KORTEWEG-de-VRIES WITH L^2 DATA

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This note is a follow-up of [B] and the reader is referred to [B], also [K-P-V] for introductory material on the subject.

Lemma 1. *Let*

$$F(x,t) = \iint \frac{c(n,\lambda)}{(1+|\lambda - n^3|)^\rho} \, e^{i(nx+\lambda t)} \, dn \, d\lambda \quad \left(\rho > \frac{1}{4}\right). \tag{0}$$

Then

$$\| \partial_x^{1/2} \, F \|_{L_x^4 L_t^2} \leq c \left(\iint c(n,\lambda)^2 \right)^{1/2}. \tag{1}$$

Proof.

$$\| \partial_x^{1/2} \, F \|_{L_t^2} = \left(\int \left| \int \frac{|n|^{1/2} \, c(n,\lambda)}{(1+|\lambda - n^3|)^\rho} \, e^{inx} \, dn \right|^2 d\lambda \right)^{1/2}.$$

By 2-convexity of L^4-norm and Hausdorff-Young

$$\| \partial_x^{1/2} \, F \|_{L_x^4 L_t^2} \leq \left(\int \left\| \int \frac{|n|^{1/2} \, c(n,\lambda)}{(1+|\lambda - n^3|)^\rho} \, e^{inx} \, dn \right\|_{L_x^4}^2 d\lambda \right)^{1/2}$$

assume $c(n,\lambda) \geq 0$

$$\leq \left[\int \left(\int \frac{|n|^{2/3} \, c(n,\lambda)^{4/3}}{(1+|\lambda - n^3|)^{\rho 4/3}} \, dn \right)^{3/2} d\lambda \right]^{1/2}. \tag{2}$$

From Hölder's inequality

$$\int \frac{|n|^{2/3} \, c(n,\lambda)^{4/3}}{(1+|\lambda - n^3|)^{4/3\rho}} \, dn \leq \left(\int c(n,\lambda)^2 \, dn \right)^{2/3} \left(\int \frac{|n|^2}{(1+|\lambda - n^3|)^{4\rho}} \, dn \right)^{1/3}$$
$$\leq c \left(\int c(n,\lambda)^2 \, dn \right)^{2/3}. \tag{3}$$

Substitution of (3) in (2) yields (1).

Lemma 2. *Let F be given by (0) with $\rho > \frac{1}{3}$. Then*

$$\| F \|_{L_x^4 L_t^4} \leq c \left(\iint c(n,\lambda)^2 \, dn \, d\lambda \right)^{1/2}. \tag{4}$$

Proof. Use interpolation between 2 and 8 $\left(\frac{1}{4}=\frac{\frac{1}{2}}{2}+\frac{\frac{3}{4}}{8}\right)$. Clearly for $F_0=\iint c(n,\lambda)\, e^{i(nx+\lambda t)}\, dn\, d\lambda$

$$\| F_0 \|_{L^2_x L^2_t} \le \left(\iint c(n,\lambda)^2\right)^{1/2} \tag{5}$$

while for $F_{\rho_1}=\iint \frac{c(n,\lambda)}{(1+|\lambda-n^3|)^{\rho_1}}\, e^{i(nx+\lambda t)}\, dn\, d\lambda$, $\rho_1>\frac{1}{2}$

$$\| F_{\rho_1} \|_{L^8_x L^8_t} \le c\left(\iint c(n,\lambda)^2\right)^{1/2}. \tag{6}$$

Inequality (6) results from

$$\left\|\int c(n)\, e^{i(nx+n^3t)}\, dn\right\|_{L^8_{x,t}} \le c\ \left(\int c(n)^2\right)^{1/2} \tag{7}$$

writing

$$\lambda=n^3+\lambda_1$$

and estimating in (6)

$$\begin{aligned}\| F_{\rho_1} \|_8 &\le \int \left\|\int \frac{c(n,n^3+\lambda_1)}{(1+|\lambda_1|)^{\rho_1}}\, e^{i(nx+n^3t)}\, dn\right\|_8 d\lambda_1\\ &\le \int \left[\int c(n,n^3+\lambda_1)^2\, dn\right]^{1/2} (1+|\lambda_1|)^{-\rho_1}\ d\lambda_1\\ &\le \left(\iint c(n,\lambda)^2\right)^{1/2}\end{aligned}$$

applying (7) and Hölder.

Lemma 3.

$$\left\|\iint \frac{c(n,\lambda)}{(1+|\lambda|)^{\rho}}\, e^{i(nx+\lambda t)}\, dn\, d\lambda\right\|_{L^2_x L^\infty_t} \le c\left(\iint c(n,\lambda)^2\, dn\, d\lambda\right)^{1/2} \quad \left(\rho>\frac{1}{2}\right). \tag{8}$$

Kdv equation writes

$$u_t+u_{xxx}+uu_x=0\ \ ,\ \ u(x,0)=\phi(x).$$

We prove wellposedness locally (on $\mathbf{R}_x\times I_t$) in space

$$\|u\| = \left(\iint (1+|\lambda-n^3|)\, |\widehat{u}(n,\lambda)|^2\ dn\, d\lambda\right)^{1/2} + \left(\int_{|n|<1}\int |\lambda|^{2\alpha}\ |\widehat{u}(n,\lambda)|^2\ dn\, d\lambda\right)^{1/2} \tag{9}$$

where $\alpha>\frac{1}{2}$ is fixed ($\alpha<1$ and in fact close to $\frac{1}{2}$).

(9) should be understood as a "restriction norm" since the Fourier series on $\mathbf{R}\times I$ is not unique.

Consider integral equation

$$u(t)=W(t)\,\phi-\int_0^t W(t-\tau)\, w(\tau)\, d\tau\ ;\ W(t)=e^{-t\partial_x^3}\ ;\ w=u\partial_x u=\frac{1}{2}\,\partial_x(u^2)$$

and in Fourier transform notation

$$u(x,t)=\int \widehat{\phi}(n)\, e^{i(nx+n^3t)}\, dn + c\iint dn\, d\lambda\ \widehat{w}(n,\lambda)\, e^{inx}\, \frac{e^{i\lambda t}-e^{in^3t}}{\lambda-n^3}\, d\lambda. \tag{10}$$

We study the induced transformation on the $\| \ \|$-space.

For $\widehat{w}(n, \lambda)$, we have

$$\widehat{w}(n,\lambda) = n\,(\widehat{u} * \widehat{u})\ (n,\lambda). \tag{11}$$

Defining

$$\begin{aligned} c(n,\lambda) = &\ \left(1 + |\lambda - n^3|\right)^{1/2}\ |\widehat{u}(n,\lambda)| \\ = &\ |\lambda|^\alpha\ |\widehat{u}(n,\lambda)| \quad \text{if} \quad |n| \le 1 \end{aligned} \tag{12}$$

one has

$$\iint c(n,\lambda)^2 \le \|u\|^2 \quad \text{from (9)}$$

and from (11)

$$|\widehat{w}(n,\lambda)| \le \tag{13}$$

$$|n| \iint dn_1\, d\lambda_1\ \frac{c(n_1,\lambda_1)}{(1 + |\lambda_1 - n_1^3|)^{1/2} + |\lambda_1|^\alpha\ \chi_{[|n_1|<1]}}\ \frac{c(n - n_1, \lambda - \lambda_1)}{(1 + |\lambda - \lambda_1 - (n - n_1)^3|)^{1/2} + |\lambda - \lambda_1|^\alpha\ \chi_{[|n-n_1|<1]}}.$$

Write

$$\frac{e^{i\lambda t} - e^{in^3 t}}{\lambda - n^3} = \frac{e^{i\lambda t} - e^{in^3 t}}{\lambda - n^3}\ (1 - \varphi)(\lambda - n^3) + e^{in^3 t}\ \frac{e^{i(\lambda - n^3)t} - 1}{\lambda - n^3}\ \varphi(\lambda - n^3) \tag{14}$$

where φ is a bump function, $\varphi = 1$ on neighborhood of 0.

In order to take care of second term of (14), use fact that $|t| < \delta < 1$ (similarly as in periodic case, cf. [B]). We will only be concerned here with contributions of first term of (14) to $\| \ \|$. They are estimated by

(I) $$\left(\iint \frac{|\widehat{w}(n,\lambda)|^2}{(1 + |\lambda - n^3|)}\ dn\ d\lambda\right)^{1/2} + \left(\iint_{|n|\le 1} \frac{|\widehat{w}(n,\lambda)|^2}{1 + \lambda^{2(1-\alpha)}}\ dn\ d\lambda\right)^{1/2}$$

(II) $$\left[\int \left(\int \frac{|\widehat{w}(n,\lambda)|}{1 + |\lambda - n^3|}\ d\lambda\right)^2\ dn\right]^{1/2}$$

again as in the periodic case.

Estimates on (I) (first term)

Proceed by duality, taking $\{d(n,\lambda)_{>0}\}$ satisfying $\iint d(n,\lambda)^2 \le 1$. From (13), we get

$$\int dn\ d\lambda\ dn_1\ d\lambda_1 \tag{15}$$

$$\frac{|n|\ c(n_1,\lambda_1)\ c(n - n_1, \lambda - \lambda_1)\ d(n,\lambda)}{(1 + |\lambda - n^3|)^{1/2}\ \left[1 + |\lambda_1 - n_1^3|^{1/2} + |\lambda_1|^\alpha\ \chi_{[|n_1|<1]}\right]\ \left[1 + |\lambda - \lambda_1 - (n - n_1)^3|^{1/2} + |\lambda - \lambda_1|^\alpha\ \chi_{[|n-n_1|<1]}\right]}.$$

Assume $|n-n_1| \geq |n_1|$ (other case is similar).

One has

$$\max\left(|\lambda - n^3|\ ,\ |\lambda_1 - n_1^3|\ ,\ |\lambda - \lambda_1 - (n-n_1)^3|\right) \geq |n|\ |n-n_1|\ |n_1| \geq |n_1|\ |n|^2. \tag{16}$$

There are 2 cases, $n\ell$

(a) $$|n_1| \geq 1$$

(b) $$|n_1| < 1.$$

Case (a) is completely ananalogous with periodic case. From (16)

$$\max\left(|\lambda - n^3|, \cdot, \cdot\right) \geq n^2 \tag{17}$$

and we assume for instance

$$|\lambda - n^3| \geq n^2. \tag{18}$$

The other cases are similar.

This contribution to (15) is bounded by

$$\int dn\ dn_1\ d\lambda\ d\lambda_1\ \ \frac{c(n_1,\lambda_1)\ c(n-n_1,\lambda-\lambda_1)\ d(n,\lambda)}{(1+|\lambda_1-n_1^3|)^{1/2}\ (1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}}. \tag{19}$$

Define

$$F(x,t) = \iint dn\ d\lambda\ e^{i(nx+\lambda t)}\ \frac{c(n,\lambda)}{(1+|\lambda-n^3|)^{1/2}} \tag{20}$$

$$G(x,t) = \iint dn\ d\lambda\ e^{i(nx+\lambda t)}\ d(n,\lambda). \tag{21}$$

Then (19) equals

$$\langle \widehat{F} * \widehat{F}, \widehat{G}\rangle = \langle FF, G\rangle \leq \|\ F\ \|_4^2\ \|\ G\ \|_2$$

and use Lemma 2, to estimate by $\iint c(n,\lambda)^2\ dn\ d\lambda = \|u\|^2$.

In **case (b)**, we proceed differently, since (16) is not useful anymore. Since $|n| \leq 2|n-n_1|$ one has

$$(15) \leq \int dn\ d\lambda\ dn_1\ d\lambda_1\ \frac{|n|^{1/2}\ d(n,\lambda)}{(1+|\lambda-n^3|)^{1/2}}\ \frac{|n-n_1|^{1/2}\ c(n-n_1,\lambda-\lambda_1)}{(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}}\ \frac{c(n_1,\lambda_1)}{1+|\lambda_1|^{\alpha}}. \tag{22}$$

Define functions F, G, H by letting

$$\widehat{F}(n,\lambda) = \frac{d(n,\lambda)}{(1+|\lambda-n^3|)^{1/2}}$$

$$\widehat{G}(n,\lambda) = \frac{c(n,\lambda)}{(1+|\lambda-n^3|)^{1/2}}$$

$$\widehat{H}(n,\lambda) = \frac{c(n,\lambda)}{1+|\lambda|^{\alpha}}.$$

Thus

$$(22) = \left\langle \left(\left(|n|^{1/2}\ \widehat{F}\right) * \left(|n|^{1/2}\ \widehat{G}\right), \widehat{H}\right\rangle \le \iint \left|D_x^{1/2}\ F\right|\ \left|D_x^{1/2}\ G\right|\ |H|\ dx\ dt$$
$$\le \|\ H\ \|_{L_x^2 L_t^\infty}\ \left\|D_x^{1/2}\ F\right\|_{L_x^4 L_t^2}\ \left\|D_x^{1/2}\ G\right\|_{L_x^4 L_t^2} \tag{23}$$

and we apply Lemma 3 for first factor, Lemma 1 to estimate second and third factor of (23).

Estimate on (I) (second term)

Estimate $\left(\iint_{|n|\le 1} |\widehat{w}(n,\lambda)|^2\ dn\ d\lambda\right)^{1/2}$ by duality, (13), since $|n| \le 1$

$$\int dn\ d\lambda\ dn_1\ d\lambda_1\ \frac{c(n_1,\lambda_1)\ c(n-n_1,\lambda-\lambda_1)\ d(n,\lambda)}{\left(1+|\lambda_1-n_1^3|\right)^{1/2}\ \left(1+|\lambda-\lambda_1-(n-n_1)^3|\right)^{1/2}}. \tag{24}$$

Put $\widehat{F} = \frac{c(n,\lambda)}{(1+|\lambda-n^3|)^{1/2}}$, $\widehat{G} = d(n,\lambda)$ be that $(24) = \left\langle \widehat{F} * \widehat{F}, \widehat{G}\right\rangle = \langle F^2, G\rangle \le \|\ F\ \|_4^2\ \|\ G\ \|_2$ and we apply Lemma 2.

Estimates on (II)

Proceed again by duality and take $\{d(n)_{>0}\}$ satisfying $\int d(n)^2 \le 1$. Consider the expression

$$\int dn\ d\lambda\ dn_1\ d\lambda_1\ \frac{|n|\ d(n)}{1+|\lambda-n^3|}\ \frac{c(n_1,\lambda_1)}{1+|\lambda_1-n_1^3|^{1/2}+|\lambda_1|\ \chi_{[|n_1|<1]}}\ \frac{c(n-n_1,\lambda-\lambda_1)}{\cdots}. \tag{25}$$

We assume again $|n-n_1| \ge |n_1|$ and distinguish the cases

(a) $$|n_1| \ge 1$$

(b) $$|n_1| < 1.$$

Case (a) is again analogous with periodic case. One has

$$\max\left(|\lambda-n^3|\ ,\ |\lambda_1-n_1^3|\ ,\ |\lambda-\lambda_1-(n-n_1)^3|\right) \ge |n|^2.$$

If $|\lambda-n^3| \ge n^2$, define

$$d(n,\lambda) = \frac{|n|\ d(n)}{|\lambda-n^3|+n^2} \tag{26}$$

clearly satisfying

$$\iint d(n,\lambda)^2 \le 1. \tag{27}$$

Letting $\widehat{F}(n,\lambda) = d(n,\lambda)$, $\widehat{G} = \frac{c(n,\lambda)}{1+|\lambda-n^3|^{1/2}}$, estimate (25) by $\left\langle \widehat{F}, \widehat{G} * \widehat{G}\right\rangle \le \|\ F\ \|_2\ \|\ G\ \|_4^2$ and Lemma 2.

If $|\lambda_1-n_1^3| > |n|^2$, estimate (25) by

$$\int dn\, d\lambda\, dn_1\, d\lambda_1\, \frac{d(n,\lambda)}{(1+|\lambda-n^3|)^{9/20}}\, \frac{c(n-n_1,\lambda-\lambda_1)}{1+|\lambda-\lambda_1-(n-n_1)^3|^{1/2}}\, c(n_1,\lambda_1) \tag{28}$$

where we denote

$$d(n,\lambda) = \frac{d(n)}{(1+|\lambda-n^3|)^{11/20}} \tag{29}$$

hence

$$\iint d(n,\lambda)^2\, dn\, d\lambda < c. \tag{30}$$

The factor 9/20 instead of 1/2 suffices to apply Lemma 2, since only $\rho > \frac{1}{3}$ is required.

Case (b). Define $d(n,\lambda)$ by (29) and estimate (25) by

$$\int dn\, d\lambda\, dn_1\, d\lambda_1\, \frac{|n|^{1/2}\, d(n,\lambda)}{(1+|\lambda-n^3|)^{9/20}}\, \frac{|n-n_1|^{1/2}\, c(n-n_1,\lambda-\lambda_1)}{(1+|\lambda-\lambda_1-(n-n_1)^3|)^{1/2}}\, \frac{c(n_1,\lambda_1)}{1+|\lambda_1|^{\alpha}}. \tag{31}$$

This expression is similar to (22), except for the exponent $\frac{9}{20}$ instead of $\frac{1}{2}$. However, the application of Lemma 1 only requires exponent $> \frac{1}{4}$.

The conclusion is an estimate by $C\|u\|^2$. If one considers a difference

$$\partial_x(u^2) - \partial_x(v^2) = \partial_x(u-v)(u+v),$$

the estimate

$$\|u+v\|\ \|u-v\| \le (\|u\| + \|v\|)\ \|u-v\|$$

follows.

Moreover, it is possible to gain a factor $\delta_1 \to 0$ by letting the interval $I = [-\delta, \delta]$ be small enough, using the same argument as in the periodic case. This yields the contraction property of the transformation.

REFERENCES

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