ON THE RADIAL VARIATION OF BOUNDED ANALYTIC FUNCTIONS ON THE DISC

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0. Introduction and statement of results.

Let F be a bounded analytic function on the disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Following [Ru], define

$$W(F, r, \theta) = \int_0^r \left|F'(\rho e^{i\theta})\right| d\rho \ (r < 1) \quad ; \quad V(F, \theta) = W(F, 1, \theta).$$

The quantity $V(F, \theta)$ corresponds of course to the variation of F on the radius of D terminating at the point $e^{i\theta}$. It is shown in [R] that the set

$$\{\theta \mid V(F, \theta) < \infty\}$$

may be of measure zero (and of first category). In fact, as shown in [R], F may be taken to be a Blaschke product or an element of the disc algebra (*i.e.* continuous up to the boundary). The problem left open in [R] is whether $V(F, \theta)$ may be infinite in *any* direction. The purpose of the note is to disprove this. More precisely

Theorem 1. *The set $\{\theta \mid V(F, \theta) < \infty\}$ is nonempty and in fact is of Hausdorff dimension 1, whenever F is bounded analytic, i.e. $F \in H^\infty(D)$.*

Let us recall at this point Zygmund's result [Zyg]

$$W(F, r, \theta) = 0\left(\log^{\frac{1}{2}} \frac{1}{1-r}\right) \qquad r \to 1$$

almost everywhere in θ. This statement is optimal, also assuming $F \in H^\infty(D)$.

Our method also yields

Theorem 2. *The statement of Theorem 1 holds assuming F a bounded* real *harmonic function on D.*

Theorem 2 has a martingale counterpart for real bounded dyadic martingales, obtained by taking conditional expectations *w.r.t.* the natural filtration on the Cantor group $G = \{1, -1\}^{\mathbb{N}}$. Writing

$$F(\varepsilon) = \sum_{k=1}^{\infty} \Delta_k F(\varepsilon_1, \ldots, \varepsilon_{k-1})\ \varepsilon_k \qquad \varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \in G$$

it is easy to see that at least for some $\varepsilon \in G$

$$\sum_{k=1}^{\infty} |\Delta_k F(\varepsilon_1, \ldots, \varepsilon_{k-1})| \leq |F(\varepsilon)|$$

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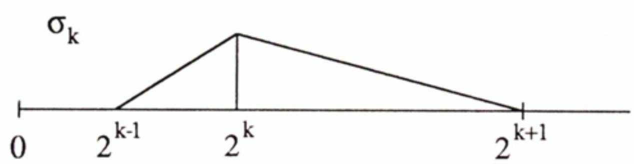
holds. In fact, N. Makarov showed the author an argument yielding a measure $\mu = \mu_{F,\varepsilon}$ on G such that $\dim \operatorname{supp} \mu > 1 - \varepsilon$ and $\int \left(\sum_k |\Delta_k F| \right) d\mu < \infty$, for all $\varepsilon > 0$.

If one tries to exploit the idea of selecting a "good" branch in a tree, which does the trick in the martingale setting, the existence is shown of a path γ is D connecting 0 with ∂D, so that $\int_\gamma |\nabla F| < \infty$, assuming $F \in H^\infty(D)$. The main problem is to replays γ by a straight line segment. Our approach here is Harmonic Analysis, *i.e.* we consider the boundary value $f(\theta) = F\left(e^{i\theta}\right)$ of F which we write as a Fourier series $\sum_{n \geq 0} \widehat{f}(n) e^{in\theta}$. Roughly speaking, estimating $V(F, \theta)$ may be achieved from an ℓ^1-estimate on the Littlewood-Paley decomposition, thus

$$\sum_k \left| \sum_n \sigma_k(n)\, \widehat{f}(n)\, e^{in\theta} \right| \tag{$*$}$$

where the $\{\sigma_k\}$ are usual dyadic multipliers.



Bounding expressions of the form $(*)$ will be our main concern below. A key point is the trivial fact that if z_1, z_2 are complex numbers, one may always find ψ such that $|z_1 + e^{i\psi} z_2| = |z_1| + |z_2|$; this is also the place where analyticity is exploited.

Finally, let us observe that Theorem 1 (and obviously Theorem 2) are "scalar". In fact P. Jones [J] constructed a pair $F_1, F_2 \in H^\infty(D)$ satisfying

$$\int_\gamma \left(|F_1'(s)| + |F_2'(s)| \right) ds = \infty$$

whenever γ is a path joining 0 with ∂D. This construction was useful to settle (negatively) some problems in complex geometry (the existence of a bounded complete $\mathbb{C}^3$-manifold).

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1. Construction of convolution kernels K_N.

Let $N = 1, 2, \ldots$.

Denote $F_N(t) = \sum_{|n|<N} \frac{N-|n|}{N} e^{2\pi int}$ the usual Fejer kernel on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.

Define

$$K_N = F_N * \frac{\chi_1}{10N} * \frac{\chi_1}{10N} \quad \text{where} \quad \chi_\varepsilon = \frac{1}{\varepsilon}\, \chi_{[-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]}.$$

The following properties are easily checked and we omit details

(1) $K_N \geq 0 \quad , \quad \int K_N = 1$

(2) $K_N(t) \sim \frac{1}{N(\|t\|^2+N^{-2})}$

(3) $|K_N'| \lesssim N K_N$ (pointwise)

(4) For $N_1 > N$

$$\int |(K_N)_\tau - K_N| \, K_{N_1}(\tau) d\tau \lesssim \frac{N}{N_1} \log\left(2 + \frac{N_1}{N}\right) K_N \quad \text{(pointwise)}$$

(5) For $N_1 > N$

$$|(K_N * K_{N_1}) - K_N| \lesssim \left(\frac{N}{N_1}\right)^{1/2} K_N \quad \text{(pointwise)}.$$

In particular

$$K_N * K_N \lesssim K_N \tag{6}$$

(7) $\widehat{K}_N \geq 0$ and $\widehat{K}_N(n) > \frac{1}{20}$ for $|n| < \frac{7}{8}N$

(8) $\widehat{K}_N$ is the restriction to $\mathbb{Z}$ of a function of $\mathbb{R}$ which derivative has variation $\lesssim \frac{1}{N}$.

2. Induction hypothesis.

Define for $k = 1, 2, \ldots$

$$f_{(k)} = f * (K_{2^k} * K_{2^k}) * (K_{2^{k+1}} * K_{2^{k+1}}) * \cdots \tag{9}$$

where f is the (boundary value of) the given H^∞-function.

We will construct inductively points $\theta_k \in \mathbb{T}$ and sequences $c_j^{(k)}$ $(j \leq k)$ of positive numbers (which will stay away from zero), satisfying following inequality

$$\sum_{j=0}^{k} c_j^{(k)} \left(\left| f_{(j)} * \left(e^{2\pi i 2^{j-1} \cdot} K_{2^{j-1}} \right) \right| * K_{2^{j-1}} \right) (\theta_k) \leq \left| f_{(k)}(\theta_k) \right|. \tag{10}$$

We will show later how to estimate the L^1-norm of the radial derivative in θ_k-direction by the left member of (10).

3. Induction step.

One has $\operatorname{supp} \widehat{f} \subset \mathbb{Z}_+$, $\operatorname{supp} \widehat{f}_{(k+1)} \subset \mathbb{Z}_+$, $\operatorname{supp} \widehat{K}_{2^k} \subset \operatorname{supp} \widehat{F}_{2^k} \subset]-2^k, 2^k[$. Thus one may write for given θ, ψ

$$\begin{aligned} &\int f_{(k+1)}(\eta) \left[K_{2^k}(\theta - \eta) \left(1 + \cos(2\pi 2^k(\theta - \eta) + \psi)\right)\right] d\eta = \\ &\left(f_{(k+1)} * K_{2^k}\right)(\theta) + \frac{1}{2}\, e^{i\psi} \left(f_{(k+1)} * \left(e^{2\pi i 2^k \cdot} K_{2^k} \right)\right)(\theta). \end{aligned} \tag{11}$$

Thus given θ, there is ψ_θ satisfying

$$\begin{aligned} &\int |f_{k+1}(\eta)| \left[K_{2^k}(\theta - \eta) \left(1 + \cos(2\pi 2^k(\theta - \eta) + \psi_\theta)\right)\right] d\eta \geq \\ &|(f_{k+1} * K_{2^k})(\theta)| + \frac{1}{2} \left| \left(f_{k+1} * \left(e^{2\pi i 2^k \cdot} K_{2^k} \right)\right)(\theta) \right|. \end{aligned} \tag{12}$$

Multiply both sides of (12) with $K_{2^k}(\theta_k - \theta)$ and integrate in θ. It follows

$$\int |f_{k+1}(\eta)|\, L(\theta_k, \eta) d\eta \geq \tag{13}$$

$$|(f_{k+1} * K_{2^k} * K_{2^k})\,(\theta_k)| + \tag{14}$$

$$\frac{1}{2}\left(\left|f_{k+1} * \left(e^{2\pi i 2^k \cdot} K_{2^k}\right)\right| * K_{2^k}\right)(\theta_k) \tag{15}$$

denoting

$$L(\theta_k, \eta) = \int K_{2^k}(\theta_k - \theta) K_{2^k}(\theta - \eta)\left(1 + \cos(2\pi 2^k(\theta - \eta) + \psi_\theta)\right) d\theta. \tag{16}$$

Clearly

$$L(\theta, \eta) \geq 0 \quad , \quad \int_{\mathbb{T}} L(\theta, \eta) d\eta = 1. \tag{17}$$

By induction hypothesis

$$(14) = |f_k(\theta_k)| \geq \sum_{j=0}^{k} c_j^{(k)} \left(\left|f_j * \left(e^{2\pi i 2^{j-1} \cdot} K_{2^{j-1}}\right)\right| * K_{2^{j-1}}\right)(\theta_k). \tag{18}$$

Denote for simplicity

$$g_j = f_{(j)} * \left(e^{2\pi i 2^{j-1} \cdot} K_{2^{j-1}}\right). \tag{19}$$

We claim that

$$\int (|g_j| * K_{2^{j-1}})(\eta)\ L(\theta_k, \eta) d\eta \leq \left(1 + c_0\ 2^{-\frac{k-j}{2}}\right)(|g_j| * K_{2^{j-1}})(\theta_k). \tag{20}$$

One has indeed by (17)

$$\begin{aligned} &\int (|g_j| * K_{2^{j-1}})(\eta)\ L(\theta_k, \eta) d\eta - (|g_j| * K_{2^{j-1}})(\theta_k) \leq \\ &\int |(|g_j| * K_{2^{j-1}})(\eta) - (|g_j| * K_{2^{j-1}})(\theta_k)|\ \ L(\theta_k, \eta) d\eta \leq \\ &\int\int |g_j|(x)\, |K_{2^{j-1}}(\eta - x) - K_{2^{j-1}}(\theta_k - x)|\ \ L(\theta_k, \eta) dx\ d\eta. \end{aligned} \tag{21}$$

Now it clearly follows from (16), (6) that

$$L(\theta_k, \eta) \leq 2(K_{2^k} * K_{2^k})(\theta_k - \eta) \leq c_1\ K_{2^k}(\theta_k - \eta). \tag{22}$$

Substitute (22) in (21) and apply (4) with $N = 2^{j-1}$, $N_1 = 2^k$ at the point $\theta_k - x$

$$\Rightarrow \int |K_{2^{j-1}}(\eta - x) - K_{2^{j-1}}(\theta_k - x)|\ \ L(\theta_k, \eta) d\eta \leq c_2\ 2^{-\frac{k-j}{2}} K_{2^{j-1}}(\theta_k - x) \tag{23}$$

and

$$(21) \leq c_2\ 2^{-\frac{k-j}{2}}\,(|g_j| * K_{2^{j-1}})(\theta_k)$$

proving inequality (20).

Similarly

$$\int (|g_{k+1}| * K_{2^k})(\eta)\ L(\theta_k,\eta)d\eta \le c_3\,(|g_{k+1}| * K_{2^k})(\theta_k). \tag{24}$$

Now collect inequalities (13)-(14)-(15); (18); (20); (24) to get

$$\int |f_{k+1}(\eta)|\ \ L(\theta_k,\eta)d\eta \ge$$

$$\int \left\{ \sum_{j=0}^{k} \left(1 + c_0\ 2^{-\frac{k-j}{2}}\right)^{-1} c_j^{(k)}\,(|g_j| * K_{2^{j-1}})(\eta) + \frac{1}{2c_3}\,(|g_{k+1}| * K_{2^k})(\eta) \right\} L(\theta_k,\eta)d\eta. \tag{25}$$

Thus putting

$$\left.\begin{array}{ll} c_j^{(k+1)} = c_j^{(k)} \left(1 + c_0\ 2^{-\frac{k-j}{2}}\right)^{-1} & j \ge k \\ c_{k+1}^{(k+1)} = \frac{1}{2c_3} & \end{array}\right\} \tag{26}$$

(25) implies in particular the existence of a point $\theta_{k+1} \in \mathbf{T}$ such that

$$|f_{k+1}(\theta_{k+1})| \ge \sum_{j \le k+1} c_j^{(k+1)}\,(|g_j| * K_{2^{j-1}})(\theta_{k+1}). \tag{27}$$

It is obvious from the way the $c_j^{(k)}$ are obtained from (26) that

$$c_j^{(k)} > c_4 > 0 \quad \text{for all} \quad j \le k.$$

4. Estimating the radial derivative in some direction.

Assume $\| f \|_\infty \le 1$. It then follows from the preceding that each of the closed subsets of $\mathbf{T}$

$$\left\{ \theta \in \mathbf{T}\ ;\ \sum_{j=0}^{k} c_4 \left(\left| f_{(j)} * \left(e^{2\pi i 2^{j-1}\cdot} K_{2^{j-1}} \right) \right| * K_{2^{j-1}} \right)(\theta) \le 1 \right\} \tag{28}$$

is nonempty. Let θ be a common point. Hence

$$\sum_{j=0}^{\infty} \left(\left| f_{(j)} * \left(e^{2\pi i 2^{j-1}\cdot} K_{2^{j-1}} \right) \right| * K_{2^{j-1}} \right)(\theta) < \infty. \tag{29}$$

We claim that

$$\int_0^1 \left| F'(re^{i\theta}) \right| dr < \infty \tag{30}$$

where

$$F(re^{i\theta}) = (f * P_r)(\theta) \qquad P_r = \tfrac{1-r^2}{1-2r\cos\theta + r^2} \text{ in the Poisson kernel.}$$

From (9), one has

$$f_{(j)} * \left(e^{2\pi i 2^{j-1}\cdot} K_{2^{j-1}} \right) = f * X_j$$

where

$$X_j = (K_{2^j} * K_{2^j}) * (K_{2^{j+1}} * K_{2^{j+1}}) * \cdots * \left(e^{2\pi i 2^{j-1}} K_{2^{j-1}}\right) \tag{31}$$

hence

$$\widehat{X}_j(n) = \prod_{j' \geq j} \widehat{K}_{2^{j'}}(n)^2 \cdot \widehat{K}_{2^{j-1}}(n - 2^{j-1}). \tag{32}$$

Thus (*cf.* (7))

$$\widehat{X}_j(n) > c_5 > 0 \quad \text{on} \quad \left[\frac{1}{4}\, 2^{j-1}, \frac{3}{2}\, 2^{j-1}\right]. \tag{33}$$

Considering $\widehat{X}_j$ as a function on $\mathbb{R}$, one gets from (8), (32) and product differentiation rule the estimates(*)

$$\left|\partial_n \widehat{X}_j\right| \lesssim \sum_{j' \geq j-1} \frac{1}{2^{j'}} \sim 2^{-j} \tag{34}$$

$$\text{Var } \partial_n\, \widetilde{X}_j \lesssim \sum_{j' \geq j-1} \text{Var } \partial_n\, \widehat{K}_{2^{j'}} + 2^j \sum_{j_1, j_2 \geq j} \frac{1}{2^{j_1} 2^{j_2}} \sim 2^{-j}. \tag{35}$$

It follows in particular from (33), (34), (35) that there is a multiplier Y_j satisfying

$$\widehat{Y}_j = \left(\widehat{X}_j\right)^{-1} \quad \text{on} \quad \left[\frac{1}{4}\, 2^{j-1}, \frac{3}{2}\, 2^{j-1}\right] \tag{36}$$

$$|Y_j(t)| \leq \frac{c_6}{2^j\, (\| t \|^2 + 4^{-j})}. \tag{37}$$

Inequality (30) is equivalent to

$$\int_0^1 \left| \sum_{n \geq 0} n \widehat{f}(n)\, r^{n-1}\, e^{2\pi i n \theta} \right| dr < \infty. \tag{38}$$

Consider multipliers $\{\sigma_{(j)}\}$ satisfying

$$\text{supp}\, \sigma_{(j)} \subset \left[\frac{1}{4}\, 2^{j-1}, \frac{3}{2}\, 2^{j-1}\right] \tag{39}$$

$$0 \leq \sigma_{(j)}(n) \leq 1 \quad , \quad \sum_j \sigma_{(j)}(n) = 1 \quad \forall n \tag{40}$$

$$\left|\partial_n^{(s)} \sigma_{(j)}\right| \leq c_7\, 2^{-js} \quad (s = 1, 2) \tag{41}$$

(obvious derivative estimates).

Estimate the left member of (38) as

$$\sum_j 2^j r^{\frac{1}{10} 2^j} \left| f * \left[\sum_n \sigma_{(j)}(n) \cdot \frac{n}{2^j}\, r^{n - \frac{1}{10} 2^j} e^{2\pi i n \theta} \right] \right|. \tag{42}$$

Denoting $Z = Z_{j,r}$ the expression in (42) between [ ], one has by (39), (41)

$$\left.\begin{aligned} \left|\partial_n \widehat{Z}\right| &\leq 2c_7\, 2^{-j} + 2^{-j} + 2 r^{n - \frac{1}{10} 2^j} \log \tfrac{1}{r} \lesssim 2^{-j} \\ \left|\partial_n^2 \widehat{Z}\right| &\lesssim 4^{-j} + r^{n - \frac{1}{10} 2^j} \left(\log \tfrac{1}{r}\right)^2 \lesssim 4^{-j} \end{aligned}\right\} \tag{43}$$

(*) ∂_n refers to the derivative with n considered as $\mathbb{R}$ (rather than $\mathbb{Z}$) variable.

implying again a shape

$$|Z_{j,r}(t)| \leq \frac{c_8}{2^j\,(\|\,t\,\|^2 + 4^{-j})} \quad \text{(independent of } r\text{)}. \tag{44}$$

Write in account of (39), (36)

$$\begin{aligned} f * Z_{j,r} &= (f * X_j) * Y_j * Z_{j,r} \\ |f * Z_{j,r}| &\leq \left| f_{(j)} * \left(e^{2\pi i 2^{j-1}\cdot} K_{2^{j-1}} \right) \right| * |Y_j| * |Z_{j,r}| \quad \text{(pointwise)}. \end{aligned} \tag{45}$$

It follows from (37), (44), (2) the pointwise inequality

$$|Y_j| * |Z_{j,r}| \lesssim K_{2^{j-1}}. \tag{46}$$

Hence

$$(42) \lesssim \sum_j 2^j r^{\frac{1}{10}2^j} \left[\left| f_{(j)} * \left(e^{2\pi i 2^{j-1}\cdot} K_{2^{j-1}} \right) \right| * K_{2^{j-1}} \right](\theta) \tag{47}$$

and since

$$\sup_j \int_0^1 \left(2^j r^{\frac{1}{10}2^j} \right) dr < \infty$$

(38) is implied by (29).

At this point, we proved the existence of at least one direction where (30) holds.

5. Dimension estimates.

The points $\theta \in \mathbf{T}$ satisfying (30) form a set of dimension 1.

We first explicit more the distribution of "good points" according to previous method.

Observe that for a fixed $\overline{k}$ the point $\theta_{\overline{k}}$ is choosen according to the density

$$\Omega_{\overline{k}}(\eta) = \int L_1(\theta_1, \theta_2) L_2(\theta_2, \theta_3) \cdots L_{\overline{k}-1}(\theta_{\overline{k}-1}, \eta) d\theta_1 d\theta_2 \cdots d\theta_{\overline{k}-1} \tag{48}$$

where (*cf.* (16))

$$L_k(\theta', \theta'') = \int K_{2^k}(\theta' - \theta) K_{2^k}(\theta - \theta'')(1 + \cos(2\pi 2^k(\theta - \theta'') + \psi_\theta) d\theta. \tag{49}$$

This easily follows from the description of the inductive step above.

Now

$$L_k(\theta', \theta'') \leq 2(K_{2^k} * K_{2^k})(\theta' - \theta'') \tag{50}$$

and substituting in (48)

$$\Omega_{\overline{k}}(\eta) \leq 2^{\overline{k}-1} \int (K_{2^1} * K_{2^1})(\theta_1 - \theta_2)(K_{2^2} * K_{2^2})(\theta_2 - \theta_3) \cdots (K_{2^{\overline{k}-1}} * K_{2^{\overline{k}-1}})(\theta_{\overline{k}-1} - \eta) d\theta_1 \cdots d\theta_{\overline{k}-1}$$

$$\Rightarrow \Omega_{\overline{k}} = |\Omega_{\overline{k}}| \lesssim 2^{\overline{k}}. \tag{51}$$

The average with respect to $\Omega_{\overline{k}}$ of

$$\sum_{j\leq \overline{k}} \left| f_{(j)} * \left(e^{2\pi i 2^{j-1}} K_{2^{j-1}}\right)\right| * K_{2^{j-1}} \tag{52}$$

remains uniformly bounded.

Choose a fixed parameter $0 < \varepsilon \leq 1$ ($\varepsilon \to 0$) and modify the definition of L_k as

$$L_k(\theta', \theta'') = \int K_{2^k}(\theta' - \theta) K_{2^k}(\theta - \theta'') \left(1 + \varepsilon \cos\left(2\pi \left(2^k + \left[\frac{2^k}{10}\right]\right)(\theta - \theta'') + \psi_\theta\right)\right) d\theta \tag{53}$$

instead of (49), so that in particular

$$L_k(\theta', \theta'') \leq (1 + \varepsilon)(K_{2^k} * K_{2^k})(\theta' - \theta''). \tag{54}$$

Replace (52) by

$$\sum_{j\leq \overline{k}} \left| f_{(j)} * \left(e^{2\pi i \left(2^{j-1} + \left[\frac{2^{j-1}}{10}\right]\right)\cdot} K_{2^{j-1}}\right)\right| * K_{2^{j-1}}. \tag{55}$$

The effect of ε is just a different $c_4 = c_4(\varepsilon)$ constant (above) and hence a different bound (only depending on ε) for

$$\int_{\mathbf{T}} (55) \cdot \Omega_{\overline{k}}\,. \tag{56}$$

The $\left[\frac{2^{j-1}}{10}\right]$-translation is introduced for a technical reason and has no effect in bounding (30) by

$$\sum_{j=0}^{\infty} \left| f_{(j)} * \left(e^{2\pi i \left(2^{j-1} + \left[\frac{2^{j-1}}{10}\right]\right)\cdot} K_{2^{j-1}}\right)\right| * K_{2^{j-1}} \tag{57}$$

(in particular (33) is preserved).
Denote μ the probability measure obtained as w^*-limit of $\{\Omega_{\overline{k}}\}$ in $M(\mathbf{T})$(*). From (56)

$$\int (57) d\mu < \infty \tag{58}$$

(54) permits to replace (51) by

$$\Omega_{\overline{k}} \leq (1 + \varepsilon)^{\overline{k}-1}. \tag{59}$$

Fix k and let γ be a trigonometric polynomial, $\operatorname{supp}\widehat{\gamma} \subset \left[\frac{-2^k}{20}, \frac{2^k}{20}\right]$. One then has

$$\left|\int \gamma d\mu\right| \lesssim (1+\varepsilon)^k \parallel \gamma \parallel_{L^1(\mathbf{T})}\,. \tag{60}$$

To check this fact, one uses formula (48) and (53), with translated frequency $2^k + \left[\frac{2^k}{10}\right]$. One finds

$$\left|\int \gamma d\mu\right| \lesssim \int \Omega_k(\eta)\,[|\gamma| * (K_{2^k} * K_{2^k}) * (K_{2^{k+1}} * K_{2^{k+1}}) * \cdots](\eta) d\eta \tag{61}$$

and hence (60) by (59).

From (60), one deduce that dim $\operatorname{supp}\mu \geq 1 - \frac{\log(1+\varepsilon)}{\log 2}$. Since (58) holds, this clearly completes the proof of Theorem 1.

(*) The space of Radon measures on $\mathbf{T}$.

6. Proof of Theorem 2.

Let F be a bounded real harmonic function on D and $f(t) = \sum_{n=-\infty}^{\infty} \widehat{f}(n)\, e^{2\pi int}$ its boundary value. Thus $\widehat{f}(-n) = \overline{\widehat{f}(n)}$, $n \in \mathbb{Z}$. The only significant change is the replacement of (11) by

$$\int f_{(k+1)}(\eta)\left[K_{2^k}(\theta-\eta)(1+\cos(2\pi 2^k(\theta-\eta)+\psi))\right] d\eta =$$
$$\left(f_{(k+1)} * K_{2^k}\right)(\theta) + \frac{1}{2}\,\mathrm{Re}\left\{e^{i\psi}\left(f_{(k+1)} * \left(e^{2\pi i 2^k \cdot} K_{2^k}\right)\right)(\theta)\right\} \tag{62}$$

which still allows one to write (12) for a suitable $\psi = \psi_\theta$.

In fact, J. Garnette pointed out to the author that Theorem 2 is a formal consequence of Theorem 1, considering the H^∞ function $F = e^{f+i\widetilde{f}}$, where $\widetilde{f}$ is the conjugate of f.

Observe that the conclusion of Theorem 1 also holds if the function is obtained as the Riesz projection of a bounded real harmonic function.

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