A new perspective on Functional Integration

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Abstract

The core of this article is a general theorem with a large number of specializations. Given a manifold N and a finite number of one-parameter groups of point transformations on N with generators $Y, X_{(1)}, \cdots, X_{(d)}$, we obtain, via functional integration over spaces of pointed paths on N (paths with one fixed point), a one-parameter group of functional operators acting on tensor or spinor fields on N. The generator of this group is a quadratic form in the Lie derivatives $\mathcal{L}_{X_{(\alpha)}}$ in the $X_{(\alpha)}$-direction plus a term linear in $\mathcal{L}_Y$.

The basic functional integral is over $L^{2,1}$ paths $x : \mathbf{T} \to N$ (continuous paths with square integrable first derivative). Although the integrator is invariant under time translation, the integral is powerful enough to be used for systems which are not time translation invariant. We give seven non trivial applications of the basic formula, and we compute its semiclassical expansion.

The methods of proof are rigorous and combine Albeverio Høegh-Krohn oscillatory integrals with Elworthy's parametrization of paths in a curved space. Unlike other approaches we solve Schrödinger type equations directly, rather than solving first diffusion equations and then using analytic continuation.

I - Introduction

We have studied many applications of functional integration looking for its *substantifique moelle*[1] . Little by little, several ideas have taken shape concerning the domains of integration, the integrators, and the integrands.

1. The domain of integration is a function space.

Working with an infinite-dimensional space is easier than working with the limit for large n of the product of n copies of a finite-dimensional space. For example, a space of pointed paths (paths with one fixed point) is contractible even when the paths take their values in a non contractible space. Over the years the advantages – often the necessity – of working with spaces of paths rather than with the discretized version of the paths have become increasingly apparent:

– Semiclassical approximations, even in the presence of caustics, are obtained by expanding a functional on the space of paths around a dominating contribution [1,2].

– If the paths take their values in a multiply-connected space the topology of the space of paths plays a central role [3].

– A change of variable of integration, regarded as a map on the domain of integration gives, in two lines, the Cameron-Martin formula [4] obtained from discretized paths via a lengthy derivation, and generalizes its applications [5].

– Computing functional determinants using properties of linear maps on Banach spaces is simpler than computing limits of finite-dimensional determinants [6].

– A major progress in the definition and computation of functional integrals was achieved by Elworthy [7] when he parametrized the space of paths on a Riemannian manifold M, with fixed initial point a, by the space of paths on T_aM starting at the origin of T_aM.

The importance of the domain of integration, noted in the above examples, is even more striking in the formulation presented here. A key point is a generalization of Elworthy's idea. Consider a finite-dimensional manifold N, and denote by $\mathbf{T}$ some finite time interval. Here we parametrize a space $\mathcal{P}_{x_0}N$ of pointed paths $x : \mathbf{T} \to N$ by a space $\mathcal{P}_0\mathbb{R}^d$ of pointed paths $z : \mathbf{T} \to \mathbb{R}^d$. A general construction for maps

$$\mathcal{P}_0\mathbb{R}^d \to \mathcal{P}_{x_0}N \qquad \text{by} \qquad z \mapsto x$$

[1] F. Rabelais. Literally "bone marrow as a producer of substance" in *Gargantua*, Prologue de l'auteur.

is given in Section II by solving suitable first-order *differential* equations, not by constructing stochastic processes. It is stated in terms of vector fields Y, $X_{(1)}, \cdots, X_{(d)}$ on N. By specializing N and the vector fields, *one basic functional integral yields, with rigor and no Ansatz, a great variety of functional integrals which are solutions of complex problems.* A number of them are presented in Section IV.

In general, to define the domain of integration, one needs to specify:
- the analytic nature of the paths[2] (continuous, $L^2, L^{2,1}, \cdots$);
- the domains of the paths and their ranges;
- the behaviour of the paths at the boundaries of their domains.

2. Integrators cannot be expected to be universal.

The naive approach to the definition of an integral in an infinite number of variables is to take a limit $d = \infty$ in a d-dimensional integral. Because of scaling problems this procedure is well known to abort. For instance, if we wish to evaluate (for $a > 0$) the integral

$$I_\infty := \int_{\mathbb{R}^\infty} d^\infty x \exp\left(-\frac{\pi}{a}|x|^2\right), \tag{I.1}$$

where $|x|^2 := \sum_{\alpha=1}^{\infty} (x^\alpha)^2$, we may first evaluate the corresponding integral

$$I_d := \int_{\mathbb{R}^d} dx \exp\left(-\frac{\pi}{a}|x|^2\right) \tag{I.2}$$

for finite d and then set $d = \infty$. Since $I_d = a^{d/2}$ we get

$$I_\infty = \begin{cases} 0 & \text{if} \quad 0 < a < 1 \\ 1 & \text{if} \quad a = 1 \\ \infty & \text{if} \quad 1 < a \end{cases} \tag{I.3}$$

and this fails to be continuous in the parameter a, as should be reasonably desired.

A way out of this difficulty is to introduce, for each value of the scaling parameter $a > 0$, an integrator $\mathcal{D}_a x$ in the d-dimensional space $\mathbb{R}^d$, namely

$$\mathcal{D}_a x = a^{-d/2} dx^1 \cdots dx^d \tag{I.4}$$

[2] A path $z : \mathbf{T} \to \mathbb{R}^d$ is said to be $L^{2,1}$ if $\int_\mathbf{T} dt\, |\dot{z}(t)|^2 < \infty$ where $\dot{z}(t) = dz(t)/dt$.

and to remark that it is characterized by the following integration formula

$$\int_{\mathbb{R}^d} \mathcal{D}_a x \cdot \exp\left(-\frac{\pi}{a}|x|^2 - 2\pi i \langle x', x\rangle\right) = \exp\left(-\pi a |x'|^2\right). \tag{I.5}$$

Here x' runs over the space $\mathbb{R}_d$ dual to $\mathbb{R}^d$ and the scalar product is given[3] by $\langle x', x\rangle = x'_\alpha x^\alpha$.

This formula is dimension-independent and hence suitable for the generalization from $\mathbb{R}^d$ to a (real) Banach space $\mathbf{X}$. Let $\mathbf{X}'$ be its dual and consider two continuous quadratic forms Q on $\mathbf{X}$ and W on $\mathbf{X}'$. Assume that *Q and W are inverse to each other* in the following sense. There exist continuous linear maps

$$D : \mathbf{X} \to \mathbf{X}', \qquad G : \mathbf{X}' \to \mathbf{X}$$

such that[4]

$$\begin{cases} DG = GD = \mathbb{1} \\ \langle Dx, y\rangle = \langle Dy, x\rangle \\ Q(x) = \langle Dx, x\rangle \ , \ W(x') = \langle x', Gx'\rangle \ . \end{cases} \tag{I.6}$$

Here $\langle x', x\rangle$ denotes the duality between $\mathbf{X}$ (elements x) and its dual $\mathbf{X}'$ (elements x').

Then we define the integrator $\mathcal{D}_{s,Q}x$ (also denoted $\mathcal{D}x$ for simplicity) by the following requirement

$$\int_{\mathbf{X}} \mathcal{D}_{s,Q}x \cdot \exp\Big(-\frac{\pi}{s}Q(x) - 2\pi i \langle x', x\rangle\Big) = \exp(-\pi s W(x')) \ . \tag{I.7}$$

Here x' runs over $\mathbf{X}'$ and there are two cases:

i) $s = 1$ and Q is positive definite, namely $Q(x) > 0$ for $x \neq 0$;
ii) $s = i$ and there is no restriction on Q except it be real.

Introduction of the parameter s enables us to treat in a unified way the diffusion and the Schrödinger equations[5]. We can introduce a suitable space

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[3] Here and in the rest of this paper we use the Einstein summation convention over repeated indices.

[4] In standard applications, D is a differential operator and G the corresponding Green operator, taking into account the boundary conditions of the domain of D.

[5] It is not true, as is still often stated, that the case $s = i$ has no mathematical foundation. See for example references [8, 9, 10, 11].

$\mathcal{F}(\mathbf{X})$ of functionals on $\mathbf{X}$ integrable by $\mathcal{D}_{s,Q}$ and a norm on $\mathcal{F}(\mathbf{X})$, and then compute integrals of the type

$$I = \int_{\mathbf{X}} \mathcal{D}_{s,Q}x \cdot F(x) \tag{I.8}$$

for F in $\mathcal{F}(\mathbf{X})$. In both cases, $s = 1$ and $s = i$, we shall call $\mathcal{D}_{s,Q}x$ a ***Gaussian integrator.***

Gaussian integrators have the following properties:

$$\mathcal{D}(x + x_0) = \mathcal{D}x, \qquad x_0 \text{ a fixed element of } \mathbf{X} \tag{I.9}$$

$$\mathcal{D}(Lx) = |\operatorname{Det} L| \cdot \mathcal{D}x, \qquad L : \mathbf{X} \to \mathbf{X} \tag{I.10}$$

where L is in a suitable class of linear changes of variables of integration, including the obvious case where $Q(Lx) = Q(x)$ and $|\operatorname{Det} L| = 1$ (cf. formula (A.140) in Appendix A).

Gaussian integrators are not the only possible integrators. In reference [11] we have developed an axiomatic for functional integrals on a Banach space Φ expressed in terms of integrators $\mathcal{D}_{\Theta,Z}$ defined by

$$\int_{\Phi} \mathcal{D}_{\Theta,Z}\varphi \cdot \Theta(\varphi, J) = Z(J) \tag{I.11}$$

for φ in Φ, J in the dual Φ' of Φ, where Θ and Z are two given continuous bounded functionals

$$\Theta : \Phi \times \Phi' \to \mathbb{C} \quad , \quad Z : \Phi' \to \mathbb{C}.$$

In quantum field theory, we interpret φ as a field and J as a source, $Z(J)$ is then the Schwinger generating functional for the n-point functions.

3. Integrands and integrators.

Splitting the quantity inside the integral sign into "integrator" and "integrand" belongs to the art of integration, but rules of thumb apply:

– When the functional integral has its origin in physics try not to break up the action into, say, kinetic and potential contributions. On the other hand, do not hesitate to work with a potential which is a functional of a path rather than a function of its *value* (e.g. in equation (III.1), V is a functional of z, not a function of $z(t)$).

– Look for a possible change of variable of integration; this may suggest a practical choice for the integrand.

– Gaussian integrators have a wealth of simple, powerful properties; look for exponentials of quadratic forms, this suggests a practical choice for the integrator.

4. A basic functional integral.

The core of this article is a theorem which provides the mathematical underpinning for a great variety of functional integrals. It consists of two parts: the definition of a functional integral, and the partial differential equation satisfied by the *value* of the functional integral, as a function of a set of parameters. Given a manifold N consider the $L^{2,1}$ paths over a finite time interval $\mathbf{T}$ with values in N, $x : \mathbf{T} \to N$, and with a fixed point $x_0 \in N$; for instance, if $\mathbf{T} = [t_a, t_b]$, the fixed point can be either $x(t_a)$ or $x(t_b)$. Given $d+1$ vector fields Y and $X_{(\alpha)}$ on N, define a map P from a space $\mathcal{P}_0\mathbb{R}^d$ of $L^{2,1}$ paths into a space $\mathcal{P}_{x_0}N$ of $L^{2,1}$ paths,

$$P : \mathcal{P}_0\mathbb{R}^d \to \mathcal{P}_{x_0}N,$$

by $P(z) = x$, where

$$\begin{cases} dx(t,z) = X_{(\alpha)}(x(t,z))dz^\alpha + Y(x(t,z))dt \\ x(t_0,z) = x_0. \end{cases} \tag{I.12}$$

Here t_0 and x_0 are fixed, with t_0 in $\mathbf{T}$ and x_0 in N, and the paths $z : \mathbf{T} \to \mathbb{R}^d$ satisfy $z(t_0) = 0$. In general, the vector fields do not commute, that is:

$$[X_{(\alpha)}, X_{(\beta)}] \neq 0 \quad , \quad [Y, X_{(\alpha)}] \neq 0\,;$$

therefore the solution of (I.12) is of the form

$$x(t,z) = x_0 \cdot \Sigma(t,z) \tag{I.13}$$

where x is a *function of* x_0 *and* t, *and a functional of* z. Here $\Sigma(t,z)$ is a transformation in N, depending on t and z as stated. Only when $[X_{(\alpha)}, X_{(\beta)}] = [Y, X_{(\alpha)}] = 0$ can one express $\Sigma(t,z)$ as a function of t and $z(t)$.

Given (I.12) one can express an integral over $\mathcal{P}_{x_0}N$ as an integral over $\mathcal{P}_0\mathbb{R}^d$, which is an integral that one can manipulate and compute. The partial differential equation satisfied by the integral on $\mathcal{P}_{x_0}N$ is expressed in terms of Lie derivatives along the vector fields Y and $X_{(\alpha)}$. Conversely, given a parabolic partial differential equation, one can construct in many cases the path integral representation of its solutions (see for instance in Section IV, the lift of a covariant Laplacian at a point of a Riemannian manifold in terms of Lie derivatives at a point of its frame bundle [13]).

The basic equations are given in the first paragraph of Section II, followed by a summary of notations used throughout the paper.

5. Examples.

In Section III we compute semiclassical approximations of the basic functional integral (II.1), and in Section IV we specialize the basic integral by making particular choices of the manifold N. We treat in detail the following examples:

– $N = \mathbb{R}^d$, in cartesian or polar coordinates.

– N is a frame bundle $O(M)$ over a Riemannian manifold M. We give the explicit functional integral representing the solution Ψ of the Schrödinger equation on M, with initial wave function ϕ.

– N is a multiply-connected space.

– N is a $U(1)$-bundle; the basic integral solves the Schrödinger equation for a particle in an electromagnetic field.

– N is a symplectic manifold; coherent-state transitions can be obtained from the basic integral.

We conclude this section by an analysis of the Bohm-Aharonov effect, where all the previous techniques are brought to bear.

6. Techniques.

In the course of computing our basic functional integral in various situations, we have used the properties of linear changes[6] of variable of integration and properties of functional determinants; they are given in two appendices. The transformation Σ in formula (I.13) is related to the Cartan development map; a third appendix is devoted to its properties.

[6] Linear changes of variable of integration in an infinite-dimensional space are sufficiently powerful and varied for the purposes of this paper. In a later publication we shall present nonlinear changes, simplifying and generalizing earlier works such as [12].

II - A general theorem

The primary goal of this section is to define functional integrals over $L^{2,1}$ pointed paths (paths with a fixed point) taking their values in a manifold N, by reducing them to functional integrals over paths taking their values in a flat space $\mathbb{R}^d$. The main definition is given as follows

$$(U_T\phi)(x_0) := \int_{\mathbf{Z}_0} \mathcal{D}_{s,Q_0}z \cdot \exp\left(-\frac{\pi}{s}Q_0(z)\right)\phi\left(x_0 \cdot \Sigma(T,z)\right). \qquad \text{(II.1)}$$

All the notations are given in paragraph 1. The previous integral, being also denoted $\Psi(T, x_0)$, is a solution of the *generalized Schrödinger equation*

$$\frac{\partial \Psi}{\partial T} = \frac{s}{4\pi} h^{\alpha\beta} \mathcal{L}_{X_{(\alpha)}} \mathcal{L}_{X_{(\beta)}} \Psi + \mathcal{L}_Y \Psi \qquad \text{(II.2)}$$

with initial condition $\Psi(0, x_0) = \phi(x_0)$.

We give also a general construction of *time-ordered products* in the form of a functional integral generalizing equation (II.1), namely

$$\left(U_T^F\phi\right)(x_0) = \int_{\mathbf{Z}_0} \mathcal{D}_{s,Q_0}z \cdot \exp\left(-\frac{\pi}{s}Q_0(z)\right) F(\mathbf{T}, z)\phi\left(x_0 \cdot \Sigma(T,z)\right). \qquad \text{(II.3)}$$

In paragraph 2, we construct the simplest functional integral of type (II.1) and prove that it satisfies equation (II.2). In paragraph 3, we study the general case.

1. The setup, and a summary of notations.

1.1. *A manifold and vector fields.*

– A finite-dimensional manifold N.

– One-parameter groups acting on N, denoted $\sigma_{(\alpha)}(r)$; here α takes the values $0, 1, 2, \cdots, d$ and r is a real parameter; the transform of a point x in N under $\sigma_{(\alpha)}(r)$ is denoted by $x \cdot \sigma_{(\alpha)}(r)$, and

$$\sigma_{(\alpha)}(r) \circ \sigma_{(\alpha)}(s) = \sigma_{(\alpha)}(r+s). \qquad \text{(II.4)}$$

– The generator of $\left\{\sigma_{(\alpha)}(r)\right\}$ is the vector field $X_{(\alpha)}$ in N such that

$$\frac{d}{dr}\left(x \cdot \sigma_{(\alpha)}(r)\right) = X_{(\alpha)}\left(x \cdot \sigma_{(\alpha)}(r)\right) \qquad \text{(II.5)}$$

and in particular

$$X_{(\alpha)}(x) = \frac{d}{dr}\left(x \cdot \sigma_{(\alpha)}(r)\right)\Big|_{r=0} \tag{II.6}$$

for any point x in N. We do not assume that the vector fields $X_{(\alpha)}$ commute, hence $[X_{(\alpha)}, X_{(\beta)}] \neq 0$ in general. We often write Y for $X_{(0)}$ emphasizing its special role.

– $\mathcal{L}_X$ denotes the Lie derivative w.r.t. the vector field X.

1.2. *Pointed paths on the flat space* $\mathbb{R}^d$.

– $\mathbf{T}$ is a time interval of length T, hence

$$\mathbf{T} = [t_a, t_b] \ , \qquad T = t_b - t_a \,. \tag{II.7}$$

– t_0 is a chosen element of $\mathbf{T}$; the standard choices are $t_0 = t_a$ or $t_0 = t_b$.

– $\mathbf{Z}_0$ (or $\mathbf{Z}_{0,\mathbf{T}}$ if we need to specify $\mathbf{T}$) consists of the real vector-valued functions $z = (z^1, \cdots, z^d)$ whose components $t \mapsto z^\alpha(t)$ are continuous functions with square-integrable derivatives $\dot{z}^\alpha$. We assume the normalization $z^\alpha(t_0) = 0$.

– $\mathbf{Z}'_0$ is the space dual to $\mathbf{Z}_0$. Its elements are interpreted as vector-valued distributions $z' = (z'_1, \cdots, z'_d)$, each component being the derivative of an L^2-function. The duality is given by

$$\langle z', z \rangle = \int_{\mathbf{T}} dt \, z'_\alpha(t) z^\alpha(t) \tag{II.8}$$

(summation over α and integration over t).

– s is a parameter equal to 1 or i; its square root $\sqrt{s}$ is normalized as follows

$$\sqrt{s} = \begin{cases} 1 & \text{if} \quad s = 1 \\ e^{\pi i/4} & \text{if} \quad s = i \,. \end{cases} \tag{II.9}$$

– $h = (h_{\alpha\beta})$ is a constant invertible symmetric real matrix of size $d \times d$. We denote by $(h^{\alpha\beta})$ the inverse matrix and we assume that h is positive definite in case s is equal to 1. By a suitable linear change of coordinates, we may take h into a diagonal form

$$h = \text{diag}(1, \cdots, 1, -1, \cdots, -1) \tag{II.10}$$

with p elements $+1$, q elements -1 and $p + q = d$. Hence $p = d$, $q = 0$ in the case $s = 1$.

– We introduce a quadratic form Q_0 on $\mathbf{Z}_0$ as follows

$$Q_0(z) = \int_{\mathbf{T}} dt \, h_{\alpha\beta} \, \dot{z}^\alpha(t) \dot{z}^\beta(t) \,. \tag{II.11}$$

The corresponding kernel is given by

$$D_{\alpha\beta}(u,r) = \tfrac{1}{2}\frac{\delta^2 Q_0(z)}{\delta z^\alpha(u)\delta z^\beta(r)}, \tag{II.12}$$

hence the representation

$$Q_0(z) = \int_{\mathbf{T}} du \int_{\mathbf{T}} dr\, D_{\alpha\beta}(u,r)\, z^\alpha(u)\, z^\beta(r). \tag{II.13}$$

– On $\mathbf{Z}_0'$ we consider a quadratic form W_0, with kernel $G^{\alpha\beta}(u,r)$. As above, we have the relations

$$G^{\alpha\beta}(u,r) = \tfrac{1}{2}\frac{\delta^2 W_0(z')}{\delta z'_\alpha(u)\delta z'_\beta(r)}, \tag{II.14}$$

$$W_0(z') = \int_{\mathbf{T}} du \int_{\mathbf{T}} dr\, G^{\alpha\beta}(u,r)\, z'_\alpha(u) z'_\beta(r). \tag{II.15}$$

– We assume that the quadratic forms Q_0 on $\mathbf{Z}_0$ and W_0 on $\mathbf{Z}_0'$ are inverse to each other in the sense of relation (I.6). In terms of kernels, this is expressed as follows

$$\int_{\mathbf{T}} dt\, D_{\alpha\beta}(s_1,t)\, G^{\beta\gamma}(t,s_2) = \delta_\alpha^\gamma\, \delta(s_1 - s_2). \tag{II.16}$$

Here are explicit formulas for the kernels:

$$D_{\alpha\beta}(u,r) = \int_{\mathbf{T}} dt\, h_{\alpha\beta}\, \delta'(t-u)\delta'(t-r) = -h_{\alpha\beta}\, \delta''(u-r), \tag{II.17}$$

that is $D : \mathbf{Z}_0 \to \mathbf{Z}_0'$ is the differential operator with matrix $\left(-h_{\alpha\beta}\frac{d^2}{dt^2}\right)$. Hence, $G^{\alpha\beta}(u,r)$ is the corresponding Green's function taking into account the boundary condition $z^\alpha(t_0) = 0$, that is

$$G^{\alpha\beta}(u,r) = \begin{cases} h^{\alpha\beta} \inf(u-t_0, r-t_0) & \text{for } u \geq t_0, r \geq t_0, \\ h^{\alpha\beta} \inf(t_0-u, t_0-r) & \text{for } u \leq t_0, r \leq t_0, \\ 0 & \text{otherwise.} \end{cases} \tag{II.18}$$

Remark. We shall refrain from integrating by parts in formulas like (II.11) in order not to have to make explicit statements about boundary conditions.

1.3. *Functional integrals on $\mathcal{P}_0\mathbb{R}^d$.*

– The space of paths $\mathbf{Z}_0$ shall also be denoted by $\mathcal{P}_0\mathbb{R}^d$ to remind us of the flat space $\mathbb{R}^d$ where the paths lie and of the fixed point 0 (origin in $\mathbb{R}^d$) of the paths z. We shall consider a variety of pairs (Q, W) consisting of a quadratic form Q on a space $\mathbf{Z}$ and a quadratic form W on its dual $\mathbf{Z}'$ satisfying the analogues of relations (II.12), (II.14) and (II.16).

– For such a pair (Q, W) we have a translation invariant integrator $\mathcal{D}_{s,Q}z$ on $\mathbf{Z}$ characterized by

$$\int_{\mathbf{Z}} \mathcal{D}_{s,Q}z \cdot \exp\left(-\frac{\pi}{s}Q(z) - 2\pi i \langle z', z\rangle\right) = \exp(-\pi s W(z')) \tag{II.19}$$

for z' in $\mathbf{Z}'$. The normalizations are chosen so that Gaussian integrators and Fourier transforms do not include powers of π depending on the dimension of their domain of definition.

We also write

$$\mathcal{D}\omega_{s,Q}(z) = \mathcal{D}_{s,Q}z \cdot \exp\left(-\frac{\pi}{s}Q(z)\right) \tag{II.20}$$

and refer to both $\mathcal{D}_{s,Q}$ and $\mathcal{D}\omega_{s,Q}$ as "Gaussian integrators". When working with the basic pair (Q_0, W_0) we simply write $\mathcal{D}_s$ and $\mathcal{D}\omega_s$. In the context of an application where s has been chosen once for all, we omit it in the notations.

1.4. *Functional integrals on $\mathcal{P}_{x_0}N$.*

– We fix a point x_0 in N and consider continuous paths $x : \mathbf{T} \to N$ with the fixed point $x(t_0) = x_0$ and square-integrable velocity[7]. The set of all such paths is denoted by $\mathcal{P}_{x_0}N$.

– The time interval $\mathbf{T}$ being given, consider an element z of $\mathbf{Z}_0$. As we shall see in Appendix C, the differential equation

$$\begin{cases} dx(t) = X_{(\alpha)}(x(t))dz^{\alpha}(t) + Y(x(t))dt \\ x(t_0) = x_0 \end{cases} \tag{II.21}$$

admits a unique solution $x(\cdot)$ in $\mathcal{P}_{x_0}N$.

– The previous construction defines a parametrization P of the space $\mathcal{P}_{x_0}N$ of pointed paths on N by the space $\mathcal{P}_0\mathbb{R}^d$ of pointed paths in $\mathbb{R}^d$, namely

$$P : \mathcal{P}_0\mathbb{R}^d \to \mathcal{P}_{x_0}N$$

by taking z into x. If necessary we shall denote by $x(t, z)$ the solution of the differential equation (II.21) for given z in $\mathcal{P}_0\mathbb{R}^d$. Hence $x(t, z)$ is *a function of t and a functional of z.*

[7] More precisely, for every smooth function f in $C^\infty(N)$, we assume that the continuous function $t \mapsto f(x(t))$ on $\mathbf{T}$ is the primitive of a function in $L^2(\mathbf{T})$.

– Assume now that $\mathbf{T} = [0, T]$ and $t_0 = 0$. With the previous definitions, define $\Sigma(T, z)$ as the transformation taking a point x_0 in N into the point $x(T, z)$.

– Take a good[8] function ϕ on N. Define a function $\widetilde{\phi}$ on $\mathbf{Z}_0$ by

$$\widetilde{\phi}(z) = \phi\left(x_0 \cdot \Sigma(T, z)\right). \tag{II.22}$$

It is integrable under the integrator $\mathcal{D}\omega_s$ on $\mathbf{Z}_0$. By integrating we get

$$\begin{aligned} I(\phi, T, x_0) &= \int_{\mathbf{Z}_0} \mathcal{D}\omega_s(z)\widetilde{\phi}(z) \\ &= \int_{\mathbf{Z}_0} \mathcal{D}_s z \cdot \exp\left(-\frac{\pi}{s} Q_0(z)\right) \cdot \phi\left(x_0 \cdot \Sigma(T, z)\right). \end{aligned} \tag{II.23}$$

We could replace Q_0 by another quadratic form on $\mathbf{Z}_0$.

– The functional operator U_T associates to the function ϕ on N the function $x_0 \mapsto I(\phi, T, x_0)$ on N.

1.5. *Variational techniques.*

A frequently used technique consists in introducing one-parameter variations in the space of paths, or in the space of functionals on the space of paths. We work in particular with the following variations.

(i) Fix a time interval $\mathbf{T} = [t_a, t_b]$ and consider the space $\mathcal{P}\mathbb{R}^d$ of paths $x : \mathbf{T} \to \mathbb{R}^d$, say of class[9] C^1. Assuming an action functional $S : \mathcal{P}\mathbb{R}^d \to \mathbb{R}$ (for instance, the time integral of a Lagrangian $L(x, \dot{x}, t)$), the critical points of S will form a $2d$-dimensional manifold $\mathcal{U}^{2d}$, the so-called *space of* (classical) *motions*, denoted by x_{cl}. We can parametrize them by a set of parameters $\mu = (\mu^1, \cdots, \mu^{2d})$; for a given k between 1 and $2d$, the derivative $\partial x_{\text{cl}}(\mu)/\partial \mu^k$ defines a variation through classical paths, and we get $2d$ of them.

(ii) If t_0 is a given epoch in $\mathbf{T}$, the pointed paths are defined by the condition $x(t_0) = 0$. They form the space $\mathcal{P}_0\mathbb{R}^d$. In paragraph III.1 we shall use a one-parameter family of pointed paths $x(\lambda)$ in $\mathcal{P}_0\mathbb{R}^d$ (for λ running over $[0, 1]$).

(iii) Introducing d independent boundary conditions at t_a, and similarly at t_b, we define the subspace $\mathcal{P}_{a,b}\mathbb{R}^d$ of $\mathcal{P}\mathbb{R}^d$. In paragraph B.2, we consider a one-parameter family of paths $x(\lambda)$ in $\mathcal{P}_{a,b}\mathbb{R}^d$, with λ in $[0, 1]$, such that $x(0)$ belongs to $\mathcal{U}^{2d} \cap \mathcal{P}_{a,b}\mathbb{R}^d$. The presence and nature of the caustics is conveniently analyzed in terms of this intersection.

(iv) In paragraph III.2 we consider a one-parameter family of action functionals $S(\nu)$ defined on a space $\mathcal{P}_0\mathbb{R}^d$ of pointed paths.

[8] Any function in $C^\infty(N)$ with compact support will do.

[9] That is, continuous with continuous first-order derivatives.

2. The one-dimensional case.

We begin by the simple case $d = 1$. Here $X = X_{(1)}$ is a vector field on N generating the one-parameter group of transformation $\sigma(r)$ on N. Hence $\sigma(r)$ obeys the differential equation

$$d(x_0 \cdot \sigma(r)) = X(x_0 \cdot \sigma(r))\, dr \qquad \text{(II.24)}$$

for a fixed x_0 in N. We set $Y = 0$ and consider the time interval $\mathbf{T} = [0, T]$. The differential equation (II.21) reads now as

$$\begin{cases} dx(t) = X(x(t)) \cdot dz(t) \\ x(0) = x_0\,. \end{cases} \qquad \text{(II.25)}$$

Substituting $r = z(t)$ in (II.24) we see that the solution to the previous equation is given by $x(t) = x_0 \cdot \sigma(z(t))$. Hence the transformation $\Sigma(T, z)$ is simply $\sigma(z(T))$.

The path space $\mathbf{Z}_0$ consists of the $L^{2,1}$ functions $z : [0, T] \to \mathbf{R}$ such that $z(0) = 0$. It is endowed with the quadratic form

$$Q_0(z) = \int_0^T dt\, \dot{z}(t)^2\,. \qquad \text{(II.26)}$$

The corresponding integrator is, for simplicity, denoted by $\mathcal{D}_s z$, hence our basic path integral specializes to

$$\Psi(T, x) = \int_{\mathbf{Z}_0} \mathcal{D}_s z \cdot \exp\left(-\frac{\pi}{s} \int_0^T dt\, \dot{z}(t)^2\right) \phi(x \cdot \sigma(z(T)))\,. \qquad \text{(II.27)}$$

Our goal is to show that we have solved the differential equation

$$\frac{\partial}{\partial T} \Psi(T, x) = \frac{s}{4\pi} \mathcal{L}_X^2 \Psi(T, x)\,. \qquad \text{(II.28)}$$

Fix a point x in N and define a function of a real variable $h(r) = \phi(x \cdot \sigma(r))$. From the group property $\sigma(r_1) \circ \sigma(r_2) = \sigma(r_1 + r_2)$, we get

$$\phi(x \cdot \sigma(r) \cdot \sigma(z(T))) = h(r + z(T))\,. \qquad \text{(II.29)}$$

Then define

$$H(T, r) := \int_{\mathbf{Z}_0} \mathcal{D}_s z \cdot e^{-\pi Q_0(z)/s} h(r + z(T))\,. \qquad \text{(II.30)}$$

Since the integrand $h(r + z(T))$ depends on the path z through $z(T)$, a linear change of variable $z \mapsto z(T)$ transforms immediately this functional integral over $\mathbf{Z}_0$ into an ordinary integral over $\mathbb{R}$ (see formula (A.38) in Appendix A). It is easier to use directly the properties of Fourier transforms of Gaussian integrators underpinning (A.38). Denoting by $\widehat{h}$ the Fourier transform of h, we obtain

$$h(r + z(T)) = \int_{\mathbb{R}} d\rho\, \widehat{h}(\rho)\, e^{2\pi i \rho (r + z(T))}\,. \tag{II.31}$$

Therefore, after changing the order of integration, we get

$$H(T, r) = \int_{\mathbb{R}} d\rho\, \widehat{h}(\rho)\, e^{2\pi i \rho r} \int_{\mathbf{Z}_0} \mathcal{D}_s z \cdot \exp\Big(-\frac{\pi}{s} Q_0(z) + 2\pi i \rho z(T)\Big)\,. \tag{II.32}$$

By the definition (II.19) and the duality $\rho z(T) = \langle \rho \delta_T, z \rangle$, we get

$$\int_{\mathbf{Z}_0} \mathcal{D}_s z \cdot \exp\Big(-\frac{\pi}{s} Q_0(z) + 2\pi i \rho z(T)\Big) = \exp(-\pi s W_0(\rho \delta_T))\,. \tag{II.33}$$

By (II.15) and (II.18) we obtain

$$W_0(\rho \delta_T) = \rho^2\, G(T, T) = \rho^2 T\,. \tag{II.34}$$

Collecting equations (II.32) to (II.34) we conclude

$$H(T, r) = \int_{\mathbb{R}} d\rho\, \widehat{h}(\rho) e^{2\pi i \rho r - \pi s \rho^2 T} \tag{II.35}$$

and by derivation under this integral sign, we conclude

$$\frac{\partial H}{\partial T} = \frac{s}{4\pi} \frac{\partial^2 H}{\partial r^2}\,. \tag{II.36}$$

The vector field X is the generator of the one-parameter group of transformations $\sigma(r)$. Hence, for every integer $m \geq 0$, we get

$$\left(\frac{\partial}{\partial r}\right)^m h(r + z(T)) = (\mathcal{L}_X^m \phi)\,(x \cdot \sigma(r + z(T))). \tag{II.37}$$

By differentiating under the integral sign in formula (II.30), we deduce

$$\left(\frac{\partial}{\partial r}\right)^m H(T, r) = \int_{\mathbf{Z}_0} \mathcal{D}_s z \cdot \exp\Big(-\frac{\pi}{s} Q_0(z)\Big)\, \mathcal{L}_X^m \phi(x \cdot \sigma(r + z(T)))\,. \tag{II.38}$$

Using definition (II.27), we get therefore

$$\left(\frac{\partial}{\partial r}\right)^m H(T,r)\bigg|_{r=0} = \mathcal{L}_X^m \Psi(T,x) \tag{II.39}$$

by applying m times the differential operator $\mathcal{L}_X$ acting on functions of x. In particular, for $m = 0$, we get

$$H(T,r)|_{r=0} = \Psi(T,x) \tag{II.40}$$

and both sides can be derivated with respect to T, giving

$$\frac{\partial}{\partial T} H(T,r)\bigg|_{r=0} = \frac{\partial}{\partial T}\Psi(T,x). \tag{II.41}$$

Setting $r = 0$ in the relation $\dfrac{\partial H}{\partial T} = \dfrac{s}{4\pi}\dfrac{\partial^2}{\partial r^2}H$ and using relation (II.39) for $m = 2$, and relation (II.41), we conclude the proof of the differential equation

$$\frac{\partial}{\partial T}\Psi(T,x) = \frac{s}{4\pi}\mathcal{L}_X^2\Psi(T,x).$$

Remark. The previous proof can be recast in the following symbolical way [5]. We begin with a consequence of equations (II.33) and (II.34), namely

$$\int_{\mathbf{Z}_0} \mathcal{D}_s z \cdot \exp\Big(-\frac{\pi}{s}Q_0(z)\Big)\exp(2\pi i \rho z(T)) = \exp(-\pi s \rho^2 T)\,, \tag{II.42}$$

one of many characterizations of our integrator $\mathcal{D}_s z$. Make the formal substitution

$$\rho = \frac{1}{2\pi i}\mathcal{L}_X,$$

and get

$$\int_{\mathbf{Z}_0} \mathcal{D}_s z \cdot \exp\Big(-\frac{\pi}{s}Q_0(z)\Big)\exp(z(T)\mathcal{L}_X) = \exp\left(\frac{sT}{4\pi}\mathcal{L}_X^2\right). \tag{II.43}$$

Apply this operator identity to the function $\phi(x)$ and notice that the operator $\exp(r\mathcal{L}_X)$ transforms $\phi(x)$ into $\phi(x \cdot \sigma(r))$. Hence we get

$$\int_{\mathbf{Z}_0} \mathcal{D}_s z \cdot \exp\Big(-\frac{\pi}{s}Q_0(z)\Big)\,\phi(x\cdot\sigma(z(T)) = \left(\exp\frac{sT}{4\pi}\mathcal{L}_X^2\right)\phi(x) \tag{II.44}$$

that is

$$\Psi(T,x) = \exp\left(\frac{sT}{4\pi}\mathcal{L}_X^2\right)\phi(x). \tag{II.45}$$

This integrated form is equivalent to the differential equation (II.28) together with the initial condition

$$\Psi(0,x) = \phi(x)\,. \tag{II.46}$$

3. The general case.

3.1. *The group property.*

The functional operator U_T defined by formula (II.1) satisfies the group property

$$U_T \circ U_{T'} = U_{T+T'} \tag{II.47}$$

for $T > 0$, $T' > 0$. The proof rests on three facts.

a) *Group property for the point-transformations* $\Sigma(T, z)$:

The relevant function space $\mathbf{Z}_{0,T}$ consists of the paths $z : [0,T] \to \mathbb{R}^d$ with L^2 derivative. For z in $\mathbf{Z}_{0,T}$ and z' in $\mathbf{Z}_{0,T'}$, we define a new path $z \times z'$ by the rule

$$(z \times z')(t) = \begin{cases} z(t) & \text{for} \quad 0 \le t \le T \\ z(T) + z'(t-T) & \text{for} \quad T \le t \le T + T'\,. \end{cases} \tag{II.48}$$

It is obvious that $z \times z'$ is an element of $\mathbf{Z}_{0,T+T'}$. Furthermore, the map $(z, z') \mapsto z \times z'$ is an isomorphism of $\mathbf{Z}_{0,T} \times \mathbf{Z}_{0,T'}$ onto $\mathbf{Z}_{0,T+T'}$ and, by the uniqueness of the solution of the differential equation (II.21), we obtain

$$x_0 \cdot \Sigma(T + T', z \times z') = x_0 \cdot \Sigma(T, z) \cdot \Sigma(T', z') \tag{II.49}$$

(see also formula (C.23) in Appendix C).

b) *Quadratic forms*:

We denote by $Q_{0,T}$ the basic quadratic form on $\mathbf{Z}_{0,T}$ given by equation (II.11), namely:

$$Q_{0,T}(z) = \int_0^T dt\, h_{\alpha\beta}\dot{z}^\alpha(t)\dot{z}^\beta(t)\,. \tag{II.50}$$

Using similar definitions for $Q_{0,T'}$ and $Q_{0,T+T'}$, one obtains immediately

$$Q_{0,T+T'}(z \times z') = Q_{0,T}(z) + Q_{0,T'}(z') \tag{II.51}$$

and by exponentiating

$$\exp\Big(-\frac{\pi}{s} Q_{0,T+T'}(z \times z')\Big) = \exp\Big(-\frac{\pi}{s} Q_{0,T}(z)\Big) \exp\Big(-\frac{\pi}{s} Q_{0,T'}(z')\Big)\,. \tag{II.52}$$

We can identify $\mathbf{Z}_{0,T+T'}$ to $\mathbf{Z}_{0,T} \times \mathbf{Z}_{0,T'}$. This identification entails an identification of the dual space $\mathbf{Z}'_{0,T+T'}$ to the product space $\mathbf{Z}'_{0,T} \times \mathbf{Z}'_{0,T'}$.

We use the notations ζ for elements of $\mathbf{Z}_{0,T}$, ζ' in $\mathbf{Z}_{0,T'}$ and denote by $\zeta \times \zeta'$ the corresponding element in $\mathbf{Z}'_{0,T+T'}$. Using simple algebra, one derives the identity

$$W_{0,T+T'}(\zeta \times \zeta') = W_{0,T}(\zeta) + W_{0,T'}(\zeta') \tag{II.53}$$

from (II.51). Here $W_{0,T}$ denotes the quadratic form on $\mathbf{Z}'_{0,T}$ inverse to $Q_{0,T}$, etc.

c) *Integrators:*

The following form of Fubini's theorem holds

$$\int_{\mathbf{Z}_{0,T+T'}} \mathcal{D}_s u \cdot F(u) = \int_{\mathbf{Z}_{0,T}} \mathcal{D}_s z \int_{\mathbf{Z}_{0,T'}} \mathcal{D}_s z' \cdot F(z \times z') . \tag{II.54}$$

According to the general method explained in Appendix A, it is enough to check this formula for a function of the form

$$F(z \times z') = \exp(-2\pi i(\langle \zeta, z\rangle + \langle \zeta', z'\rangle)) . \tag{II.55}$$

But then our contention follows from the characterization (II.19) of the integrator, and the relations (II.51) and (II.53).

We combine this formula with equation (II.52) and obtain another form of Fubini's theorem

$$\int_{\mathbf{Z}_{0,T+T'}} \mathcal{D}\omega_s(u)\, G(u) = \int_{\mathbf{Z}_{0,T}} \mathcal{D}\omega_s(z) \int_{\mathbf{Z}_{0,T'}} \mathcal{D}\omega_s(z')\, G(z \times z') . \tag{II.56}$$

We conclude the proof of equation (II.47). Indeed

$$\begin{aligned}
(U_T(U_{T'}\phi))(x_0) &= \int_{\mathbf{Z}_{0,T}} \mathcal{D}\omega_s(z)\,(U_{T'}\phi)(x_0 \cdot \Sigma(T,z)) \\
&= \int_{\mathbf{Z}_{0,T}} \mathcal{D}\omega_s(z) \int_{\mathbf{Z}_{0,T'}} \mathcal{D}\omega_s(z')\phi(x_0 \cdot \Sigma(T,z) \cdot \Sigma(T',z')) \\
&= \int_{\mathbf{Z}_{0,T}} \mathcal{D}\omega_s(z) \int_{\mathbf{Z}_{0,T'}} \mathcal{D}\omega_s(z')\phi(x_0 \cdot \Sigma(T+T',z \times z')) \\
&= \int_{\mathbf{Z}_{0,T+T'}} \mathcal{D}\omega_s(u)\phi(x_0 \cdot \Sigma(T+T',u)) \\
&= (U_{T+T'}\phi)(x_0) .
\end{aligned}$$

3.2. *The differential equation.*

From the group property (II.47) it follows that we need to establish the partial differential equation (II.2) at the time $T = 0$, and that the general case will follow. That is, we want to prove

$$(U_T\phi)(x_0) = \phi(x_0) + T\Big(\frac{s}{4\pi}h^{\alpha\beta}\mathcal{L}_{X_{(\alpha)}}\mathcal{L}_{X_{(\beta)}}\phi(x_0) + \mathcal{L}_Y\phi(x_0)\Big) + o(T)\,. \quad \text{(II.57)}$$

This relation implies also the initial condition

$$\lim_{T=0}(U_T\phi)(x_0) = \phi(x_0)\,. \quad \text{(II.58)}$$

For the proof, we rely on the scaling properties of paths as described in paragraph A.3.6. For z in $\mathbf{Z}_{0,1}$ we define the scaled path z_T in $\mathbf{Z}_{0,T}$ by

$$z_T(t) = T^{1/2}z(t/T)\,. \quad \text{(II.59)}$$

By scaling the differential equation (II.21), we get

$$dx(t/T) = T^{1/2}X_{(\alpha)}(x(t/T))dz_T^{\alpha}(t) + TY(x(t/T))d(t/T)\,. \quad \text{(II.60)}$$

Hence the transformation $\Sigma(T, z_T)$ in N defined using the vector fields $X_{(\alpha)}$ and Y is the same as the transformation $\Sigma(1, z)$ using the vector fields $T^{1/2}X_{(\alpha)}$ and TY. Moreover we can transfer the path integration from the variable space $\mathbf{Z}_{0,T}$ to the fixed space $\mathbf{Z}_{0,1}$, that is

$$(U_T\phi)(x_0) = \int_{\mathbf{Z}_{0,1}} \mathcal{D}\omega_s(z)\,\phi(x_0 \cdot \Sigma(T, z_T))\,. \quad \text{(II.61)}$$

We use then a limited expansion of ϕ around $\phi(x_0)$, namely

$$\begin{aligned}\phi(x_0 \cdot \Sigma(T, z_T)) = {} & \phi(x_0) + T^{1/2}\mathcal{L}_{X_{(\alpha)}}\phi(x_0)z^{\alpha}(1) \\ & + \frac{T}{2}\mathcal{L}_{X_{(\alpha)}}\mathcal{L}_{X_{(\beta)}}\phi(x_0)z^{\alpha}(1)z^{\beta}(1) \\ & + T\mathcal{L}_Y\phi(x_0) + O\Big(T^{3/2}\Big)\,.\end{aligned} \quad \text{(II.62)}$$

Recall the integration formulas

$$\int_{\mathbf{Z}_{0,1}} \mathcal{D}\omega_s(z)\,z^{\alpha}(1) = 0 \quad \text{(II.63)}$$

and

$$\int_{\mathbf{Z}_{0,1}} \mathcal{D}\omega_s(z)\,z^{\alpha}(1)z^{\beta}(1) = sh^{\alpha\beta}/2\pi\,. \quad \text{(II.64)}$$

Collecting equations (II.61) to (II.64), we conclude the proof of (II.57).

The previous calculation can be extended to give the complete Taylor expansion of $(U_T\phi)(x_0)$ around $T = 0$. Since the vector fields $X_{(\alpha)}$ and Y do not commute, we have to use a time-ordered exponential to express the solution of the differential equation (II.21).

3.3. *About the construction of* $\Sigma(T, z)$.

For simplicity, assume $Y = 0$ and $h_{\alpha\beta} = \delta_{\alpha\beta}$. Consider again a path $z : [0, T] \to \mathbb{R}^d$ of class $L^{2,1}$ with $z(0) = 0$. For each t in $[0, T]$, denote by $z^1(t), \cdots, z^d(t)$ the components of the vector $z(t)$ and define the transformation $\mathbf{s}(z(t))$ on N by

$$x_0 \cdot \mathbf{s}(z(t)) = x_0 \cdot \sigma_{(1)}(z^1(t)) \cdot \ldots \cdot \sigma_{(d)}(z^d(t)) . \tag{II.65}$$

Here $\{\sigma_{(\alpha)}(r)\}$ denotes the one-parameter group of transformations in N with generator $X_{(\alpha)}$. In general, the vector fields $X_{(\alpha)}$ do not commute, hence the transformations $\mathbf{s}(z(t))$ do not form a group. In the general case, we can use a *multistep method* to solve the differential equation (II.21), hence

$$x_0 \cdot \Sigma(T, z) = \lim_{n=\infty} x_0 \cdot \mathbf{s}(z(T/n)) \cdot \mathbf{s}(z(2T/n) - z(T/n)) \cdot \ldots \cdot \mathbf{s}(z(T) - z(T - T/n)) . \tag{II.66}$$

Putting this into the integral (II.1), we obtain after some calculations the following variant of the Lie, Trotter, Kato, Nelson formula

$$U_T\phi = \lim_{n=\infty} \left(U_{T/n}^{(1)} \cdots U_{T/n}^{(d)} \right)^n \phi . \tag{II.67}$$

Here $U_T^{(\alpha)}$ for α in $\{1, \cdots, d\}$ corresponds to the one-dimensional case studied in paragraph II.2, hence

$$U_T^{(\alpha)} = \exp\left(\frac{sT}{4\pi} \mathcal{L}_{X_{(\alpha)}}^2 \right) . \tag{II.68}$$

These relations are in agreement with

$$U_T = \exp\left(\frac{sT}{4\pi} \sum_\alpha \mathcal{L}_{X_{(\alpha)}}^2 \right) . \tag{II.69}$$

4. Some generalizations.

4.1. *Including a potential.*

Consider the Schrödinger equation with potential

$$\frac{\partial \Psi}{\partial t} = \frac{s}{4\pi} h^{\alpha\beta} \mathcal{L}_{X_{(\alpha)}} \mathcal{L}_{X_{(\beta)}} \Psi + V\Psi . \tag{II.70}$$

A path integral solution is obtained as follows

$$\Psi(T,x_0)=\int_{\mathbf{Z}_{0,T}}\mathcal{D}_s z\cdot\exp\Big(-\frac{\pi}{s}Q_0(z)+\int_{\mathbf{T}}dt\,V(x_0\cdot\Sigma(t,z))\Big)\,\phi(x_0\cdot\Sigma(T,z))\,. \tag{II.71}$$

The proof can be obtained by a slight generalization of the arguments presented in paragraph II.3. We can also use the following trick: we add one variable Θ, considering the manifold $N\times\mathbb{R}$ The vector fields $X_{(\alpha)}$ in N give vector fields, also denoted by X_α, in $N\times\mathbb{R}$ which have a zero component on the factor $\mathbb{R}$ Moreover $Y=V(x)\partial/\partial\Theta$. The function $\Psi(T,x)$ satisfies the equation (II.70) if, and only if, the function $\Psi(T,x)\exp\Theta$ satisfies the equation (II.2) on $N\times\mathbb{R}$ The differential system (II.21) is now written as

$$\begin{cases} dx = X_{(\alpha)}(x)dz^\alpha \\ d\Theta = V(x)dt\,. \end{cases} \tag{II.72}$$

Hence the transformation $\Sigma(T,z)$ takes (x_0,Θ_0) into

$$\Big(x_0\cdot\Sigma(T,z),\ \Theta_0+\int_{\mathbf{T}}dt\ V(x_0\cdot\Sigma(t,z))\Big)\,,$$

hence

$$\phi((x_0,0)\cdot\Sigma(T,z))=\phi(x_0\cdot\Sigma(T,z))\exp\Big(\int_{\mathbf{T}}dt\,V(x_0\cdot\Sigma(t,z))\Big)\,. \tag{II.73}$$

Equation (II.71) follows easily from these remarks. Notice that the exponent in this formula does not have a simple dynamical interpretation in general – but see our illustrations in section IV.

4.2. *Time-ordered product.*

Suppose that F is a suitable functional on the space $\mathbf{Z}_{0,T}$. We generalize equation (II.1) as follows

$$\left(U_T^F\phi\right)(x_0)=\int_{\mathbf{Z}_{0,T}}\mathcal{D}\omega_s(z)\,F(z)\,\phi(x_0\cdot\Sigma(T,z))\,. \tag{II.74}$$

By imitating the proof of (II.47), we get

$$U^G_{T+T'}=U^F_TU^{F'}_{T'} \tag{II.75}$$

for functionals F on $\mathbf{Z}_{0,T}$ and F' on $\mathbf{Z}_{0,T'}$, where the functional G on $\mathbf{Z}_{0,T+T'}$ is defined by

$$G(z\times z')=F(z)F'(z')\,. \tag{II.76}$$

III - Semiclassical Expansions

We shall compute the semiclassical approximation of the expression in the basic equation (II.1). The scalar potential V is included via coupled equations rather than via an additional variable, so that the equations will take readily their familiar forms; i.e. we compute the wave function:

$$\Psi(t_b, x_b) = \int_{\mathbf{Z}} \mathcal{D}_{s,Q_0} z \cdot \exp\left(-\frac{\pi}{s} Q_0(z)\right) \cdot \exp\left(\frac{1}{s\hbar} \int_{t_a}^{t_b} dt\, V(x_b \cdot \Sigma(t, z))\right) \cdot \phi\left(x_b \cdot \Sigma(t_a, z)\right). \tag{III.1}$$

For ready use of our results in quantum physics, we compute $\Psi(t_b, x_b)$. Therefore in the general set up, we set $t_0 = t_b$, $x_0 = x_b$, hence $\mathbf{Z}$ is the ***space*** $\mathbf{Z}_b$ ***of paths vanishing at*** t_b and $x(t_b, z) = x_b$. In paragraph 3, we give the corresponding results for $\Psi(t_b, x_a)$.

We choose an initial wave function given by

$$\phi(x) = \exp\left(-\frac{1}{s\hbar} \mathcal{S}_0(x)\right) \cdot \mathcal{T}(x), \tag{III.2}$$

where $\mathcal{T}$ is a smooth function on N with compact support, and $\mathcal{S}_0$ is an arbitrary, but reasonable function on N.

The initial wave function given by (III.2) generalizes plane waves on $\mathbb{R}^d$, but is, obviously, not a momentum eigenstate. We can nevertheless call the semiclassical expansion of $\Psi(t_b, x_b)$ with this initial wave function a "*momentum-to-position transition amplitude*" for three reasons:

(i) In the limit $h = 0$, assuming $s = i$, the current density corresponding to the initial wave function ϕ is

$$\lim_{\hbar=0} \frac{\hbar}{2i} \left(\phi^* \nabla \phi - (\nabla \phi)^* \phi\right) = |\mathcal{T}|^2 p\,, \quad \text{where } p(x) = \nabla \mathcal{S}_0(x)\,. \tag{III.3}$$

Consequently, $\Psi(t_b, x_b)$ is the amplitude corresponding to the transition from momentum $\nabla \mathcal{S}_0(x)$ to position x_b.

(ii) We shall expand (III.1) around a "classical path" z_{cl} characterized by initial momentum and final position.

(iii) The leading terms (henceforth labeled WKB) of semiclassical approximations combine as if they were transitions between eigenstates, e.g.

$$\langle \text{position} \mid \text{position} \rangle_{\text{WKB}} = \int d \text{ momentum} \langle \text{position} \mid \text{momentum} \rangle_{\text{WKB}} \langle \text{momentum} \mid \text{position} \rangle_{\text{WKB}}$$

[see reference 15, §7 in the Appendix].

Another useful initial wave function is

$$\phi(x) = \delta_{x_a}(x) . \tag{III.4}$$

The corresponding expression $\psi(t_b, x_b)$ gives a *position-to-position transition* amplitude. In general, this case is more complicated than the previous one, because now the initial wave function restricts the domain of integration $\mathbf{Z}$. Moreover the paths $z \in \mathbf{Z}$ such that $x(t_a, z) = x_a$ do not usually have a common origin. Even at the very best, when $N = \mathbb{R}^d$, and $z(t_a) \simeq x_a - x_b$, one needs to be careful because the domain of integration, say $\mathbf{Z}_{a,b} \subset \mathbf{Z}$, is not a vector space but an affine space. Therefore, we shall compute position-to-position transitions in section IV where we specialize the basic equation (III.1). We shall treat in detail the case $N = \mathbb{R}^d$ and give the necessary indications and references when $N = O(M)$, a frame bundle over a Riemannian space M^d.

Semiclassical expansions are best analyzed in the broader context of spaces of paths with no requirement on their boundary values. For instance let $\mathcal{P}\mathbb{R}^d$ be a space of paths z with no requirement on $z(t_a)$, nor $z(t_b)$, and let $\mathcal{P}_{a,b}\mathbb{R}^d$ be the subspace of $\mathcal{P}\mathbb{R}^d$ with d requirements at t_a and d requirements at t_b. Let $\{S(\nu, z)\}_\nu$ be a one-parameter family of actions, and let $U^{2d}(\nu)$ be the $2d$-dimensional space of motions made of the critical points of $S(\nu, z)$. A classical paths $z_{\text{cl}} \in \mathcal{P}_{a,b}\mathbb{R}^d$ for the action $S(\nu, z)$ is at the intersection of $\mathcal{P}_{a,b}\mathbb{R}^d$ with $U^{2d}(\nu)$. If the intersection is transversal, no Jacobi field of z_{cl} is in the tangent space to $\mathcal{P}_{a,b}\mathbb{R}^d$. In this paper we assume that such is the case. Otherwise there are caustics and we refer the reader to the literature [e.g. 2].

1. General strategy.

We introduce a Lagrangian

$$L(t, z) = \tfrac{1}{2} h_{\alpha\beta} \dot{z}^\alpha(t) \dot{z}^\beta(t) - V(x(t, z)) \tag{III.5}$$

where as usual $x(t, z)$ is the solution of the differential equation (II.21) with boundary condition $x(t_b, z) = x_b$. It is a function of $\dot{z}(t)$ and a functional of z. From this Lagrangian we deduce the *action functional*

$$S(z) = \int_{\mathbf{T}} dt \, L(t, z) + \mathcal{S}_0(x(t_a, z)) \tag{III.6}$$

on the space $\mathbf{Z}_b = \mathcal{P}_0\mathbb{R}^d$ of paths z obeying the boundary condition $z(t_b) = 0$. Let z_{cl} in $\mathbf{Z}_b$ be a critical point of the action functional S.

As it is customary in the calculus of variations, we take a one-parameter variation

$$z(\lambda) = z_{\text{cl}} + \lambda\zeta \tag{III.7}$$

with ζ in $\mathbf{Z}_b$ and the equation

$$\frac{d}{d\lambda} S(z(\lambda))\bigg|_{\lambda=0} = 0 \tag{III.8}$$

has to be satisfied for all ζ. That yields, after integrating by parts, a functional differential equation for $z_{\text{cl}}(t_a)$.

Under the affine change of variable from z to ζ given by (III.7), we obtain

$$Q_0(z) = Q_0(z_{\text{cl}} + \lambda\zeta) = Q_0(z_{\text{cl}}) + 2\lambda Q_0(z_{\text{cl}}, \zeta) + \lambda^2 Q_0(\zeta) \tag{III.9}$$

and

$$\mathcal{D}_{s,Q_0} z = \mathcal{D}_{s,\lambda^2 Q_0}\zeta. \tag{III.10}$$

The expansion of $x(\cdot, z(\lambda))$ around $x(\cdot, z_{\text{cl}})$ reads

$$P(z_{\text{cl}} + \lambda\zeta) = P(z_{\text{cl}}) + \lambda P'(z_{\text{cl}}) \cdot \zeta + \lambda^2 P''(z_{\text{cl}}) \cdot \zeta\zeta + O(\lambda^3). \tag{III.11}$$

Here $P'(z_{\text{cl}})$ and $P''(z_{\text{cl}})$ are the first and the second derivative mappings of P at z_{cl}, where $P : \mathbf{Z}_b \to \mathcal{P}_{x_b} N$ takes z into $x(\cdot, z)$. They are of the form

$$\begin{aligned}(P'(z_{\text{cl}}) \cdot \zeta)^\alpha(t) &= -\int_t^{t_b} ds\, \frac{\delta x^\alpha_{\text{cl}}(t)}{\delta x^\beta_{\text{cl}}(s)}\, \zeta^\beta(s) \\ &= -\int_t^{t_b} ds\, k^\alpha{}_\beta(t,s)\, \zeta^\beta(s) =: \xi^\beta(t)\end{aligned} \tag{III.12}$$

$$\begin{aligned}(P''(z_{\text{cl}}) \cdot \zeta\zeta)^\alpha(t) &= -\int_t^{t_b} ds \left(-\int_t^{t_b} du\right) \frac{\delta^2 x^\alpha_{\text{cl}}(t)}{\delta z^\beta_{\text{cl}}(s)\delta z^\gamma_{\text{cl}}(u)} \zeta^\beta(s)\zeta^\gamma(u) \\ &= \int_t^{t_b} ds \int_t^{t_b} du\, k^\alpha{}_{\beta\gamma}(t,s,u)\zeta^\beta(s)\zeta^\gamma(u).\end{aligned} \tag{III.13}$$

We could obtain the short time propagator by expanding the quantity $\phi(x_b \cdot \Sigma(t,z))$ in equation (III.1) around $\phi(x_b)$ with $x_b \in N$. Here, we shall expand $\phi(x_b \cdot \Sigma(t,z))$ around $\phi(x_b \cdot \Sigma(t, z_{\text{cl}}))$ with $z_{\text{cl}} \in \mathbf{Z}_b$. When we choose for z_{cl} a critical point of the action, we obtain the WKB approximation. These are two different expansions, and in general, the short time propagator is different from the WKB approximation [14]. But in some simple cases, they are equal [1].

Remark. A simple and powerful argument of Stephen A. Fulling clarifies this issue: The Schrödinger operator $\exp(-itH/\hbar)$ can be written in terms of dimensionless terms $\frac{tH}{\hbar} = -\frac{1}{2}A\Delta + BV$, where $A = \frac{\hbar t}{m}$ and $B = \frac{\lambda t}{\hbar}$ with λ a coupling constant. There are 4 *different* possible expansions of physical interest:

- expansion in B, equivalently expansion in λ;
- expansion in A, equivalently expansion in m^{-1};
- expansion in $A \cdot B$, equivalently expansion in t;
- expansion in A/B, equivalently expansion in $\hbar$.

2. Momentum-to-position transitions.

In this paragraph we take $s = i$ and write $\mathcal{D}_{s,Q_0}$ simply as $\mathcal{D}_{Q_0}$. We expand the integrand in (III.1) by taking z in the form $z_{\text{cl}} + \zeta$ where z_{cl} is a critical point of the action functional S, with boundary condition $z_{\text{cl}}(t_b) = 0$. The initial wave function is given by (III.2).

The terms independent of ζ combine to make the action function

$$\mathcal{S}(t_b, x_b) = \frac{1}{2}Q_0(z_{\text{cl}}) - \int_{t_a}^{t_b} dt\, V(x_{\text{cl}}(t)) + \mathcal{S}_0(x_{\text{cl}}(t_a)) \tag{III.14}$$

where $x_{\text{cl}}(t)$ is by definition $x\,(t, z_{\text{cl}}.)$. The terms linear in ζ

$$\begin{aligned}\int_{\mathbf{T}} dt\, \Big(h_{\alpha\beta}\dot{z}^{\alpha}_{\text{cl}}(t)\dot{z}^{\beta}_{\text{cl}}(t) - \nabla_\alpha V\,(x_{\text{cl}}(t))\,(P'\,(z_{\text{cl}})\cdot\zeta)^{\alpha}\,(t)\Big)\\ + \nabla_\alpha\, \mathcal{S}_0\,(x_{\text{cl}}(t_a))\,(P'\,(z_{\text{cl}})\cdot\zeta)^{\alpha}\,(t_a)\end{aligned} \tag{III.15}$$

vanish since z_{cl} is a critical point of S such that $z_{\text{cl}}(t_b) = 0$. The quadratic terms can be written in the form $(Q_0 + Q)(\zeta)$. Thanks to the equations (A.127) and (A.132), the corresponding integration is:

$$\int_{\mathbf{Z}} \mathcal{D}_{Q_0}\zeta \cdot \exp\Big(-\frac{\pi}{s}\,(Q_0 + Q)\,(\zeta)\Big) = \operatorname{Det}\,(Q_0/(Q_0 + Q))^{1/2}\,. \tag{III.16}$$

Thus the dominating term of the semiclassical expansion of $\Psi(t_b, x_b)$ is:

$$\Psi_{\text{WKB}}\,(t_b, x_b) = \exp\Big(\frac{i}{\hbar}\mathcal{S}(t_b, x_b)\Big) \cdot \operatorname{Det}\,(Q_0/(Q_0 + Q))^{1/2} \cdot \mathcal{T}(x_{\text{cl}}(t_a))\,. \tag{III.17}$$

We proceed to calculate $\mathrm{Det}(Q_0/Q_\nu)$, with $Q_\nu = Q_0 + \nu Q$, in the case $\nu = 1$. Physical arguments as well as previous calculations (see [6] and the appendix of [15]) suggest that $\mathrm{Det}(Q_0/Q_1)$ is equal to $\det K^\alpha{}_\beta(t_b, t_a)$, where the Jacobi matrix $K^\alpha{}_\beta(t_b, t_a)$ is defined by:

$$K^\alpha{}_\beta(t_b, t_a) = \frac{\partial z^\alpha_{\mathrm{cl}}(t_b)}{\partial z^\beta_{\mathrm{cl}}(t_a)}. \tag{III.18}$$

Therefore we shall construct a one-parameter family of actions $S(\nu, z)$ and a one-parameter family of Jacobian matrices:

$$K^\alpha{}_\beta(\nu; t_b, t_a) := \frac{\partial z^\alpha_{\mathrm{cl}}(\nu; t_b)}{\partial z^\beta_{\mathrm{cl}}(\nu; t_a)}. \tag{III.19}$$

We shall show that:

$$\det\Big(K(\nu; t_b, t_a) \cdot K(0; t_b, t_a)^{-1}\Big) =: c(\nu) \tag{III.20}$$

satisfies the same boundary condition as $\mathrm{Det}\,(Q_0/Q_\nu)$, namely

$$c(0) = \mathrm{Det}\,(Q_0/Q_0) = 1, \tag{III.21}$$

and the same differential equation as $\mathrm{Det}\,(Q_0/Q_\nu)$, namely,

$$\frac{d}{d\nu}\ell n\ \mathrm{Det}\,(Q_\nu/Q_0) = \mathrm{Tr}\bigg(Q_\nu^{-1}\frac{d}{d\nu}Q_\nu\bigg) = \mathrm{Tr}(Q_\nu^{-1}Q) \tag{III.22}$$

(See equation (A.124) in Appendix A).

2.1. *Differential equation satisfied by* $c(\nu)$.

We choose the family $S(\nu, z)$ such that its second variation evaluated at z_{cl} be

$$S''(\nu, z_{\mathrm{cl}}) \cdot \zeta\zeta = Q_0(\zeta) + \nu Q(\zeta) =: Q_\nu(\zeta). \tag{III.23}$$

It follows from its definition (III.18) that the β-column $K_\beta(\nu; t, t_a)$ of the Jacobi matrix $K(\nu; t, t_a)$ is the following Jacobi field along the classical path $z_{\mathrm{cl}}(\nu, t)$ of the system governed by the action $S(\nu, z)$

$$Q_\nu K_{(\beta)}(\nu; t, t_a) = 0 \tag{III.24}$$

$$K^\alpha{}_\beta(\nu; t_a, t_a) = \delta^\alpha_\beta. \tag{III.25}$$

Relation (III.24) encodes a functional differential equation for K and an implicit equation for $dK(\nu; t, t_a)/dt$ at $t = t_a$. It also gives, after differentiating w.r.t. ν

$$Q_\nu \frac{d}{d\nu} K(\nu) = -QK(\nu) \tag{III.26}$$

which can be solved with the *retarded Green's function* G_ν^{ret} of Q_ν, namely:

$$\frac{d}{d\nu} K(\nu; t, t_a) = -\int_{\mathbf{T}} G_\nu^{\text{ret}}(t, s) Q(s) K(\nu; s, t_a)\, ds\,. \tag{III.27}$$

The retarded Green's function can be expressed in terms of Jacobi fields, namely:

$$G_\nu^{\text{ret}}(t, s) = \theta(t - s) J(\nu; t, s) \tag{III.28}$$

where θ is the step function, equal to 1 for $t > s$ and 0 for $t < s$; moreover $J(\nu; t, s)$ is a Jacobi matrix, *i.e.* each column is a Jacobi field. The α-component of the β-Jacobi field $J^{(\beta)}(\nu; t, t_a)$ is

$$J^{\alpha\beta}(\nu; t, t_a) = \frac{\partial z_{\text{cl}}^\alpha(\nu, t)}{\partial p_{\text{cl},\beta}(\nu, t_a)} \tag{III.29}$$

where $p_{\text{cl}}(\nu, t) := \delta S(\nu, z)/\delta z_{\text{cl}}(\nu, t)$, here $p_{\text{cl},\alpha}(\nu, t) = h_{\alpha\beta} \dot{z}_{\text{cl}}^\beta(\nu, t)$.

From (III.20), we get the equation for $c(\nu)$:

$$\frac{d}{d\nu} \ell n\, c(\nu) = \text{tr}\left(N(\nu; t_a, t_b) \frac{d}{d\nu} K(\nu; t_b, t_a) \right) \tag{III.30}$$

where the matrix N is the inverse of K

$$N(\nu; t_a, t_b) K(\nu; t_b, t_a) = \mathbb{1}. \tag{III.31}$$

Using (III.27) and (III.28), we obtain the sought-for differential equation:

$$\frac{d}{d\nu} \ell n\; c(\nu) = -\,\text{tr}\left(\int_{\mathbf{T}} ds\, J(\nu; t_b, s)\, Q(s) K(\nu; s, t_a)\, N(\nu; t_a, t_b) \right). \tag{III.32}$$

Within focal distance of $z_{\text{cl}}(\nu; t_a)$, the d Jacobi fields $K_{(\beta)}(\nu)$ and the d Jacobi fields $J^{(\beta)}(\nu)$ are linearly independent, and $c(\nu)$ is defined. It gives the rate at which the flow of classical paths $\{z_{\text{cl}}(\nu)\}$ diverges or converges.

2.2. *Comparison between the two differential equations* (III.32) *and* (III.22).

We shall express Q_ν^{-1} in terms of Jacobi fields, as this is always possible within focal distance of $z_{\rm cl}(t_a)$. The unique inverse G_ν of Q_ν satisfies the equation

$$Q_\nu G_\nu = \mathbb{1} \tag{III.33}$$

with

$$G_\nu(t_b, s) = 0 \tag{III.34}$$

by virtue of (A.40) together with the specialization used in (A.62) when applied to the space $\mathbf{Z}_b$ of paths vanishing at t_b. Therefore (see formula (B.26) in Appendix B)

$$\begin{aligned} G_\nu(t,s) = {} & \theta(s-t) K(\nu; t, t_a)\, N(\nu; t_a, t_b)\, J(\nu; t_b, s) \\ & - \theta(t-s) J(\nu; t, t_b)\, \widetilde{N}(\nu; t_b, t_a)\, \widetilde{K}(\nu; t_a, s) \end{aligned} \tag{III.35}$$

where $\widetilde{K}$ is the transpose of K, and $\widetilde{N}$ is $\widetilde{K}$'s inverse.

Substituting $Q_\nu^{-1} = G_\nu$ in (III.22) gives:

$$\frac{d}{d\nu} \ell n \, \mathrm{Det}(Q_\nu/Q_0) = \mathrm{tr}\left(\int_{\mathbf{T}} dt\, K(\nu; t, t_a)\, N(\nu; t_a, t_b)\, J(\nu; t_b, t)\, Q(t) \right) . \tag{III.36}$$

Comparison with (III.32) shows that

$$\ell n \, \mathrm{Det}\,(Q_\nu/Q_0) = -\ell n \, c(\nu) \tag{III.37}$$

i.e.[10]

$$\mathrm{Det}\,(Q_0/Q_\nu) = \det\Big(K(\nu; t_b, t_a) \cdot K(0; t_b, t_a)^{-1} \Big) . \tag{III.38}$$

Here $K(0; t, t_a)$ satisfies

$$Q_0 K(0; t, t_a) = 0 \tag{III.39}$$

$$K(0; t_a, t_a) = \mathbb{1} \tag{III.40}$$

where

$$Q_0(z) = - \int_{\mathbf{T}} dt\, h_{\alpha\beta} \dot{z}^{\alpha}(t) z^{\beta}(t) - h_{\alpha\beta} \dot{z}^{\alpha}(t_a)\, z^{\beta}(t_a) .$$

[10] The credit for the technique used in deriving (III.38) is due to B. Nelson and B. Sheeks [6]. It is made simpler and more general here by not integrating Q_ν by parts. The first calculation giving the ratio of functional determinants in terms of finite determinants can be found in [16, 5].

Hence we have

$$\frac{d^2}{dt^2}K(0;t,t_a) = 0 \quad , \quad \frac{d}{dt}K(0;t,t_a)\bigg|_{t=t_a} = 0 \tag{III.41}$$

therefore

$$K^\alpha{}_\beta\,(0;t,t_a) = \delta^\alpha{}_\beta\,. \tag{III.42}$$

Inserting $\mathrm{Det}(Q_0/Q_\nu) = \det K^\alpha{}_\beta\,(t_b,t_a)$ into (III.17) we obtain:

$$\Psi_{\mathrm{WKB}}\,(t_b,x_b) = \left(\det_{\alpha,\beta}\frac{\partial z^\alpha_{\mathrm{cl}}\,(t_b)}{\partial z^\beta_{\mathrm{cl}}\,(t_a)}\right)^{1/2}\cdot\exp\left(\frac{i}{\hbar}\mathcal{S}(t_b,x_b)\right)\cdot\mathcal{T}(x_{\mathrm{cl}}(t_a))\,. \tag{III.43}$$

If $z_{\mathrm{cl}}\,(t_b)$ is conjugate to $z_{\mathrm{cl}}\,(t_a)$, in the sense of caustic theory, one needs [2] to include terms in ζ of order higher than 2 in the calculation of Ψ.

2.3. *End of calculation.*

Equation (III.43) is not the end of the calculation; Ψ_{WKB} must be expressed in terms of $x_{\mathrm{cl}} : T \to N$, where

$$x_{\mathrm{cl}}(t) = x(t,z_{\mathrm{cl}}) = x_b\cdot\Sigma\,(t,z_{\mathrm{cl}}) \tag{III.44}$$

but not in terms of $z_{\mathrm{cl}} : \mathbf{T} \to \mathbb{R}^d$. Two techniques present themselves: we know the evolution equation satisfied by the wave function Ψ on N, therefore we can construct an action on N and find its critical points. But we do not need to find x_{cl} corresponding to z_{cl}, we only need the expression corresponding to the prefactor of (III.43). This can be obtained by a simpler technique.

Recall the parametrization $P : \mathcal{P}_0\mathbb{R}^d \to \mathcal{P}_{x_b}N$ given by $P(z) = x$. Corresponding to the action functional S on $\mathcal{P}_0\mathbb{R}^d$ we get a functional $\overline{S}$ on $\mathcal{P}_{x_b}N$ such that $S = \overline{S}\circ P$, i.e. $\overline{S}(x) = S(z)$. Making the substitution $z = z_{\mathrm{cl}} + \zeta$ and expanding up to second order terms in ζ, we get

$$\overline{S}(x_{\mathrm{cl}}) = S(z_{\mathrm{cl}}) \tag{III.45}$$

$$\overline{S}'(x_{\mathrm{cl}})\cdot\frac{\delta x_{\mathrm{cl}}}{\delta z_{\mathrm{cl}}} = S'(z_{\mathrm{cl}}) \tag{III.46}$$

$$\overline{S}''(x_{\mathrm{cl}})\cdot\frac{\delta x_{\mathrm{cl}}}{\delta z_{\mathrm{cl}}}\frac{\delta x_{\mathrm{cl}}}{\delta z_{\mathrm{cl}}} = S''(z_{\mathrm{cl}})\,. \tag{III.47}$$

In the beginning of this paragraph we calculated the expansion of $S(z_{\mathrm{cl}}+\zeta)$ up to second order and obtained the expression

$$S''(z_{\mathrm{cl}})\cdot\zeta\zeta = Q_0(\zeta) + Q(\zeta) = Q_1(\zeta) \tag{III.48}$$

for the second order variation. We also know that the infinite-dimensional determinant $\mathrm{Det}(Q_0/Q_1)$ is equal to the determinant of the Jacobi matrix $\partial z_{\rm cl}(t_b)/\partial z_{\rm cl}(t_a)$.

Similarly we write

$$\overline{S}''(x_{\rm cl})\cdot\frac{\delta x_{\rm cl}}{\delta z_{\rm cl}}\zeta\frac{\delta x_{\rm cl}}{\delta z_{\rm cl}}\zeta = \overline{Q}_0(\zeta)+\overline{Q}(\zeta) \tag{III.49}$$

and identify $\overline{Q}_0$ as follows. Let

$$S_0(z) := \tfrac{1}{2}Q_0(z) \tag{III.50}$$

and

$$\overline{S}_0(x(z)) := S_0(z) \tag{III.51}$$

then

$$\overline{Q}_0(\zeta) := \overline{S}_0''(x_{\rm cl})\cdot\frac{\delta x_{\rm cl}}{\delta z_{\rm cl}}\zeta\frac{\delta x_{\rm cl}}{\delta z_{\rm cl}}\zeta \tag{III.52}$$

$$\det\frac{\partial x_{\rm cl}(t_b)}{\partial z_{\rm cl}(t_a)} = \mathrm{Det}\big(\overline{Q}_0/(\overline{Q}_0+\overline{Q})\big) = \mathrm{Det}(Q_0/(Q_0+Q))\,. \tag{III.53}$$

Therefore

$$\det\frac{\partial x_{\rm cl}(t_b)}{\partial x_{\rm cl}(t_a)} = \det\frac{\partial z_{\rm cl}(t_b)}{\partial z_{\rm cl}(t_a)}\,. \tag{III.54}$$

In conclusion, we obtain the sought-for semiclassical expansion

$$\Psi_{\rm WKB}(t_b,x_b) = \det\left(\frac{\partial x_{\rm cl}(t_b)}{\partial x_{\rm cl}(t_a)}\right)^{1/2}\cdot\exp\left(\frac{i}{\hbar}\mathcal{S}(t_b,x_b)\right)\cdot\mathcal{T}(x_{\rm cl}(t_a))\,. \tag{III.55}$$

According to equation (III.46), the path $x_{\rm cl}$ in N is a critical point for the action functional $\overline{S}:\mathcal{P}_{x_b}N\to\mathbb{R}$ By construction, we have

$$\overline{S}'(x(\cdot,z)) = S'(z)\,. \tag{III.56}$$

Using equation (III.5) and (III.6) this can be made more explicit as follows

$$\overline{S}(x) = \tfrac{1}{2}\int_{\mathbf{T}}dt\,h_{\alpha\beta}\,\dot{z}^\alpha(t)\dot{z}^\beta(t) - \int_{\mathbf{T}}V(x(t))dt + \mathcal{S}_0(x(t_a))\,. \tag{III.57}$$

In our general setup , the first integral remains somewhat implicit, but can be used for practical calculations in the applications given in section IV.

The prefactor in (III.55) also gives the volume expansion or contraction of a congruence of classical paths originating in the neighborhood of $\mathcal{S}_0(x_{\rm cl}(t_a))$

with momentum $\nabla \mathcal{S}_0(m)$, $m \in N$. This determinant depends both on the choice of the initial wave function and the dynamics of the system.

Set $d\omega_a$ the volume element on N at $x_{\rm cl}(t_a)$ and $d\omega_b = \det\left(\frac{\partial x^\alpha_{\rm cl}(t_b)}{\partial x^\beta_{\rm cl}(t_a)}\right) d\omega_a$; then

$$\lim_{\hbar=0} \int_{C_{t_b}\Omega} |\Psi(t_b, x_b)|^2 \, d\omega_b = \int_\Omega |\phi(x_a)|^2 \, d\omega_a$$

where C_t belongs to the group of transformations generated by the classical flow (see details in [5 p. 299]).

3. Diffusion problems.

In a diffusion problem, one is interested in computing the functional integral

$$\begin{aligned}\Psi(t_b, x_a) = \int_{\mathbf{Z}} \mathcal{D}_{s,Q_0} z \exp\left(-\frac{\pi}{s} Q_0(z)\right) \cdot \\ \exp\left(\frac{1}{s\hbar} \int_{t_a}^{t_b} dt\, V(x_a \cdot \Sigma(t, z))\right) \cdot \phi(x_a \cdot \Sigma(t_b, z))\end{aligned} \tag{III.58}$$

where $\mathbf{Z}$ denotes now the space of paths z vanishing at $t = t_a$. If we choose ϕ to be of the form (III.2), namely

$$\phi(x_a \cdot \Sigma(t_b, z)) = \phi(x(t_b, z)) = \exp\left(-\frac{1}{s\hbar} \mathcal{S}_0(x(t_b, z))\right) \mathcal{T}(x(t_b, z)) \tag{III.59}$$

then computing (III.58), with $s = i$, can be said to be computing the probability amplitude of a transition from a position x_a to a momentum of the form

$$p(x_b) = \nabla \mathcal{S}_0(x_b) \tag{III.60}$$

for the end point $x_b = x(t_b, z)$ of the path x.

With ϕ given by (III.59), one obviously expects the WKB approximation of (III.58) to be

$$\Psi_{\rm WKB}(t_b, x_a) = \det\left(\frac{\partial x^\alpha_{\rm cl}(t_a)}{\partial x^\beta_{\rm cl}(t_b)}\right)^{1/2} \exp\left(\frac{i}{\hbar} \mathcal{S}(t_b, x_a)\right) \cdot \mathcal{T}(x_{\rm cl}(t_b)) \,. \tag{III.61}$$

Nevertheless, it is gratifying to derive (III.61) by the method followed in paragraph 2. The only necessary changes are described as follows:

$$c(\nu) = \det\left(K(\nu; t_a, t_b) \cdot K(0; t_a, t_b)^{-1}\right) \tag{III.20$^{\rm bis}$}$$

$$G_\nu^{\operatorname{ad}\nu}(t, s) = \theta(t - s) J(\nu; s, t) = -\theta(s - t) J(\nu; t, s) \tag{III.28$^{\rm bis}$}$$

$$\begin{aligned}G_\nu(t, s) = \theta(s - t) J(\nu; t, t_a)\, \widetilde{N}(\nu; t_a, t_b)\, \widetilde{K}(\nu; t_b, s) \\ - \theta(t - s) K(\nu; t, t_b)\, N(\nu; t_b, t_a)\, J(\nu; t_a, s)\end{aligned} \tag{III.35$^{\rm bis}$}$$

(see formula (B.27) in Appendix B).

IV - Illustrations

The specializations presented in this section illustrate and develop the general formulas derived in sections II and III. They differ by the choice of the manifold N and of the initial wave function ϕ. We treat the case of scalar wave functions, but the case of a tensor field is easily accommodated by using the Lie derivative of tensor fields.

We are basically interested in the Schrödinger equation. So the reader should substitute i to s in the formulas. Moreover, we want to evaluate the value of the wave function Ψ at the *final time* t_b, and *final position* x_b. This dictates the choice of the space $\mathbf{Z}_b$ of paths, determined by $z(t_b) = 0$.

1. Point-to-point transitions in a flat space.

The basic manifold N is the euclidean space $\mathbb{R}^d$ of dimension d, in cartesian coordinates (see paragraph IV.2 for other coordinates). For a vector x, with coordinates $x^1, \cdots, x^d$, we set its length to be $|x| = \left(\sum_{\alpha=1}^d (x^\alpha)^2\right)^{1/2}$ as usual. We consider a particle of mass m moving in the field of a potential V. The Lagrangian and action are given as usual by

$$L(x, \dot{x}) = \frac{m}{2} |\dot{x}|^2 - V(x), \tag{IV.1}$$

$$S(x(\cdot)) = \int_{\mathbf{T}} dt\, L(x(t), \dot{x}(t))\,. \tag{IV.2}$$

In classical mechanics, we solve the equations of motion with suitable boundary conditions:

$$m\, \ddot{x}_{\text{cl}} = -\nabla V(x_{\text{cl}}) \tag{IV.3}$$

$$x_{\text{cl}}(t_a) = x_a \quad , \quad x_{\text{cl}}(t_b) = x_b\,, \tag{IV.4}$$

where x_a, x_b are points in $\mathbb{R}^d$. In quantum mechanics, we want to solve the Schrödinger equation

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V \Psi \tag{IV.5}$$

with initial condition

$$\Psi(t_a, x) = \phi(x)\,. \tag{IV.6}$$

The *point-to-point transition amplitudes* are given by

$$\langle t_b, x_b \mid t_a, x_a \rangle = \Psi(t_b, x_b)\ , \tag{IV.7}$$

where $\phi(x)$ is a delta function $\delta(x - x_a)$.

The original claim of Feynman was that we can solve the Schrödinger equation by the following path integral

$$\Psi(t_b, x_b) = \int_{\mathcal{P}_b} \mathcal{D}x \cdot e^{iS(x)/\hbar} \cdot \phi(x(t_a)) , \tag{IV.8}$$

where $\mathcal{P}_b$ is the space of all paths $x : \mathbf{T} \to \mathbb{R}^d$ with endpoint at x_b, namely $x(t_b) = x_b$. The questionable part was the rigorous definition of the integrator $\mathcal{D}x$.

To fit within our general framework, we consider translations acting on $\mathbb{R}^d$, namely

$$x \cdot \sigma_{(\alpha)}(r) = \left(x^1, \cdots, x^{\alpha-1}, x^\alpha + \lambda r, x^{\alpha+1}, \cdots, x^d\right) . \tag{IV.9}$$

The corresponding Lie derivative is given by

$$\mathcal{L}_{X_{(\alpha)}} f = \lambda \frac{\partial f}{\partial x^\alpha} \tag{IV.10}$$

and the parameter λ is chosen equal to $(h/m)^{\frac{1}{2}}$. The general differential equation $dx = X_{(\alpha)}(x) \cdot dz^\alpha$ reduces to $dx^\alpha = \lambda dz^\alpha$ (for α in $\{1, \cdots, d\}$). Hence the solution

$$x(t, z) = x_b + \lambda z(t) \tag{IV.11}$$

describes the parametrization of the space $\mathcal{P}_b$ of paths x with $x(t_b) = x_b$ by the space $\mathbf{Z}_b$ of paths z with $z(t_b) = 0$. Accordingly, we obtain for the action

$$-\frac{1}{s\hbar} S(x(\cdot, z)) = -\frac{\pi}{s} \int_{\mathbf{T}} dt \, |\dot{z}(t)|^2 + \frac{1}{s\hbar} \int_{\mathbf{T}} dt \, V(x(t, z)) . \tag{IV.12}$$

Substituting z for the integration variable x in equation (IV.8), we obtain the path integral (remember that $s = i$ in quantum mechanics)

$$\Psi(t_b, x_b) = \int_{\mathbf{Z}_b} \mathcal{D}z \cdot \exp\left(-\frac{1}{s\hbar} S(x(\cdot, z))\right) \cdot \phi(x(t_a, z)) . \tag{IV.13}$$

From our general results in section II, the function Ψ is a solution of the differential equation

$$\frac{\partial \Psi}{\partial t} = \frac{s}{4\pi} \sum_\alpha \mathcal{L}^2_{X_{(\alpha)}} \Psi + \frac{1}{s\hbar} V \Psi , \tag{IV.14}$$

that is

$$s\hbar\frac{\partial\Psi}{\partial t}=\frac{s^2\hbar^2}{2m}\Delta\Psi+V\Psi\,. \tag{IV.15}$$

For $s=i$, this is the sought-for Schrödinger equation. The integrator $\mathcal{D}z$ in the space $\mathbf{Z}_b$ is invariant under translations and is normalized by[11]

$$\int_{\mathbf{Z}_b}\mathcal{D}z\cdot\exp\left(-\frac{\pi}{s}\int_{\mathbf{T}}dt\,|\dot{z}(t)|^2\right)=1\,. \tag{IV.16}$$

We derive now the *semiclassical approximation*[12]. We use the initial wave function $\phi(x)=\delta(x-x_a)$ and reparametrize the paths *around the classical path* x_{cl}, that is

$$x(t,\zeta)=x_{\text{cl}}(t)+\lambda\zeta(t)\,, \tag{IV.17}$$

with ζ in $\mathbf{Z}_b$. Since the integrator in $\mathbf{Z}_b$ is invariant under translations, and $x_{\text{cl}}(t_a)=x_a$, we transform equation (IV.13) into

$$\Psi(t_b,x_b)=\int_{\mathbf{Z}_b}\mathcal{D}\zeta\cdot\exp\left(-\frac{1}{s\hbar}S(x_{\text{cl}}+\lambda\zeta)\right)\delta(\lambda\zeta(t_a))\,. \tag{IV.18}$$

In expanding $S(x_{\text{cl}}+\lambda\zeta)$ in powers of λ, there is no term linear in λ, since x_{cl} is a critical point of the action functional S *and* because $\zeta(t_a)=\zeta(t_b)=0$. Hence we obtain

$$\begin{aligned}-\frac{1}{s\hbar}S(x_{\text{cl}}+\lambda\zeta)=&-\frac{1}{s\hbar}S(x_{\text{cl}})-\frac{\pi}{s}\int_{\mathbf{T}}dt\;|\dot{\zeta}(t)|^2\\&+\frac{\pi}{ms}\int_{\mathbf{T}}dt\,\nabla_\alpha\nabla_\beta V(x_{\text{cl}}(t))\,\zeta^\alpha(t)\zeta^\beta(t)+O(\hbar^{1/2})\,.\end{aligned} \tag{IV.19}$$

The action $S(x_{\text{cl}})$ corresponding to the classical path x_{cl} with endpoints $x_{\text{cl}}(t_a)=x_a$, $x_{\text{cl}}(t_b)=x_b$ is nothing else than the *classical action function* $\mathcal{S}(t_b,x_b;t_a,x_a)$.

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[11] The scaling factor $\lambda=(h/m)^{1/2}$ has been chosen in such a way that no physical constant enters in this normalization. With the notations of paragraph A.3.6, we have the dimensional equations

$$[z^\alpha]=\mathcal{T}^{1/2}\quad,\quad[t]=\mathcal{T}\,.$$

[12] Since $\lambda=(h/m)^{1/2}$, the semiclassical expansion will proceed according to the powers of $h^{1/2}$.

Omitting terms of order $\hbar^{1/2}$, we obtain the WKB approximation $\Psi_{\text{WKB}}(t_b, x_b)$ to $\Psi(t_b, x_b)$. Using formulas (IV.18) and (IV.19), we derive

$$\Psi_{\text{WKB}}(t_b, x_b) = \exp\Big(-\frac{1}{s\hbar}\mathcal{S}(t_b, x_b; t_a, x_a)\Big) \cdot I, \tag{IV.20}$$

with the integral

$$I = \int_{\mathbf{Z}_b} \mathcal{D}\zeta \cdot \exp\Big(-\frac{\pi}{s}Q_1(\zeta)\Big)\, \delta(\lambda\zeta(t_a)) \,. \tag{IV.21}$$

Besides the quadratic form

$$Q_0(\zeta) = \int_{\mathbf{T}} dt\ |\dot{\zeta}(t)|^2 \tag{IV.22}$$

corresponding to the free particle, we need the quadratic form

$$Q_V(\zeta) = -\frac{1}{m}\int_{\mathbf{T}} dt\, h_{\alpha\beta}(t)\, \zeta^{\alpha}(t)\zeta^{\beta}(t) \tag{IV.23}$$

with $h_{\alpha\beta}(t) = \nabla_\alpha \nabla_\beta V(x_{\text{cl}}(t))$. In (IV.21), we use the quadratic form $Q_1 = Q_0 + Q_V$.

The easiest method to calculate the functional integral I consists of the following steps.

a) *Changing the integrator*: according to formulas (A.126), (A.132) and (A.134) in Appendix A, we obtain $I = I_1 I_2$ where

$$I_1 = |\text{Det}(Q_0/Q_1)|^{1/2}\, s^{-\text{Ind}(Q_1)}, \tag{IV.24}$$

$$I_2 = \int_{\mathbf{Z}_b} \mathcal{D}_{s,Q_1}\zeta \cdot \exp\Big(-\frac{\pi}{s}Q_1(\zeta)\Big)\, \delta(\lambda\zeta(t_a)) \,. \tag{IV.25}$$

Here $\text{Ind}(Q_1)$ is the number of negative directions for the quadratic form Q_1.

b) *Restricting the domain of integration*: to treat the δ factor in I_2, we use the linear change of variables $\zeta \mapsto \zeta(t_a)$ from $\mathbf{Z}_b$ to $\mathbb{R}^d$. By a method similar to the one used in paragraph A.3.8, we obtain

$$I_2 = (\det sW_1(\delta_{t_a}))^{-1/2}\, \lambda^{-d}, \tag{IV.26}$$

where W_1 is the quadratic form on $\mathbf{Z}'_b$ inverse to Q_1. Using the Green's function G given by equation (B.26), we evaluate the matrix $W_1(\delta_{t_a})$ as follows

$$W_1(\delta_{t_a}) = G(t_a, t_a) = N(t_a, t_b)\, J(t_b, t_a) \,. \tag{IV.27}$$

c) *Reducing a functional determinant to a finite determinant*: using the same strategy as in paragraph III.2 we obtain

$$\mathrm{Det}(Q_0/Q_1) = \det\Big(K(1;t_b,t_a)^{-1}\,K(0;t_b,t_a)\Big)\ . \tag{IV.28}$$

Here $K(\nu;t,t_a)$ is the Jacobi field defined by

$$Q_\nu K(\nu;t,t_a) = 0 \quad , \quad K(\nu;t_a,t_a) = \mathbb{1} \tag{IV.29}$$

for ν equal to 0 or 1.

Finally, collecting the previous equations (IV.20) to (IV.29) and using equations (B.11), (B.12), (B.7), (B.8), (B.19) and (III.42), we obtain *the* WKB *approximation to the point-to-point transition amplitude* (in the case $s = i$):

$$\big\langle t_b, x_b \mid t_a, x_a\big\rangle_{WKB} = c\,h^{-d/2}\left|\det \partial^2\mathcal{S}/\partial x_a^\alpha\cdot\partial x_b^\beta\right|^{1/2} e^{i\mathcal{S}/\hbar}\ . \tag{IV.30}$$

Here $\mathcal{S} = \mathcal{S}(t_b,x_b;t_a,x_a)$ is the classical action function and the phase factor c is $e^{\pi i(p-q)/4}$ where p (q) is the number of positive (negative) eigenvalues of the van Vleck-Morette matrix $\Big(\partial^2\mathcal{S}/\partial x_a^\alpha\cdot\partial x_b^\beta\Big)_{\substack{1\le\alpha\le d\\ 1\le\beta\le d}}$.

For the higher-order terms in the semiclassical approximation, whether x_{cl} is or is not a degenerate critical point of the action, we refer the reader to the literature [5] or the references at the end of Appendix B. The reader can easily transfer these old results using the simpler and more general formalism of this paper.

2. Polar coordinates.[13]

Our basic integral is formulated in terms of a transformation $\Sigma(t,z)$ on a manifold N, in a form valid for an arbitrary system of coordinates. Before considering the case of a general Riemannian manifold in paragraph IV.3, we consider polar coordinates in a plane. The case of cylindrical coordinates in $\mathbb{R}^3$ is very similar.

"Path integrals in polar coordinates", the 1964 paper by S.F. Edwards and Y.V. Gulyaev [17] has been, and still is, at the origin of many investigations. To the best of our knowledge, all the papers on the subject deal with the discretized version of the path integral, and propose various path integral prescriptions when the discretized values of the paths are expressed in coordinates other than cartesian.

[13] Contributed by John La Chapelle.

The basic manifold is $N = \mathbb{R}^2\backslash\{0\}$ with coordinates x^1 and x^2. We also consider the manifold $\widetilde{N} =]0, +\infty[\times\mathbb{R}$ with coordinates r, θ (sole restriction $r > 0$) and the covering map $\Pi : \widetilde{N} \to N$ taking (r, θ) into (x^1, x^2) with

$$x^1 = r\cos\theta \quad , \quad x^2 = r\sin\theta\,. \tag{IV.31}$$

Two points of $\widetilde{N}$ map onto the same point of N if, and only if, their θ-coordinates differ by an integral multiple of 2π.

Let $\Delta = \frac{\partial^2}{(\partial x^1)^2} + \frac{\partial^2}{(\partial x^2)^2}$ be the Laplacian in cartesian coordinates. We know how to solve the Schrödinger equation

$$\frac{\partial\Psi}{\partial t} = \frac{s}{4\pi}\Delta\Psi \tag{IV.32}$$

by means of the path integral

$$\Psi(t_b, x_b) = \int_{\mathbf{Z}_b} \mathcal{D}z \cdot e^{-\pi Q_0(z)/s}\phi(x(t_a, z)) \tag{IV.33}$$

provided that $x(t, z)$ is a solution of the differential system

$$dx^1 = dz^1 \quad , \quad dx^2 = dz^2 \tag{IV.34}$$

with boundary condition $x(t_b, z) = x_b$. To solve the Schrödinger equation in polar coordinates, we need only to transform the system (IV.34) in polar coordinates with the help of the transformation equations (IV.31). Hence

$$\begin{cases} dr = \cos\theta \cdot dz^1 + \sin\theta \cdot dz^2 = X^1_{(1)}dz^1 + X^1_{(2)}dz^2 \\ d\theta = -\frac{\sin\theta}{r} \cdot dz^1 + \frac{\cos\theta}{r} \cdot dz^2 = X^2_{(1)}dz^1 + X^2_{(2)}dz^2\,. \end{cases} \tag{IV.35}$$

The vector fields $X_{(1)}$ and $X_{(2)}$ can be read off from the above equations:

$$\mathcal{L}_{X_{(1)}} = \cos\theta \cdot \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \cdot \frac{\partial}{\partial\theta}\,, \tag{IV.36}$$

$$\mathcal{L}_{X_{(2)}} = \sin\theta \cdot \frac{\partial}{\partial r} + \frac{\cos\theta}{r} \cdot \frac{\partial}{\partial\theta}\,. \tag{IV.37}$$

Hence, if $(r(t, z), \theta(t, z))$ is the solution of the differential system (IV.35) such that $r(t_b, z) = r_b$, $\theta(t_b, z) = \theta_b$, the path integral

$$\Psi(t_b, r_b, \theta_b) := \int_{\mathbf{Z}_b} \mathcal{D}z \cdot e^{-\pi Q_0(z)/s}\phi(r(t_a, z), \theta(t_a, z)) \tag{IV.38}$$

solves the Schrödinger equation (IV.32). As expected, the operator

$$\Delta = \mathcal{L}^2_{X_{(1)}} + \mathcal{L}^2_{X_{(2)}} = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}$$

is the Laplacian on $\mathbb{R}^2 \backslash \{0\}$ in polar coordinates.

We want to evaluate the point-to-point transition amplitudes in polar coordinates, denoted by $\langle t_b, r_b, \theta_b \mid t_a, r_a, \theta_a \rangle$. To obtain then, it suffices to put $\phi(r,\theta) = \delta(r-r_a)\delta(\theta-\theta_a)$ in equation (IV.38). Solving the system (IV.35) is easy by reverting to cartesian coordinates, hence

$$\begin{cases} r(t,z)\cos\theta(t,z) = r_b \cos\theta_b + z^1(t) \\ r(t,z)\sin\theta(t,z) = r_b \sin\theta_b + z^2(t)\,. \end{cases} \tag{IV.39}$$

For given x^1, x^2, the equations (IV.31) in r, θ have infinitely many solutions, and we derive easily

$$\delta(r\cos\theta - r_a\cos\theta_a)\,\delta(r\sin\theta - r_a\sin\theta_a) = \frac{1}{r_a}\sum_{n\in\mathbb{Z}} \delta(r - r_a)\,\delta(\theta - \theta_a - 2n\pi)\,. \tag{IV.40}$$

Substituting $r(t,z)$ to r and $\theta(t,z)$ to θ and taking into account equations (IV.39), we obtain the path integral representation

$$\frac{1}{r_a}\sum_{n\in\mathbb{Z}} \langle t_b, r_b, \theta_b \mid t_a, r_a, \theta_a + 2n\pi \rangle = I\,. \tag{IV.41}$$

Here we use the definitions

$$\phi_1(u) = \delta\big(r_b\cos\theta_b - r_a\cos\theta_a - u^1\big)\,\delta\big(r_b\sin\theta_b - r_a\sin\theta_a - u^2\big) \tag{IV.42}$$

for $u = \big(u^1, u^2\big)$ in $\mathbb{R}^2$, and

$$I = \int_{\mathbf{Z}_b} \mathcal{D}z \cdot e^{-\pi Q_0(z)/s} \phi_1(z(t_a))\,. \tag{IV.43}$$

This integral is nothing else than a point-to-point transition amplitude in cartesian coordinates. We give now a derivation of the well-known formula for this amplitude, by relying on our methods.

The integrand in equation (IV.43) depends only on $z^1(t_a)$, $z^2(t_a)$, hence as explained in paragraph A.2.2, we introduce the linear map $L : \mathbf{Z}_b \to \mathbb{R}^2$ mapping z into $z(t_a)$. The image of the integrator $\mathcal{D}z \cdot e^{-\pi Q_0(z)/s}$ on $\mathbf{Z}_b$ is a Gaussian integrator on $\mathbb{R}^2$; we need only the value of the corresponding covariance matrix

$$W^{\alpha\beta} = \int_{\mathbf{Z}_b} \mathcal{D}z \cdot e^{-\pi Q_0(z)/s} z^\alpha(t_a) z^\beta(t_a) . \tag{IV.44}$$

According to formulas (A.57) and (A.62), we obtain

$$W^{\alpha\beta} = \frac{s}{2\pi} G_b^{\alpha\beta}(t_a, t_a) = \frac{s}{2\pi} \delta^{\alpha\beta}(t_b - t_a) . \tag{IV.45}$$

According to formulas (A.27) and (A.32), we transform the integral (IV.43) into

$$I = (t_b - t_a)^{-1} \int_{\mathbb{R}} du^1 \int_{\mathbb{R}} du^2 \exp\left(-\frac{\pi}{s} \frac{|u|^2}{t_b - t_a} \right) \phi_1(u) . \tag{IV.46}$$

The function $\phi_1(u)$ is a δ-factor and the final result is

$$\begin{aligned} I &= (t_b - t_a)^{-1} \exp\left(-\frac{\pi}{s} \frac{r_a^2 + r_b^2}{t_b - t_a} \right) \exp\left(\frac{2\pi}{s} \frac{r_a r_b \cos(\theta_b - \theta_a)}{t_b - t_a} \right) \\ &= \left(t_b - t_a^{-1} \right) \exp\left(-\frac{\pi}{s} \frac{|x_b - x_a|^2}{t_b - t_a} \right) . \end{aligned} \tag{IV.47}$$

From the equations (IV.41) and (IV.47) we cannot derive directly the point-to-point transition amplitudes in polar coordinates. We defer to paragraph IV.7 a further discussion of this point.

3. Frame bundles over Riemannian manifolds.[14]

We consider a Riemannian (or pseudo-Riemannian) manifold M of dimension d, with metric g, and the orthonormal frame bundle $N = O(M)$ over M, with projection $\pi : O(M) \to M$. We want to choose vector fields $X_{(1)}, \cdots, X_{(d)}$ on $O(M)$ such that the *equation on the bundle* $O(M)$

$$\frac{\partial}{\partial t_b} \Psi(t_b, \rho_b) = \frac{s}{4\pi} h^{\alpha\beta} X_{(\alpha)} X_{(\beta)} \Psi(t_b, \rho_b) , \tag{IV.48}$$

[14] For a discussion of functional integrals when paths take their values in a Riemannian space, see [5], [14] and [18].

where ρ_b belongs to $O(M)$ and the first-order differential operators are given by $X_{(\alpha)} = X^{\lambda}_{(\alpha)}(\rho_b)\frac{\partial}{\partial \rho_b^\lambda}$, gives by projection *an equation on the base space* M, namely:

$$\begin{aligned} \frac{\partial}{\partial t_b}\psi(t_b, x_b) &= \frac{s}{4\pi} g^{\lambda\mu} D_\lambda D_\mu \psi(t_b, x_b) \\ &= \frac{s}{4\pi}\Delta\psi(t_b, x_b)\ . \end{aligned} \tag{IV.49}$$

Here D_λ is the covariant derivative defined by the Riemannian connection, and $\Psi = \psi \circ \pi$; moreover Δ is the Laplace-Beltrami operator on M.

It has been shown in [13] that the covariant Laplacian Δ at a point x_b of M can be lifted to a sum of products of Lie derivatives $h^{\alpha\beta}X_{(\alpha)}X_{(\beta)}$ at the frame ρ_b in $O(M)$; the integral curves of the set of vector fields $\{X_{(\alpha)}\}$ starting from ρ_b at time t_b are the horizontal lifts of a set of geodesics at x_b, tangent to the basis $\{e_\alpha\}$ of $T_{x_b}M$ corresponding to the frame ρ_b. The constant matrix $(h_{\alpha\beta})$ has been chosen with the same signature as the metric g on M, and $g(e_\alpha, e_\beta) = h_{\alpha\beta}$.

An explicit construction of $\{X_{(\alpha)}\}$ goes as follows. Let $\rho(t)$ be the horizontal lift of a path $x(t)$ in M defined by the Riemannian connection map[15] $\sigma : O(M) \to L(TM, TO(M))$; it satisfies the differential equation

$$\dot\rho(t) = \sigma(\rho(t)) \cdot \dot x(t) \quad , \quad \rho(t_b) = \rho_b\,. \tag{IV.50}$$

If we put $\rho(t) = (x(t), u(t))$ and $\rho_b = (x_b, u_b)$, a solution $\rho(t)$ of the previous equation corresponds to the frame $u(t)$ obtained by parallel transport of u_b, along the path x from x_b to $x(t)$. The frame $u(t)$ is also an admissible map

$$u(t) : \mathbb{R}^d \to T_{x(t)}M$$

i.e. $u(t)$ maps a d-tuple into a vector whose components in $u(t)$ are the chosen d-tuple; equivalently $\left(u(t)^{-1}\dot x(t)\right)^\alpha$ is the α-coordinate of $\dot x(t)$ in the $u(t)$ frame. Set

$$\dot z(t) := u(t)^{-1}\dot x(t) = \dot z^\alpha(t)\overline{e}_\alpha\,, \tag{IV.51}$$

where $\{\overline{e}_\alpha\}$ denotes the canonical basis of the model space $\mathbb{R}^d$. Then we can express (IV.50) in the canonical form (II.21)

$$\dot\rho(t) = X_{(\alpha)}(\rho(t))\,\dot z^\alpha(t) \quad , \quad \rho(t_b) = \rho_b \tag{IV.52}$$

where $X_{(\alpha)}$ is defined by

$$X_{(\alpha)}(\rho) = \sigma(\rho)\cdot e_\alpha = (\sigma(\rho)\circ u)\cdot \overline{e}_\alpha\,. \tag{IV.53}$$

[15] More explicitly, for a given frame ρ at a point x of M, $\sigma(\rho)$ is a linear map from T_xM to $T_\rho O(M)$, mapping u into $\sigma(\rho)\cdot u$ for u in T_xM.

Here x is a point of M and $\rho = (x, u)$ a frame, where $u : \mathbb{R}^d \to T_x M$ is an admissible map.

If $\dot{z}(t) = \overline{e}_\alpha$, i.e. $\dot{z}^\beta(t) = \delta^\beta_\alpha$, the coordinates of $\dot{x}(t)$ are constant in the frame $u(t)$ parallel transported along $x(t)$. Therefore $x(t)$ is the geodesic defined by

$$x(t_b) = x_b \quad , \quad \dot{x}(t_b) = e_\alpha \,. \tag{IV.54}$$

With $\dot{z}(t) = \overline{e}_\alpha$, equation (IV.52) reads

$$\dot{\rho}_{(\alpha)}(t) = X_{(\alpha)}\big(\rho_{(\beta)}(t)\big)\, \delta^\beta_\alpha = X_{(\alpha)}\big(\rho_{(\alpha)}(t)\big) \;.$$

The horizontal lift $\rho_{(\alpha)}(t)$ *of the geodesic* (IV.54) *is, as desired, the integral curve of* $X_{(\alpha)}$ *going through* ρ_b *at time* t_b. With $X_{(\alpha)}$ defined by (IV.53), ρ can be expressed in terms of the *Cartan development map*

$$(\pi \circ \rho)(t) = x(t) = (\mathrm{Dev}\, z)(t)\,. \tag{IV.55}$$

The Cartan development is a bijection from a space of pointed paths (paths with a fixed end point) on $T_{x_b}M$ (identified to $\mathbb{R}^d$ via the frame ρ_b) into a space of pointed paths on M – or vice versa. Here

$$\mathrm{Dev} : \mathcal{P}_0 T_{x_b} M \to \mathcal{P}_{x_b} M \quad \text{by} \quad z \mapsto x\,. \tag{IV.56}$$

The path x is said to be the development of z, if $\dot{x}(t)$ parallel transported along x from $x(t)$ to x_b is equal to $\dot{z}(t)$ trivially transported to the origin of $T_{x_b}M$, for every $t \in \mathbf{T}$.

The path integral solution of (IV.48) is

$$\Psi(t_b, \rho_b) = \int_{\mathbf{Z_b}} \mathcal{D}z \cdot \exp\Big(-\frac{\pi}{s} Q_0(z)\Big)\, \Phi(\rho(t_a, z)) \tag{IV.57}$$

with $\rho(t, z)$ solution of (IV.52). The path integral solution of (IV.49) is

$$\psi(t_b, x_b) = \int_{\mathbf{Z_b}} \mathcal{D}z \cdot \exp\Big(-\frac{\pi}{s} Q_0(z)\Big)\, \phi((\mathrm{Dev}\, z)(t_a))\,. \tag{IV.58}$$

If a scalar potential is desired in (IV.48) or (IV.49), one can proceed along either of the methods outlined in paragraph II.4.

The semiclassical approximation [5] of Ψ, with or without scalar potential, is considerably more complicated to compute than the semiclassical approximation of Ψ given in section III. For an initial wave function ϕ of type

(III.2) the blue-print given in Section III.2 is complete. If one wishes to compute the point-to-point propagator on a Riemannian space M, one chooses the initial wave function on M to be

$$\phi(x) = \delta_{x_a}(x) . \tag{IV.59}$$

The detailed calculation can be found in reference [5, pp. 309-311]. The development map cannot parametrize spaces of paths with two fixed points[16] but (IV.59) with $(\mathrm{Dev}\, z)(t_a)$ substituted to x restricts the domain of integration appropriately.

The following remarks simplify the calculations.

Remark 1. According to the formulas

$$\begin{aligned} \int_{\mathbf{T}} dt\, h_{\alpha\beta} \dot{z}^\alpha(t) \dot{z}^\beta(t) &= \int_{\mathbf{T}} dt\, g_{\mu\nu}(x(t)) \dot{x}^\mu(t) \dot{x}^\nu(t) \\ &= \int_{\mathbf{T}} dt\, g_{\mu\nu}(x(t)) \dot{\rho}^\mu(t) \dot{\rho}^\nu(t) , \end{aligned} \tag{IV.60}$$

development map and horizontal lift preserve lengths and angles.

Remark 2. If z develops into a classical paths x_{cl}, the determinant of the derivative mapping $\mathrm{Dev}'(z)$ is unity. For the proof see, for instance, reference [5], p. 308.

4. A multiply connected manifold.

The domain of integration, a space of pointed paths $\mathcal{P}_x N$, is the union of disjoint sets made of paths in different homotopy classes. We recall in paragraphs a) and b) earlier calculations of propagators on multiply connected spaces. Then, in paragraph c), we explain how these methods fit into our general framework.

a) It was shown [3] in 1971 that, for a system with a multiply connected configuration space N, the propagator K is a linear combination of propagators $K_{(\alpha)}$

$$|K^A| = \left| \sum_{g_\alpha \in \pi_1(N)} \chi^A(g_\alpha) K_{(\alpha)} \right| . \tag{IV.61}$$

Each $K_{(\alpha)}$ is obtained by summing over paths in the same homotopy class, say α; the set $\{\chi^A(g_\alpha\}_\alpha$ forms a representation, labeled A, of the fundamental

[16] For instance, two geodesics on S^2 intersect at two antipodal points; they are the developments of two halflines with one common origin.

group $\pi_1(N)$. Since a homotopy class cannot be identified uniquely with an element of $\pi_1(N)$, the propagator is defined modulo an overall phase factor. There are as many propagators K^A as there are inequivalent representations of $\pi_1(N)$.

The proof of (IV.61) uses two facts:

i) the superposition principle of quantum propagators implies the linear combination of partial propagators;

ii) the fundamental group based at a point is isomorphic to the fundamental group based at another point, but not canonically so. Therefore the pairing $(g_\alpha, K_{(\alpha)})$ is done by choosing an homotopy mesh (choosing a point for the fundamental group, and pairing one group element with one homotopy class), then requiring that the result be independent of the homotopy mesh.

b) Later on [14, see also 18, p. 65] the same result (worked out for a different example) was obtained from stochastic processes on fibre bundles. The basic steps are as follows[17].

i) A universal covering $\widetilde{N}$ is a principal G-bundle with projection $\Pi : \widetilde{N} \to N$, where $N = \widetilde{N}/G$ and G is a discrete group of automorphisms of $\widetilde{N}$ isomorphic to the fundamental group of N. For example, $N = S^1 = \mathbb{R}/\mathbb{Z}$ and $\widetilde{N} = \mathbb{R}$ is a $\mathbb{Z}$-principal bundle over S^1.

ii) The wave function for a system with configuration space N is a section of a bundle *weakly* associated to $\widetilde{N}$. i.e. a bundle whose typical fibre is associated to a not necessarily faithful representation of G, hence to a representation of a group homomorphic to G, says G_0. For example if $N = S^1$, a vector bundle over S^1 with structure group $U(1)$ is weakly associated to a $\mathbb{Z}$-principal bundle over S^1 by a homomorphism h_α of $\mathbb{Z}$ into $U(1)$ mapping n into $e^{in\alpha}$.

iii) There is a unique connection on $\widetilde{N}$: the horizontal lift $\widetilde{x}$ of a pointed path x with fixed point $\widetilde{x}(t_0) = \widetilde{x}_0$ is uniquely defined by the lift $\widetilde{x}_0$ of x_0. This unique connection defines the parallel transport of the wave function $\phi(x(t))$ back to x_0.

Consider for instance the case where $\widetilde{N} = \mathbb{R}$, $N = S^1$ and $\Pi(x) = e^{ix}$. Take for ϕ a section of a $U(1)$-bundle defined by the homomorphism h_α as above. Let $x(t)$ be a map into S^1, lifted to a map $\widetilde{x}_0(t)$ into $\mathbb{R}$ in such a way that $x(t) = e^{i\widetilde{x}_0(t)}$. The other liftings are given by

$$\widetilde{x}_k(t) = \widetilde{x}_0(t) + 2\pi k \tag{IV.62}$$

[17] We refer the reader to paragraph IV.7 for an explicit example where we use this strategy.

for k in $\mathbb{Z}$. The parallel transport of $\phi(\widetilde{x}_k(t))$ to x_0 is given by the formulas

$$\tau_{t_0}^t \phi(\widetilde{x}_k(t)) = \widetilde{h}_\alpha(\widetilde{x}_0)\widetilde{h}_\alpha(\widetilde{x}_k(t))^{-1}\,\phi(x(t)) \tag{IV.63}$$

and

$$\widetilde{h}_\alpha(\widetilde{x}_k(t))^{-1} = \widetilde{h}_\alpha(\widetilde{x}_0(t))^{-1}\,h_\alpha(k)^{-1} = e^{-ik\alpha}\widetilde{h}_\alpha(\widetilde{x}_0(t))\ . \tag{IV.64}$$

The map $\widetilde{h}_\alpha$ is the bundle map from the $\mathbb{Z}$-bundle $\mathbb{R}$ over S^1 to the $U(1)$-bundle corresponding to the map $h_\alpha : \mathbb{Z} \to U(1)$.

iv) Summing the wave function $\tau_{t_0}^t \phi(\widetilde{x}_k(t))$ over all k's gives the solution $\Psi(t, x_0)$ of a parabolic equation with initial value $\Psi(t_0, x_0) = \phi(x_0)$.

In both computations outlined above, one chooses a representation of the fundamental group, $\{\chi^A(g_\alpha)\}$, or the homomorphism $h : G \to G_0$.

c) We specialize our basic formula (II.1)

$$\Psi(t_b, x_b) = \int_{\mathbf{Z}_b} \mathcal{D}z \cdot e^{-\pi Q_0(z)/s}\phi(x_b \cdot \Sigma(t_a, z)) \tag{IV.65}$$

to the case where N is multiply-connected. To calculate the point-to-point transition amplitudes, we select $\phi(x)$ of the form $\delta_{x_a}(x)$ with a δ-factor centered at a point x_a of N. Denote the evaluation map taking z into $x_b \cdot \Sigma(t_a, z)$ by $\varepsilon : \mathbf{Z}_b \to N$. Since $\mathbf{Z}_b$ is contractible, we can lift ε to a map $\widetilde{\varepsilon} : \mathbf{Z}_b \to \widetilde{N}$ into the universal covering $\widetilde{N}$ of N and hence $\varepsilon = \Pi \circ \widetilde{\varepsilon}$. In the path integral (IV.65), the domain of integration is restricted by the δ-factor $\delta_{x_a}(x)$ to the inverse image $\varepsilon^{-1}(x_a)$. It consists of paths such that $x_b \cdot \Sigma(t_a, z) = x_a$ and *splits as the union of domains* $\mathbf{Z}_\Gamma = \widetilde{\varepsilon}^{-1}(\widetilde{x}_\Gamma)$ where $\widetilde{x}_\Gamma$ runs over the various points of $\widetilde{N}$ mapping to x_a by Π. The labels Γ correspond to the various homotopy classes of paths $x : \mathbf{T} \to N$ such that $x(t_a) = x_a$, $x(t_b) = x_b$. Hence the integral (IV.65) splits into a sum of integrals over the various subdomains

$$\Psi(t_b, x_b) = \sum_\Gamma \int_{\mathbf{Z}_\Gamma} \mathcal{D}z \cdot e^{-\pi Q_0(z)/s}\delta_{x_a}(x_b \cdot \Sigma(t_a, z))\ . \tag{IV.66}$$

This equation is the justification of the heuristic idea used in (IV.61) that the building blocks of K are the propagators $K_{(\alpha)}$ obtained by summing over paths in the same homotopy class. To explain the coefficients $\chi^A(g_\alpha)$ we can proceed as follows: using the previous notations $\widetilde{N}$, Π, G, consider a homomorphism χ^A of G into $U(1)$. The corresponding wave functions are functions $\widetilde{\phi}$ on $\widetilde{N}$ such that $\widetilde{\phi}(\widetilde{x}g) = \chi^A(g)\widetilde{\phi}(\widetilde{x})$ for g in G and $\widetilde{x}$ in $\widetilde{N}$. We denote by $\widetilde{\Sigma}(t,z)$ the lifting of $\Sigma(t,z)$ to $\widetilde{N}$ and generalize equation (IV. 65) by

$$\widetilde{\Psi}(t_b,\widetilde{x}_b) = \int_{\mathbf{Z}_b} \mathcal{D}z \cdot e^{-\pi Q_0(z)/s}\widetilde{\phi}\Big(\widetilde{x}_b \cdot \widetilde{\Sigma}(t,z)\Big) \,. \tag{IV.67}$$

The function $\widetilde{\Psi}$ will satisfy the same transformation property as $\widetilde{\phi}$ and when $\widetilde{\phi}$ is a δ-factor, we can split the integration domain into subdomains $\mathbf{Z}_\Gamma$ as above.

5. Gauge fields.

We begin with the case of an abelian gauge group. In physical terms, we consider a particle of mass m and electric charge e moving under the influence of a magnetic potential A, with components $A_\alpha(x)$ at the point x. We consider generally a d-dimensional space $\mathbb{R}^d$ in cartesian coordinates x^α ($\alpha \in \{1,\cdots,d\}$) and metric $|x|^2 = \delta_{\alpha\beta}x^\alpha x^\beta$. The classical Lagrangian is given by

$$L(x,\dot{x}) = \frac{m}{2}\,|\dot{x}|^2 + e\,A_\alpha(x)\dot{x}^\alpha \,, \tag{IV.68}$$

hence the action functional

$$S(x) = \int_{\mathbf{T}} dt\ L(x,\dot{x}) = \frac{m}{2}\int_{\mathbf{T}} \frac{|dx|^2}{dt} + e\int_{\mathbf{T}} A_\alpha dx^\alpha \,. \tag{IV.69}$$

The equation of motion can be derived from this Lagrangian, and can be put into the Hamiltonian form with the following definitions

$$p_\alpha = m\dot{x}^\alpha + eA_\alpha \quad , \quad H = |p - eA|^2/2m \,. \tag{IV.70}$$

The corresponding Schrödinger equation is obtained in the standard way by replacing p_α by the operator $\frac{\hbar}{i}\frac{\partial}{\partial x^\alpha}$ in the definition of H, and reads as

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\sum_\alpha \left(\frac{\hbar}{i}\frac{\partial}{\partial x^\alpha} - eA_\alpha(x)\right)^2 \psi \,. \tag{IV.71}$$

Our goal in this paragraph is to fit the well-known path integral solution of this equation into our general framework.

It has long been recognized [19] that it is desirable to treat ψ as a section of a complex line bundle (here over $\mathbb{R}^d$) associated to a principal $U(1)$-bundle via the canonical representation of $U(1)$ acting on $\mathbb{C}$ by multiplication. We describe the main steps of this construction.

5.1. *The invariant formalism.*

The *base space* is $M = \mathbb{R}^d$. The *gauge group* G is the set $U(1)$ of complex numbers of modulus one $g = e^{i\Theta}$. We consider a *principal bundle* P, with G acting from the right via $(p, g) \mapsto p \cdot g$, and projection $\Pi : P \to M$.

The *connection* is a differential form ω on P with the transformation rule

$$\omega(p \cdot g) = \omega(p) + g^{-1} dg\,. \tag{IV.72}$$

With the angular coordinate Θ such that $g = e^{i\Theta}$, one obtains $g^{-1}dg = id\Theta$ for the invariant differential form on $U(1)$, hence the Lie algebra $\mathfrak{g}$ of $U(1)$ is naturally identified with the set of pure imaginary numbers, and ω is pure imaginary.

For any path $x : \mathbf{T} \to M$ and any point p_b in P with $\Pi(p_b) = x(t_b)$, the *horizontal lifting* is a curve $\xi : \mathbf{T} \to P$ satisfying the following conditions:

$$\Pi\xi(t) = x(t) \quad , \quad \xi(t_b) = p_b \tag{IV.73}$$

$$\left\langle \omega_{\xi(t)}, \dot{\xi}(t) \right\rangle = 0\,, \tag{IV.74}$$

where $\omega_{\xi(t)}$ is the value of ω at the point $\xi(t)$ of P, that is a linear form on the tangent space $T_{\xi(t)}P$.

Let L be the *associated line bundle.* For any point x in M, a point p of P with $\Pi(p) = x$ corresponds to an admissible map $\widehat{p} : \mathbb{C} \to L_x$ where L_x is the fiber of L above the point x. If ψ is a *section* of L over M, its value at point x is an element $\psi(x)$ of L_x, hence $\widehat{p}^{-1}(\psi(x))$ is a complex number $\Psi(p)$. In this way (see e.g. [20], vol. I, p. 404), we identify the section ψ of L to a function $\Psi : P \to \mathbb{C}$ with the transformation rule

$$\Psi(p \cdot g) = g^{-1} \cdot \Psi(p) \tag{IV.75}$$

for p in P and g in G.

5.2. *Fixing the gauge.*

A fixing of the gauge corresponds to a section $s : M \to P$ of the principal bundle. When s is chosen, we may identify P with $M \times G$ in such a way that $p = s(x) \cdot g$ in P correspond to (x, g) in $M \times G$. A section ψ of the line bundle L corresponds now to a *wave function*, that is to a complex-valued function on M and the corresponding function Ψ on $M \times G$ is given by $\Psi(x, g) = g^{-1}\psi(x)$ and conversely $\psi(x) = \Psi(x, 1)$, or intrinsically $\psi = \Psi \circ s$.

The differential form ω on P gives by pull-back via $s : M \to P$ a purely imaginary differential form on M, to be written as $-\frac{ie}{\hbar}A$ to fit with standard physical dimensions. Hence the differential form A on M can be written as $A_\alpha(x)dx^\alpha$ and the functions $A_\alpha(x)$ are the components of the magnetic potential. On $M \times G$ the differential form ω is given by

$$\omega = g^{-1}dg - \frac{ie}{\hbar}A = i\left(d\Theta - \frac{e}{\hbar}A_\alpha dx^\alpha\right) \tag{IV.76}$$

(for $g = e^{i\Theta}$).

Let $x : \mathbf{T} \to M$ be a path. The horizontal lifting ξ of x is now described by $\xi(t) = \big(x(t), e^{i\Theta(t)}\big)$ and since ω induces a zero form on the image $\xi(\mathbf{T}) \subset M \times G$, we obtain the differential equation

$$\dot{\Theta} = \frac{e}{\hbar}A_\alpha(x(t))\dot{x}^\alpha(t) \tag{IV.77}$$

(see e.g. [18, pp. 64-65]).

Changing the gauge corresponds to choosing another section $s_1 : M \to P$. There exists then a function $R : M \to \mathbb{R}$ such that $s_1(x) = s(x) \cdot e^{-ieR(x)/\hbar}$. In the new gauge, the section of the line bundle L corresponds to a new wave function

$$\psi_1(x) = e^{ieR(x)/\hbar}\psi(x) \tag{IV.78}$$

and the new components of the magnetic potential are given by

$$A^1_\alpha(x) = A_\alpha(x) + \frac{\partial R(x)}{\partial x^\alpha}\,. \tag{IV.79}$$

5.3. *Path integrals.*

We revert to the notations in paragraph IV.1. The constant λ is again $(h/m)^{1/2}$ and we parametrize the paths x with $x(t_b) = x_b$ by

$$x(t,z) = x_b + \lambda z(t) \tag{IV.80}$$

where z runs over the space $\mathbf{Z}_b$. The horizontal lift of the previous path is given by

$$\xi(t,z) = \left(x(t,z), e^{i\Theta(t,z)}\right)\,. \tag{IV.81}$$

Taking into account the differential equation (IV.77), we obtain the following differential system

$$\begin{cases} dx^\alpha(t) = \lambda dz^\alpha(t) \quad \text{for } \alpha \text{ in } \{1,\cdots,d\} \\ d\Theta(t) = \frac{e\lambda}{\hbar}A_\alpha(x(t))dz^\alpha(t)\,. \end{cases} \tag{IV.82}$$

This system has the canonical form (II.21) where the vector fields $X_{(1)}, \cdots, X_{(d)}$, Y are given by

$$\mathcal{L}_{X_{(\alpha)}} = \lambda\left(\frac{\partial}{\partial x^\alpha} + \frac{e}{\hbar}A_\alpha(x)\frac{\partial}{\partial\Theta}\right) \quad , \quad Y = 0\,. \tag{IV.83}$$

Notice that for $\Psi(x,\Theta) = e^{-i\Theta}\psi(x)$, we have

$$\mathcal{L}_{X_{(\alpha)}}\Psi(x,\Theta) = e^{-i\Theta}\cdot\lambda D_\alpha\psi(x) \tag{IV.84}$$

with the differential operator (see e.g. [20, p. 405]):

$$D_\alpha = \frac{\partial}{\partial x^\alpha} - \frac{ie}{\hbar}A_\alpha(x)\,. \tag{IV.85}$$

Our general partial differential equation

$$\frac{\partial\Psi}{\partial t} = \frac{s}{4\pi}\sum_\alpha \mathcal{L}^2_{X_{(\alpha)}}\Psi \tag{IV.86}$$

translates now as

$$\frac{\partial\psi}{\partial t} = \frac{s}{4\pi}\frac{h}{m}\sum_\alpha D^2_\alpha\psi \tag{IV.87}$$

for the ψ-compoment of Ψ. For $s = i$, this equation coincides with the Schrödinger equation (IV.71).

Feynman path integral solution to this equation reads as follows:

$$\psi(t_b, x_b) = \int_{\mathcal{P}_b}\mathcal{D}x\cdot e^{iS(x)/\hbar}\phi(x(t_a)) \tag{IV.88}$$

where the action $S(x)$ is given by equation (IV.69). Parametrizing the paths x in $\mathcal{P}_b$ by the paths z in $\mathbf{Z}_b$ (see equation (IV. 80)), we can rewrite the previous path integral as[18]

$$\psi(t_b, x_b) = \int_{\mathbf{Z}_b}\mathcal{D}z\cdot e^{\pi iQ_0(z)}\exp\left(\frac{ie}{\hbar}\int_{\mathbf{T}}A_\alpha(x(t,z))dx^\alpha(t,z)\right)\phi(x(t_a,z))\,. \tag{IV.89}$$

[18] We give the formula in the oscillatory case $s = i$. The reader is invited to work out the formulas for the case $s = 1$. The definition of $Q_0(z)$ is given in equation (IV.22).

Replacing $\psi(x)$ by $\Psi(x,\Theta) = e^{-i\Theta}\psi(x)$ and similarly $\phi(x)$ by $\Phi(x,\Theta) = e^{-i\Theta}\phi(x)$, we can absorb the phase factor and obtain

$$\Psi(t_b, x_b, \Theta_b) = \int_{\mathbf{Z}_b} \mathcal{D}z \cdot e^{\pi i Q_0(z)} \Phi((x_b, \Theta_b) \cdot \Sigma(t_a, z)) \,. \tag{IV.90}$$

The transformation $\Sigma(t,z)$ of the bundle space $M \times G$ takes (x_b, Θ_b) into $(x_b + \lambda z(t),\ \Theta_b - \frac{e\lambda}{\hbar}\int_t^{t_b} A_\alpha(x_b + \lambda z(t))\, dz^\alpha(t))$ and corresponds to the integration of the differential system (IV.82).

5.4. *Various generalizations.*

a) It is easy to incorporate an *electric potential* V. The complete action functional is now

$$S(x) = \frac{m}{2}\int_{\mathbf{T}} \frac{|dx|^2}{dt} + e\int_{\mathbf{T}} A_\alpha dx^\alpha - V dt \,. \tag{IV.91}$$

Feynman solution (IV.88) is still valid and can be made explicit as

$$\psi(t_b, x_b) = \int_{\mathbf{Z}_b} \mathcal{D}z \cdot e^{\pi i Q_0(z)} \exp\left(\frac{ie}{\hbar}\int_{\mathbf{T}} A_\alpha dx^\alpha - V dt\right) \phi(x(t_a, z)) \tag{IV.92}$$

where the line integral $\int_{\mathbf{T}} A_\alpha dx^\alpha - V dt$ is calculated along the path $x(\cdot, z)$. The Schrödinger equation reads as follows:

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}\sum_\alpha \left(\frac{\hbar}{i}\frac{\partial}{\partial x^\alpha} - eA_\alpha(x)\right)^2 \psi + eV\psi \,. \tag{IV.93}$$

b) A non-abelian gauge group is for instance $G = U(N)$; a more general compact gauge group can always be realized as a closed subgroup of some unitary group $U(N)$. We mention a few of the required changes. The Lie algebra $\mathfrak{g}$ is the set of matrices of the form iE where E is an N-by-N hermitian matrix. The connection form ω on the principal bundle P takes its values in $\mathfrak{g}$ and the transformation rule is now

$$\omega(p \cdot g) = g^{-1}\omega(p)g + g^{-1}dg \tag{IV.94}$$

where $\omega(p)$, g, dg are N-by-N matrices and the products are given by matrix multiplication.

For the associated vector bundle L, we consider the natural action of $U(N)$ on the complex vector space $\mathbf{C}^N$. Hence a section ψ of L corresponds to a function $\Psi : P \to \mathbb{C}^N$ such that $\Psi(p \cdot g) = g^{-1} \cdot \Psi(p)$ for p in P and g in $G = U(N)$.

In a given gauge the form ω is given by $\omega(x, g) = g^{-1}dg - \frac{ie}{\hbar}A(x)$, where $A(x) = A_\alpha(x)dx^\alpha$ is a hermitian N-by-N matrix of differential forms on M. A gauge transformation is given by the formulas

$$\psi_1(x) = U(x)^{-1}\psi(x) \tag{IV.95}$$

$$A^1_\alpha(x) = U(x)^{-1}A_\alpha(x)U(x) + \frac{i\hbar}{e}U(x)^{-1}\frac{\partial}{\partial x^\alpha}U(x), \tag{IV.96}$$

where $U(x)$ is a unitary matrix depending on the point x of M.

The horizontal lift of a path $x : T \to M$ is of the form $\xi(t) = (x(t), U(t))$ where the unitary matrix $U(t)$ satisfies the differential equation

$$\dot{U}(t) = \frac{ie}{\hbar}U(t) \cdot A_\alpha(x(t))\dot{x}^\alpha(t)\,. \tag{IV.97}$$

We can solve this equation in the standard way using time-ordered exponentials $\mathcal{T}\exp$.

The Schrödinger equation is still written in the form (IV.87), but D_α is now a matrix of differential operators, namely $\frac{\partial}{\partial x^\alpha} \cdot \mathbb{1} - \frac{ie}{\hbar}A_\alpha(x)$ where $A_\alpha(x)$ is an N-by-N hermitian matrix. In the path integral (IV.89) replace the exponential factor by

$$\mathcal{T}\exp\left(\frac{ie\lambda}{\hbar}\int_{t_a}^{t_b} A_\alpha(x_b + \lambda z(t))\, dz^\alpha(t)\right)\,. \tag{IV.98}$$

c) We could consider gauge groups over a curved manifold and combine the results of paragraph IV.3 with those of the present paragraph.

6. A symplectic manifold.[19]

Let N be a symplectic manifold $\mathcal{M}$ of dimension $d = 2n$. The manifold $\mathcal{M}$ represents the classical phase space of a physical system – usually the cotangent bundle T^*Q of a configuration space Q but not necessarily[20].

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[19] Contributed by John LaChapelle.

[20] The fact that the quadratic form Q_0 is not related to the Jacobi operator of the physical system under consideration is illustrated again when one considers phase space: the Jacobi operator in phase space is a *first* order differential operator whereas the kernel of Q_0 is a *second* order differential operator.

Paths in $\mathcal{M}$ have n initial and n final boundary conditions. These boundary conditions are consistent with the requirement of quantum uncertainty, and they imply a choice of polarization[21]. Hence, it is possible to cover $\mathcal{M}$ with a family of open subsets $\{U_i\}$ such that each U_i is diffeomorphic to a product of two transverse Lagrangian submanifolds $L_i \times L'_i$. For simplicity consider a symplectic manifold which admits global transverse Lagrangian submanifolds L and L', and identity $\mathcal{M}$ with $L \times L'$. Set $x(t)$ in the form $(x_L(t), x_{L'}(t))$ with $x_L(t)$ in L and $x_{L'}(t)$ in L', in such a way that $x_L(t_b) = x_b$ and that $x_{L'}(t_a) = x_a$.

We require the group generated by the transformations $\Sigma(\mathbf{T}, z)$ to be a subgroup of the group of symplectomorphisms which leave the polarization invariant. This implies that the set $\{X_{(\alpha)}\}$ is of the form $\{X_{(a)}, X_{(a')}\}$ such that $X_{(a)}(x_{L'}(t)) = 0$, $X_{(a')}(x_L(t)) = 0$, and $[X_{(a)}, X_{(a')}] = 0$. Here $a \in \{1, \cdots, k\}$, $a' \in \{k+1, \cdots, d\}$, and k is a fixed integer between 1 and d, possibly, but not necessarily, equal to $d/2$. Consequently, a path satisfies the differential equations

$$\begin{cases} dx_L(t) = X_{(a)}(x_L(t))\, dz^a + Y(x_L(t))\, dt \\ dx_{L'}(t) = X_{(a')}(x_{L'}(t))\, dz^{a'} + Y(x_{L'}(t))\, dt\,, \end{cases} \tag{IV.99}$$

and the general formula (II.1) becomes

$$\begin{aligned} (U_{t_b,t_a}\phi)(x_b, x_a) := & \int_{\mathbf{Z}_L} \int_{\mathbf{Z}_{L'}} \mathcal{D}z_L\, \mathcal{D}z_{L'} \cdot e^{-\pi Q_0(z_L, z_{L'})/s} \\ & \times \phi(x_b \cdot \Sigma(t_a, z_L)\,, x_a \cdot \Sigma(t_b, z_{L'}))\;, \end{aligned} \tag{IV.100}$$

where now $h^{\alpha\beta} = \begin{pmatrix} h^{ab} & 0 \\ 0 & h^{a'b'} \end{pmatrix}$. Here $\mathbf{Z}_L$ is the space of paths $z_L : \mathbf{T} \to \mathbb{R}^k$ such that $z_L(t_b) = 0$, and $\mathbf{Z}_{L'}$ is the space of paths $z_{L'} : \mathbf{T} \to \mathbb{R}^{d-k}$ such that $z_{L'}(t_a) = 0$. Each functional integral separately satisfies a partial differential equation:

$$\begin{cases} \dfrac{\partial \Psi_L}{\partial t_a} = \frac{s}{4\pi} h^{ab} \mathcal{L}_{X_{(a)}} \mathcal{L}_{X_{(b)}} \Psi_L + \mathcal{L}_Y \Psi_L \\ \dfrac{\partial \Psi_{L'}}{\partial t_b} = \frac{s}{4\pi} h^{a'b'} \mathcal{L}_{X_{(a')}} \mathcal{L}_{X_{(b')}} \Psi_{L'} + \mathcal{L}_Y \Psi_{L'}\,, \end{cases} \tag{IV.101}$$

where $\Psi_L := \Psi|_L$ and $\Psi_{L'} := \Psi|_{L'}$.

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[21] Roughly speaking, a polarization is a foliation of $\mathcal{M}$ whose leaves are Lagrangian submanifolds of dimension n.

6.1. *The case of a cotangent bundle.*

Choosing an initial function ϕ is choosing a transition amplitude. We consider the case where $\mathcal{M}$ is the cotangent bundle of a flat configuration space Q, hence $\mathcal{M} = Q \times P$ where Q and P are finite-dimensional vector spaces in duality. In order to define ***position-to-position transition amplitudes***, we choose the initial function of the form $\phi(q,p) = \delta(q - q_a)$. Equation (IV.100) yields

$$\mathcal{K}(q_b, t_b; q_a, t_a) = \int_{\mathbf{Z}_Q}\int_{\mathbf{Z}_P} \mathcal{D}z_Q\,\mathcal{D}z_P \cdot \exp\Big(-\frac{\pi}{s}Q_0(z_Q, z_P)\Big)\, h(z_Q, z_P) \tag{IV.102}$$

where the integrand is given by

$$h(z_Q, z_P) = \delta(q_b \cdot \Sigma(t_a, z_Q) - q_a)\ . \tag{IV.103}$$

In order for the transition amplitudes to be consistent with the initial wave function, we require $\lim\limits_{t_a \to t_b} \mathcal{K}(q_b, t_b; q_a, t_a)$ to be equal to $\delta(q_b - q_a)$. But the integrand $h(z_Q, z_P)$ tends to $\delta(q_b \cdot \Sigma(t_b, z_Q) - q_a) = \delta(q_b - q_a)$ when t_a tends to t_b, a limit independent of z_Q, z_P. Hence, provided we can interchange limit and integration, we get

$$\lim_{t_a \to t_b} \mathcal{K}(q_b, t_b; q_a, t_a) = \int_{\mathbf{Z}_Q}\int_{\mathbf{Z}_P} \mathcal{D}z_Q\,\mathcal{D}z_P \cdot \exp\Big(-\frac{\pi}{s}Q_0(z_Q, z_P)\Big)\, \delta(q_b - q_a)\ .$$

Hence this is equal to $\delta(q_b - q_a)$ by the normalization of our integrator.

We handle the other cases in a similar way.

a) *Momentum-to-position amplitude* $\mathcal{K}(q_b, t_b; p_a, t_a)$: use the initial function $h^{-n/2}e^{iq_b \cdot p/\hbar}$ and the integrand

$$h^{-n/2}\exp\bigg(\frac{i}{\hbar}q_b \cdot (p_a \cdot \Sigma(t_b, z_P))\bigg)\ . \tag{IV.104$_\mathrm{a}$}$$

b) *Position-to-momentum amplitude* $\mathcal{K}(p_a, t_a; q_b, t_b)$: use the initial function $h^{-n/2}e^{-iq \cdot p_a/\hbar}$ and the integrand

$$h^{-n/2}\exp\bigg(-\frac{i}{\hbar}\,(q_b \cdot \Sigma(t_a, z_Q)) \cdot p_a\bigg)\ . \tag{IV.104$_\mathrm{b}$}$$

c) *Momentum-to-momentum amplitude* $\mathcal{K}(p_b, t_b; p_a, t_a)$: use the initial function $\delta(p - p_b)$ and the integrand

$$\delta(p_a \cdot \Sigma(t_b, z_P) - p_b)\ . \tag{IV.104$_\mathrm{c}$}$$

It follows easily from these definitions that

$$\mathcal{K}(p_b, t_b; p_a, t_a) = h^{-n/2} \int_Q dq_b \; e^{-iq_b \cdot p_b/\hbar} \mathcal{K}(q_b, t_b; p_a, t_a) \qquad \text{(IV.105)}$$

$$\mathcal{K}(q_b, t_b; q_a, t_a) = h^{-n/2} \int_P dp_b \; e^{iq_b \cdot p_b/\hbar} \mathcal{K}(p_b, t_b; q_a, t_a)\,. \qquad \text{(IV.106)}$$

6.2. *Coherent states.*

More general transition amplitudes, which cannot be interpreted as position-to-momentum transitions, are possible by choosing more complicated initial wave functions, different polarizations, and/or by having non-trivial phase spaces. For instance, coherent state transitions can be calculated – given a suitable characterization of coherent states.

Choose a Kähler polarization on a non-trivial symplectic manifold, so that $\mathcal{M}$ is a Kähler manifold M of complex dimension n. It is convenient (though not necessary) to take the phase space to be the product manifold $M \times \overline{M}$ with coordinates $(\zeta, \overline{z})$ and complex dimension $2n$. Following the work of Berezin [21] and Bar-Moshe and Marinov [22], we use generalized coherent state wave functions $\phi_{\zeta'}(\zeta) = \exp K(\zeta, \overline{\zeta}')$ where $K(\zeta, \overline{\zeta}')$ is the Kähler potential. Consider the function $\phi : M \times \overline{M} \to \mathbf{C}$ given by $\phi(\zeta, \overline{\zeta}') = e^{K(\zeta, \overline{\zeta}_a)}$ and take $z_M : \mathbf{T} \to \mathbf{C}^n$ such that $z_M(t_b) = 0$ and $z_{\overline{M}} : \mathbf{T} \to \mathbf{C}^n$ such that $z_{\overline{M}}(t_a) = 0$. Then

$$\mathcal{K}(\zeta_b, t_b; \overline{\zeta}_a, t_a) := \int_{\mathbf{Z}_M} \int_{\mathbf{Z}_{\overline{M}}} \mathcal{D}z_M \, \mathcal{D}z_{\overline{M}} \cdot e^{-\pi Q_0(z_M, z_{\overline{M}})/s} e^{K(\zeta_b \cdot \Sigma(t_a, z_M), \overline{\zeta}_a)}$$

are generalized coherent state transition amplitudes. Note that $\mathcal{K}(\overline{\zeta}_b, t_b; \zeta_a, t_a) = \overline{\mathcal{K}(\zeta_b, t_b; \overline{\zeta}_a, t_a)}$, and $\lim_{t_a \to t_b} \mathcal{K}(\zeta_b, t_b; \overline{\zeta}_a, t_a) = \exp K(\zeta_b, \overline{\zeta}_a)$ provided the limit can be taken inside the integral.

7. A simple model of the Bohm-Aharonov effect.

We illustrate on an elementary example the various techniques expounded in this section. Our basic manifold N is the euclidean plane $\mathbb{R}^2$ with the origin removed. As in paragraph IV.2, we denote the cartesian coordinates by x^1, x^2. The universal covering of N is the set $\widetilde{N}$ of pairs of real numbers r, θ with $r > 0$, and the covering map $\Pi : \widetilde{N} \to N$ is described by equation (IV.31), namely $x^1 = r \cos\theta$, $x^2 = r \sin\theta$.

We introduce a *magnetic potential* A with components

$$A_1 = -\frac{Fx^2}{2\pi\,|x|^2} \quad , \quad A_2 = \frac{Fx^1}{2\pi\,|x|^2}\,. \qquad \text{(IV.107)}$$

The magnetic field has one component B perpendicular to our plane, given by

$$B = \frac{\partial A_1}{\partial x^2} - \frac{\partial A_2}{\partial x^1}, \tag{IV.108}$$

and an obvious calculation gives $B = 0$ *outside of the origin.* But the circulation $\oint A_1 dx^1 + A_2 dx^2$ on any loop around the origin is equal to F, hence

$$B = F\delta(x). \tag{IV.109}$$

Physically, we have a wire perpendicular to the plane through its origin, carrying a magnetic flux equal to F.

Denote by m the mass and by e the electric charge of a particle. For a path $x : \mathbf{T} \to N$, with components $x^1(t)$, $x^2(t)$ at time t, the action is given by

$$S(x) = S_0(x) + S_M(x), \tag{IV.110}$$

with the *kinetic action*

$$S_0(x) = \frac{m}{2} \int_{\mathbf{T}} \frac{|dx|^2}{dt} \tag{IV.111}$$

and *the magnetic action*

$$S_M(x) = \frac{eF}{2\pi} \int_{\mathbf{T}} \frac{x^1 dx^2 - x^2 dx^1}{(x^1)^2 + (x^2)^2}. \tag{IV.112}$$

We fix a point x_b in N, and denote by $\mathbf{X}_b$ the set of paths $x : \mathbf{T} \to N$ for which the kinetic action $S_0(x)$ is finite (that is the $L^{2,1}$ paths) with the boundary condition $x(t_b) = x_b$. On $\mathbf{X}_b$, we denote by $\mathcal{D}x$ the translation-invariant integrator normalized by

$$\int_{\mathbf{X}_b} \mathcal{D}x \cdot \exp\left(\frac{i}{\hbar} S_0(x)\right) = 1. \tag{IV.113}$$

The free position-to-position transition amplitudes are defined by

$$\langle t_b, x_b \mid t_a, x_a \rangle_0 = \int_{\mathbf{X}_b} \mathcal{D}x \cdot \exp\left(\frac{i}{\hbar} S_0(x)\right) \delta(x(t_a) - x_a). \tag{IV.114}$$

According to the calculation made at the end of paragraph IV.2, we get[22]

$$\langle t_b, x_b \mid t_a, x_a \rangle_0 = \frac{-mi}{h(t_b - t_a)} \exp\left(\frac{i}{\hbar} S_0(x_{\text{cl}})\right), \tag{IV.115}$$

[22] We express our formulas directly in terms of the path x, and have no need for the scaling $x = \lambda z$ introduced earlier.

where x_{cl} is the classical path of a free particle, namely:

$$x_{\text{cl}}(t) = \frac{x_a(t_b - t) + x_b(t - t_a)}{t_b - t_a} . \tag{IV.116}$$

Hence, we obtain explicitely

$$S_0(x_{\text{cl}}) = \frac{m}{2} \frac{|x_b - x_a|^2}{t_b - t_a} , \tag{IV.117}$$

that is, the WKB approximation (IV.30) is exact in our case.

We consider now the transition amplitudes in the given magnetic potential:

$$\langle t_b, x_b \mid t_a, x_a \rangle_F = \int_{\mathbf{X}_b} \mathcal{D}x \cdot \exp\left(\frac{i}{\hbar} S(x)\right) \delta(x(t_a) - x_a) . \tag{IV.118}$$

According to (IV.87), we consider the Schrödinger equation

$$\frac{\partial \psi_F}{\partial t} = \frac{i\hbar}{2m} \left(D_1^2 + D_2^2\right) \psi_F \tag{IV.119}$$

with the differential operators $D_\alpha = \frac{\partial}{\partial x^\alpha} - \frac{ie}{\hbar} A_\alpha$. Explicitly, we obtain

$$\frac{\partial \psi_F}{\partial t} = \frac{i\hbar}{2m} L \psi_F , \tag{IV.120}$$

with

$$L = \left(\frac{\partial}{\partial x^1}\right)^2 + \left(\frac{\partial}{\partial x^2}\right)^2 - \frac{2ci}{|x|^2} \left(x^1 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^1}\right) - \frac{c^2}{|x|^2} . \tag{IV.121}$$

The constant c is equal to eF/h; it is dimensionless. The solution to the equation (IV.119) is given by

$$\psi_F(t_b, x_b) = \int_N dx_a \langle t_b, x_b \mid t_a, x_a \rangle_F \psi_F(t_a, x_a) , \tag{IV.122}$$

a formula equivalent to our more familiar one

$$\psi_F(t_b, x_b) = \int_{\mathbf{X}_b} \mathcal{D}x \cdot \exp\left(\frac{i}{\hbar} S(x)\right) \phi(x(t_a)) \tag{IV.123}$$

if we take into account the initial wave function $\phi(x) = \psi_F(t_a, x)$ at time t_a.

We lift now everything to the universal covering $\widetilde{N}$. By means of the formulas (IV.31), the wave function is now a function $\widetilde{\psi}_F(t,r,\theta)$ with the restriction

$$\widetilde{\psi}_F(t,r,\theta+2\pi) = \widetilde{\psi}_F(t,r,\theta) \tag{IV.124}$$

that is, a function on $\mathbf{T} \times \widetilde{N}$ invariant under the group $\mathbb{Z}$ acting on $\widetilde{N}$ by $(r,\theta)\cdot n = (r,\theta+2\pi n)$. The Schrödinger equation (IV.120) keeps its form with L changed into the new operator

$$\widetilde{L} = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{2ci}{r^2}\frac{\partial}{\partial \theta} - \frac{c^2}{r^2}. \tag{IV.125}$$

This operator can be written as $\widetilde{L} = e^{ci\theta}\Delta e^{-ci\theta}$ where Δ is the Laplacian in polar coordinates, namely:

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{1}{r}\frac{\partial}{\partial r}. \tag{IV.126}$$

Hence the new wave function

$$\widetilde{\psi}_0(t,r,\theta) = e^{-ci\theta}\widetilde{\psi}_F(t,r,\theta) \tag{IV.127}$$

satisfies the equation

$$\frac{\partial\widetilde{\psi}_0}{\partial t} = \frac{i\hbar}{2m}\Delta\widetilde{\psi}_0\,, \tag{IV.128}$$

that is the Schrödinger equation for a free particle, written in polar coordinates. As we shall see in a moment, equation (IV.127) expresses a *gauge transformation* which apparently removes the magnetic potential. But the periodicity condition (IV.124) transforms into

$$\widetilde{\psi}_0(t,r,\theta+2\pi) = e^{-2\pi i c}\widetilde{\psi}_0(t,r,\theta)\,, \tag{IV.129}$$

hence the wave function $\widetilde{\psi}_0$ on $\widetilde{N}$ is not the lifting of a function ψ_0 on N. Rather, the linear representation $n \mapsto e^{2\pi i n c}$ of the group $\mathbb{Z}$ into $U(1)$ defines a line bundle on $N = \mathbb{R}^2\backslash\{0\}$ and $\widetilde{\psi}_0$ *corresponds to a section* ψ *of this line bundle.* Otherwise stated, despite the fact that the magnetic field $B = \partial A_1/\partial x^2 - \partial A_2/\partial x^1$ is identically zero on N, there is no single-valued function R on N such that $A_\alpha = -\partial R/\partial x^\alpha$: *the space N is multiply connected.*

To explain gauge transformations, let us introduce the principal bundles $P = N \times U(1)$ and $\widetilde{P} = \widetilde{N} \times U(1)$. According to the general formulas the connection form ω_F on P is given by

$$\omega_F = i\left(d\Theta - \frac{e}{\hbar}A\right) = i\left(d\Theta - c\frac{x^1dx^2 - x^2dx^1}{(x^1)^2 + (x^2)^2}\right), \tag{IV.130}$$

where Θ is the angular coordinate on $U(1)$ (taken modulo 2π). When there is no magnetic field, it reduces to $\omega_0 = id\Theta$. We cannot transform ω_F into ω_0 by a gauge transformation, but if we lift these differential forms to $\widetilde{N}$, we obtain

$$\widetilde{\omega}_F = i(d\Theta - cd\theta) \quad , \quad \widetilde{\omega}_0 = id\Theta\,. \tag{IV.131}$$

The transformation $\widetilde{U}$ of $\widetilde{P}$ into $\widetilde{P}$ taking (r, θ, Θ) into $(r, \theta, \Theta + c\theta)$ is an automorphism of $U(1)$-bundle, and $\widetilde{U}^*\widetilde{\omega}_F = \widetilde{\omega}_0$, but $\widetilde{U}$ is not the lifting of an automorphism U of P, *except when c is an integer.*

To the wave function $\psi_F(x)$ on N we associate the function

$$\Psi_F(x, \Theta) = e^{-i\Theta}\psi_F(x) \tag{IV.132}$$

on P. The Schrödinger equation (IV.119) translates into our standard form

$$\frac{\partial \Psi_F}{\partial t} = \frac{i}{4\pi} h^{\alpha\beta} \mathcal{L}_{X^F_{(\alpha)}} \mathcal{L}_{X^F_{(\beta)}} \Psi_F \tag{IV.133}$$

provided we take $h_{\alpha\beta}$ equal to $m\delta_{\alpha\beta}/h$, with the vector fields

$$\begin{cases} \mathcal{L}_{X^F_{(1)}} = \dfrac{\partial}{\partial x^1} - \dfrac{cx^2}{|x|^2}\dfrac{\partial}{\partial \Theta} \\[2ex] \mathcal{L}_{X^F_{(2)}} = \dfrac{\partial}{\partial x^2} + \dfrac{cx^1}{|x|^2}\dfrac{\partial}{\partial \Theta}\,. \end{cases} \tag{IV.134}$$

We lift now everything to $\widetilde{P}$, so ψ_F lifts to $\widetilde{\psi}_F$, Ψ_F to $\widetilde{\Psi}_F$ and the lifted vector fields are given by[23]

$$\begin{cases} \mathcal{L}_{\widetilde{X}^F_{(1)}} = \cos\theta \cdot \dfrac{\partial}{\partial r} - \dfrac{\sin\theta}{r} \cdot \left(\dfrac{\partial}{\partial \theta} + c\dfrac{\partial}{\partial \Theta}\right) \\[2ex] \mathcal{L}_{\widetilde{X}^F_{(2)}} = \sin\theta \cdot \dfrac{\partial}{\partial r} + \dfrac{\cos\theta}{r} \cdot \left(\dfrac{\partial}{\partial \theta} + c\dfrac{\partial}{\partial \Theta}\right)\,. \end{cases} \tag{IV.135}$$

The Schrödinger equation (IV.133) remains the same with Ψ_F replaced by $\widetilde{\Psi}_F$ and $X^F_{(\alpha)}$ by $\widetilde{X}^F_{(\alpha)}$.

We describe the effect of the gauge transformation $\widetilde{U}$ given by $\widetilde{U}(r, \theta, \Theta) = (r, \theta, \Theta + c\theta)$. We noticed that $\widetilde{U}^*\widetilde{\omega}_F$ is equal to $\widetilde{\omega}_0$, and similarly, we get $\widetilde{\Psi}_F \circ \widetilde{U} = \widetilde{\Psi}_0$ where $\widetilde{\Psi}_0$ is defined by analogy to $\widetilde{\Psi}_F$, namely

$$\widetilde{\Psi}_0(r, \theta, \Theta) = e^{-i\Theta}\widetilde{\psi}_0(r, \theta)\,. \tag{IV.136}$$

[23] For $F = 0$, they reduce to the vector fields given in equations (IV.36) and (IV.37).

Dually, $\widetilde{U}$ transforms the vector field $\widetilde{X}^0_{(\alpha)}$ into $\widetilde{X}^F_{(\alpha)}$ for $\alpha = 1$ or 2. Hence $\widetilde{\Psi}_F$ is a solution of the Schrödinger equation (IV.133) if, and only if, $\widetilde{\Psi}_0$ is a solution of the corresponding equation for $F = 0$.

We conclude this paragraph by a discussion of path integrals. We parametrize paths in $\mathbf{X}_b$ by paths in $\mathbf{Z}_b$, using the correspondence

$$x(t) = x_b + z(t). \tag{IV.137}$$

Fix r_b and θ_b in such a way that

$$x_b = (r_b \cos\theta_b, r_b \sin\theta_b) . \tag{IV.138}$$

Then x can be lifted in a unique way to a path $\widetilde{x} : \mathbf{T} \to \widetilde{N}$ of the form $\widetilde{x}(t) = (r(t), \theta(t))$ such that $\widetilde{x}(t_b) = (r_b, \theta_b)$, that is $r(t_b) = r_b$ and $\theta(t_b) = \theta_b$. The initial point $x_a = x(t_a)$ is lifted to $\widetilde{x}(t_a) = (r_a, \theta_a)$ where $r(t_a) = r_a$, $\theta(t_a) = \theta_a$. Moreover, the magnetic action of the original path is given by

$$S_M(x)/\hbar = c \cdot (\theta_b - \theta_a) . \tag{IV.139}$$

Consider the space $\mathbf{X}_{a,b}$ of paths $x : \mathbf{T} \to N$ such that $x(t_a) = x_a$, $x(t_b) = x_b$. *Then two such paths are in the same homotopy class if, and only if, the lifted paths in $\widetilde{N}$ correspond to the same determination of the angular coordinate θ_a of x_a. An equivalent condition is that they have the same magnetic action.*

Using the connection of the principal bundle $\widetilde{P}$ over $\widetilde{N}$, we can define the horizontal lifting of $\widetilde{x}$; it is the unique path of the form $y(t) = (r(t), \theta(t), \Theta(t))$ on which $\widetilde{\omega}_F$ induces a zero form, that is $\Theta(t) - c\theta(t)$ is a constant in time. For the phase factors, we get

$$e^{i\Theta(t_b)} = e^{i\Theta(t_a)} e^{iS_M(x)/\hbar} . \tag{IV.140}$$

Moreover, the horizontal lifting is a solution of our standard differential equation $dy = X_{(\alpha)}(y) \cdot dz^\alpha$.

From all this, it follows that the transition amplitudes as defined in the Feynman way (IV.118) agree with our standard expression (see for instance (IV.38)). The result obtained in paragraph IV.2 (see formulas (IV.41) and (IV.47)) can be restated as

$$\langle t_b, x_b \mid t_a, x_a \rangle_0 = \frac{1}{r_a} \sum_{n \in \mathbb{Z}} \langle t_b, r_b, \theta_b \mid t_a, r_a, \theta_a + 2n\pi \rangle . \tag{IV.141}$$

A similar reasoning yields a more general formula:

$$\langle t_b, x_b \mid t_a, x_a \rangle_F = \frac{1}{r_a} \sum_{n \in \mathbb{Z}} e^{ic(\theta_b - \theta_a - 2n\pi)} \langle t_b, r_b, \theta_b \mid t_a, r_a, \theta_a + 2n\pi \rangle . \tag{IV.142}$$

We can invert this formula[24] and obtain

$$\left\langle t_b, r_b, \theta_b \;\middle|\; t_a, r_a, \theta_a \right\rangle = r_a \int_0^1 dc\, e^{-ic(\theta_b - \theta_a)} \left\langle t_b, x_b \;\middle|\; t_a, x_a \right\rangle_{hc/e} . \qquad \text{(IV.143)}$$

Hence, *from the knowledge of the transition amplitude* $\left\langle t_b, x_b \;\middle|\; t_a, x_a \right\rangle_F$ *as a function of the magnetic flux* F*, we can infer the value of the transition amplitudes in polar coordinates.*

[24] Notice that the transition probabilities are functions of F which admit a period h/e, a well-known physical effect.

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Appendix A
Functional integration

In this appendix, we develop the basic properties of our integrators. In the oscillating case ($s = i$), our theory is, up to some inessential changes in notation, the same as the one expounded by Albeverio and Høegh-Krohn in [9]. The introduction of a parameter s equal to 1 or i enables us to treat in a unified way the oscillating integrators of Fresnel type $e^{\pi i Q(x)}\mathcal{D}x$ and the Gaussian integrators of type $e^{-\pi Q(x)}\mathcal{D}x$ (with $Q(x) > 0$ for $x \neq 0$ in the latter case). We content ourselves by giving here the *basic formulas* and the *computational tools.* In the Gaussian case, we would have to justify these formal manipulations, since, as it is well-known, the $L^{2,1}$ *functions are a set of measure* 0 *for the Wiener measure*, and *we are considering functions on this set of measure* 0. Our claims can be fully vindicated, but we defer the complete justification to another publication.

1. Gaussian integrators.

1.1. *The setup.*

We denote by $\mathbf{X}$ a real, separable, Banach space, by $\mathbf{X}'$ its dual and by $\langle x', x\rangle$ (or sometimes $\langle x', x\rangle_{\mathbf{X}}$) the duality between $\mathbf{X}$ and $\mathbf{X}'$. We suppose given a continuous linear map $D : \mathbf{X} \to \mathbf{X}'$ with the following properties:

– (symmetry) $\langle Dx, y\rangle = \langle Dy, x\rangle$ for x, y in $\mathbf{X}$;

– (invertibility) there exists a continuous linear map $G : \mathbf{X}' \to \mathbf{X}$ inverse of D, that is $DG = \mathbb{1}$ and $GD = \mathbb{1}$.

Out of these data one constructs two quadratic forms, Q on $\mathbf{X}$ and W on $\mathbf{X}'$, by the rules

$$Q(x) = \langle Dx, x\rangle \quad , \quad W(x') = \langle x', Gx'\rangle . \tag{A.1}$$

They are related to each other as follows:

$$Q(x) = W(Dx) \quad , \quad W(x') = Q(Gx') \tag{A.2}$$

for x in $\mathbf{X}$ and x' in $\mathbf{X}'$.

We denote by s a parameter equal to 1 or i. The function $e^{-\pi s W}$ on $\mathbf{X}'$ is continuous. It is bounded in the following cases:

– $s = i$;

– $s = 1$ and $W(x') > 0$ for $x' \neq 0$ in $\mathbf{X}'$. Equivalently, by (A.2), $s = 1$ and $Q(x) > 0$ for $x \neq 0$ in $\mathbf{X}$.

1.2. *The oscillatory case* ($s = i$).

The integrator $\mathcal{D}x$ is characterized by the following integration formula:

$$\int_{\mathbf{X}} \mathcal{D}x \cdot \exp(\pi i Q(x) - 2\pi i \langle x', x\rangle) = \exp(-\pi i W(x')) \tag{A.3}$$

for every x' in $\mathbf{X}'$. This relation should be interpreted as follows.

Since the metric space $\mathbf{X}'$ is complete and separable, we know the notion of a complex bounded Borel measure[25] μ on $\mathbf{X}'$. The *Fourier-Stieltjes transform* $\mathcal{F}\mu$ of μ is given the customary definition:

$$(\mathcal{F}\mu)(x) = \int_{\mathbf{X}'} d\mu(x')\, e^{-2\pi i \langle x', x\rangle}. \tag{A.4}$$

It is a continuous bounded function on $\mathbf{X}$.

We denote by[26] $\mathcal{F}(\mathbf{X})$ the set of functions on $\mathbf{X}$ of the form $\Phi_\mu = e^{\pi i Q} \cdot \mathcal{F}\mu$, where μ runs over the measures on $\mathbf{X}'$. Since the map $\mu \mapsto \mathcal{F}\mu$ is injective, $\mathcal{F}(\mathbf{X})$ is a Banach space, the norm of the function Φ_μ being taken equal to the total variation[27] $\mathrm{Var}(\mu)$ of μ. On this Banach space, one defines a continuous linear form, denoted as an integral, by

$$\int_{\mathbf{X}} \mathcal{D}x \cdot \Phi_\mu(x) = \int_{\mathbf{X}'} d\mu(x')\, e^{-\pi i W(x')}. \tag{A.5}$$

Hence by definition, we have

$$\int_{\mathbf{X}} \mathcal{D}x \int_{\mathbf{X}'} d\mu(x') \exp(\pi i Q(x) - 2\pi i \langle x', x\rangle) = \int_{\mathbf{X}'} d\mu(x') \exp(-\pi i W(x')). \tag{A.6}$$

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[25] That is: a σ-additive functional from the σ-algebra of Borel subsets of $\mathbf{X}'$ into the complex numbers (see [23] for instance). We use a simplified terminology by refering to μ as a "measure" on $\mathbf{X}'$.

[26] The initial $\mathcal{F}$ stands for Fresnel or Feynman according to the worshipping habits of the reader.

[27] According to the standard definition, this is the l.u.b. of the set of numbers $\sum_{i=1}^{p} |\mu(A_i)|$ where $(A_1, \cdots, A_p)$ runs over the set of all partitions of $\mathbf{X}'$ into a finite number of Borel subsets.

Formally, this is the equation obtained by integrating equation (A.3) w.r.t. $d\mu(x')$ and then interchanging the integrations:

$$\int_{\mathbf{X}'} d\mu(x') \int_{\mathbf{X}} \mathcal{D}x = \int_{\mathbf{X}} \mathcal{D}x \int_{\mathbf{X}'} d\mu(x') .$$

The space $\mathcal{F}(\mathbf{X})$ of Feynman-integrable functionals on $\mathbf{X}$ is invariant under translations by elements of $\mathbf{X}$, and so is the integral, namely:

$$\int_{\mathbf{X}} \mathcal{D}x \cdot F(x) = \int_{\mathbf{X}} \mathcal{D}x \cdot F(x + x_0) \qquad \text{(A.7)}$$

or, in shorthand notation, $\mathcal{D}x = \mathcal{D}(x + x_0)$, for any fixed element x_0 of $\mathbf{X}$.

According to equations (A.1) and (A.2), formula (A.3) can be rewritten as

$$\int_{\mathbf{X}} \mathcal{D}x \cdot e^{\pi i Q(x - Gx')} = 1. \qquad \text{(A.8)}$$

Hence *assuming the invariance under translations of the integral*, the normalization of $\mathcal{D}x$ is achieved by

$$\int_{\mathbf{X}} \mathcal{D}x \cdot e^{\pi i Q(x)} = 1. \qquad \text{(A.9)}$$

1.3. *The positive case* ($s = 1$).

Henceforth, we assume that $s = 1$ and that Q is positive-definite (that is $Q(x) > 0$ for $x \neq 0$ in $\mathbf{X}$). We take as basic integration formula

$$\int_{\mathbf{X}} \mathcal{D}x \cdot \exp(-\pi Q(x) - 2\pi i \langle x', x \rangle) = \exp(-\pi W(x')) . \qquad \text{(A.10)}$$

We can interpret this relation as above. The conclusion is the integration formula

$$\int_{\mathbf{X}} \mathcal{D}x \int_{\mathbf{X}'} d\mu(x') \exp(-\pi Q(x) - 2\pi i \langle x', x \rangle) = \int_{\mathbf{X}'} d\mu(x') \exp(-\pi W(x')) . \qquad \text{(A.11)}$$

But the space $\mathcal{F}(\mathbf{X})$ is no longer invariant under translations. To restore this invariance, we have to replace Fourier-Stieltjes transforms by ***Laplace-Stieltjes transforms***. For that purpose, we have to consider the complex dual space $\mathbf{X}'_{\mathbb{C}}$ consisting of the continuous real-linear maps $x' : \mathbf{X} \to \mathbb{C}$ and measures μ on $\mathbf{X}'_{\mathbb{C}}$. We leave the details to a forthcoming publication.

Formulas (A.3) and (A.10) are obtained as the specializations of formula (I.7) for $s = i$ and $s = 1$ respectively.

2. Linear changes of variables.

2.1. *The case of Lebesgue integrals.*

Let $\mathbf{Y}$ be another real, separable, Banach space, and let $L : \mathbf{X} \to \mathbf{Y}$ be a Borel-measurable map. If $\omega_{\mathbf{X}}$ is a measure on $\mathbf{X}$, its image under L is the measure $\omega_{\mathbf{Y}}$ on $\mathbf{Y}$ defined by $\omega_{\mathbf{Y}}(B) = \omega_{\mathbf{X}}(L^{-1}(B))$ for any Borel subset B of $\mathbf{Y}$. In functional terms, this definition is tantamount to

$$\int_{\mathbf{X}} d\omega_{\mathbf{X}}(x)\, g(L(x)) = \int_{\mathbf{Y}} d\omega_{\mathbf{Y}}(y)\, g(y) \tag{A.12}$$

for any bounded (or non-negative) Borel-measurable function g on $\mathbf{Y}$.

Assume now that L is linear and continuous. We consider the transpose $\widetilde{L}$ of L, that is the map $\widetilde{L} : \mathbf{Y}' \to \mathbf{X}'$ such that

$$\langle \widetilde{L}y', x\rangle_{\mathbf{X}} = \langle y', Lx\rangle_{\mathbf{Y}} \tag{A.13}$$

for x in $\mathbf{X}$ and y' in $\mathbf{Y}'$. The measures $\omega_{\mathbf{X}}$ on $\mathbf{X}$ and $\omega_{\mathbf{Y}}$ on $\mathbf{Y}$ being as above, introduce their Fourier-Stieltjes transforms $\mathcal{F}\omega_{\mathbf{X}}$ and $\mathcal{F}\omega_{\mathbf{Y}}$ respectively. Hence $\mathcal{F}\omega_{\mathbf{X}}$ is a bounded continuous function on $\mathbf{X}'$, and similarly for $\mathcal{F}\omega_{\mathbf{Y}}$ on $\mathbf{Y}'$. By definition, we have

$$\mathcal{F}\omega_{\mathbf{X}}(x') = \int_{\mathbf{X}} d\omega_{\mathbf{X}}(x) \exp(-2\pi i \langle x', x\rangle_{\mathbf{X}}) \tag{A.14}$$

$$\mathcal{F}\omega_{\mathbf{Y}}(y') = \int_{\mathbf{Y}} d\omega_{\mathbf{Y}}(y) \exp(-2\pi i \langle y', y\rangle_{\mathbf{Y}})\,. \tag{A.15}$$

By specializing $g(y) = \exp(-2\pi i \langle y', y\rangle_{\mathbf{Y}})$ into formula (A.12) and taking into account formulas (A.13) to (A.15), we get $\mathcal{F}\omega_{\mathbf{Y}}(y') = \mathcal{F}\omega_{\mathbf{X}}(\widetilde{L}y')$ for every y' in $\mathbf{Y}'$. Hence the composition formula:

$$\mathcal{F}\omega_{\mathbf{Y}} = \mathcal{F}\omega_{\mathbf{X}} \circ \widetilde{L}\,. \tag{A.16}$$

The case of a translation is similar. Assume now that x_0 is a given vector in $\mathbf{X}$ and denote by T the translation taking x into $x + x_0$ in $\mathbf{X}$. If ω is any measure on $\mathbf{X}$ and ω_{x_0} its image under the translation T, a suitable specialization of formula (A.12) gives the Fourier transform of ω_{x_0}, namely

$$(\mathcal{F}\omega_{x_0})(x') = \exp(-2\pi i \langle x', x_0\rangle) \cdot \mathcal{F}\omega(x'). \tag{A.17}$$

2.2. *Infinite-dimensional integrators.*

We go back to the setup of paragraph A.1.1. We denote by $\mathcal{D}x$ the integrator characterized by

$$\int_{\mathbf{X}} \mathcal{D}x \cdot \exp\Big(-\frac{\pi}{s}Q(x)\Big) \cdot \exp(-2\pi i \langle x', x\rangle) = \exp(-\pi s W(x')). \tag{A.18}$$

Formally, this means that the integrator $\mathcal{D}\omega$ defined by

$$\mathcal{D}\omega(x) = \exp\Big(-\frac{\pi}{s}Q(x)\Big) \cdot \mathcal{D}x \tag{A.19}$$

has a Fourier transform equal to $\exp(-\pi s W)$, namely:

$$\int_{\mathbf{X}} \mathcal{D}\omega(x) \exp(-2\pi i \langle x', x\rangle) = \exp\left(-\pi s W(x')\right). \tag{A.20}$$

This can be interpreted as follows: the integrator $\mathcal{D}\omega$ is a continuous linear form on the Banach space of Fourier-Stieltjes transforms $\mathcal{F}\mu$, given by

$$\int_{\mathbf{X}} \mathcal{D}\omega(x)\mathcal{F}\mu(x) = \int_{\mathbf{X}'} d\mu(x') \exp\left(-\pi s W(x')\right). \tag{A.21}$$

This is another form of the Parseval relation.

We proceed to study the image of the integrator $\mathcal{D}\omega$ under a linear map. We use the following notations:

- $Q_{\mathbf{X}}$ is a quadratic form on $\mathbf{X}$, with inverse $W_{\mathbf{X}}$ on $\mathbf{X}'$;
- $Q_{\mathbf{Y}}$ is a quadratic form on $\mathbf{Y}$, with inverse $W_{\mathbf{Y}}$ on $\mathbf{Y}'$;
- L is a continuous linear map from $\mathbf{X}$ into $\mathbf{Y}$.

We assume that L is surjective, hence $\widetilde{L}$ is injective, and that the quadratic forms $W_{\mathbf{X}}$ and $W_{\mathbf{Y}}$ are related by

$$W_{\mathbf{Y}} = W_{\mathbf{X}} \circ \widetilde{L}\,. \tag{A.22}$$

Consider now the integrators[28]

$$\mathcal{D}\omega_{\mathbf{X}}(x) = \exp\Big(-\frac{\pi}{s}Q_{\mathbf{X}}(x)\Big)\cdot\mathcal{D}x \tag{A.23}$$

$$\mathcal{D}\omega_{\mathbf{Y}}(y) = \exp\Big(-\frac{\pi}{s}Q_{\mathbf{Y}}(y)\Big)\cdot\mathcal{D}y\,. \tag{A.24}$$

The Fourier transforms are given respectively by

$$\mathcal{F}\omega_{\mathbf{X}} = \exp(-\pi s W_{\mathbf{X}}) \quad , \quad \mathcal{F}\omega_{\mathbf{Y}} = \exp(-\pi s W_{\mathbf{Y}}) \tag{A.25}$$

and according to formula (A.22), we obtain

$$\mathcal{F}\omega_{\mathbf{Y}} = \mathcal{F}\omega_{\mathbf{X}}\circ\widetilde{L}\,. \tag{A.26}$$

This is the same as formula (A.16), hence $\omega_{\mathbf{Y}}$ is the image of $\omega_{\mathbf{X}}$ under the linear mapping L. Explicitly stated, we obtain the integration formula:

$$\int_{\mathbf{X}}\mathcal{D}x\cdot\exp\Big(-\frac{\pi}{s}Q_{\mathbf{X}}(x)\Big)\cdot g(Lx) = \int_{\mathbf{Y}}\mathcal{D}y\cdot\exp\Big(-\frac{\pi}{s}Q_{\mathbf{Y}}(y)\Big)\cdot g(y)\,. \tag{A.27}$$

This holds if g is a Fourier-Stieltjes transform $\mathcal{F}\nu$ for some measure ν on $\mathbf{Y}'$.

The relationship between the quadratic forms $Q_{\mathbf{X}}$ on $\mathbf{X}$ and $Q_{\mathbf{Y}}$ on $\mathbf{Y}$ is expressed by the relation (A.22) between their inverses. Let $D_{\mathbf{X}}:\mathbf{X}\to\mathbf{X}'$ be the continuous linear map such that $Q_{\mathbf{X}}(x) = \langle D_{\mathbf{X}}x, x\rangle_{\mathbf{X}}$ and $\langle D_{\mathbf{X}}x_1, x_2\rangle_{\mathbf{X}} = \langle D_{\mathbf{X}}x_2, x_1\rangle_{\mathbf{X}}$, and define similarly $D_{\mathbf{Y}}$. Between the inverses $G_{\mathbf{X}}$ of $D_{\mathbf{X}}$ and $G_{\mathbf{Y}}$ of $D_{\mathbf{Y}}$ there holds the relation:

$$G_{\mathbf{Y}} = L\circ G_{\mathbf{X}}\circ\widetilde{L}\,. \tag{A.28}$$

We mention a few particular cases:

– if L is invertible, with inverse L^{-1}, then $Q_{\mathbf{Y}} = Q_{\mathbf{X}}\circ L^{-1}$;

– if $Q_{\mathbf{X}}$ is positive-definite, then $Q_{\mathbf{Y}}(y)$ is the infimum of $Q_{\mathbf{X}}$ over the set of elements x in $\mathbf{X}$ such that $Lx = y$;

– for any given y in $\mathbf{Y}$ the equation

$$D_{\mathbf{X}}x = \Big(\widetilde{L}\circ D_{\mathbf{Y}}\Big)(y) \tag{A.29}$$

has a unique solution $x = x(y)$ in $\mathbf{X}$ and we obtain

$$Q_{\mathbf{Y}}(y) = Q_{\mathbf{X}}(x(y))\,. \tag{A.30}$$

[28] In the space $\mathbf{X}$, the integrator $\mathcal{D}x$ depends on $Q_{\mathbf{X}}$ and s, and should be written more explicitly as $\mathcal{D}_{s,Q_{\mathbf{X}}}x$. Similarly for $\mathcal{D}y$ in space $\mathbf{Y}$.

3. Examples and applications.

3.1. *The finite-dimensional case.*

We record here the basic formulas. Assume that $\mathbf{X}$ is of finite dimension d; after choosing a linear frame, we represent a vector x by a column matrix (x^α) where $\alpha \in \{1, \cdots, d\}$. The elements of $\mathbf{X}'$ are represented by row matrices and the duality is given by $\langle x', x\rangle = x'_\alpha x^\alpha$ (Einstein's summation convention). The volume element is given by $dx = dx^1 \cdots dx^d$.

The quadratic forms Q and W correspond to symmetric matrices, namely:

$$Q(x) = h_{\alpha\beta} x^\alpha x^\beta \quad , \quad W(x') = h^{\alpha\beta} x'_\alpha x'_\beta \tag{A.31}$$

with $h_{\alpha\beta} h^{\beta\gamma} = \delta^\gamma_\alpha$. The integrators are given by

$$\mathcal{D}x = |\det h_{\alpha\beta}|^{1/2}\, dx^1 \cdots dx^d \tag{A.32}$$

$$\mathcal{D}\omega(x) = \exp\bigl(-\pi h_{\alpha\beta} x^\alpha x^\beta\bigr) \cdot \mathcal{D}x \tag{A.33}$$

when $s = 1$ and the matrix $(h_{\alpha\beta})$ is positive-definite. In the oscillating case, we have to multiply $\mathcal{D}x$ by $e^{\pi i(q-p)/4}$ where the symmetric matrix $(h_{\alpha\beta})$ has p positive and q negative eigenvalues.

The quadratic form W gives the covariance. More precisely, one obtains

$$\int_{\mathbf{X}} \mathcal{D}\omega(x)\, \langle x', x\rangle^2 = \frac{s}{2\pi} W(x') \tag{A.34}$$

hence

$$\int_{\mathbf{X}} \mathcal{D}\omega(x) x^\alpha x^\beta = \frac{s}{2\pi} h^{\alpha\beta}\,. \tag{A.35}$$

3.2. *Image under a linear form.*

Let x' in the dual $\mathbf{X}'$ of $\mathbf{X}$. By specializing the results of paragraph A.2.2 to the linear map $L : x \mapsto \langle x', x\rangle$ from $\mathbf{X}$ into $\mathbb{R}$, we obtain the following result. We identify $\mathbb{R}$ with its dual, hence $\widetilde{L}$ takes a number λ to $\lambda x'$ in $\mathbf{X}'$. With $Q_{\mathbf{X}}$ equal to Q, hence $W_{\mathbf{X}}$ to W , we obtain

$$W_{\mathbb{R}}(\lambda) = \lambda^2 W(x'), \tag{A.36}$$

hence

$$Q_{\mathbb{R}}(u) = u^2 / W(x'). \tag{A.37}$$

From formula (A.27), we deduce the following integration formula:

$$\int_{\mathbf{X}} \mathcal{D}x \cdot e^{-\pi Q(x)/s} g\left(\langle x', x\rangle\right) = C \int_{\mathbb{R}} du\, e^{-\pi u^2/sW(x')} g(u) \tag{A.38}$$

where the normalization constant C is $1/(sW(x'))^{1/2}$ (principal branch of the square root). More explicitly:

– for $s = 1$, hence $W(x') > 0$, then $C = \dfrac{1}{\sqrt{W(x')}}$;

– for $s = i$, and $W(x') > 0$, then $C = \dfrac{e^{-\pi i/4}}{\sqrt{W(x')}}$;

– for $s = i$, and $W(x') < 0$, then $C = \dfrac{e^{\pi i/4}}{\sqrt{|W(x')|}}$.

If we take in particular $g(u) = u^2$, we get

$$\int_{\mathbf{X}} \mathcal{D}x \cdot e^{-\pi Q(x)/s} \langle x', x\rangle^2 = \frac{s}{2\pi} W(x'). \tag{A.39}$$

By polarization, we obtain the more general formula

$$\int_{\mathbf{X}} \mathcal{D}x \cdot e^{-\pi Q(x)/s} \langle x', x\rangle \langle y', x\rangle = \frac{s}{2\pi} \langle x', Gy'\rangle \tag{A.40}$$

for x' and y' in $\mathbf{X}'$, thus giving the covariance of our integrator.

Remark. In the application IV.1 to point-to-point transitions, we need to restrict the domain of integration (see also paragraph A.3.8). This can be achieved by inserting a delta function $\delta(\langle x'_0, x\rangle)$ and then using equation (A.38): we restrict the domain of integration from the space $\mathbf{X}$ to the hyperplane $\mathbf{X}_0$ with equation $\langle x'_0, x\rangle = 0$.

In general, any integral of the form

$$\int_{\mathbf{X}} \mathcal{D}x \cdot e^{-\pi Q(x)/s} f(\langle x'_1, x\rangle, \cdots, \langle x'_n, x\rangle)$$

with fixed elements $x'_1, \cdots, x'_n$ in $\mathbf{X}'$ can be reduced to an n-dimensional integral. For instance, combining formulas (A.38) and (A.40), the Green's function $G_{a,b}(t,u)$ on $\mathbf{Z}_{a,b}$ given by formulas (A.58) and (B.28), can be evaluated using an integral

$$\int_{\mathbf{Z}_b} \mathcal{D}_b z \cdot \exp\left(-\frac{\pi}{s}\int_{\mathbf{T}} dt\, z(t)^2\right) \delta(z(t_a))\, z(t) z(u)$$

and reducing it to a 3-dimensional integral.

3.3. *Space of paths of finite action.*

The time interval $\mathbf{T} = [t_a, t_b]$ being given, we denote by $L^{2,1}$ (or more accurately $L^{2,1}(\mathbf{T})$) the space of real-valued functions $z(\cdot)$ on $\mathbf{T}$ with square-integrable derivative $\dot{z}(\cdot)$ and we define the quadratic form Q_0 on $L^{2,1}$ by

$$Q_0(z) = \int_{\mathbf{T}} dt\, \dot{z}(t)^2 . \tag{A.41}$$

The definition of $L^{2,1}$ can be rephrased as follows: the function z belongs to $L^{2,1}$ if and only if there exists a function $\dot{z}$ in $L^2(\mathbf{T})$ such that

$$z(t') - z(t) = \int_t^{t'} du\, \dot{z}(u) \tag{A.42}$$

whenever t, t' are epochs in $\mathbf{T}$ such that $t < t'$. The function $\dot{z}$ is then defined up to a null set; indeed by Lebesgue's derivation theorem, one gets

$$\dot{z}(t) = \lim_{\tau = 0} \left(z(t+\tau) - z(t) \right) / \tau \tag{A.43}$$

for almost all t in $\mathbf{T}$. By Cauchy-Schwarz inequality, one deduces from (A.42) the inequality

$$|z(t') - z(t)|^2 \leq Q_0(z) \cdot |t' - t| \tag{A.44}$$

for t, t' in $\mathbf{T}$. Hence any function z in $L^{2,1}$ satisfies a Lipschitz condition of order $1/2$, and *a fortiori* it is a continuous function.

The quantity $Q_0(z)$ in equation (A.41) can be calculated using a *discretization of time*. Consider a subdivision $\mathcal{T}$ of the time interval $\mathbf{T}$ by epochs

$$t_a \leq t_0 < t_1 < \ldots < t_{N-1} < t_N \leq t_b .$$

Set $\Delta t_i = t_i - t_{i-1}$ for $0 \leq i \leq N+1$ with the convention $t_{-1} = t_a$, $t_{N+1} = t_b$ and denote by $\delta(\mathcal{T})$ the largest among the increments Δt_i, that is the *mesh* of the subdivision $\mathcal{T}$. For any function $z : \mathbf{T} \to \mathbb{R}$ set

$$z_i = z(t_i) \quad , \quad \Delta z_i = z_i - z_{i-1} \tag{A.45}$$

for $0 \leq i \leq N+1$. The *quadratic variation of* z w.r.t. the subdivision $\mathcal{T}$ is defined as

$$Q_{\mathcal{T}}(z) = \sum_{i=1}^{N} (\Delta z_i)^2 / \Delta t_i . \tag{A.46}$$

Then the function z belongs to $L^{2,1}$ if and only if the set of quadratic variations $Q_{\mathcal{T}}(z)$ is bounded when $\mathcal{T}$ runs over all subdivisions of $\mathbf{T}$. Then

$$Q_0(z) = \text{l.}\underset{\mathcal{T}}{\text{u}}\text{.b. } Q_{\mathcal{T}}(z). \tag{A.47}$$

More precisely, for any sequence of subdivisions $\mathcal{T}(n)$ (for $n \in \mathbb{N}$) whose mesh $\delta(\mathcal{T}(n))$ tends to 0, one gets

$$Q_0(z) = \lim_{n=\infty} Q_{\mathcal{T}(n)}(z). \tag{A.48}$$

It is therefore justified to write $Q_0(z)$ in the form $\displaystyle\int_{\mathbf{T}} \frac{(dz)^2}{dt}$.

3.4. *Green's functions.*

Fix an element t_0 in $\mathbf{T}$ and denote by $\mathbf{Z}_0$ the space of functions z in $L^{2,1}$ such that $z(t_0) = 0$. This is a (real) Hilbert space, with scalar product

$$\langle z_1 \mid z_2 \rangle = \int_{\mathbf{T}} dt\, \dot{z}_1(t)\dot{z}_2(t), \tag{A.49}$$

hence $Q_0(z)$ is equal to $\langle z \mid z \rangle$. From (A.44), one deduces the inequality

$$|z(t)|^2 \leq |t - t_0| \cdot \langle z \mid z \rangle, \tag{A.50}$$

and therefore there is an element δ_t in the dual $\mathbf{Z}_0'$ of $\mathbf{Z}_0$ such that $z(t) = \langle \delta_t, z \rangle$ (that is: δ_t is a Dirac "function" centered at t). According to the general theory, one introduces a linear continuous and invertible map $G_0 : \mathbf{Z}_0' \to \mathbf{Z}_0$ corresponding to the quadratic form Q_0 on $\mathbf{Z}_0$. The *Green's function* is defined as follows:

$$G_0(t, u) = \langle \delta_t, G_0 \delta_u \rangle. \tag{A.51}$$

That is, the function $t \mapsto G_0(t, u)$ belongs to $\mathbf{Z}_0$ and is equal to $G_0\delta_u$. By definition, one gets

$$z(u) = \langle G_0 \delta_u, z \rangle \tag{A.52}$$

for every z in $\mathbf{Z}$. More explicitly, these conditions on $G_0(t, u)$ can be expressed as follows:

$$\begin{cases} G_0(t_0, u) = 0 & \text{(A.53)} \\ \displaystyle\int_{\mathbf{T}} dt\, \frac{\partial}{\partial t} G_0(t, u) \frac{\partial}{\partial t} z(t) = z(u). & \text{(A.54)} \end{cases}$$

The unique solution to these equations is given by

$$G_0(t,u) = \begin{cases} \inf(t-t_0, u-t_0) & \text{for } t \geq t_0,\ u \geq t_0, \\ \inf(t_0-t, t_0-u) & \text{for } t \leq t_0,\ u \leq t_0, \\ 0 & \text{otherwise}. \end{cases} \tag{A.55}$$

We consider some special cases:

– space $\mathbf{Z}_a$ of functions z in $L^{2,1}$ with $z(t_a) = 0$;
– space $\mathbf{Z}_b$ of functions z in $L^{2,1}$ with $z(t_b) = 0$;
– space $\mathbf{Z}_{a,b} = \mathbf{Z}_a \cap \mathbf{Z}_b$ with boundary conditions $z(t_a) = z(t_b) = 0$.

The corresponding Green's functions are as follows:

$$G_a(t,u) = \theta(t-u)(u-t_a) + \theta(u-t)(t-t_a) \tag{A.56}$$

$$G_b(t,u) = \theta(t-u)(t_b-t) + \theta(u-t)(t_b-u) \tag{A.57}$$

$$\begin{aligned} G_{a,b}(t,u) = {} & \theta(t-u)(t-t_b)(t_b-t_a)^{-1}(t_a-u) \\ & - \theta(u-t)(t-t_a)(t_a-t_b)^{-1}(t_b-u)\,. \end{aligned} \tag{A.58}$$

Remark. Equation (A.58) can be obtained by integrating

$$\int_{\mathbf{Z}_b} \mathcal{D}\omega(z)\,\delta(z(t_a))\; z(t)\,z(u)\,.$$

3.5. *Vector-valued functions.*

As in paragraph II.1.1, we consider vector functions $z = (z^1, \cdots, z^d)$ with components in $L^{2,1}$. We introduce a real symmetric matrix $(h_{\alpha\beta})$ with an inverse denoted by $(h^{\alpha\beta})$. The basic quadratic form is given by

$$Q_0(z) = \int_{\mathbf{T}} dt\, h_{\alpha\beta}\, \dot{z}^\alpha(t)\, \dot{z}^\beta(t)\,. \tag{A.59}$$

The auxiliary condition $z(t_0) = 0$ defines the space $\mathbf{Z}_0$. Similarly, the spaces $\mathbf{Z}_a$, $\mathbf{Z}_b$ and $\mathbf{Z}_{a,b}$ are described by the respective boundary conditions:

– $z(t_a) = 0$ for $\mathbf{Z}_a$;
– $z(t_b) = 0$ for $\mathbf{Z}_b$;
– $z(t_a) = z(t_b) = 0$ for $\mathbf{Z}_{a,b}$.

In each case, the linear form δ_t^α taking z into $z^\alpha(t)$ belongs to the dual of the corresponding of paths, and the Green's function is characterized by

$$G_0^{\alpha\beta}(t,u) = \left\langle \delta_t^\alpha, G_0 \delta_u^\beta \right\rangle\,. \tag{A.60}$$

Explicitly, we obtain

$$G_0^{\alpha\beta}(t,u) = h^{\alpha\beta} G_0(t,u) \tag{A.61}$$

where the Green's function G_0 refers to the boundary condition of the relevant space of paths (see formulas (A.55) to (A.58)). The Green's function can also be obtained from formula (A.40) by specialization:

$$\int_{\mathbf{Z}} \mathcal{D}z \cdot e^{-\pi Q_0(z)/s}\, z^\alpha(t)\, z^\beta(u) = \frac{s}{2\pi} G_0^{\alpha\beta}(t,u)\,. \tag{A.62}$$

Here $\mathcal{D}z$ stands for $\mathcal{D}_{s,Q_0} z$.

Remark. In general, the Green's functions G satisfying a second-order differential equation $DG = \mathbb{1}$ are uniquely determined by d conditions at $t = t_a$ and d conditions at $t = t_b$. These conditions are obvious from equation (A.62) for the space $\mathbf{Z}_{a,b}$, namely $G^{\alpha\beta}(t,u) = 0$ if t (or u) is one of the end points t_a, t_b.

In the case of the space $\mathbf{Z}_a$, the boundary conditions are

$$G|_{t=t_a} = 0 \quad , \quad \partial G/\partial t|_{t=t_b} = 0\,.$$

According to equation (A.62), these conditions can be expressed as

$$\int_{\mathbf{Z}_a} \mathcal{D}\omega(z) \cdot z^\alpha(t_a) z^\beta(u) = 0,$$
$$\int_{\mathbf{Z}_a} \mathcal{D}\omega(z) \cdot \dot{z}^\alpha(t_b) z^\beta(u) = 0\,.$$

The interpretation is as follows: any path z in $\mathbf{Z}_a$ satisfies $z^\alpha(t_a) = 0$; the time derivative $\dot{z}^\alpha(t_b)$ at t_b is totally unspecified, nevertheless it vanishes in a statistical sense being uncorrelated to the position and velocity at any other time.

On the space $\mathbf{Z}_0'$ dual to $\mathbf{Z}_0$, we have defined the quadratic form W_0 inverse to Q_0. The Greens's function can be expressed as follows:

$$G_0^{\alpha\beta}(t,u) = \tfrac{1}{2}\left[W_0(\delta_t^\alpha + \delta_u^\beta) - W_0(\delta_t^\alpha) - W_0(\delta_u^\beta)\right]\,. \tag{A.63}$$

Conversely, given an element z' of $\mathbf{Z}'$ represented by

$$\langle z', z\rangle = \int_{\mathbf{T}} dt\, z'_\alpha(t) z^\alpha(t), \tag{A.64}$$

the quadratic form is given by

$$W_0(z') = \int_{\mathbf{T}} dt \int_{\mathbf{T}} dt\, du\, G_0^{\alpha\beta}(t,u) z'_\alpha(t)\, z'_\beta(u)\,. \tag{A.65}$$

3.6. *Scaling the paths.*

Here is a concise *dimensional analysis* of our quantities.
Conventions : $\mathcal{L}$ ($\mathcal{T}$) stands for the dimension of length (time) and $[X]$ for the dimensional content of a quantity X.

Since $Q_0(z)$ appears in $e^{-\pi Q_0(z)/s}$, it has to be a pure number, hence we can use formula (A.59) to deduce $[h_{\alpha\beta}] = \mathcal{L}^{-2}\mathcal{T}$. From formulas (A.55) to (A.58) we infer the dimension of $G_0(t,u)$ to be $\mathcal{T}$. From (A.62) we infer that $\left[G_0^{\alpha\beta}\right] = \mathcal{L}^2$, hence $\left[h^{\alpha\beta}\right] = \mathcal{L}^2\mathcal{T}^{-1}$ from (A.61). This is in accordance with the matrix relation $h_{\alpha\beta}h^{\beta\gamma} = \delta_\alpha{}^\gamma$. Notice also that for a particle of mass m in a flat space, we have

$$h_{\alpha\beta} = m\delta_{\alpha\beta}/h \quad , \quad h^{\alpha\beta} = h\delta_{\alpha\beta}/m \tag{A.66}$$

(where h is the Planck constant and $\delta_{\alpha\beta}$ the Kronecker delta), and that the dimension of h/m is $\mathcal{L}^2\mathcal{T}^{-1}$. We summarize our findings :

TABLE 1

Quantity	z^α	t	$h_{\alpha\beta}$	$G_0(t,u)$	$G_0^{\alpha\beta}(t,u)$	$h^{\alpha\beta}$
Dimension	$\mathcal{L}$	$\mathcal{T}$	$\mathcal{L}^{-2}\mathcal{T}$	$\mathcal{T}$	$\mathcal{L}^2$	$\mathcal{L}^2\mathcal{T}^{-1}$

We can confirm these results by *scaling* our paths. Let $\Lambda > 0$ be a numerical constant, and denote by Λz the path with components $\Lambda z^\alpha(t)$ at time t. The basic integrator being written as $\mathcal{D}\omega(z) = e^{-\pi Q_0(z)/s}\mathcal{D}z$, we scale it into an integrator $\mathcal{D}\omega^\Lambda(z)$ according to the formula

$$\int_{\mathbf{Z}_0} \mathcal{D}\omega^\Lambda(z)\, F(z) = \int_{\mathbf{Z}_0} \mathcal{D}\omega(z)\, F(\Lambda z)\,. \tag{A.67}$$

By suitably specializing $F(z)$, we obtain the new Green's function:

$$\int_{\mathbf{Z}} \mathcal{D}\omega^\Lambda(z)\, z^\alpha(t) z^\beta(u) = \Lambda^2 G_0^{\alpha\beta}(t,u)\,, \tag{A.68}$$

in accordance with $\left[G_0^{\alpha\beta}\right] = \mathcal{L}^2$. Similarly any n-point function scales as $\mathcal{L}^n$. The map taking z into Λz is a linear map from $\mathbf{Z}_0$ to $\mathbf{Z}_0$. By the general theory in paragraph A.2.2, $\mathcal{D}\omega^\Lambda$ is another Gaussian integrator. From the formulas (A.65) and (A.68), it corresponds to the quadratic form $\Lambda^2 W_0(z')$ on $\mathbf{Z}'_0$, with an inverse quadratic form on $\mathbf{Z}_0$ given by

$$\Lambda^{-2}Q_0(z) = \int_{\mathbf{T}} dt\, \Lambda^{-2} \cdot h_{\alpha\beta}\, \dot{z}^\alpha(t)\dot{z}^\beta(t)\,. \tag{A.69}$$

This relation confirms $[h_{\alpha\beta}] = \mathcal{L}^{-2}$. A similar analysis applies to the scaling of time.

We can rewrite formula (A.67) in the following form:

$$\int_{\mathbf{Z}_0} \mathcal{D}^\Lambda z \cdot e^{-\pi\Lambda^{-2}Q_0(z)/s} H\left(\Lambda^{-1}z\right) = \int_{\mathbf{Z}_0} \mathcal{D}z \cdot e^{-\pi Q_0(z)/s} H(z)\,, \tag{A.70}$$

with a new integrator $\mathcal{D}^\Lambda z$ which is invariant under translations. It is justified to summarize the previous formula by $\mathcal{D}^\Lambda z = \mathcal{D}\left(\Lambda^{-1}z\right)$, hence $\mathcal{D}\omega^\Lambda(z) = \mathcal{D}\omega\left(\Lambda^{-1}z\right)$ according to formula (A.67). In the standard heuristic derivations, one writes $\mathcal{D}z$ in the form $C \cdot \prod_{t,\alpha} dz^\alpha(t)$. According to our normalization

$$\int_{\mathbf{Z}_0} \mathcal{D}z \cdot e^{-\pi Q_0(z)/s} = 1, \tag{A.71}$$

we should write

$$\mathcal{D}z = \frac{\displaystyle\prod_{t,\alpha} dz^\alpha(t)}{\displaystyle\int \prod_{t,\alpha} dz^\alpha(t) \cdot \exp\left(-\pi Q_0(z)/s\right)}\,, \tag{A.72}$$

and similarly

$$\mathcal{D}^\Lambda z = \frac{\displaystyle\prod_{t,\alpha} dz^\alpha(t)}{\displaystyle\int \prod_{t,\alpha} dz^\alpha(t) \cdot \exp\left(-\pi Q_0\left(\Lambda^{-1}z\right)/s\right)}\,. \tag{A.73}$$

Replacing z by $\Lambda^{-1}z$ amounts to replacing $dz^\alpha(t)$ by $\Lambda^{-1}dz^\alpha(t)$, hence the volume element $\prod_{t,\alpha} dz^\alpha(t)$ is multiplied by Λ^{-N} where N is the (infinite) number of degrees of freedom t, α. If we calculate $\mathcal{D}\left(\Lambda^{-1}z\right)$ in accordance with (A.72), both numerator and denominator acquire a factor Λ^{-N} which drops out, and the correct formula $\mathcal{D}^\Lambda z = \mathcal{D}\left(\Lambda^{-1}z\right)$ is obtained from the heuristic formulas (A.72) and (A.73).

The heuristic constant Λ^{-N} is equal to ∞, 1 or 0 according to the three cases $0 < \Lambda < 1$, $\Lambda = 1$, $\Lambda > 1$. This is reflected in the rigorous theory by the following fact : *for* $\Lambda \neq 1$ *no functional* $F(z)$, *except the constant* 0, *is such that both* $F(z)$ *and* $F(\Lambda z)$ *can be simultaneously integrated* w.r.t. $\mathcal{D}z$ In a finite-dimensional space $\mathbb{R}^N$, the volume element $dx^1 \cdots dx^N$ is the only one, up to a multiplicative constant, which is invariant under translations. In our infinite-dimensional setup there exist many translation-invariant integrators, but *they act in different functional sectors.*

3.7. *White noise representations.*

We consider now the Hilbert space $L^2(\mathbf{T})$ (or L^2) consisting of the square-integrable functions $\xi : \mathbf{T} \to \mathbb{R}$, with the standard quadratic form

$$H(\xi) = \int_{\mathbf{T}} dt\, \xi(t)^2 . \tag{A.74}$$

As usual we identify $L^2(\mathbf{T})$ with its own dual by means of the following scalar product

$$\langle \xi', \xi \rangle = \int_{\mathbf{T}} dt\, \xi'(t)\xi(t) . \tag{A.75}$$

Let us denote for a while the space $L^2(\mathbf{T})$ by $\mathbf{H}$. Under our general conventions, both $D : \mathbf{H} \to \mathbf{H}'$ and $G : \mathbf{H}' \to \mathbf{H}$ are identity operators, hence the quadratic form $\mathbf{H}$ is its own inverse. The corresponding covariance is given by the kernel of the identity operator, namely $\delta(t-u)$. Hence denoting by $\mathcal{D}\xi$ the basic integrator, we get from equation (A.40):

$$\int_{L^2(\mathbf{T})} \mathcal{D}\xi \cdot \exp\left(-\frac{\pi}{s}\int_{\mathbf{T}} dt\, \xi(t)^2\right) \xi(t)\xi(u) = \frac{s}{2\pi}\delta(t-u) . \tag{A.76}$$

Let $L^{2,1}_0(\mathbf{T})$ (or $L^{2,1}_0$) be the subspace of functions z in $L^{2,1}(\mathbf{T})$ such that $z(t_0) = 0$. The spaces $L^2(\mathbf{T})$ and $L^{2,1}_0(\mathbf{T})$ are isomorphic under a correspondance $\xi \leftrightarrow z$ where $\xi = \dot{z}$ and conversely $z(t) = \int_{t_0}^t dt'\, \xi(t')$. This can be expressed by the formulas

$$z(t) = \int_{\mathbf{T}} dt'\, \Theta(t', t)\xi(t') , \tag{A.77}$$

$$\Theta(t', t) = \begin{cases} 1 & \text{for} \quad t_0 \leq t' \leq t, \\ -1 & \text{for} \quad t \leq t' \leq t_0, \\ 0 & \text{otherwise.} \end{cases} \tag{A.78}$$

The following transformation rule holds for functional integrals:

$$\int_{L_0^{2,1}} \mathcal{D}z \cdot e^{-\pi Q_0(z)/s} F(z) = \int_{L^2} \mathcal{D}\xi \cdot e^{-\pi H(\xi)/s} \Phi(\xi)\,, \tag{A.79}$$

where $\Phi(\xi)$ is equal to $F(z)$ if ξ corresponds to z by (A.77). The covariance in $L_0^{2,1}$ is obtained by specializing $F(z)$ to $z(t)z(u)$ in (A.79). Using (A.76), we obtain

$$G_0(t,u) = \int_{\mathbf{T}} dt'\, \Theta(t',t)\Theta(t',u) \tag{A.80}$$

in agreement with formula (A.55).

To make contact with the heuristic definitions, we introduce the "coordinates" $X_t = \xi(t)\sqrt{dt}$ for the function ξ, hence

$$H(\xi) = \sum_t X_t^2 \quad , \quad \mathcal{D}\xi = \prod_t \left(dX_t/\sqrt{s}\right)\,. \tag{A.81}$$

We urge the reader to substantiate these claims by resorting to a subdivision $\mathcal{T}$ of the time interval $\mathbf{T}$, like in paragraph A.3.3.

Let $\mathbf{X}$ be any closed vector subspace of $L^2(\mathbf{T})$, and let Π be the orthogonal projector from $L^2(\mathbf{T})$ onto $\mathbf{X}$, represented as an integral operator with kernel $\Pi(t,u)$. The quadratic form $H_{\mathbf{X}}$ on $\mathbf{X}$ is obtained by restriction of H, namely:

$$H_{\mathbf{X}}(\xi) = \int_{\mathbf{T}} dt\, \xi(t)^2 \tag{A.82}$$

for ξ in $\mathbf{X}$. Denote by $\mathcal{D}_{\mathbf{X}}\xi$ the corresponding integrator on $\mathbf{X}$. Then the covariance is expressed as follows:

$$\int_{\mathbf{X}} \mathcal{D}_{\mathbf{X}}\xi \cdot \exp\left(-\frac{\pi}{s}\int_{\mathbf{T}} dt\, \xi(t)^2\right) \cdot \xi(t)\xi(u) = \frac{s}{2\pi}\Pi(t,u)\,. \tag{A.83}$$

For instance, if we denote by $L_0^2(\mathbf{T})$ the subspace of $L^2(\mathbf{T})$ defined by the condition $\int_{\mathbf{T}} dt\, \xi(t) = 0$, the orthogonal projector is given by

$$(\Pi\xi)(u) = \xi(u) - \frac{1}{t_b - t_a}\int_{\mathbf{T}} dt\, \xi(t), \tag{A.84}$$

hence

$$\Pi(t,u) = \delta(t-u) - \frac{1}{t_b - t_a}\,. \tag{A.85}$$

The map $\xi \mapsto z$ where $z(t) = \int_{t_a}^{t} dt'\, \xi(t')$ takes isomorphically the space $L_0^2(\mathbf{T})$ onto the space denoted by $\mathbf{Z}_{a,b}$ in paragraph A.3.4. It follows that the Green's function $G_{a,b}$ corresponding to the space $\mathbf{Z}_{a,b}$ is given by

$$G_{a,b}(t,u) = \int_{t_a}^{t} dv \int_{t_a}^{u} dv'\, \Pi(v,v'). \tag{A.86}$$

Using (A.85), we obtain easily (A.58).

Explicit formulas for transformations, mapping a given quadratic form on $L^2(\mathbf{T})$ into any given quadratic form on $L^{2,1}(\mathbf{T})$, can be found in [5, p. 274]. The reader is urged to extend the results of this paragraph to vector-valued functions.

3.8. *Fixing the endpoints.*

We denote by $\mathcal{D}_a z$, $\mathcal{D}_b z$ and $\mathcal{D}_{a,b} z$ respectively the integrators on the spaces $\mathbf{Z}_a$, $\mathbf{Z}_b$ and $\mathbf{Z}_{a,b}$ (see paragraph A.3.5.). These integrators are related by the following formula:

$$\int_{\mathbf{Z}_a} \mathcal{D}_a z \cdot e^{-\pi Q_0(z)/s} F(z) \delta(z(t_b)) = C^{-1} \int_{\mathbf{Z}_{a,b}} \mathcal{D}_{a,b}\, z \cdot e^{-\pi Q_0(z)/s} F(z)\,, \tag{A.87}$$

where the constant C is given by

$$\begin{aligned} C &= \left(\det s G_a^{\alpha,\beta}(t_b,t_b)\right)^{1/2} \\ &= \left(\det s h^{\alpha\beta}\right)^{1/2} (t_b - t_a)^{d/2}\,. \end{aligned} \tag{A.88}$$

A similar formula (where the roles of t_a and t_b are exchanged) can be found in [5, pp. 284, 357].

We give a new derivation of this formula to illustrate our methods. We simplify the notations by putting $\mathcal{D}\omega_a(z)$ equal to $\mathcal{D}_a z \cdot e^{-\pi Q_0(z)/s}$, and defining $\mathcal{D}\omega_{a,b}(z)$ similarly. According to our general strategy, we need only to prove formula (A.87) in the special case

$$F(z) = e^{-2\pi i \langle z', z\rangle} \tag{A.89}$$

where z' is an element of $\mathbf{Z}'_a$. That is, we have to establish the following equality:

$$\int_{\mathbf{Z}_a} \mathcal{D}\omega_a(z) \cdot e^{-2\pi i \langle z', z\rangle} \delta(z(t_b)) = C^{-1} e^{-\pi s W_{a,b}(z')}\,, \tag{A.90}$$

where $W_{a,b}$ is the quadratic form on $\mathbf{Z}'_{a,b}$ inverse of the restriction of Q_0 to the subspace $\mathbf{Z}_{a,b}$ of $\mathbf{Z}_a$.

To prove (A.90), we use the well-known formula

$$\delta(x) = \int_{\mathbb{R}} e^{-2\pi i u x}\, du\,. \tag{A.91}$$

We shall consider in detail the scalar case $d = 1$ and leave the general case to the reader. The left-hand side L of (A.90) can be rewritten as

$$L = \int_{\mathbb{R}} du \int_{\mathbf{Z}_a} \mathcal{D}\omega_a(z) \exp\left(-2\pi i \langle z' + u\delta_{t_b}, z\rangle\right), \tag{A.92}$$

hence

$$L = \int_{\mathbb{R}} du \exp\left(-\pi s W_a\left(z' + u\delta_{t_b}\right)\right) \tag{A.93}$$

by (A.18). We develop now the exponent

$$W_a\left(z' + u\delta_{t_b}\right) = u^2 G_a(t_b, t_b) + 2u \int_{\mathbf{T}} dt\, z'(t) G_a(t_b, t) + W_a(z') \tag{A.94}$$

and use the integration formula

$$\int_{\mathbb{R}} du\, e^{-\pi s\left(au^2 + 2bu + c\right)} = \frac{1}{\sqrt{as}} \exp\left(-\pi s\left(c - b^2/a\right)\right) \tag{A.95}$$

to obtain

$$L = \frac{1}{\sqrt{sG_a(t_b, t_b)}} \cdot \tag{A.96}$$

$$\exp\left(-\pi s\left(W_a(z') - G_a(t_b, t_b)^{-1} \int_{\mathbf{T}} dt \int_{\mathbf{T}} du\, z'(t) z'(u) G_a(t_b, t)\, G_a(t_b, u)\right)\right).$$

This can be transformed in the desired right-hand side of (A.90) with $C = \sqrt{sG_a(t_b, t_b)}$ provided we establish the following identity:

$$G_{a,b}(t, u) = G_a(t, u) - G_a(t, t_b)\, G_a(t_b, t_b)^{-1}\, G_a(t_b, u)\,. \tag{A.97}$$

We leave it as an exercice (see formulas (A.56) and (A.58)).

We can prove a more general formula by a similar reasoning. We fix points z_a and z_b and consider $L^{2,1}$ paths $z(\cdot)$ with $z(t_a) = z_a$. We obtain the for an operator T given by (A.99).

We introduce a power series in λ, namely:

$$\sum_{p\geq 0} \sigma_p(T)\lambda^p := \exp\left(\lambda\,\mathrm{Tr}(T) - \frac{\lambda^2}{2}\,\mathrm{Tr}(T^2) + \frac{\lambda^3}{3}\,\mathrm{Tr}(T^3) - \cdots\right). \qquad \text{(A.101)}$$

By using Hadamard's inequality about determinants, we obtain the ***basic estimate***:

$$|\sigma_p(T)| \leq p^{p/2}\,\|T\|_1^p\,/p!\,. \qquad \text{(A.102)}$$

It follows that the power series $\sum_{p\geq 0}\sigma_p(T)\lambda^p$ has an infinite radius of convergence. We can therefore define the determinant as follows:

$$\mathrm{Det}(1+T) := \sum_{p\geq 0} \sigma_p(T) \qquad \text{(A.103)}$$

for any nuclear operator T. By homogeneity, we obtain more generally

$$\mathrm{Det}(1+\lambda T) = \sum_{p\geq 0} \sigma_p(T)\lambda^p\,. \qquad \text{(A.104)}$$

The fundamental property of determinants is, as expected, the ***multiplicative rule***:

$$\mathrm{Det}\,(U_1 \circ U_2) = \mathrm{Det}\,(U_1)\,\mathrm{Det}\,(U_2)\;, \qquad \text{(A.105)}$$

where U_i is of the form $1+T_i$, with T_i nuclear (for $i=1,2$). From this, and the relation $\sigma_1(T) = \mathrm{Tr}(T)$, we get a *variation formula* (for $U-1$ and δU nuclear):

$$\frac{\mathrm{Det}(U+\delta U)}{\mathrm{Det}(U)} = 1 + \mathrm{Tr}\left(U^{-1}\cdot\delta U\right) + O\left(\|\delta U\|_1^2\right)\,. \qquad \text{(A.106)}$$

Otherwise stated, if $U(\nu)$ is an operator of the form $1+T(\nu)$, where $T(\nu)$ is nuclear, depending smoothly on the parameter ν, we get the ***derivation formula***:

$$\frac{d}{d\nu}\ell n\,\mathrm{Det}(U(\nu)) = \mathrm{Tr}\left(U(\nu)^{-1}\frac{d}{d\nu}U(\nu)\right)\,. \qquad \text{(A.107)}$$

Remark. For any other norm $\|\cdot\|^1$ defining the topology of $\mathbf{X}$, we have an estimate

$$C^{-1}\,\|x\| \leq \|x\|^1 \leq C\,\|x\|\;, \qquad \text{(A.108)}$$

with a finite numerical constant $C > 0$. It follows easily that the previous definitions are independent of the choice of the particular norm $\|x\| = H(x)^{1/2}$ in $\mathbf{X}$.

4.2. *Explicit formulas.*

Introduce a basis $(e_n)_{n\geq 1}$ of $\mathbf{X}$ orthonormal for the quadratic form H, hence $H(\sum_n t_n e_n) = \sum_n t_n^2$. An operator T in $\mathbf{X}$ has a matrix (t_{mn}) such that

$$Te_n = \sum_m e_m \cdot t_{mn} \,. \tag{A.109}$$

Assume that T is nuclear. Then the series $\sum_n t_{nn}$ of diagonal terms in the matrix converges absolutely and the trace $\mathrm{Tr}(T)$ is equal to $\sum_n t_{nn}$, as it should be. Furthermore $\sigma_p(T)$ is the sum of the series made of the principal minors of order p:

$$\sigma_p(T) = \sum_{i_1<\cdots<i_p} \det\left(t_{i_\alpha, i_\beta}\right)_{\substack{1\leq\alpha\leq p\\ 1\leq\beta\leq p}} \,. \tag{A.110}$$

For the operator $U = 1 + T$, with the matrix with elements $u_{mn} = \delta_{mn} + t_{mn}$, we obtain the determinant as a *limit of finite-size determinants*:

$$\mathrm{Det}(U) = \lim_{N=\infty} \det\left(u_{mn}\right)_{\substack{1\leq m\leq N\\ 1\leq n\leq N}} \,. \tag{A.111}$$

As a special case, suppose that the basic vectors e_n are *eigenvectors* for T, namely

$$Te_n = \lambda_n e_n \,. \tag{A.112}$$

Then we get

$$\mathrm{Tr}(T) = \sum_n \lambda_n \quad , \quad \mathrm{Det}(1+T) = \prod_n (1+\lambda_n)$$

where both the series and the infinite product converge absolutely.

The nuclear norm $\|T\|_1$ can also be computed as follows: there exists an orthonormal basis (e_n) such that the vectors Te_n are mutually orthogonal (for the quadratic form H) and then $\|T\|_1 = \sum_n \|Te_n\|$.

Remark. Let T be a continuous linear operator in $\mathbf{X}$. Assume that the series of diagonal terms $\sum_n t_{nn}$ converges absolutely for *every* orthonormal basis. Then T is nuclear. When T is symmetric and positive, it is enough to assume that this statement holds for *one* given orthonormal basis, and then it holds for all. There are counterexamples when T is not symmetric and positive.

4.3. *The case of quadratic forms.*

Contrary to a widespread misbelief, *there is no such thing like the determinant of a quadratic form.* Consider for instance a quadratic form Q on some finite-dimensional space with coordinates $x^1, \cdots, x^d$, namely

$$Q(x) = h_{\alpha\beta} x^\alpha x^\beta . \tag{A.113}$$

If we introduce a new system of coordinates $\overline{x}^1, \cdots, \overline{x}^d$ such that $x^\alpha = u^\alpha_\lambda \overline{x}^\lambda$, then we obtain

$$Q(x) = \overline{h}_{\lambda\mu} \overline{x}^\lambda \overline{x}^\mu \tag{A.114}$$

with a new matrix

$$\overline{h}_{\lambda\mu} = u^\alpha_\lambda u^\beta_\mu h_{\alpha\beta} . \tag{A.115}$$

The determinants $D = \det(h_{\alpha\beta})$ and $\overline{D} = \det\left(\overline{h}_{\lambda\mu}\right)$ are connected by the *scaling relation:*

$$\overline{D} = D \cdot (\det(u^\alpha_\lambda))^2 . \tag{A.116}$$

Hence, what makes sense is the *ratio of determinants*

$$\det\left(h^{(1)}_{\alpha\beta}\right) / \det\left(h^{(0)}_{\alpha\beta}\right) \tag{A.117}$$

associated to two quadratic forms

$$Q_0(x) = h^{(0)}_{\alpha\beta} x^\alpha x^\beta \quad , \quad Q_1(x) = h^{(1)}_{\alpha\beta} x^\alpha x^\beta \tag{A.118}$$

on the same space. We shall denote it by $\det(Q_1/Q_0)$.

To obtain an intrinsic definition, let us consider two continuous quadratic forms Q_0 and Q_1 on a Banach space $\mathbf{X}$ and assume that Q_0 is invertible. We associate to Q_0 and Q_1 two continuous linear maps $D_i : \mathbf{X} \to \mathbf{X}'$ such that

$$\langle D_i x_1, x_2 \rangle = \langle D_i x_2, x_1 \rangle \quad , \quad Q_i(x) = \langle D_i x, x \rangle , \tag{A.119}$$

for x, x_1 and x_2 in $\mathbf{X}$. By assumption, D_0 is invertible, hence there exists a unique continuous operator U in $\mathbf{X}$ such that $D_1 = D_0 \circ U$. We denote U by[30] Q_1/Q_0. In case the determinant of U is defined, namely when $U - 1$ is nuclear, we denote it by $\mathrm{Det}\,(Q_1/Q_0)$. Of course, when $\mathbf{X}$ is finite-dimensional, this definition agrees with the previous one and

$$\mathrm{Det}\,(Q_1/Q_0) = \det(Q_1/Q_0) \tag{A.120}$$

[30] A better notation should perhaps be $Q_0 \backslash Q_1$.

in this case.

A procedure to calculate this determinant is as follows. Let V be a finite-dimensional subspace of $\mathbf{X}$. By restricting Q_0 and Q_1 to V we obtain two quadratic forms $Q_{0,V}$ and $Q_{1,V}$ on V. Assume now that V runs through an increasing sequence of subspaces, whose union is dense in $\mathbf{X}$, and that $Q_{0,V}$ be invertible for every V. Then

$$\operatorname{Det}(Q_1/Q_0) = \lim_{V} \det(Q_{1,V}/Q_{0,V}) . \tag{A.121}$$

4.4. *A pencil of quadratic forms.*

Here are our assumptions:

– *there exists an invertible positive-definite quadratic form on* $\mathbf{X}$;

– Q_0 *and* Q_1 *are continuous quadratic forms on* $\mathbf{X}$;

– Q_0 *is invertible*;

– *if we set* $Q = Q_1 - Q_0$, *the operator* $T = Q/Q_0$ *on* $\mathbf{X}$ *is nuclear.*

The last condition is an intrinsic property of the quadratic form Q, namely the existence of a representation like

$$Q(x) = \sum_{n\geq 1} \alpha_n \langle x'_n, x\rangle^2 , \tag{A.122}$$

where the series $\sum_n \alpha_n$ converges absolutely, and the sequence of numbers $(\langle x'_n, x\rangle)_{n\geq 1}$ is bounded for every vector x in $\mathbf{X}$. We express this property by saying that the *quadratic form* Q *is nuclear.*

We say that λ is an *eigenvalue* of a quadratic form Q' w.r.t. Q_0 if it is an eigenvalue of the operator Q'/Q_0. This is tantamount to saying that the quadratic form $Q' - \lambda Q_0$ is not invertible.

We interpolate between the quadratic forms Q_0 and $Q_1 = Q_0 + Q$ by putting $Q_\nu = Q_0 + \nu Q$ where ν is a real or complex parameter. The determinant

$$\Delta(\nu) = \operatorname{Det}(Q_\nu/Q_0) \tag{A.123}$$

is defined, being equal to $\operatorname{Det}(1 + \nu T)$. This is an entire function of the complex variable ν, hence it vanishes for a discrete set of values of ν (possibly empty). Furthermore, λ is an eigenvalue of Q w.r.t. Q_0 if and only if the quadratic form $Q_{-1/\lambda}$ is non invertible, that is if and only if $\Delta(-1/\lambda) = 0$.

As a consequence, the function of real variable $\nu \mapsto \Delta(\nu)$ has, at most, a finite number of zeroes $\nu_{(1)}, \cdots, \nu_{(p)}$ in the interval $[0, 1]$ with $0 < \nu_{(1)} <$

$\cdots < \nu_{(p)} \leq 1$. According to the relation (A.107), between two such zeroes, the following differential equation holds:

$$\frac{d}{d\nu} \ell n \operatorname{Det}(Q_\nu / Q_0) = \operatorname{Tr}(Q/Q_\nu) \tag{A.124}$$

where $Q_\nu = Q_0 + \nu Q$.

4.5. *Some integration formulas.*

With the previous notations, consider any ν in $[0,1]$ distinct from $\nu_{(1)}, \cdots, \nu_{(p)}$. Hence the quadratic form Q_ν on $\mathbf{X}$ is invertible, with an inverse quadratic form W_ν on $\mathbf{X}'$. There exists an integrator $\mathcal{D}_\nu x$ on $\mathbf{X}$ characterized by the formula

$$\int_{\mathbf{X}} \mathcal{D}_\nu x \cdot \exp\left(-\frac{\pi}{s} Q_\nu(x) - 2\pi i \langle x', x \rangle\right) = \exp\left(-\pi s\, W_\nu(x')\right). \tag{A.125}$$

Our claim is that $\mathcal{D}_\nu x$ *is proportional to* $\mathcal{D}_0 x$, namely that there exists a constant $I(\nu)$ such that $\mathcal{D}_\nu x = I(\nu)\mathcal{D}_0 x$. More explicitly, we assert the formula

$$\int_{\mathbf{X}} \mathcal{D}_\nu x \cdot F(x) = I(\nu) \int_{\mathbf{X}} \mathcal{D}_0 x \cdot F(x), \tag{A.126}$$

and that a functional $F(\cdot)$ on $\mathbf{X}$ is Feynman-integrable for $\mathcal{D}_\nu x$ if and only if it is Feynman-integrable for $\mathcal{D}_0 x$. The constant $I(\nu)$ can be obtained by putting $F(x) = \exp\left(-\frac{\pi}{s} Q_\nu(x)\right)$ into equation (A.126), hence

$$I(\nu)^{-1} = \int_{\mathbf{X}} \mathcal{D}_0 x \cdot \exp\left(-\frac{\pi}{s}\left(Q_0(x) + \nu Q(x)\right)\right). \tag{A.127}$$

First case $s = 1$:

According to our conventions, the quadratic forms Q_0 and Q_1 are positive-definite and invertible. Since the quadratic form $Q = Q_1 - Q_0$ is nuclear, it follows from the spectral theory that Q_0 and Q can be simultaneously diagonalized. Hence there exists a basis $(e_n)_{n \geq 1}$ for $\mathbf{X}$ such that

$$Q_0\left(\sum_{n=1}^{\infty} t_n e_n\right) = \sum_{n=1}^{\infty} (t_n)^2 \tag{A.128}$$

$$Q\left(\sum_{n=1}^{\infty} t_n e_n\right) = \sum_{n=1}^{\infty} \lambda_n (t_n)^2 \tag{A.129}$$

with real constants λ_n such that $\sum_{n=1}^{\infty} |\lambda_n| < \infty$. Since $Q_1 = Q_0 + Q$ is positive-definite, we have $1 + \lambda_n > 0$, hence

$$Q_\nu\left(\sum_{n=1}^{\infty} t_n e_n\right) = \sum_{n=1}^{\infty} \left((1-\nu) + \nu(1+\lambda_n)\right) \cdot (t_n)^2 \tag{A.130}$$

is again positive-definite and invertible for ν in $[0,1]$. Hence the determinant of Q_ν/Q_0 is defined and

$$\mathrm{Det}(Q_\nu/Q_0) = \prod_{n=1}^{\infty} (1+\nu\lambda_n) > 0\,. \tag{A.131}$$

The main result is given by the following formula:

$$I(\nu) = \mathrm{Det}(Q_\nu/Q_0)^{\frac{1}{2}}\,. \tag{A.132}$$

The proof is obtained without difficulty using equations (A.127) to (A.131) and the *approximation formula*:

$$\int_{\mathbf{X}} \mathcal{D}_0 x \cdot F(x) = \lim_{N=\infty} \int_{\mathbb{R}^N} d^N t\, F(t_1 e_1 + \cdots + t_N e_n)\,. \tag{A.133}$$

Second case $s = i$:

Here the basic formula is given by

$$I(\nu) = |\mathrm{Det}\,(Q_\nu/Q_0)|^{1/2}\, i^{\mathrm{Ind}(Q_\nu/Q_0)} \tag{A.134}$$

where the *index* $\mathrm{Ind}\,(Q_\nu/Q_0)$ is the number of negative eigenvalues of Q_ν w.r.t. Q_0. To simplify the statements, we shall assume that Q_0 is positive-definite. When ν runs over the interval $[0,1]$, this index remains constant except when ν is crossing a singular value $\nu_{(k)}$, where it experiences a jump. Hence the index $\mathrm{Ind}\,(Q_\nu/Q_0)$ is a sum of local contributions from the exceptional values $\nu_{(1)}, \cdots, \nu_{(p)}$, a phenomenon reminiscent of caustics.

The easiest proof of formula (A.134) is obtained by following a strategy initiated by Nelson and Sheeks in [6]. It works in both cases $s = 1$ and $s = i$. Starting from equation (A.127), we obtain by derivation

$$\begin{aligned} \frac{d}{d\nu} I(\nu)^{-1} &= \int_{\mathbf{X}} \mathcal{D}_0 x \cdot \exp\left(-\frac{\pi}{s}\,(Q_0(x) + \nu Q(x))\right)\left(-\frac{\pi}{s} Q(x)\right) \\ &= I(\nu)^{-1} \int_{\mathbf{X}} \mathcal{D}_\nu x \cdot \exp\left(-\frac{\pi}{s} Q_\nu(x)\right)\left(-\frac{\pi}{s} Q(x)\right), \end{aligned} \tag{A.135}$$

that is

$$\frac{d}{d\nu}\, \ell n\, I(\nu) = \frac{\pi}{s} \int_{\mathbf{X}} \mathcal{D}\omega_\nu(x)\, Q(x)\,. \tag{A.136}$$

Expanding $Q(x)$, according to formula (A.122) and using formula (A.39) for the covariance, we obtain

$$\frac{d}{d\nu}\, \ell n\, I(\nu) = \frac{\pi}{s} \cdot \sum_{n=1}^{\infty} \alpha_n \cdot \frac{s}{2\pi} W_\nu(x'_n) \tag{A.137}$$

which can be transformed easily into

$$\frac{d}{d\nu}\,\ell n\, I(\nu) = \tfrac{1}{2}\,\mathrm{Tr}\,(Q/Q_\nu)\,. \tag{A.138}$$

According to equation (A.124), we conclude

$$\frac{d}{d\nu}\,\ell n\, I(\nu) = \tfrac{1}{2}\frac{d}{d\nu}\,\ell n\ \mathrm{Det}\,(Q_\nu/Q_0)\,. \tag{A.139}$$

It remains to study the shift in phase when ν goes through an exceptional value $\nu_{(k)}$.

Remark. To simplify matters, assume again that Q_0 is positive-definite. Let L be an invertible operator in $\mathbf{X}$, of the form $1+T$ where T is nuclear. Putting $Q_1 = Q_0 \circ L$, it can be shown that the quadratic form $Q = Q_1 - Q_0$ is nuclear and that $\mathrm{Det}\,(Q_1/Q_0)$ is equal to $\mathrm{Det}(L)^2$. The quadratic from Q_1 is positive-definite, hence $\mathrm{Ind}\,(Q_1/Q_0) = 0$. From our main formulas (A.132) and (A.134), we derive

$$\mathcal{D}_{s,Q_0}(Lx) = |\mathrm{Det}(L)| \cdot \mathcal{D}_{s,Q_0}x\,. \tag{A.140}$$

Coming back to the general situation, where $Q_\nu = Q_0 + \nu Q$, the reader will find in [5, p. 277-279] examples of operators $L(\nu)$ such that $Q_\nu = Q_0 \circ L(\nu)$. In such a case, we obtain $\mathcal{D}_\nu x = |\mathrm{Det}\, L(\nu)| \cdot \mathcal{D}_0 x$.

Let us summarize the main results obtained in this paragraph:

Let Q_0 and Q_1 be continuous quadratic forms on the Banach space $\mathbf{X}$. We assume that Q_0 and Q_1 are invertible and that $Q_1 - Q_0$ is nuclear.

(A) Assuming that both Q_0 and Q_1 be positive-definite, we obtain

$$\int_{\mathbf{X}} \mathcal{D}_{Q_0}x \cdot e^{-\pi Q_1(x)} = \mathrm{Det}(Q_1/Q_0)^{-1/2}\,. \tag{A.141}$$

(B) For the the oscillatory integral, we obtain

$$\int_{\mathbf{X}} \mathcal{D}_{i,Q_0}x \cdot e^{\pi i Q_1(x)} = |\mathrm{Det}(Q_1/Q_0)|^{-1/2}\, i^{-\mathrm{Ind}(Q_1/Q_0)} \tag{A.142}$$

where the index $\mathrm{Ind}(Q_1/Q_0)$ *counts the number of negative eigenvalues of* Q_1 *w.r.t.* Q_0.

Equations (A.141) *and* (A.142) *justify the basic formula* (A.126)

$$\int_{\mathbf{X}} \mathcal{D}_\nu x \cdot F(x) = I(\nu) \int_{\mathbf{X}} \mathcal{D}_0 x \cdot F(x)$$

with $I(\nu)$ *given by* (A.132) *or* (A.134).

Appendix B
Functional determinants of Jacobi operators

Many functional determinants of Jacobi operators have been computed in works on semiclassical expansions. They are conveniently expressed in terms of finite-dimensional determinants of Jacobi matrices. We give here the backbone of the method, and applications to Jacobi operators along a classical path x_{cl} characterized by d initial conditions (position or momentum at t_a) and d final conditions (position or momentum at t_b). Details, proofs, applications and generalizations of the equations presented here are scattered in the literature and we give a few selected references at the end of this appendix. A general presentation on the properties of Jacobi operators will be found in the Ph.D dissertation of John La Chapelle [University of Texas (Austin) Ph. D. expected in 1995].

1. Jacobi fields and Jacobi matrices.

A *Jacobi field* is a vector field along a classical path obtained by variation through classical paths. A $d \times d$ Jacobi matrix is built up from Jacobi fields; each column consists of the components of a Jacobi field. We give explicit constructions when the action functional is $S(x) = \int_{\mathbf{T}} dt\, L(x(t), \dot{x}(t))$.

Let $\{x_{\mathrm{cl}}(\mu)\}$ be a $2d$-parameter family of classical paths (critical paths of the functional S) with values in a manifold M^d

$$x_{\mathrm{cl}}(\mu) : \mathbf{T} \to M^d \quad , \quad \mathbf{T} = [0, T] .$$

We introduce the notations

$$x_{\mathrm{cl}}(t; \mu) := (x_{\mathrm{cl}}(\mu))(t) \tag{B.1}$$

$$\dot{x}_{\mathrm{cl}}(t; \mu) := \partial x_{\mathrm{cl}}(t; \mu)/\partial t \tag{B.2}$$

$$x'_{\mathrm{cl},\alpha}(t; \mu) := \partial x_{\mathrm{cl}}(t; \mu)/\partial \mu^{\alpha}. \tag{B.3}$$

We choose $\mu = (\mu^1, \cdots, \mu^{2d})$ to stand for $2d$ initial conditions that characterize the classical path $x_{\mathrm{cl}}(\mu)$, assumed to be unique for the time being; namely,

$$S'(x_{\mathrm{cl}}(\mu)) = 0 \tag{B.4}$$

has a unique solution for a given set μ of parameters.

By varying successively the $2d$ initial conditions, one obtains $2d$ Jacobi fields $\{\partial x_{\text{cl}}/\partial\mu^\alpha\}$ for $\alpha \in \{1, \cdots, 2d\}$. It is easy to show that

$$S''(x_{\text{cl}}(\mu)) \cdot \frac{\partial x_{\text{cl}}}{\partial \mu^\alpha} = 0\,. \tag{B.5}$$

We assume for the time being that the quadratic form $S''(x_{\text{cl}}(\mu)) \cdot \xi\xi$ is not degenerate for variations ξ which respect the boundary conditions specified by μ. Therefore the $2d$ Jacobi fields are linearly independent.

There are several convenient basis for the $2d$-dimensional space of Jacobi fields. Choose as initial condition

$$\mu = (x_a, p_a) \tag{B.6a}$$

where

$$x_a := x(t_a) \tag{B.6b}$$

$$p_a := \partial L/\partial \dot{x}(t)\big|_{t=t_a}\,. \tag{B.6c}$$

The corresponding Jacobi fields are

$$j^{\bullet\beta}(t) = \partial x^{\bullet}_{\text{cl}}\,(t; x_a, p_a)\,/\partial p_{a,\beta} \tag{B.7}$$

$$k^{\bullet}{}_{\beta}(t) = \partial x^{\bullet}_{\text{cl}}\,(t; x_a, p_a)\,/\partial x^{\beta}_{a}\,. \tag{B.8}$$

Having introduced p_a as one of the initial conditions, the reader suspects that we shall also use[31]

$$\widetilde{k}_{\bullet}{}^{\beta}(t) = \partial p_{\text{cl},\bullet}\,(t; x_a, p_a)\,/\partial p_{a,\beta} \tag{B.9}$$

$$\ell_{\bullet\beta}(t) = \partial p_{\text{cl},\bullet}\,(t; x_a, p_a)\,/\partial x^{\beta}_{a}\,. \tag{B.10}$$

Let the *Jacobi matrices* $J, K, \widetilde{K}, L$ be made of the Jacobi fields $j, k, \widetilde{k}, \ell$ respectively

$$J^{\alpha\beta}(t, t_a) = j^{\alpha\beta}(t) \tag{B.11}$$

$$K^{\alpha}{}_{\beta}(t, t_a) = k^{\alpha}{}_{\beta}(t) \tag{B.12}$$

$$\widetilde{K}_{\alpha}{}^{\beta}(t, t_a) = \widetilde{k}_{\alpha}{}^{\beta}(t) \tag{B.13}$$

$$L_{\alpha\beta}(t, t_a) = \ell_{\alpha\beta}(t)\,. \tag{B.14}$$

The $2d \times 2d$ matrix constructed from the four $d \times d$ blocks $J, K, \widetilde{K}, L$ is a solution of the equation defined by the Jacobi operator in phase space. The

[31] The classical momentum $p_{\text{cl},\bullet}\,(t; x_a, p_a)$ is obtained by evaluating $\partial L/\partial \dot{x}(t)$ at the point $x(t) = x_{\text{cl}}\,(t; x_a, p_a)$.

properties of the Jacobi matrices are numerous. We note only their properties at $t = t_a$, which can be read off from their definitions :

$$J(t_a, t_a) = 0 \quad , \quad L(t_a, t_a) = 0 \tag{B.15}$$

$$K(t_a, t_a) = \mathbb{1} \quad , \quad \widetilde{K}(t_a, t_a) = \mathbb{1} \,. \tag{B.16}$$

The matrices K and $\widetilde{K}$ are indeed the transposed of each other

$$\widetilde{K}_\alpha{}^\beta(t, t_a) = K^\beta{}_\alpha(t_a, t). \tag{B.17}$$

Let $M, N, \widetilde{N}, P$ be the matrix inverses of $J, K, \widetilde{K}, L$ respectively. Hence we have

$$J^{\alpha\beta}(t, t_a) M_{\beta\gamma}(t_a, t) = \delta^\alpha_\gamma \tag{B.18}$$

and three similar equations expressed in condensed form as follows

$$KN = \mathbb{1}, \quad \widetilde{K}\widetilde{N} = \mathbb{1}, \quad LP = \mathbb{1} \,. \tag{B.19}$$

The matrices $M, N, \widetilde{N}, P$ are the hessians of the corresponding action functions (Van Vleck matrices):

$$M_{\beta\alpha}(t_b, t_a) = \frac{\partial^2 \mathcal{S}}{\partial x^\beta_{\text{cl}}(t_b) \partial x^\alpha_{\text{cl}}(t_a)}, \tag{B.20}$$

and similarly N and P are the hessians of $\mathcal{S}(x_{\text{cl}}(t_b), p_{\text{cl}}(t_a))$ and $\mathcal{S}(p_{\text{cl}}(t_b), p_{\text{cl}}(t_a))$ respectively.

2. Jacobi operators and their Greens's functions.

The Green's functions of the Jacobi operators in phase space can be expressed in terms of the Jacobi matrices $J, K, \widetilde{K}, L$; they include the Green's functions of the Jacobi operators in configuration space. For the sake of brevity, but at the cost of elegance, we consider here only the Green's functions of the Jacobi operators in configuration space.

In the previous paragraph, Jacobi fields were obtained by variation through classical paths. In this paragraph, we consider a one-parameter family of paths $x(\lambda)$ satisfying d fixed boundary conditions at t_a and d fixed boundary conditions at t_b, referred in brief as "a" and "b". The corresponding path space is denoted by $\mathcal{P}_{a,b}M$, hence

$$x(\lambda) \in \mathcal{P}_{a,b}M \subset \mathcal{P}M \,. \tag{B.21}$$

We assume that for $\lambda = 0$, the path $x_{\text{cl}} = x(0)$ is the unique critical point of the action functional S restricted to $\mathcal{P}_{a,b}M$. We set

$$\dot{x}(\lambda, t) := \partial x(\lambda, t)/\partial t \ , \quad x'(\lambda, t) := \partial x(\lambda, t)/\partial \lambda \ , \quad x'(0, t) = \xi(t).$$

The second variation of the action functional gives the Jacobi operator at x_{cl}:

$$\begin{aligned} S''(x_{\text{cl}}) \cdot \xi\xi &:= \frac{d^2}{d\lambda^2} S(x(\lambda))|_{\lambda=0} \\ &= \int_{\mathbf{T}} dt \left(L_{1\alpha,1\beta}\xi^\alpha(t) + L_{2\alpha,1\beta}\dot{\xi}^\alpha(t) \right) \xi^\beta(t) \\ &+ \int_{\mathbf{T}} dt \left(L_{1\alpha,2\beta}\xi^\alpha(t) + L_{2\alpha,2\beta}\dot{\xi}^\alpha(t) \right) \dot{\xi}^\beta(t) \end{aligned} \qquad (B.22)$$

where $L_{2\alpha,2\beta}$ is $\partial^2 L/\partial\dot{x}^\alpha(t)\partial\dot{x}^\beta(t)$ evaluated at x_{cl}, etc...

If we integrate (B.22) by parts, we obtain the Jacobi operator in its differential garb (second order differential operator on the space of vector fields ξ along x_{cl} together with boundary terms at t_a and t_b). It is often simpler to work with the quadratic form $S''(x_{\text{cl}}) \cdot \xi\xi$ as written in (B.22). We call functional Jacobi operator the kernel corresponding to the quadratic form $S''(x_{\text{cl}}) \cdot \xi\xi$, namely:

$$\frac{1}{2} \frac{\delta^2 S''(x_{\text{cl}}) \cdot \xi\xi}{\delta\xi^\alpha(s)\delta\xi^\beta(t)} =: \mathcal{J}_{\alpha\beta}(x_{\text{cl}}, s, t)\,. \qquad (B.23)$$

Its inverse will be called its *Green's function* $G^{\bullet\bullet}(x_{\text{cl}}, t, u)$, namely

$$\int_{\mathbf{T}} dt\, \mathcal{J}_{\alpha\beta}(x_{\text{cl}}, s, t) G^{\beta\gamma}(x_{\text{cl}}, t, u) = \delta_\alpha^\gamma \delta(s-u)\,. \qquad (B.24)$$

Provided the quadratic form $S''(x_{\text{cl}}) \cdot \xi\xi$ is non degenerate, the functional Jacobi operator has a unique inverse. We say $S''(x_{\text{cl}})$ is *non degenerate* if for any $\xi \neq 0$ in $T_{x_{\text{cl}}}\mathcal{P}_{a,b}M$ there exists η in this vector space with

$$S''(x_{\text{cl}}) \cdot \xi\eta \neq 0\,. \qquad (B.25)$$

This equation says that there are no (nonzero) Jacobi field in the tangent space $T_{x_{\text{cl}}}\mathcal{P}_{a,b}M$ to the space of paths with boundary conditions a and b.

We list below the Green's functions of Jacobi operators at classical paths with different boundary conditions. A more abstract formula could encode all cases. Explicit formulas may be more useful for applications.

i) *The classical path is characterized by* $p_{\text{cl}}(t_a) = p_a$, $x_{\text{cl}}(t_b) = x_b$

$$
\begin{aligned}
G(t,s) = {} & \theta(s-t)K(t,t_a)N(t_a,t_b)J(t_b,s) \\
& - \theta(t-s)J(t,t_b)\widetilde{N}(t_b,t_a)\widetilde{K}(t_a,s)\,.
\end{aligned}
\tag{B.26}
$$

ii) *The classical path is characterized by* $x_{\text{cl}}(t_a) = x_a$, $p_{\text{cl}}(t_b) = p_b$

$$
\begin{aligned}
G(t,s) = {} & \theta(s-t)J(t,t_a)\widetilde{N}(t_a,t_b)\widetilde{K}(t_b,s) \\
& - \theta(t-s)K(t,t_b)N(t_b,t_a)J(t_a,s)\,.
\end{aligned}
\tag{B.27}
$$

iii) *The classical path is characterized by* $x_{\text{cl}}(t_a) = x_a$, $x_{\text{cl}}(t_b) = x_b$

$$
\begin{aligned}
G(t,s) = {} & \theta(s-t)J(t,t_a)M(t_a,t_b)J(t_b,s) \\
& - \theta(t-s)J(t,t_b)M(t_b,t_a)J(t_a,s)\,.
\end{aligned}
\tag{B.28}
$$

iv) *The classical path is characterized by* $p_{\text{cl}}(t_a) = p_a$, $p_{\text{cl}}(t_b) = p_b$

$$
\begin{aligned}
G(t,s) = {} & \theta(s-t)K(t,t_a)P(t_a,t_b)\widetilde{K}(t_b,s) \\
& - \theta(t-s)K(t,t_b)\widetilde{P}(t_b,t_a)\widetilde{K}(t_a,s)\,.
\end{aligned}
\tag{B.29}
$$

3. Semiclassical expansions.

We have given examples of WKB approximations in two cases:

i) The classical path of reference is characterized by initial momentum, final position (paragraph III.2).

ii) The classical path is characterized by initial position, final position (paragraph IV.1).

Two other cases are often needed:

iii) The classical path is characterized by initial position and final momentum. The transposition from case i) is straightforward.

iv) The classical path is characterized by initial momentum and final momentum. The critical points of the action functional are degenerate when the classical system is constrained by a conservation law. We refer the reader to the reference [15] for this case.

The computation of semiclassical expansions along the lines of paragraphs III.2 and IV.1, but with general action functionals S, brings in a second variation

$$
S''(x_{\text{cl}}) \cdot \xi\xi = Q_0(\xi) + Q(\xi) \tag{B.30}
$$

where

$$Q_0(\xi) = \int_{\mathbf{T}} dt\, L_{2\alpha,2\beta} \dot{\xi}^\alpha(t) \dot{\xi}^\beta(t)\,. \tag{B.31}$$

Provided the Legendre matrix

$$L_{2\alpha,2\beta}(x_{\mathrm{cl}}(t), \dot{x}_{\mathrm{cl}}(t)) = \partial^2 L / \partial \dot{x}^\alpha_{\mathrm{cl}} \partial \dot{x}^\beta_{\mathrm{cl}} \tag{B.32}$$

is invertible, the Gaussian integrator defined by the quadratic form(B.30) can be handled by the same techniques as the Gaussian integrator defined by the quadratic form

$$\int_{\mathbf{T}} dt\, h_{\alpha\beta} \dot{\xi}^\alpha(t) \dot{\xi}^\beta(t)\,. \tag{B.33}$$

The contribution of the second variation $S''(x_{\mathrm{cl}}) \cdot \xi\xi$ to the semiclassical expansion of $\Psi(t_b, x_b)$ is

$$\int \mathcal{D}_{s,Q_0}\xi \; \exp\Big(-\frac{\pi}{s}(Q_0(\xi) + Q(\xi)\Big) = \mathrm{Det}\,(Q_0/(Q_0+Q))^{1/2}\,. \tag{B.34}$$

Then one uses the Green's functions (B.26-29) to identify the ratio of Jacobi matrices whose determinant is equal to $\mathrm{Det}(Q_0/(Q_0+Q))$. The results are identical to the results obtained by discretizing the functional determinants, and reported in [16].

4. References for Appendix B.

The most complete *summary*, to date, of Jacobi fields in phase space, including degenerate critical points of the action, can be found in Appendix A of

- C. DeWitt-Morette "Feynman path integrals" Acta Physica Austriaca, Suppl. XXVI, 101-170 (1984).

For the *proofs* of the results presented in the present paper, and for properties of Jacobi matrices used in the present paper, see [2, 5, 15, 16] and

- C. DeWitt-Morette and T.-R. Zhang "Feynman-Kac formula in phase space with application to coherent state transitions" Phys. Rev. **D 28**, 2517-2525 (1983).

More on degenerate critical points can be found in

- C. DeWitt-Morette, B. Nelson and T.-R. Zhang "Caustic problems in quantum mechanics with applications to scattering theory" Phys. Rev. **D 28**, 2526-2546 (1983).

- C. DeWitt-Morette and B.L. Nelson "Glories and other degenerate points of the action" Phys. Rev. **D 29**, 1663-1668 (1984).

A short introduction to the Hamiltonian techniques underlying the above results can be found in

- P. Cartier "Some fundamental techniques in the theory of integrable systems" in *Lectures on Integrable Systems* (O. Babelon, P. Cartier and Y. Kosmann-Schwarzbach edit.), World Scientific, Singapore (1994).

Appendix C

A new class of ordinary differential equations

The purpose of this Appendix is to extend to the $L^{2,1}$ case the familiar theorems about the existence and uniqueness of solutions of differential equations, and to describe the parametrization of paths in a curved space by means of paths in a flat space.

1. Solutions of differential equations: the classical case.

We shall follow the usual strategy, as described in any standard textbook, for instance Bourbaki [27]. More precisely, consider a domain Ω in the Euclidean space $\mathbb{R}^n$ and a vector field in Ω associating to every epoch t in the time interval $\mathbf{T} = [t_a, t_b]$ and to every point x in Ω a velocity vector $v(t,x)$ in $\mathbb{R}^n$. We assume that $v(t,x)$ is a continuous function of t and x. The two basic remarks are as follows:

a) *any trajectory of the vector field v can be prolongated as long as it does not reach the boundary of Ω;*

b) *by the mean value theorem, if the absolute velocity $|v|$ is bounded by a constant V along a given trajectory leading from x_a at time t_a to x_b at time t_b, then the mean velocity $\frac{|x_b - x_a|}{t_b - t_a}$ is bounded by V.*

From these remarks follows the following existence theorem (Peano):

Assume that Ω is a closed ball centered at x_a of radius L, and that

$$|v(t,x)| < L/T \tag{C.1}$$

holds uniformly for t in $\mathbf{T}$ and x in Ω, where $T = t_b - t_a$ is the length of $\mathbf{T}$. Then there exists a solution $x : \mathbf{T} \to \Omega$ of the differential equation $\dot{x} = v(t,x)$ with the initial condition $x(t_a) = x_a$.

For the proof, we construct first an approximate solution by the Euler method. We select epochs $t_1, \cdots, t_{N-1}$ such that $t_a < t_1 < \cdots < t_{N-1} < t_b$ and set $t_0 = t_a$, $t_N = t_b$. Then we define inductively points $x_0, x_1, \cdots, x_N$ in Ω by $x_0 = x_a$ and

$$x_i = x_{i-1} + (t_i - t_{i-1})\, v(t_{i-1}, x_{i-1}) \tag{C.2}$$

for $1 \leq i \leq N$; the estimate (C.1) guarantees that the points $x_0, x_1, \cdots, x_N$ are in Ω. We then interpolate linearly in each subinterval $[t_{i-1}, t_i]$, and generate a function $x_{\mathcal{T}} : \mathbf{T} \to \Omega$ depending on the subdivision $\mathcal{T} = (t_0 < t_1 < \cdots < t_N)$ of $\mathbf{T}$. By the mean value theorem and the estimate (C.1) we obtain

$$|x_{\mathcal{T}}(t) - x_{\mathcal{T}}(t')| \leq V\,|t - t'| \tag{C.3}$$

for t, t' in $\mathbf{T}$, where $V = L/T$. We then use Ascoli's theorem: it asserts the existence of a sequence of subdivisions $\mathcal{T}(n)$ of $\mathbf{T}$, whose mesh $\Delta(n)$ tends to 0, such that $x_{\mathcal{T}(n)}(t)$ tends to a limit $x(t)$ uniformly for t in $\mathbf{T}$. Then, one shows that $x_{\mathcal{T}(n)}$ satisfies an approximate integral equation

$$\left| x_{\mathcal{T}(n)}(t) - x_a - \int_{t_a}^{t} ds\; v(s, x(s)) \right| \leq \varepsilon_n\,, \tag{C.4}$$

where $\lim_{n=\infty} \varepsilon_n = 0$. By uniform convergence, the limit function $x(\cdot)$ satisfies the integral equation

$$x(t) = x_a + \int_{t_a}^{t} ds\; v(s, x(s))\,, \tag{C.5}$$

fully equivalent to the differential equation

$$\begin{cases} \dot{x}(t) = v(t, x(t)) \\ x(t_a) = x_a\,. \end{cases} \tag{C.6}$$

Both the *uniqueness* of a solution to the previous equation and the *continuous dependence of the unique solution on the initial position* x_a are established by an analysis of the *stability* of trajectories. Suppose given another trajectory $y : \mathbf{T} \to \Omega$ with initial position $y_a = y(t_a)$; such a trajectory exists if y_a is sufficiently close to x_a. Hence we have the integral relation

$$y(t) = y_a + \int_{t_a}^{t} ds\, v(s, y(s))\,. \tag{C.7}$$

Furthermore, assume that the velocity field satisfies a *Lipschitz condition*:

$$|v(t,x) - v(t,x')| \leq k\,|x - x'| \tag{C.8}$$

for x, x' in Ω and t in $\mathbf{T}$, with a fixed constant $k > 0$. Denote by $\delta(t)$ the distance between $x(t)$ and $y(t)$. From (C.5), (C.7) and (C.8) one derives

$$|\delta(t) - \delta(t_a)| \leq k \int_{t_a}^{t} ds\, \delta(s)\,. \tag{C.9}$$

By using an iteration procedure reminiscent of Picard's method, one derives the following estimate:

$$\delta(t) \leq \delta(t_a) e^{k(t-t_a)} \tag{C.10}$$

("*Gronwall's lemma*"). If $x_a = y_a$, then $\delta(t_a) = 0$, hence $\delta(t) = 0$ for all t and $x(t) = y(t)$: uniqueness. Furthermore if $\delta(t_a) = |y_a - x_a|$ tends to 0, then $y(t)$ tends uniformly to $x(t)$ for t in the finite interval $\mathbf{T}$.

The results obtained so far are local. To get a global existence theorem, assume that N is a compact manifold, of dimension n, and that X is a time-dependent vector field on N, namely $X(t,x)$ belongs to $T_x N$ for t in $\mathbf{T}$, and x in N. Assume furthermore that $X(t,x)$ is continuous in t, x, with continuous first derivatives in x. Choose a point x_a in N and a coordinate system around x_a. Then X is expressed in this coordinate system by a function $v : \mathbf{T} \times \Omega \to \mathbb{R}^n$, where Ω is a closed ball centered at x_a, with radius L. Since a continuous function is bounded on any compact set, there exists a constant $\tau > 0$ such that $|v(t,x)| \leq L/2\tau$ uniformly for t in $\mathbf{T}$ and x in Ω. From the local existence theorem, it follows that for any point x_0 in the open ball centered at x_a with radius $L/2$, and any t_0 in $\mathbf{T}$, there exists a unique trajectory $x : [t_0, t_0 + \tau] \to N$ such that $x(t_0) = x_0$. By compacity of N, we can cover N by a finite number of such balls. Defining σ as the minimum of the life-times τ, we conclude that the existence and uniqueness of a trajectory $x : [t_0, t_0 + \sigma] \to N$ with $x(t_0) = x_0$ hold for every point x_0 in N.

Consider now the time interval $\mathbf{T} = [t_a, t_b]$ and subdivide it into subintervals $[t_i, t_{i+1}]$ of lengths $\leq \sigma$ (for i in $\{0, 1, \cdots, N\}$) where $t_0 = t_a$, $t_{N+1} = t_b$. So given any point x_a in N, there exists a trajectory $x_0(t)$ (for $t_0 \leq t \leq t_1$) such that $x_0(t_0) = x_a$. Set $x_1 = x_0(t_1)$ and consider the trajectory $x_1(t)$ (for $t_1 \leq t \leq t_2$) with initial point x_1. By repeating this procedure, we construct step by step the required trajectory $x : \mathbf{T} \to N$ such that $x(t_a) = x_a$.

Proceeding backwards rather than forwards, we conclude that given any point x_b in N, there exists a unique solution $x(t) = x(t; x_b)$ of the differential equation $\dot{x}(t) = X(t, x(t))$ with final position $x(t_b) = x_b$. For fixed t, we define the transformation $\Sigma(t)$ in N taking x_b to $x(t; x_b)$, hence

$$x(t; x_b) = x_b \cdot \Sigma(t)\,. \tag{C.11}$$

2. Solutions of differential equations: the $L^{2,1}$ case.

We sketch the necessary modifications. A more detailed account will appear elsewhere.

Consider an $L^{2,1}$ path $x : \mathbf{T} \to \Omega$; its action is defined by

$$A = \int_{\mathbf{T}} \frac{|dx|^2}{dt}\,. \tag{C.12}$$

The mean value theorem is replaced by the following estimate

$$\frac{|x_b - x_a|^2}{t_b - t_a} \leq A\,, \tag{C.13}$$

which follows easily from Cauchy-Schwarz inequality.

We make the following assumptions about the velocity field $v(t,x)$.

(A) *The function $v(t,x)$ is jointly-measurable in (t,x), and continuous in x for a given t, and furthermore there exists a numerical L^2 function $V : \mathbf{T} \rightarrow [0,+\infty[$ such that*

$$|v(t,x)| \leq V(t) \tag{C.14}$$

holds for x in Ω and t in $\mathbf{T}$.

(B) *Denoting by A the action $\int_{\mathbf{T}} dt\, V(t)^2$, the radius L of the ball Ω centered at x_a, and the length T of the time interval $\mathbf{T}$ obey the estimate*

$$A < L^2/T\,. \tag{C.15}$$

Under these conditions, one proves an existence theorem for the differential equation (C.6) with an $L^{2,1}$ solution $x : \mathbf{T} \rightarrow \Omega$. As a first step, one replaces Euler's approximation (C.2) by the following approximate solution

$$x_{\mathcal{T}}(t) = x_{\mathcal{T}}(t_{i-1}) + \int_{t_i}^{t} ds\, v(s, x_{\mathcal{T}}(t_{i-1})) \tag{C.16}$$

for t in the subinterval $[t_{i-1}, t_i]$. The inequalities (C.13) to (C.15) guarantee that this trajectory $x_{\mathcal{T}}(t)$ remains in the closed ball Ω. Furthermore, the velocity $\dot{x}_{\mathcal{T}}(t)$ of this trajectory satisfies the estimate $|\dot{x}_{\mathcal{T}}(t)| \leq V(t)$ for t in $\mathbf{T}$, and by (C.13), we obtain the estimate

$$|x_{\mathcal{T}}(t) - x_{\mathcal{T}}(t')|^2 \leq A\,|t - t'|\,. \tag{C.17}$$

We can then invoke Ascoli's theorem, and find a sequence of approximate solutions $x_{\mathcal{T}(n)}$ converging uniformly on $\mathbf{T}$ towards an $L^{2,1}$ function x. Each approximation $x_{\mathcal{T}(n)}$ satisfies an approximate integral equation, and in the limit the integral equation (C.5) is obtained using Lebesgue's dominated convergence.

For the uniqueness, we need a Lipschitz condition of the type

$$|v(t,x) - v(t,x')| \leq k(t)\,|x - x'| \tag{C.18}$$

where the integral $\int_{\mathbf{T}} dt\, k(t)^2$ is finite. We use again a variant of Gronwall's lemma.

The global results are obtained for a compact manifold N by the reasonings used at the end of paragraph C.1. Only minor modifications are needed.

3. Parametrization of paths.

We consider a manifold N and d vector fields $X_{(1)}, \cdots, X_{(d)}$ on N. We assume that they are of class C^1 and linearly independent at each point of N. For x in N denote by H_x the vector subspace of T_xN generated by $X_{(1)}(x), \cdots, X_{(d)}(x)$. The collection of these vector spaces is a subbundle[32] H of the tangent bundle TN to N. We consider also a real symmetric invertible matrix $(h_{\alpha\beta})$ of size $d \times d$, and define a field of quadratic forms h_x on H_x by

$$h_x\left(X_{(\alpha)}(x), X_{(\beta)}(x)\right) = h_{\alpha\beta}\,. \tag{C.19}$$

Fix a point x_b in N. We denote by $\mathcal{P}^H_{x_b}N$ the set of $L^{2,1}$ paths $x : \mathbf{T} \to N$ which satisfy the following conditions:

(A) *The endpoint $x(t_b)$ is equal to x_b.*

(B) *For each epoch t in $\mathbf{T}$, the velocity vector $\dot{x}(t)$ lies in the subspace $H_{x(t)}$ of $T_{x(t)}N$.*

We define the *action* of such a path by

$$A(x) = \int_{\mathbf{T}} dt\, h_{x(t)}(\dot{x}(t), \dot{x}(t))\,. \tag{C.20}$$

Since the vectors $X_{(\alpha)}(x(t))$ (for $1 \leq \alpha \leq d$) form a basis of $H_{x(t)}$, and from the hypothesis that the path x is of class $L^{2,1}$, we infer that there exist functions $\dot{z}_\alpha$ in $L^2(\mathbf{T})$ such that

$$\dot{x}(t) = X_{(\alpha)}(x(t))\dot{z}^\alpha(t)\,. \tag{C.21}$$

Furthermore, the function $\dot{z}^\alpha$ is the derivative of a function z^α in $L^{2,1}(\mathbf{T})$ normalized by $z^\alpha(t_b) = 0$. The vector function $z = \left(z^1, \cdots z^d\right)$ is an element of the space denoted by $\mathbf{Z}_b$ in paragraph A.3.8. *This construction associates to a path x in $\mathcal{P}^H_{x_b}N$ a path z in $\mathbf{Z}_b$ with conservation of action:*

$$A(x) = Q_0(z)\,, \tag{C.22}$$

where as usual $Q_0(z)$ is equal to $\int_{\mathbf{T}} dt\, h_{\alpha\beta}\dot{z}^\alpha(t)\dot{z}^\beta(t)$.

Assume now that N is compact. By using the theory of $L^{2,1}$ differential equations sketched in paragraph C.2, it can be shown that we can invert the

[32] The letter H stands for "horizontal".

transformation $x \mapsto z$. Given any z in $\mathbf{Z}_b$, the differential equation (C.21) has a unique solution x in $\mathcal{P}^H_{x_b}N$, and we obtain *a parametrization*

$$P : \mathbf{Z}_b \to \mathcal{P}^H_{x_b}N$$

of a space of paths in a curved space N by a space of paths in a flat space $\mathbb{R}^d$. For z in $\mathbf{Z}_b$, we denote by $x(t, z)$ the solution of the differential equation (C.21) with endpoint $x(t_b, z) = x_b$. We can also introduce a global transformation $\Sigma(t, z) : N \to N$ taking x_b into $x(t, z)$. If necessary, we include t_b in the notation and denote this transformation by $\Sigma(\mathbf{T}; z)$ or $\Sigma(t_b, t_a; z)$ in the case $t = t_a$. The chain rule

$$\Sigma(t_b, t_a; z_{ba}) = \Sigma(t_b, t_c; z_{bc}) \cdot \Sigma(t_c, t_a; z_{ca}) \tag{C.23}$$

is a consequence of the uniqueness of the solution of the differential equation (C.21). Here z_{ba} is z, t_c is an intermediate epoch, and the paths $z_{bc} : [t_c, t_b] \to \mathbb{R}^d$ and $z_{ca} : [t_a, t_c] \to \mathbb{R}^d$ are given by

$$z_{bc}(t) = z(t) \quad , \quad z_{ca}(t) = z(t) - z(t_c)\,. \tag{C.24}$$

Remark. The more general differential equation

$$\dot{x}(t) = X_{(\alpha)}(x(t))\dot{z}^{\alpha}(t) + Y(x(t)) \tag{C.25}$$

can be handled in a similar way. In this case, we replace $\mathcal{P}^H_{x_b}N$ by the space $\mathcal{P}^{H,Y}_{x_b}N$ of $L^{2,1}$ paths $x : \mathbf{T} \to N$ such that $x(t_b) = x_b$ and $\dot{x}(t) - Y(t)$ belong to $H_{x(t)}$ for every t in $\mathbf{T}$.

These constructions are related to the *Cartan development map.* Take for M a compact Riemannian manifold, and let N be the corresponding bundle of orthonormal frames; it is a compact manifold. Fix x_b in M, and a frame $\rho_b = (x_b, u_b)$ at x_b. Then, by the Riemannian connection, there is defined a "horizontal" subspace $T^H_{\rho_b}N$ of the tangent space $T_{\rho_b}N$, and the projection $\Pi : N \to M$ induces an identification of $T^H_{\rho_b}N$ with $T_{x_b}M$. Since u_b defines an orthonormal basis of $T_{x_b}M$, we obtain a basis $X_{(1)}(\rho_b), \cdots, X_{(d)}(\rho_b)$ of $T^H_{\rho_b}N$. This construction is valid for every point in N, hence we define vector fields $X_{(1)}, \cdots, X_{(d)}$ on N. By using the previous construction, we get a parametrization

$$P : \mathbf{Z}_b \to \mathcal{P}^H_{\rho_b}N\,.$$

But the paths in $\mathcal{P}^H_{\rho_b}N$ are nothing else than the horizontal liftings of the $L^{2,1}$ paths in M. More precisely denote by $\mathcal{P}_{x_b}M$ the set of $L^{2,1}$ paths $x : \mathbf{T} \to M$, such that $x(t_b) = x_b$. Then, the projection $\Pi : N \to M$ induces a mapping $\widetilde{x} \mapsto \Pi \circ \widetilde{x}$ of $\mathcal{P}^H_{\rho_b}N$ into $\mathcal{P}_{x_b}M$, and this map is a bijection. To conclude, we get a diagram

$$\mathbf{Z}_b \to \mathcal{P}^H_{\rho_b}N \to \mathcal{P}_{x_b}M$$

and by composition a parametrization of $\mathcal{P}_{x_b}M$ by $\mathbf{Z}_b$. This is the Cartan development map for $L^{2,1}$ paths. The standard theory works for C^1 paths.

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Figure captions

Figure A.1: "Linear change of variables"

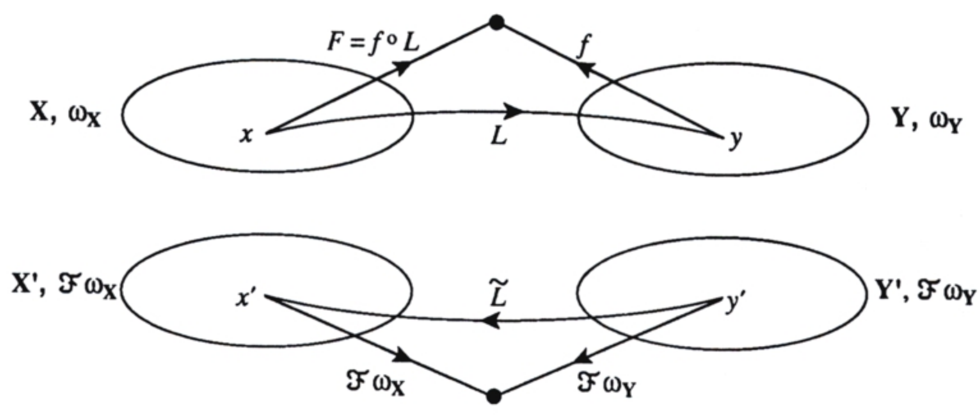

Fig A.1