INVARIANT ELLIPTIC EQUATIONS AND DISCRETE SERIES REPRESENTATIONS

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Introduction. For certain classes of Lie groups, their square-integrable (modulo the center) irreducible unitary representations have been concretely realized as spaces of L^2-solutions of invariant elliptic equations of Dirac, or Laplace-Beltrami, type (see [2] , [5] and the references given there). It is the purpose of the present note to put in evidence the converse phenomenon : subject to some restrictions on the group, the space of L^2-solutions of any invariant elliptic system of equations, defined over a homogeneous space whose isotropy subgroups are compact modulo the center, decomposes as a finite sum of irreducible square-integrable representations. This result clearly embodies a vanishing theorem which, in a certain sense, supplements the existence criterion for L^2-solutions provided by our L^2-index theorem [3] .

Statement of the results. Let G be a connected unimodular Lie group whose center Z has finitely many connected components, and let H be a closed subgroup of G , such that Z is contained in H as a co-compact subgroup. Further, let $\mathbb{E}$ and $\mathbb{F}$ be two Hermitian homogeneous vector bundles over $M = G/H$, arising from the finite-dimensional unitary H-modules E and F respectively. Consider an elliptic differential operator $D : C^\infty(M, \mathbb{E}) \to C^\infty(M, \mathbb{F})$ which is invariant in the sense that it commutes with the action of G on C^∞-sections. After fixing an invariant measure on M , one can form the Hilbert spaces $L^2(M, \mathbb{E})$ and $L^2(M, \mathbb{F})$ of L^2-sections, on which G acts unitarily. The operator $D : C_c^\infty(M, \mathbb{E}) \to C_c^\infty(M, \mathbb{F})$ extends to a closed operator, also denoted D, from $L^2(M, \mathbb{E})$ to $L^2(M, \mathbb{F})$. As mentioned from the beginning, we are interested in the structure of the Hilbert space $\mathcal{H}(D)$ of all L^2-solutions for the equation $D\varphi = 0$, regarded as a unitary G-module. For this reason, there is no loss of generality in assuming that E and F are irreducible H-modules, and that Z acts on both of them by the same unitary character $\zeta \in \hat{Z}$ however, dividing by the kernel of ζ, we can assume that H is compact.

As in the case when H is trivial, it can be shown that there is no ambiguity about the adjoint of D ; more precisely, one has :

LEMMA 1. If $D' : C_c^\infty(M, \mathbb{F}) \to C_c^\infty(M, \mathbb{E})$ denotes the formal adjoint of D, then the closure of D' coincides with the Hilbert space adjoint D^* of D.

In view of this result, $\mathcal{H}(D) = \mathcal{H}(D^*D) = \mathcal{H}((D^*D)^p)$ for any positive integer $p \geq 1$, so that we can assume, whenever convenient, that D is a self-adjoint operator of arbitrarily high order.

Now $L^2(M, \mathbb{E})$ can be regarded as the closed subspace $(L^2(G) \otimes E)_H$ of all H-invariant elements in the Hilbert tensor product $L^2(G) \otimes E$, H acting on $L^2(G)$ by right translations and in E by the given representation. For $f \in L^1(G) \otimes \mathrm{End}(E)$, $f = \Sigma_i f_i \otimes L_i$, let us define the operator $r(f) : L^2(G) \otimes E \to L^2(G) \otimes E$ by $r(f) = \Sigma_i r(f_i) \otimes L_i$, where r stands for the right regular representation of G. The group $H \times H$ acts in an obvious way on $\mathrm{End}(E)$, and it also acts on $L^1(G)$ by left translations in the first component and right translations in the second one. We denote by $(L^1(G) \otimes \mathrm{End}(E))_{H\times H}$ the subspace of all elements in $L^1(G) \otimes \mathrm{End}(E)$ which are invariant under the action of $H \times H$. If f is such an invariant element, $r(f)$ restricts to an operator

$r(f)_H : (L^2(G) \otimes E)_H \to (L^2(G) \otimes E)_H$, which commutes with the action of G by left translations.

LEMMA 2. There exist $h \in (C_c^{n-m-1}(G) \otimes \mathrm{End}(E))_{H\times H}$, where n is the order of D and m is the dimension of M , and $k \in (C_c(G) \otimes \mathrm{End}(E))_{H\times H}$, such that $r(h)_H D \subset I - r(k)_H$.

Suppose now that G is a connected linear Lie group. It is then known that it can be decomposed as a semi-direct product R.S , where R is a connected reductive Lie group and S is a connected, simply connected, solvable Lie group, normal in G . We require the existence of such a decomposition with the following additional properties :

(i) the center of S coincides with the identity component of the center Z of G ;

(ii) there exists a linear functional σ on the Lie algebra $\mathfrak{s}$ of S whose orbit under the coadjoint representation of S is precisely $\sigma + z^\perp$, where z denotes the Lie algebra of Z and $z^\perp = \{\tau \in \mathfrak{s}^* \,;\, \tau|z = 0\}$.

THEOREM. Let G be a connected linear Lie group satisfying the conditions (i) and (ii), H a closed subgroup containing the center Z and such that H/Z is compact, $\mathbb{E}$, $\mathbb{F}$ two Hermitian homogeneous vector bundles over $M = G/H$, and let $D : C^\infty(M, \mathbb{E}) \to C^\infty(M, \mathbb{F})$ be an invariant elliptic operator. Then the unitary representation of G on the corresponding Hilbert space $\mathcal{H}(D)$ of L^2-solutions decomposes as a finite direct sum of irreducible unitary representations of G , which are all square-integrable modulo Z .

This result was previously known, even with precise information about the decomposition, for more particular classes of Lie groups and only for Laplace-Beltrami or Dirac-type operators (cf. [2], [5] , where other relevant papers are also quoted).

The significance of the class of Lie groups satisfying the structural properties (i), (ii) is put in evidence by the recent work of Anh [1] . Combining his results and Harish-Chandra's criterion for the existence of discrete series representations for reductive Lie groups, one gets, as an immediate consequence of the above theorem, the following vanishing criterion for L^2-solutions of invariant elliptic operators :

COROLLARY. Let K be the maximal compact subgroup of the reductive factor R in the decomposition of G . If rank $K <$ rank R , then $\mathcal{H}(D) = \{0\}$ for any elliptic D .

<u>Sketch of the proof</u>. As mentioned earlier, we are allowed to assume that H is compact, $E = F$, and D is self-adjoint of arbitrarily high order. In view of Lemma 2, $\operatorname{Ker} D \subset \operatorname{Ker}(I-r(k)_H)$, so that it is enough to prove that $\operatorname{Ker}(I-r(k)_H)$ is a finite sum of discrete series representations.

Let us first remark that the hypothesis concerning G imply that it is unimodular and its regular representation is of type I. Thus, the space of L^2-sections of $\mathbb{E}$ decomposes as a direct integral

$$(L^2(G) \otimes E)_H = \int_{\hat{G}} \mathcal{H}(\pi) \otimes (\mathcal{H}(\pi^*) \otimes E)_H \, d\mu(\pi) \ ;$$

here $\hat{G}$ is the unitary dual of G , μ is the Plancherel measure, $\mathcal{H}(\pi)$ stands for the Hilbert space of a representative in the class $\pi \in \hat{G}$, and π^* denotes the contragredient representation of π .

For any unitary representation π of G , the operator

$\pi(k)_H : (\mathcal{H}(\pi) \otimes E)_H \to (\mathcal{H}(\pi) \otimes E)_H$ can be defined exactly in the same way as in the case when π was the right regular representation r of G. It then follows from the above direct integral decomposition that

$$\mathrm{Ker}(I-r(k)_H) = \int_{\hat{G}} \mathcal{H}(\pi) \otimes \mathrm{Ker}(I-\pi(k^*)_H)\,d\mu(\pi) \quad .$$

Let $\hat{G}_d$ denote the set of all equivalence classes of irreducible square-integrable representations. Our main task is to show that there is no continuous part in the decomposition of $\mathrm{Ker}(I-r(k)_H)$, that is to prove :

LEMMA 3. The set of all $\pi \in G^\wedge \setminus \hat{G}_d$ such that $\mathrm{Ker}(I-\pi(k^*)_H) \neq 0$ has Plancherel measure zero.

The basic argument we use in the proof, which is reminiscent of the well-known fact that the Fourier transform of a C^∞-function with compact support is analytic, can be best illustrated in the reductive case. So, we will sketch the proof only for the situation when the solvable factor S is trivial, and thus G is reductive with compact center. Due to the work of Harish-Chandra, the Plancherel measure is explicitly known in this case (cf. [4]). Its support consists of (finitely many) "non-degenerate" series of representations, and each such series arises from a conjugacy class of Cartan subgroups, the discrete series corresponding to the conjugacy class of compact Cartan subgroups (if any). Thus, it is enough to concentrate our attention to the contribution given by a single conjugacy class. To this end, after choosing a Cartan involution θ (or, equivalently, a maximal compact subgroup K), let us fix a θ-stable, non-compact, Cartan subgroup C. Then $C = T \times A$, where T is the anisotropic part and A is the vector part, and the corresponding non-degenerate series is parametrized by the orbits of the appropriate Weyl groups acting on $\Lambda \times \mathfrak{a}_0^*$; here Λ is a subset of the lattice $\hat{T}$ and

$\mathfrak{a}_o^*$ is a Zariski open dense subset of the dual vector space $\mathfrak{a}^*$ of the Lie algebra $\mathfrak{a}$ of A. In terms of this parametrization, the contribution to the Plancherel measure coming from C can be described as $d\mu_c = \mu(\lambda,\nu)d\lambda \otimes d\nu$, where $\mu(\lambda,\nu)$ is an explicitly known function on $\Lambda \times \mathfrak{a}_o^*$, invariant under the Weyl group, $d\lambda$ is the counting measure on the discrete set Λ, and $d\nu$ is the Lebesgue measure on $\mathfrak{a}^*$. Thus, if $\pi_{\lambda,\nu}$ denotes the irreducible unitary representation of G corresponding to $(\lambda,\nu) \in \Lambda \times \mathfrak{a}_o^*$, it is enough to prove that, for a fixed $\lambda \in \Lambda$, the set of all $\nu \in \mathfrak{a}_o^*$ for which $\mathrm{Ker}(I - \pi_{\lambda,\nu}(k^*)_H) \neq 0$ is of Lebesgue measure 0. At his stage, let us recall that the family $\{\pi_{\lambda,\nu} \in \hat{G} ; \nu \in {}_o^*$ can be analitically continued in the parameter ν to the complexification $\mathfrak{a}_{\mathbb{C}}^*$ of $\mathfrak{a}^*$, and that all these representations can be realized in the same Hilbert space (although not necessarily as unitary representations). Further, since k^* is C^∞ with compact support, $\{\pi_{\lambda,\nu}(k^*) ; \nu \in \mathfrak{a}_{\mathbb{C}}^*$ is an analytic family of compact operators (cf. [4]). It follows that $\Sigma = \{\nu \in \mathfrak{a}^* ;$ $\dim \mathrm{Ker}(I-\pi_{\lambda,\nu}(k^*)_H) > d = \min\limits_{\nu} \dim \mathrm{Ker}(I-\pi_{\lambda,\nu}(k^*)_H)\}$ is an analytic set in $\mathfrak{a}^*$. We are thus left to prove that $d = 0$. Let $\chi_{\lambda,\nu}$ be the infinitesimal character of $\pi_{\lambda,\nu}$ and Ω the Casimir operator. Then $\chi_{\lambda,\nu}(\Omega)\pi_{\lambda,\nu}(k^*) = \pi_{\lambda,\nu}(\Omega k^*)$. Since $\chi_{\lambda,\nu}(\Omega) \to \infty$ when $\nu \to \infty$ while $\|\pi_{\lambda,\nu}(\Omega k^*)\|$ remains bounded by the L^1-norm of Ωk^*, it follows that $\pi_{\lambda,\nu}(k^*)$ is invertible when ν is suffiently far away from 0. This completes the proof of the lemma.

To finish the proof of the theorem we have to show that only finitely many irreducibles can occur in the decomposition of $\mathrm{Ker}(I-r(k)_H)$ and that each of them has finite multiplicity. Let us denote by $\underline{\mathrm{tr}}_G$ the natural trace on the commutant of the left regular representation of G in $L^2(G) \otimes E$ and by $\underline{\dim}_G$ the corresponding dimension function. Since $k \in C_c^\infty$ it can be shown that $\mathrm{tr}_G r(k)_H < \infty$, which in turn implies that $\dim_G \mathrm{Ker}(I-r(k)_H) < \infty$.

On the other hand, $\dim_G \mathrm{Ker}(I-r(k)_H) = \Sigma_{\pi\in\hat{G}} \dim \mathrm{Ker}(I-\pi(k^*))\mu(\{\pi\})$. But, for a suitable normalization of the Haar measure μ , the formal degrees $\mu(\{\pi\}) = \dim_G \mathcal{H}(\pi)$ are strictly positive integers.

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