mathbb{R}$ maps K_0 in the integers $\mathbb{Z}$.

<u>Proof</u>. By [22], $K_0(A_\theta)$ is isomorphic to $\mathbb{Z}^2$ and is generated by the class of the trivial projection 1 and the class of the above projection e. Q.E.D.

The above computation proves the <u>integrality</u> of $C_1(e)$ but gives little understanding of the reason of this integrality, we are at about the same stage as if we had computed the integral of the Gaussian curvature of a Riemann surface and found $2\pi \times$ integer, but without any understanding of this integer.

We shall first begin by doing the computation of $C_1 : K_0(A_\theta) \to \mathbb{R}$ in a much better way. Instead of representing elements of $K_0^+(A_\theta)$ by classes of projections like e, we shall represent them by classes of finite projective modules over A_θ^∞. Now by lemma 100, A_θ^∞ is very easy to describe, so one expects that the same will be true for finite projective modules over this algebra. (In a differential geometric language, what we are doing is representing elements of K_0^+ by classes of smooth vector bundles). To realise each element of $K_0^+(A_\theta)$ as a finite projective module over A_θ^∞ we note that A_θ is stably isomorphic to the C^* algebra of the Kronecker foliation F_θ, $dx = \theta\, dy$ of the two torus $V = \mathbb{R}^2/\mathbb{Z}^2$.



Then each closed transversal N to (V, F_θ) determines canonically a finite projective module over A_θ. Choosing the smooth transversal given by the closed geodesic $N_{p,q}$; p,q relatively prime integers, one gets a finite projective module $\mathcal{E}^\infty_{p,q}$ over A^∞_θ which we now describe. The total space of $\mathcal{E}^\infty_{p,q}$ is the sum of q copies of the Schwartz space $\mathcal{S}(\mathbb{R})$, so we consider an element ξ of $\mathcal{E}^\infty_{p,q}$ as a function of two variables, one real variable $s \in \mathbb{R}$ and one variable $h \in \mathbb{Z}/q$. The action of the two generators U and V of A^∞_θ is now defined as follows :

$$(\xi . V)(s,h) = \xi(s-\varepsilon, h+1) \qquad \text{where} \quad \varepsilon = \frac{p}{q} - \theta$$

$$(\xi . f)(s,h) = \xi(s,h)\, f(s+(p/q)h)$$

where f is a periodic function : $f \in C^\infty(\mathbb{R}/\mathbb{Z})$, and U is the function $U(s) = \exp(2\pi i s)$.

One checks the compatibility of this action with the equality $VU = e^{2\pi i\theta}\, UV$, and if $\theta \notin \mathbb{Q}$ one can show that $\mathcal{E}^\infty_{p,q}$ is now a <u>finite projective module</u> over A_θ^∞.

LEMMA. The operators $(\nabla_1 \xi)(s,h) = \frac{d}{ds}\, \xi(s,h)$ and $(\nabla_2 \xi)(s,h) = 2\pi i\, s/\varepsilon\;\; \xi(s,h)$, $\varepsilon = \frac{p}{q} - \theta$ define a connection on the finite projective module $\mathcal{E}_{p,q}^\infty$.

<u>Proof</u>. One checks that $\nabla_1(\xi V) = (\nabla_1 \xi)\, V$, $\nabla_1(\xi f) = (\nabla_1 \xi) f + \xi f'$ and that $\nabla_2(\xi V) = (\nabla_2 \xi) V + 2\pi i\, \xi V$.
$\nabla_2(\xi f) = (\nabla_2 \xi) f \quad , \ \forall \xi \in \mathcal{E}_{p,q}^\infty ,\ f \in C^\infty(S^1)$. Q.E.D.

It follows now that the endomorphism of $\mathcal{E}_{p,q}^\infty$ given by the curvature of this connection, is a multiple of the identity operator :

$$\frac{1}{2i\pi}\left(\nabla_1 \nabla_2 - \nabla_2 \nabla_1\right) = \frac{1}{\varepsilon}\, id = \frac{1}{p/q - \theta}\, id$$

Thus this connection has constant curvature, whose value is the irrational number $(p/q - \theta)^{-1}$. But the dimension (in the sense of Murray and von Neumann) of the finite projective module $\mathcal{E}_{p,q}^\infty$ is also irrational, it is given by the transverse measure of the closed geodesic $N_{p,q}$, i.e. $\dim\left(\mathcal{E}_{p,q}^\infty\right) = \int_0^1 |(p\,dt - \theta q\, dt)| = |p - \theta q|$

So the computation of $c_1\left(\mathcal{E}_{p,q}^\infty\right)$ is now straightforward, $c_1\left(\mathcal{E}_{p,q}^\infty\right)$ = constant value of the curvature $\times \dim \mathcal{E}_{p,q}^\infty =$ $\left(\frac{1}{p/q - \theta}\right) |p - q\theta| = \pm q$.

In this way we get a much better and convincing prooof of the integrality of C_1. In classical differential geometry most integrality results, such as the integrality of $(2\pi^{-1})\int$Gaussian curvature, are corollaries of the Atiyah-Singer index theorem, the left hand side <u>Index D</u> being obviously an integer. In our context the integrality of C_1 is also a corollary of an index theorem. Instead of stating it in general we shall specialize to the above situation where the role of the algebra $C^\infty(X)$ of smooth functions on the manifold X is played by the (highly non commutative) algebra A_θ^∞ . We shall consider differential operators from the "bundle" $\mathcal{E}^\infty_{(0,1)}$ to itself, and of the form :

$$D = \sum a_{\alpha,\beta} \nabla_1^\alpha \nabla_2^\beta \qquad a_{\alpha,\beta} \in \operatorname{End} \mathcal{E}^\infty_{(0,1)}$$

Here, since $p=0, q=1$, the space $\mathcal{E}^\infty_{(0,1)}$ is identical with the Schwartz space $\mathcal{S}(\mathbb{R})$ and ∇_1 is ordinary differentiation , ∇_2 is $\frac{2\pi}{i\theta}$ x multiplication by the variable. The operators $a_{\alpha,\beta}$ are arbitrary endomorphisms of $\mathcal{E}^\infty_{(0,1)}$, the two basic endomorphisms being the multiplication by the periodic function $\exp\left(\frac{2\pi i}{\theta} s\right)$ and the translation of 1. Thus typically D is an operator like :

The order n of $D = \sum a_{\alpha,\beta} \nabla_1^\alpha \nabla_2^\beta$ is defined as the supremum of $\alpha+\beta$ and its principal symbol $\sigma_D(\xi)$ for $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ is the endomorphism $\sum_{\alpha+\beta=n} a_{\alpha,\beta}\, \xi_1^\alpha \xi_2^\beta$ of $\mathcal{E}^\infty_{(0,1)}$. D is called elliptic iff $\sigma_D(\xi)$ is invertible for any $\xi \neq 0$. Restricting ξ to $\|\xi\| = 1$, one thus gets a loop of invertible elements of $\operatorname{End}\left(\mathcal{E}^\infty_{(0,1)}\right)$ and hence by Bott periodicity an element of K_0 of this algebra. The meaning of $c_1(\sigma_D)$ should then be clear.

THEOREM. Let D be an elliptic operator of the above form. Then the difference differential equation : $Df=0$ has a finite dimensional space of solutions $f \in \mathcal{S}(\mathbb{R})$ and Index $D = \dim \operatorname{Ker} D - \dim \operatorname{Ker} D^*$ is equal to the scalar $c_1(\sigma_D)$.

Thus we see that from C^* algebra theory (the C^* algebra being A_θ) one obtains a highly non trivial information about difference differential equations. This is in fact a special case (the case of the Kronecker foliation F_θ) of the transversal elliptic theory for foliations.

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