RETURN TIMES OF DYNAMICAL SYSTEMS

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ABSTRACT : Let $(\Omega, \mathcal{B}, \mu, T)$ be a dynamical system (DS), μ finite, T ergodic and $A \in \mathcal{B}$ of positive measure $\mu(A) > 0$. The main result is that for almost all $\omega \in \Omega$, the return time sequence $\Lambda_\omega = \{n \in \mathbb{Z}_+ | T^n\omega \in A\}$ is a good sequence for the pointwise ergodic theorem. As in [B1], methods of Fourier Analysis are used. This is perhaps not surprising because of the closed connection between this problem and the Wiener-Wintner ergodic theorem. Some of the maximal function estimates have further applications, namely in the context of [B1],[B2],[B3] on the pointwise ergodic theorem for "arithmetic" sets. The main results of this paper were announced in [B5].

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1. STATEMENT

Definition. *$\Lambda \subset \mathbf{Z}_+$ is a good sequence for the mean (resp. pointwise) ergodic theorem provided for any DS $(\Omega, \mathcal{B}, \mu, T)$ and $f \in L^1(\mu)$,*

$$\frac{1}{|\Lambda_N|} \sum_{n \in \Lambda_N} T^n f \text{ converges in the mean (resp. almost surely)}$$

where $\Lambda_N = \Lambda \cap [0, N]$.

Remark. *If Λ has a density $d(\Lambda) > 0$, this is equivalent to mean (resp. almost sure) convergence of $\frac{1}{N}\sum_{n\in\Lambda_N} T^n f$ for $f \in L^1(\mu) \cap L^\infty(\mu)$.*

RETURN-TIME SEQUENCES : Let $(\Omega, \mathcal{B}, \mu, T)$ be DS , μ probability-measure, T ergodic. Fix $A \in \mathcal{B}, \mu(A) > 0$ and define

$$\Lambda_\omega = \{n \in \mathbf{Z}_+ | T^n\omega \in A\}$$

It follows from Birkhoff's theorem that *a.s* , $\Lambda = \Lambda_\omega$ has density

$$d(\Lambda) = \lim \frac{|\Lambda_N|}{N} = \lim \frac{1}{N} \sum_{n \leq N} (T^n \chi_A)(\omega) = \mu(A)$$

Theorem. *Λ_ω satisfies a.s. the pointwise ergodic theorem*

Remarks. *The fact that Λ_ω satisfies a.s. the mean ergodic theorem is a consequence of the* **Wiener-Wintner ergodic theorem** *(see next section for a brief discussion)*

- *Two particular cases where known (cf. [B-L]):*
- *T has discrete spectrum*
- *T has Lebesgue spectrum (see [B-L] for details)*

2. The Wiener-Wintner Theorem.

This is the following refinement of Birkhoff's theorem

Theorem. *Let $g \in L^\infty(\mu)$. There is a set $\Omega' \subset \Omega$ of full measure such that*

$$\frac{1}{N} \sum_{n=1}^{N} z^n T^n g(\omega) \tag{1}$$

converges for all $\omega \in \Omega'$ and $z \in \mathbf{C}_1$ = unit circle ($\leftrightarrow \mathbb{T} = \mathbf{R}/\mathbb{Z}$)

The proof only uses spectral-theory. It follows from Birkhoff's theorem that *a.s.* in ω

$$(2) \qquad \frac{1}{N}\Big|\sum_{n=1}^{N} z^n T^n g(\omega)\Big|^2 \longrightarrow \lambda_g \quad (\omega^*)$$

where $\lambda_g \in M_+(\Pi)$ is defined by $\hat{\lambda}_g(k) = < g, T^k g >$

Affinity Principle : Let $\mu, \nu \in M_+(\Pi)$ and (f_n), (g_n) sequences in $L^1_+(\Pi)$ such that $f_n \to \mu, g_n \to \nu(\omega^*)$. If $\mu \perp \nu$, then

$$\int f_n^{1/2} g_n^{1/2} \to 0.$$

Write indeed

$$\frac{1}{N}\sum_{n=1}^{N} z^n T^n g(\omega) = \int_0^1 < \frac{1}{\sqrt{N}}\sum_{n\leq N} z^n e^{2\pi i n\theta} , \frac{1}{\sqrt{N}}\sum_{n\leq N} T^n g(\omega) e^{2\pi i n\theta} > d\theta$$

$$\frac{1}{N}\Big|\sum_{n\leq N} z^n T^n g(\omega)\Big| \leq \int \frac{1}{\sqrt{N}}\Big|\sum_{n\leq N} z^n e^{2\pi i n\theta}\Big|\frac{1}{\sqrt{N}}\Big|\sum_{n\leq N} T^n g(\omega) e^{2\pi i n\theta}\Big| d\theta.$$

where

$$\frac{1}{N}\Big|\sum_{n\leq N} z^n e^{2\pi i n\theta}\Big|^2 \longrightarrow \delta_z(\omega^*).$$

Hence, if ω satisfies (2), then the limit is 0, except if z is in the atomic part of λ_g, hence in the (point) spectrum of T. Since the point-spectrum of T is at most countable, it remains to impose the convergence of (1) for z in a fixed countable set. Again by the ergodic theorem, this will be satisfied for ω in a set of measure 1.

Remark. *If T has no nontrivial point-spectrum, then a.s. in ω*

$$\frac{1}{N}\sum_{n\leq N} z^n T^n g(\omega) \longrightarrow 0 \text{ for } z \in \mathbf{C}_1 \setminus \{1\}.$$

If moreover $\int g = 0$, then a.s. is ω

$$\frac{1}{N}\sum_{n\leq N} z^n T^n g(\omega) \longrightarrow 0 \text{ uniformly for } z \in \mathbf{C}_1.$$

From spectral theory, Λ will be a good sequence for mean convergence (w.r.t. L^2-functions) provided the associate sequence of polynomials

$$\frac{1}{|\Lambda_N|}\sum_{n\in\Lambda_N} z^n$$

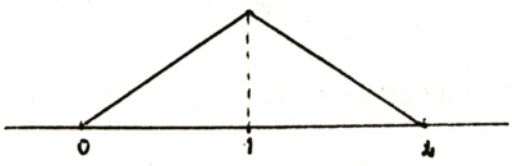
pointwise converges for $|z| = 1$. For $\Lambda = \Lambda_\omega$, we are considering the sequence

$$p_N(z) = \frac{1}{N}\sum_{n\leq N} T^n \chi_A(\omega) z^n.$$

Consequently, from the Wiener-Wintner theorem

Corollary: $\Lambda = \Lambda_\omega$ *is a good sequence for the mean ergodic theorem, a.s.*

We will use this fact in the following way. Let σ be the following function on $\mathbf{R}$



Denote for $t > 0\,, \sigma_t(x) = \frac{1}{t}\sigma(\frac{x}{t})$. The reader will easily verify

Lemma: $\chi_{[0,1]}$ *is in norm-closure of* $[\sigma_t | t > 0]$.

Assume $\Lambda, d(\Lambda) > 0$ satisfies the mean ergodic theorem. Let $(\Omega, \mathcal{B}, \mu, T)$ be $D.S.$ and $|f| \leq 1$. Assume each of the sequences, $r = 1, 2, \ldots$

$$\{\frac{1}{r2^s}\sum_{n\in\Lambda}\sigma(\frac{n}{r2^s})T^n f | s = 0, 1, 2, \ldots\} \tag{3}$$

converges $a.s.$ Since, from hypothesis

$$\{\frac{1}{N}\sum_{n\in\Lambda}\sigma(\frac{n}{N})T^n f | N = 1, 2, \ldots\} \tag{4}$$

converges in the mean, the limit of the sequences (3) are the same and hence also (4) converges $a.s.$ This remains true when r is replaced by $\sigma_t (t > 0)$ (even with same limit as (4)). Hence the lemma clearly implies $a.s$ convergence of

$$\frac{1}{N}\sum_{n\in\Lambda}\chi_{[0,1]}(\frac{n}{N})T^n f,$$

which are the usual ergodic averages.

From this discussion it appears that it suffices to show that for almost all ω the return time sequence $\Lambda = \Lambda_\omega$ satisfies

$$2^{-s}\sum_{n\in\Lambda}\sigma(2^{-s}n)T^n f \quad \text{is a.s. convergent} \tag{5}$$

for any $D.S.\,(\Omega, \mathcal{B}, \mu, T)\,, f \in L^\infty(\Omega)$.

In order to have (5) for any $D.S.$, it suffices to consider only the shift $(\mathbf{Z}, S)$ provided the convergence property is expressed as an inequality on the variation of the sequence. In the context of the shift, harmonic analysis methods may be used. In particular, by the Fourier-inversion formula

$$2^{-s}\sum_{n\in\Lambda}\sigma(2^{-s}n)S^n f = f*(2^{-s}\sum_{n\in\Lambda}\sigma(2^{-s}n)\delta_{\{n\}})$$
$$= \mathcal{F}^{-1}[\varphi_s \mathcal{F}f]$$

where (φ_s) is the sequence of multipliers on Π

(6)
$$\varphi_s(\theta) = 2^{-s}\sum_{n\in\Lambda}\sigma(2^{-s}n)e^{2\pi i n\theta}$$

and
$$\mathcal{F}f(\theta) = \sum_{x\in\mathbf{Z}} f(x)e^{-2\pi i x\theta}$$

is the Fourier transform.

Here $\Lambda = \Lambda_\omega = \{n \in \mathbf{Z}_+ | T^n\omega \in A\}$ and φ_s is thus also

(7)
$$\varphi_s(\theta) = 2^{-s}\sum \chi_A(T^n\omega)\sigma(2^{-s}n)e^{2\pi i n\theta}.$$

It will be shown that for most ω the sequence (φ_s) is behaving such that a (quantitative) convergence property for $\mathcal{F}^{-1}[\varphi_s \mathcal{F}f]$ holds.

Remark. *At this point, methods from spectral theory do not seem to suffice to obtain the theorem in the general case. Harmonic analysis methods will be exploited in the context of the shift but now estimates are needed in order to derive a.s. convergence of (5) for general D.S., from statements related to* $(\mathbf{Z}, S)$.

- The function σ will appear in an averaging argument, when passing from the averages on a sequence of diadic intervals to a convolution operator.

3. FIRST REDUCTION TO SHIFT

Introduce the following definition. Let $\Phi = (\varphi_j)$ be a sequence of multipliers on Π, N a positive integer. Let $B_\tau^{(N)}(\Phi)$ be the largest integer r (possibly ∞) s.t. there is function f on $\mathbf{Z}$, $|f| \leq 1$, $supp\, f \subset [0, N]$ and integers $j_1 < \cdots < j_r$ satisfying

(1)
$$\| \sup_{j_{s-1}\leq j\leq j_s} |\mathcal{F}^{-1}[(\varphi_j - \varphi_{j_{s-1}})\mathcal{F}f)|\|_2 > \tau N^{1/2}.$$

The idea here is to quantizise convergence.

The following remark is straightforward and will be used later.

Lemma. *Assume a collection of multiplier sequences $\Phi = (\varphi_j)$ given, satisfying a uniform maximal inequality*

(2) $$\|\sup_{\varphi\in\Phi}[\mathcal{F}^{-1}[\varphi\mathcal{F}f]|\|_2 \leq c\|f\|_2 \qquad \forall\Phi.$$

Let $\Psi = (\Psi_j)$ be a convex combination, i.e.

$$\forall_j, \quad \Psi_j = \sum_{\Phi} \lambda_\Phi \varphi_j \; , \quad \text{where} \quad \sum |\lambda_\Phi| \leq 1.$$

Then the following holds

(3) $$\mathcal{B}_{8\tau}(\Psi) > t \Rightarrow \sum_{\Phi} |\lambda_\Phi| [B_\tau(\Phi) \wedge t] > c^{-1}\tau t.$$

Assume now $\Lambda \subset \mathbf{Z}_+$ satisfying the condition

(4) $$\sup_n B_\tau^{(N)}(2^{-s}\sum_{n\in\Lambda}\sigma(2^{-s}n)z^n | s = 1, 2, \dots) < \infty \qquad \forall\tau > 0.$$

From this, (2.5) may be derived. Let thus $(\Omega, \mathcal{B}, \mu, T)$ be a D.S. and $|f| \leq 1$. Assume $s_1 < \cdots < s_J$ such that for $i = 1, \dots, J-1$

$$\|\sup_{s_i<s<s_{i+1}} |2^{-s}\sum_\Lambda \sigma(n2^{-s})T^n f - 2^{-s_i}\sum_\Lambda \sigma(n2^{-s_i})T^n f|\|_{L^2(\Omega)} > \epsilon \qquad (1 \leq i < J)$$

Choose N a large integer (depending on s_J) and consider for fixed $\omega \in \Omega$ the orbit function

$$\begin{aligned} g_\omega(n) &= T^n f(\omega) \quad \text{if} \quad 1 \leq n \leq N \\ &= 0 \quad \text{otherwise} \end{aligned}$$

It then follows from the definition of $B_\tau^{(N)}$ that

$$\sum_{i=1}^{J-1} \|\sup_{s_i<s<s_{i+1}} |2^{-s}\sum_\Lambda \sigma(n2^{-s})g_\omega(a+n) - 2^{-s_i}\sum_\Lambda \sigma(n2^{-s_i})g_\omega(a+n)|\|_{\ell^2(\mathbf{Z})}^2 \leq$$

$$\{\int_0^C \delta[B_\delta^{(N)}(\dots) \wedge J]d\delta\}N < \epsilon(J)JN$$

where $\epsilon(J) \to 0$ for $J \to \infty$, as a consequence of (4).

Take $2^{s_J} < a < N - 2^{s_J}$ and substitute g_ω to get

$$\sum_{i=1}^{J-1} \sum_{2^{s_J} < a < N - 2^{s_J}} \sup_{s_i < s < s_{i+1}} |2^{-s} \sum_{\Lambda} \sigma(n2^{-s}) T^n f(T^a \omega) - 2^{-s_i} \sum_{\Lambda} \sigma(n2^{-s_i}) T^n f(T^a \omega)|^2 < \epsilon(J) JN.$$

Integrating in ω and using the fact that T is measure preserving yields

$$\sum_{i=1}^{J-1} \| \sup_{s_i < s < s_{i+1}} |2^{-s} \sum_{\Lambda} \sigma(n2^{-s}) T^n f - 2^{-s_i} \sum_{\Lambda} \sigma(n2^{-s_i}) T^n f| \|_2^2 \leq 2\epsilon(J) J$$

hence

$$\epsilon^2 \leq 2\epsilon(J) \to 0 \quad \text{for} \quad J \to \infty,$$

thus a precise formulation of (2.5)

4. SECOND REDUCTION TO SHIFT

Consider condition (3.4)

$$\sup_{N} B_\tau^{(N)}(2^{-s}\sum_{n\in\Lambda}\sigma(2^{-s}n)z^n|s=1,2,\dots)<\infty \qquad \forall\tau>0 \tag{1}$$

where Λ is a return time sequence Λ_ω for some $DS(\Omega,\mathcal{B},\mu,T)$ and $A\in\mathcal{B}$, $\mu(A)>0$. Notice that $B_\tau^{(N)}$ is essentially increasing in N. Thus it suffices to show that

$$\mu[B_\tau^{(N)}(2^{-s}\sum\chi_A(T^n\omega)\sigma(2^{-s}n)z^n)>t]\leq$$

$$t^{-1}\int[t\wedge B_\tau^{(N)}(2^{-s}\sum\chi_A(T^n\omega)\sigma(2^{-s}n)z^n)]d\omega<\xi_{\tau(t)} \tag{2}$$

where $\xi_\tau(t)\to 0$ for $t\to\infty$ (independently of N). An estimate (2) may again be derived from the shift model $(\mathbf{Z},S)$. Fix $\omega\in\Omega$ and let $\epsilon_n=\chi_A(T^n\omega)$. Assume there is a (uniform) estimate

$$\left.\begin{aligned}&B_\tau^{(N)}(2^{-s}\sum\epsilon_n\sigma(2^{-s}(n-a))z^n|s=1,2,\dots)\leq t\\ &\text{for } 1\leq a\leq N, \text{ except on a set of size } <\zeta_\tau(t)N, \text{ where } \zeta_\tau(t)\xrightarrow{t\to\infty}0\end{aligned}\right] \tag{3}$$

For an appropriate $\xi_\tau(t)$, it then follows

$$\frac{1}{N}\sum_{1\leq a\leq N}t^{-1}[t\wedge B_\tau^{(N)}(2^{-s}\sum_n\epsilon_n\sigma(2^{-s}(n-a))z^{n-a})]\leq\xi_\tau(t).$$

Making the substitution $n\leftrightarrow n-a$ and writing $\epsilon_{n+a}=T^n\chi_A(T^a\omega)$,

$$\frac{1}{N}\sum_{1\leq a\leq N}t^{-1}[t\wedge B_\tau^{(N)}(2^{-s}\sum_n T^n\chi_A(T^a\omega)\sigma(2^{-s}n)z^n)]\leq\xi_{\tau(t)}$$

5. REDUCTION TO DIADIC INTERVALS

Take N of the form 2^n and let $(\epsilon_1,\dots,\epsilon_N)=\bar\epsilon$ be a sequence of $0,1$. Partition the interval $I_\phi=[1,\cdots,N]$ in diadic intervals $I_c, |I_c|=2^{-|c|}N$ where $c\in\cup_{j\leq n}\{0,1\}^j$

and consider the polynomials

$$p_c = |I_c|^{-1} \sum_{k \in I_c} \epsilon_k z^k$$

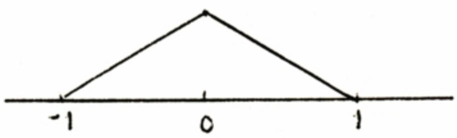
which form a tree in the sense that

$$p_c = \frac{1}{2}(p_{c,o} + p_{c,1})$$

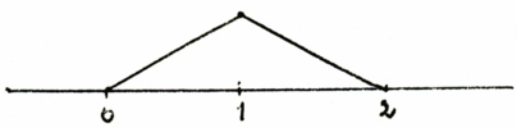
It is then possible to derive (4.3) form the following statement

(1) $$\frac{1}{N} \sharp \{c \in \{0,1\}^n | B_\tau(p_{c|j} | j = 0, \ldots, n) > t\} \overset{t \to \infty}{\longrightarrow} 0$$

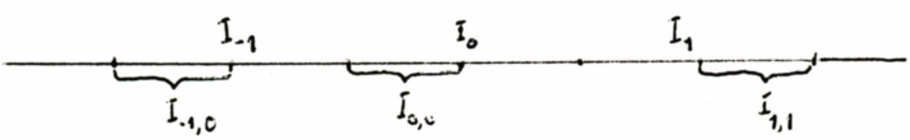
(uniformly in N and independently from the sequence $\overline{\epsilon}$).
This is done by averaging translations. The argument simplifies if σ is replaced by its *symmetric* version



(which eventually leads to a pointwise ergodic theorem for the symmetric sums). Thus most of the complications in what follows have to do with the fact that the desired σ is *one-sided*



We first introduce some notations. Consider the N-periodic extension of the partition (I_c) of $[0, N]$ to a partition of $\mathbf{Z}$. For $a \in \mathbf{Z}$ and $1 \leq j \leq n$, denote $c(a,j)$ the complex in $\mathbf{Z} \times \{0,1\}^j$ satisfying $a \in I_{c(a,j)}$



Writing

$$a \equiv N \sum_{j=1}^{n} a_j 2^{-j} \quad (\text{mod} \quad N = 2^n), \; a_j = 0,1$$

define for $1 \leq i \leq n$

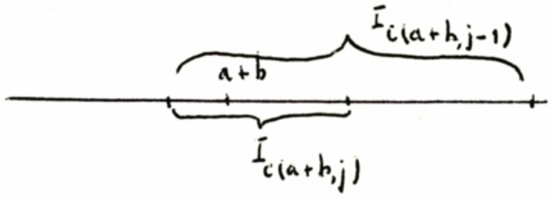
$$j(a,i) = \quad \min\{j \geq i | a_j = 0\}.$$

Define also

$$q^i_{a,b} = 2^j N^{-1} \sum_{b \in I_{c(a+b,j-1)} \setminus I_{c(a+b,j)}} \epsilon_{k-b} z^{k-b}$$

where

$$j = j(a+b,i)$$



Then, because of subsequence considerations

$$B_\tau(q^i_{a,b} | 1 \leq i \leq n) \leq 2B_\tau(2^j N^{-1} \sum_{k \in I_{c(a+b,j)}} \epsilon_{k-b} z^{k-b} | j = j(a+b,\underline{i}))$$

$$\leq 2B_\tau(2^j N^{-1} \sum_{k \in I_{c(a+b,j)}} \epsilon_{k-b} z^{k-b} | j = 1, \ldots, \quad n).$$

Notice that (1) may be reformulated as

$$\frac{1}{N} \sharp\{1 \leq a \leq N | B_\tau(2^j N^{-1} \sum_{k \in I_{c(a,j)}} \epsilon_k z^k | j = 1, \ldots n) > t\} \leq \zeta'_\tau(t)$$

with $\zeta'_\tau(t) \to 0$ for $t \to \infty$ (uniformly).
Hence, for fixed b

(2) $$\frac{1}{N} \sharp\{1 \leq a \leq N | B_\tau(q^i_{a,b}) > t\} \leq c\zeta'_\tau(t).$$

Define next

$$q^i_a = B^{-1} \sum_{|b| \leq B} q^i_{a,b}$$

where B is some large integer.
Hence, from the averaging property (3.3) (noticing that the maximal inequality (3.2) is valid)

$$\frac{1}{N} \sharp\{1 \leq a \leq N | B_\tau(q^i_a) > t\} \leq$$

$$(3) \qquad c\tau^{-1}B^{-1}\sum_{|b|\le B}\frac{1}{N}\int_{[1,N]} t^{-1}[B_\tau(q^i_{a,b})\wedge t]da \le \zeta''_\tau(t)$$

for some ζ''_τ still satisfying $\zeta''_\tau(t) \overset{t\to\infty}{\longrightarrow} 0$. Next we explicit $q^i_a = \sum \widehat{q^i_a}(\ell)z^\ell$. We have by construction

$$\widehat{q^i_a}(\ell) = B^{-1}\sum_{|b|\le B}\widehat{q^i_{a,b}}(\ell)$$

$$= \epsilon_\ell N^{-1}B^{-1}\sum\{2^{j(a+b,i)}||b|\le B \quad \text{and} \quad b+\ell\in I_{c(a+b,j(a+b,i)-1)}\setminus I_{c(a+b,j(a+b,i))}\}$$

$$\approx \quad \epsilon_\ell N^{-1}B^{-1}\sum\{2^{j(b,i)}||b|\le B \quad \text{and} \quad b+\ell-a\in I_{c(b,j(b,i)-1)}\setminus I_{c(b,j(b,i))}\}$$

provided that B is large.

From the definition of $j(b,i)$, this gives

$$\epsilon_\ell N^{-1}B^{-1}.$$

$$2^i\sharp\{|b|\le B, b_i=0|b+\ell-a\in I_{c(b,i-1)}\setminus I_{c(b,i)}\}+$$

$$2^{i+1}\sharp\{|b|\le B, b_i=1, b_{i+1}=0|b+\ell-a\in I_{c(b,i)}\setminus I_{c(b,i+1)}\}+\cdots \quad \sim$$

$$\epsilon_\ell(2^{-i}N)^{-2}.$$

$$\sharp([-\frac{N}{2^i},0]\cap[k,\frac{N}{2^i}+k])+$$

$$2\sharp([-\frac{N}{2^{i+1}},0]\cap[k,\frac{N}{2^{i+1}}+k])+$$

$$4\sharp([-\frac{N}{2^{i+2}},0]\cap[k,\frac{N}{2^{i+2}}+k])+\dots$$

letting $k=a-\ell$.

For σ defined as above, this equals

$$\epsilon_\ell(2^{-i}N)^{-1}(\sigma(x)+\sigma(2x)+\sigma(4x)+\dots) \qquad x=(2^{-i}N)^{-1}(\ell-a).$$

Hence (3) gives for $\varphi(x)=\sigma(x)+\sigma(2x)+\dots$ and $s=n-i$

$$\frac{1}{N}\sharp\{1\le a\le N \;\; |B_\tau(\sum\epsilon_\ell 2^{-s}\varphi(2^{-s}(\ell-a))z^\ell|1\le s\le n)>t\}\le\zeta''_\tau(t).$$

Since $\sigma(x)=\varphi(x)-\varphi(2x)$, also

$$\frac{1}{N}\sharp\{1\le a\le N|B_\tau(2^{-s}\sum\epsilon_\ell\sigma(2^{-s}(\ell-a))z^\ell|s=1,2,\dots)>t\}\le 2\zeta''_\tau(t).$$

which is property (4.3).

6. MARTINGALE CONVERGENCE

In this section, some results on martingales are recalled for later use. These martingales will appear in various contexts (diadic trees, convolution by approximate identity). Let $\mathbf{E}_n$ be an increasing sequence of expectations and $f_n = \mathbf{E}_n f$ a scalar martingale.

For $\lambda > 0$, denote $N_\lambda(\omega)$ the number of λ-jumps in the sequence $\{f_n(\omega)\}$. The following inequality for $1 < p < \infty$ is proved by methods of square-functions and stopping times

Lemma:

(1) $$\|\lambda N_\lambda^{1/2}\|_p \leq c_p \|f\|_p \ , \forall \lambda > 0$$

Let

$$\|x\|_{v_s} = \sup_{J;n_1<\cdots<n_J} \left(\sum |x_{n_j} - x_{n_{j+1}}|^s\right)^{1/s}.$$

Using (1) and methods of interpolation, one then proves

Proposition (Lépingle):

(2) $$\|\{f_n\}\|_{L^2_{v_s}} \leq c(s-2)^{-1}\|f\|_2 \quad \textit{for} \quad s > 2,$$

which is a quantitative form of the martingale convergence theorem. We will need the following corollary on the pointwise entropy of an H-valued martingale $f_n = \mathbf{E}_n[f]$, f taking values in a Hilbert space H. Define for $\lambda > 0$

$$M_\lambda(\omega) = \lambda\text{- entropy number of } \{f_n(\omega)\} \subset H,$$

i.e. the maximal number of points in the set with separation $\geq \lambda^{(*)}$

Lemma: *For $s > 2$*

(3) $$\|\sup_{\lambda>0}(\lambda M_\lambda^{1/s})\|_2 \leq c(s-2)^{-1}\|f\|_2$$

Proof: Clearly there is $\forall\lambda > 0$ the pointwise inequality

(4) $$\lambda M_\lambda^{1/s} \leq 2 \sup_{J;n_1<\cdots<n_J} \left(\sum_j \|f_{n_{j-1}} - f_{n_j}\|_H^s\right)^{1/s}.$$

Writing $f_n = \sum_\alpha < f_n, e_\alpha > e_\alpha$, $\{e_\alpha\}$ an ONB for H, it then follows from (4), (2) and convexity

$$\begin{aligned}\|\sup_{\lambda>0}\lambda M_\lambda^{1/s}\|_2 &\leq 2\left(\sum_\alpha \|\{< f_n, e_\alpha >\}\|^2_{L^2_{v_s}}\right)^{1/2} \\ &\leq c(s-2)^{-1}\left(\sum_\alpha \| < f, e_\alpha > \|_2^2\right)^{1/2},\end{aligned}$$

thus (3).

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*$M_\lambda = 0$ if the set is of diameter $< \lambda$.

7. A MAXIMAL INEQUALITY FOR CERTAIN MULTIPLIERS (1)

In the first part of this section, the following fact will be shown

Lemma. *Assume* $\lambda_1 < \cdots < \lambda_K \in [0,1] \simeq \Pi$ *and define for* $j \in \mathbf{Z}_+$

$$R_j = \{\lambda \in [0,1] | \min_{1 \leq k \leq K} |\lambda - \lambda_k| \leq 2^{-j}\}.$$

Then

$$\|\sup_j |\mathcal{F}^{-1}[\chi_{R_j} \mathcal{F} f]|\|_{\ell^2(\mathbf{Z})} \leq (\log K)^2 \|f\|_{\ell^2(\mathbf{Z})} \tag{1}$$

where $\mathcal{F}$ *refers to the Fourier-transform*

Remarks: - *The estimate (1) is not optimal concerning dependence on* K *but suffices for our purpose. In fact, any estimate* $K^{(1/2)-\delta}\|f\|_2$, *better than the "trivial"estimate* $K^{1/2}\|f\|_2$, *would do as well.*

- *(1) has further application (besides to the return-times) to the pointwise ergodic theorem for "arithmetic sets"(improvement of the* L^p *-estimates,* $p < 2$*) and sets of the form* $\{[p(n)]; n \in \mathbf{Z}_+\} = \Lambda$, *where* $p(x)$ *is an arbitrary polynomial. These matters will be discussed at the end of this exposé.*

The proof of the Lemma is presented in several steps.

(a) Recall the following tranference property from $L^2(\mathbf{R})$ to $\ell^2(\mathbf{Z})$-inequalities.

Lemma. *Let* Φ *be a set of multipliers on [0,1] satisfying*

$$\|\sup_{\varphi \in \Phi} |\mathcal{F}^{-1}[\varphi \mathcal{F} f]|\|_{L^2(\mathbf{R})} \leq B \|f\|_{L^2(\mathbf{R})}. \tag{2}$$

Then

$$\|\sup_{\varphi \in \Phi} |\mathcal{F}^{-1}[\varphi \mathcal{F} f]|\|_{\ell^2(\mathbf{Z})} \leq cB \|f\|_{\ell^2(\mathbf{Z})} \tag{3}$$

where C *is an absolute constant*

Proof: Denote B_1 dthe best constant satisfying (3). Writing for $x \in \mathbf{Z}$, $u \in [0, \rho]$ ($\rho < 1$ to be specified later)

$$\mathcal{F}^{-1}[\varphi \mathcal{F} f](x) = \mathcal{F}^{-1}[\varphi \mathcal{F} f](x+u) + \mathcal{F}^{-1}[(1 - e^{2\pi i \lambda u})\varphi f \mathcal{F}](x)$$

and averaging in u gives

$$\|\sup_{\varphi} |\mathcal{F}^{-1}[\varphi \mathcal{F} f]|\|_{\ell^2(\mathbf{Z})} \leq \rho^{-1/2} \|\sup_{\varphi} |\mathcal{F}^{-1}[\varphi \mathcal{F} f]|\|_{L^2(\mathbf{R})} +$$

$$(4) \qquad \sup_{0<u<\rho} \|\sup_{\varphi}|\mathcal{F}^{-1}[(1-e^{2\pi i\lambda u})\varphi\mathcal{F}f]|\|_{\ell^2(\mathbf{Z})}$$

By (2), the first turn in (4) is bounded by

$$\rho^{-1/2}B\|\mathcal{F}f\|_2 \sim \rho^{-1/2}B\|f\|_{\ell^2(\mathbf{Z})}.$$

By definition of B_1, the second termin (4) is bounded by

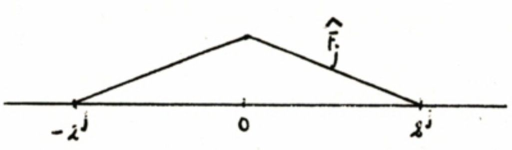
$$\begin{aligned} B_1\|f * \mathcal{F}^{-1}[1-e^{2\pi i\lambda u}]\|_{\ell^2(\mathbf{Z})} &\sim B_1\|\mathcal{F}f\cdot[1-e^{2\pi i\lambda u}]\|_{L^2[0,1]} \\ &\leq c\rho B_1\|\mathcal{F}f\|_2 \\ &\sim c\rho B_1\|f\|_{\ell^2(\mathbf{Z})}. \end{aligned}$$

Thus an appropriate choice of ρ yields

$$B_1 \leq \rho^{-1/2}B + \frac{1}{2}B_1 \Rightarrow B_1 \leq CB.$$

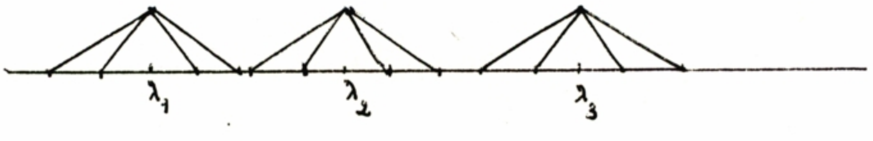
Remark: *In the preceeding, $\mathcal{F}_{\mathbf{R}}$ and $\mathcal{F}_{\mathbf{Z}}$ where both denoted by $\mathcal{F}$.*

(b) Lemma. *Denote F_j the Fé jer-Kernel on* **R**



Let $J \in \mathbf{Z}$ and $\lambda_1, \ldots \lambda_K \in \mathbf{R}$ frequencies s.t. $|\lambda_k - \lambda_{k'}| > 2^J$ for $k \neq k'$. Then the following inequality holds

$$(5) \qquad \|\sup_{j\leq J}|\sum_{k=1}^{K} e^{2\pi i\lambda_k x}(f_k * F_j)|\|_2 \leq c(\log K)^2(\sum_{k=1}^{K}\|f_k\|_2^2)^{1/2}$$



Proof: Denote $\mathcal{D}_j(j \in \mathbf{Z})$ the intervals of length 2^{-j} obtained by diadic partitioning of **R** and let $\mathcal{E}_j$ be the corresponding expectation operator. Using again (standard) arguments to pass from expectations to convolutions (see [B4] for instance), (5) may be derived (and is essentially equivalent to) from

(6)
$$\|\sup_{j\le J}|\sum_{1\le k\le K} e^{2\pi i\lambda_k x}\mathbf{E}_j[f_k]|\|_2 \le c(\log K)^2(\sum \|f_k\|_2^2)^{1/2}.$$

Denote $g_k = \sup_{j\le J}|\mathbf{E}_j f_k|$ and $G = (\sum_{k=1}^K g_k^2)^{1/2}$, which are constant on the $\mathcal{D}_J$-intervals. Also, by Doob's maximal-inequality

(7)
$$\|G\|_2 = (\sum \|g_k\|_2^2)^{1/2} \le c(\sum \|f_k\|_2^2)^{1/2}.$$

Fix an interval I in D_J. Denote $N_\delta(I)$ the $\delta G|_I$-entropy number of the set

$$A_I = \{(\mathbf{E}_j f_k|_I)_{1\le k\le K} | j\le J\} \supset \ell_K^2$$

($\delta < 1$). Thus each element $\overline{a}$ of A_I has a representation

(8)
$$\overline{a} = \sum_{0\le t\le \log K} \overline{a}_t + \overline{b}$$

where

$$|\overline{b}| < K^{-1/2}G|_I$$

$\overline{a}_t$ belongs to a set $\mathcal{E}_t(I)$ of vectors of norm $< 2^{-t}G|_I$, with

$$|\mathcal{E}_t(I)| \le N_{2^{-t}}(I).$$

From (8), triangle inequality and Hölder's inequality for the last term, estimate on I

(9)
$$\sup_{j\le J}|\sum_{k\le K} e^{8\pi i\lambda_k x}\mathcal{E}_j[f_k]| \le \sum_{t\le \log K}(\sup_{\overline{a}\in\mathcal{E}_t(I)}|\sum_{k\le K} a_k e^{2\pi i\lambda_k x}|) + G|_I$$

where further

$$\sup_{\overline{a}\in\ _t(I)}|\sum a_k e^{2\pi i\lambda_k x}| \le \quad \min \quad \{K^{1/2}2^{-t}G|_I, (\sum_{\overline{a}\in\mathcal{E}_t(I)}|\sum a_k e^{2\pi i\lambda_k x}|^2)^{1/2}\}.$$

Hence, from the separation hypothesis on the λ_k

$$(\int_I \sup_{\overline{a}\in\mathcal{E}_t(I)}|\sum a_k e^{2\pi i\lambda_k x}|^2 dx)^{1/2} \le$$

$$\min \quad (K^{1/2}2^{-t}G|_I\cdot|I|^{1/2}, N_{2^{-t}}(I)^{1/2}2^{-t}G|_I \sup_{|\overline{b}|=1}\|\sum b_k e^{2\pi i\lambda_k x}\|_{L^2(I)})$$

(10) $$\le c \quad \min \quad (K^{1/2}, N_{2^{-t}}(I)^{1/2}) 2^{-t} G|_I |I|^{1/2}$$

which substituted in (5) yields

$$\| \sup_{j \le J} | \sum_{k \le K} e^{2\pi i \lambda_k x} \mathbf{E}_j[f_k] | \|_{L^2(I)} \le$$

(11) $$c \sum_{t \le \log K} \min \quad (K, N_{2^{-t}}(I))^{1/2} 2^{-t} G|_I |I|^{1/2} + G|_I |I|^{1/2}.$$

Choosing $s > 2$, write

$$\min \quad (K, N_{2^{-t}}(I))^{1/2} \le K^{1/2 - 1/s} N_{2^{-t}}(I)^{1/s}$$

and with the notation of the previous section, by definition of $N_\delta(I)$

$$2^{-t} G|_I \cdot N_{2^{-t}}(I)^{1/s} \le \sup_{\lambda > 0} \lambda (M_\lambda|_I)^{1/s}.$$

Substitution in (11) gives

$$\| \sup_{j \le J} | \sum_{k \le K} e^{2\pi i \lambda_k x} \mathbf{E}_j[f_k] | \|_2 =$$

$$\Big(\sum_{I \in \mathcal{D}_J} \| \sup_{j \le J} | \dots | \|^2_{L^2(I)} \Big)^{1/2} \le$$

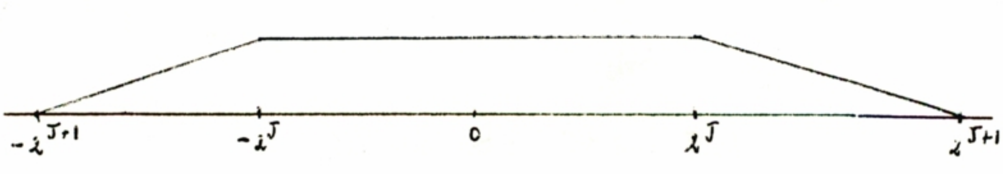
$$C K^{1/2 - 1/s} \log K \cdot \Big\{ \sum_{I \in \mathcal{D}_J} \int_I [\sup \lambda M_\lambda(\omega)^{1/s}]^2 d\omega \Big\}^{1/2} + \Big(\int G^2 \Big)^{1/2}.$$

By (6.3), applied to the ℓ^2_k-valued martingale $\{(\mathbf{E}_j[\mathcal{F}_k])_{1 \le k \le K} | j \le J\}$ and (7)

$$\| \sup_{j \le J} | \sum_{k \le K} e^{2\pi i \lambda_k x} \mathbf{E}_j[f_k] | \|_2 \le c K^{1/2 - 1/s} (\log K)(s-2)^{-1} \Big(\sum \|f_k\|^2 \Big)^{1/2}.$$

Choosing $\frac{1}{2} - \frac{1}{s} \sim \frac{1}{\log K}$, (6) follows

(c) Let V_J be the de la Vallée-Poussin kernel



and let

$$f_k = \mathcal{F}^{-1}[\mathcal{F} f(\lambda + \lambda_k) \hat{V}_J(\lambda)] = (f \cdot e^{-2\pi i \lambda_k x}) * V_J.$$

It follows from (5)

(12) $$\|\sup_{j\le J}| \mathcal{F}^{-1}[\mathcal{F}f\sum_{k\le K}\hat{F}_j(\lambda-\lambda_k)]|\|_2 \le c(\log K)^2\|f\|_2$$

(d) Let

$$R_j = \{\lambda\in\mathbf{R}|\min_{1\le k\le K}|\lambda-\lambda_k|\le 2^j\},\quad \text{for}\quad j\in\mathbf{Z}.$$

Then

(13) $$\|\sup_{j\in S}|\mathcal{F}^{-1}[\chi_{R_j}\mathcal{F}f]|\|_2 \le c(\log|S|)\|f\|_2$$

for S a finite subset of $\mathbf{Z}$.

Proof: The argument is inspired from the Burkholder-Davis-Gundi-Stein proof of the dual version of Doob's maximal inequality. The only difference here is that the operators are not positive. We only use the fact that the R_j's are increasing. Assume thus

$$R_{j-1}\subset R_j\,,\,1\le j\le 2^s \quad\text{where}\quad s\sim\log|S|.$$

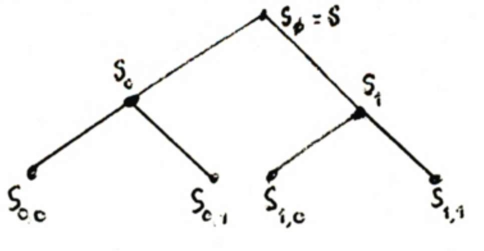
Denote B the st constant satisfying the inequality

(14) $$\|\sup_{1\le j\le 2^s}|\mathcal{F}^{-1}[\chi_{R_j}\mathcal{F}f]|\|_2 \le B\|f\|_2$$

or equivalently (by dualization)

(15) $$\|\sum_{j\le 2^s}\mathcal{F}^{-1}[\chi_{R_j}\mathcal{F}g_j]\|_2 \le B\|\sum_{j\le 2^s}|g_j|\|_2.$$

Identify S and $\{1,2,\dots,2^s\}$ and let $(S_c)_{|c|\le s}$ be a diadic partitioning



Denoting $\tilde{g}_j = \mathcal{F}^{-1}[\chi_{R_j}\mathcal{F}g_j]$, clearly

$$<\tilde{g}_j,\tilde{g}_k> = <\tilde{g}_j, g_k> \quad\text{for}\quad j\le k.$$

Using this fact and Hölder's inequality, one gets from definition of B

$$
\begin{aligned}
\|\sum_{j\in S}\tilde g_j\|_2^2 &= \sum\|\tilde g_j\|_2^2 + 2\sum_{j<k} <\tilde g_j, \tilde g_k> \\
&\le \sum\|g_j\|_2^2 + 2\sum_{|c|<s} |< \sum_{j\in S_{c,0}}\tilde g_j, \sum_{k\in S_{c,1}} g_k >| \\
&\le \sum\|g_j\|_2^2 + 2B\sum_{|c|<s}\|\sum_{j\in S_{c,0}}|g_j|\|_2\|\sum_{j\in S_{c,1}}|g_j|\|_2 \\
&\le (1+2Bs)\|\sum|g_j|\|_2^2.
\end{aligned}
$$

Consequently, $B^2 \le 1+2Bs \Rightarrow B \le cs$, proving (13).

(e) Proof of (1).

Define

$$
S = \{j\in \mathbf{Z}\ K^{-1}2^j < |\lambda_k - \lambda_{k'}| < K2^j \quad \text{for some} \quad 1\le k\ne k'\le K\}.
$$

Thus

$$
|S| \le K^3.
$$

Define further

$$
Z_r = \{j\in\mathbf{Z}\setminus S | R_j \quad \text{has} \quad r \quad \text{components}\}
$$

for $1\le r\le K$.

Thus

$$
Z_K \ < Z_{k-1} < \cdots < Z_2 < Z_1
$$

where Z_1, Z_K are half-lines and Z_r a finite segment, for $1<r<K$.
For $r>1$, let $j_r = \max Z_r$. By construction, there is a set $\Lambda_r \subset \{\lambda_k\}$ satisfying

(16) $$|\lambda-\lambda'| > 2^{j_r} \quad \text{for} \quad \lambda\ne\lambda' \quad \text{in} \quad \Lambda_r$$

(17) $$R_j \subset \cup_{\lambda\in\Lambda_r}[\lambda - 2^{j+1}, \lambda+2^{j+1}] \quad \text{for} j\in Z_r.$$

To prove (1) we proceed again by duality and estimate the best B fulfilling

$$
\|\sum_j \tilde g_j\|_2 \le B\|\sum|g_j|\|_2 \qquad \tilde g_j = \mathcal{F}^{-1}[\chi_{R_j}\mathcal{F}g_j].
$$

Estimate using (13)

$$\|\tilde{g}_j\|_2 \le \|\sum_{j\in S}\tilde{g}_j\|_2 + \|\sum_r(\sum_{j\in Z_r}\tilde{g}_j)\|_2 \le$$

$$(\log K)\|\sum|g_j|\|_2 + \|\sum_r \overline{G}_r\|_2$$

denoting

$$G_r = \sum_{j\in Z_r} g_j \quad \text{and} \quad \overline{G}_r = \sum_{j\in Z_r}\tilde{g}_j.$$

Because $Z_r > Z_{r'}$ for $r < r'$, we have for $j \in Z_r$, $j' \in Z_{r'}$

$$< \tilde{g}_j, \tilde{g}_{j'} > = < g_j, \tilde{g}_{j'} >$$

Hence

$$< \overline{G}_r, \overline{G}_{r'} > = < G_r, \overline{G}_{r'} >$$

and

$$\|\sum \overline{G}_r\|_2^2 = \sum \|\overline{G}_r\|_2^2 + 2\sum_{r<r'} < G_r, \overline{G}_{r'} > .$$

The same argument as in (d) then shows that

$$B^2 \le (\log K)^2 + B_1^2 + B(\log K) \tag{19}$$

where B_1 has to satisfy

$$\|\sup_{j\in Z_r}|\mathcal{F}^{-1}[\chi_{R_j}\mathcal{F}f]|\|_2 \le B_1\|f\|_2. \tag{20}$$

To estimate B_1, use (12) and a standard *square-function* argument. Put

$$\varphi_j(\lambda) = \sum_{\lambda'\in\Lambda_r}\hat{F}_j(\lambda-\lambda')$$

for which, by (16) and (12)

$$\|\sup_{j\in Z_r}\|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f]\|\|_2 \le c(\log\Lambda_r)^2\|f\|_2 \le c(\log K)^2\|f\|_2 \tag{21}$$

Estimating

$$\sup_{j\in Z_r}|\mathcal{F}^{-1}[\chi_{R_j}\mathcal{F}f]| \le \sup_{j\in Z_r}|\mathcal{F}^{-1}[\varphi_j\mathcal{F}f]| + (\sum_{j\in Z_r}|\mathcal{F}^{-1}[(\varphi_j-\chi_{R_j})\mathcal{F}f]|^2)^{1/2}$$

and applying Parseval's identity to estimate the L^2-norm of the second term, it remains to evaluate

$$\|(\sum_{j\in Z_r} |\varphi_j - \chi_{R_j}|^2)^{1/2}\|_{L^\infty(\mathbf{R})} \tag{22}$$

For $\lambda \in \mathbf{R}$ fixed, $(\varphi_j - \chi_{R_j})(\lambda) = 0$ unless $|\lambda - \lambda'| \leq 2^{j+1}$ for some $\lambda' \in \Lambda_r$. This follows from (17). Take $\lambda' \in \Lambda_r$ and $j_o \in \mathbf{Z}$ such that

$$|\lambda - \lambda'| = \quad \text{dist} \quad (\lambda, \Lambda_r) \sim 2^{j_o}.$$

Estimate (22) at λ as

$$C + (\sum_{\substack{j\geq j_o,\\ j\in Z_r}} |1 - \varphi_j(\lambda)|^2)^{1/2} \leq C + C\{\sum_{j\geq j_o} (\frac{|\lambda - \lambda'|}{2^j})^2\}^{1/2} < C.$$

Thus (22) is bounded and, by (21), B_1 may be estimated by $C(\log K)^2$. By (19), this also gives the bound on B and completes the proof of (1).

8. A MAXIMAL INEQUALITY FOR CERTAIN MULTIPLIERS (2)

In this section, the first main ingredient in proving (5.1) will be given.

Lemma. *Let* $\lambda_1 < \cdots < \lambda_K$, $\Lambda = \{\lambda_1, \ldots, \lambda_K\}$. *Assume* $(\varphi_j)_{j \geq j_o > 0}$ *multipliers satisfying the conditions* $(0 < \tau < 1\,, L \geq 1)$

(i) $|\varphi_j| \leq 1\,, |\varphi_j'| \leq L2^j$.

(ii) *The subset of* ℓ^2_K

$$\{\varphi_j|_\Lambda\,;\, j \geq j_o\}$$

has τ*-metrical entropy* $< A_\tau$.

Denote $\sigma = \sigma_\delta$ *a smooth function on* $\mathbf{R}$ *satisfying*

(iii) $0 \leq \sigma \leq 1$

(iv) $|\sigma'| \leq c\delta^{-1}$

(v) $\sigma(t) = 0$ *for* $t < 0$ *or* $t > \delta$.

Let $R \geq 1$ *be fixed and for each* j, *consider a localizing function* η_j *such that*

(vi) $0 \leq \eta_j \leq 1\,, \eta_j = 1$ *on* Λ, $\eta_j = 0$ *outside* $\Lambda + [-R2^{-j}, R2^{-j}]$

(vii) $|\eta_j'| \leq c2^j$.

Define $\overline{\varphi}_j = \varphi_j \sigma_\delta(|\varphi_j|)\eta_j$. Then there is the inequality

$$\|\sup_{j \geq j_o} |\mathcal{F}^{-1}[\overline{\varphi}_j \mathcal{F} f]|\|_2 \leq C(\tau + \delta A_\tau^{1/2}(\log \frac{KLR}{\delta})^2)\|f\|_2. \tag{1}$$

Here and in the sequel, $\| \quad \|_2$ stands for the $\ell^2(\mathbf{Z})$-norm.

Proof: The inequality is derived from (7.1) using square-function methods and the reader a bit familiar with this technique will find it routine.

For $\overline{a} \in \ell^2_K$, define as follows the function $\Psi_{j,\overline{a}}$. Let $\lambda_k \leq \lambda \leq \lambda_{k+1}$

- If $\lambda_{k+1} - \lambda_k \leq 2^{-j+1}$, put $\psi_{j,\overline{a}}(\lambda) = a_k$.
- If $\lambda_{k+1} - \lambda_k > 2^{-j+1}$, put

$$\begin{aligned} \Psi_{j\overline{\mu}}(\lambda) &= a_k \quad \text{if} \quad \lambda < \lambda_k + 2^{-j} \\ &= a_{k+1} \quad \text{if} \quad \lambda > \lambda_{k+1} - 2^{-j} \\ &= 0 \quad \text{otherwise} \end{aligned}$$

Define also $\gamma_j = \Psi_{j,\overline{a}}$, letting $\overline{a} = \overline{\varphi}_j|_\Lambda$. We claim that

$$\|(\sum_{j \geq j_o} |\overline{\varphi}_j - \gamma_j|^2)^{1/2}\|_\infty \leq c\delta \log \frac{RL}{\delta} \tag{2}$$

Notice that as a consequence of (iii), (iv), (v), (vi), (vii)

(3) $$|\overline{\varphi}_j| \le \delta \quad \text{and} \quad |(\overline{\varphi}_j)'| \le CL2^j.$$

To verify (2), let $\lambda_k < \lambda < \lambda_{k+1}$. By construction, $(\overline{\varphi}_j - \gamma_j)(\lambda) = 0$ except if $dist(\lambda, \Lambda) = \min \ (|\lambda - \lambda_k|, |\lambda - \lambda_{k+1}|) < R2^{-j}$. Assume for instance $2^{-j_o} \sim \text{dist}\ (\lambda, \Lambda) = |\lambda - \lambda_k|$. Then, by (3)

$$\sum |\overline{\varphi}_j(\lambda) - \gamma_j(\lambda)| \le \delta \log R + \sum_{j<j_o} |\overline{\varphi}_j(\lambda) - \overline{\varphi}_j(\lambda_k)| \le$$

$$\delta \log R + C \sum_{j<j_o} (2^{-j_o}(L2^j) \wedge \delta) < C\delta(\log R + \log \frac{L}{\delta})$$

proving (2). Hence

(4) $$\| \sup_{j \ge j_o} |\mathcal{F}^{-1}[\overline{\varphi}_j \mathcal{F} f]| \|_2 \le \| \sup_j |\mathcal{F}^{-1}[\gamma_j \mathcal{F} f]| \|_2 + \delta \log \frac{RL}{\delta} \|f\|_2$$

Next, partition $\{j \ge j_o\}$ in at most A_τ sets S_α, $\min S_\alpha = j_\alpha$ such that for each of them

$$\|\varphi_j|_\Lambda - \varphi_{j_\alpha}|_\Lambda\|_{\ell^2_K} < \tau, \quad j \in S_\alpha.$$

This is possible by the entropy hypothesis (ii) in the Lemma. Put $\overline{a}_\alpha = \overline{\varphi}_{j_\alpha}|_\Lambda$. It follows from (iv) and the inequality

$$|a\sigma(|a|) - b\sigma(|b|)| \le c|a - b| \quad (a, b \in \mathbf{R})$$

that

(5) $$\|\overline{\varphi}_j|\Lambda - \overline{a}_\alpha\|_{\ell^2_K} < C\tau \quad \text{for } j \in S_\alpha.$$

Estimate

$$\sup|\mathcal{F}^{-1}[\gamma_j \mathcal{F} f]| \le \sup_{\alpha, j \in S_\alpha} [|\mathcal{F}^{-1}[\Psi_{j,\overline{a}_\alpha} \mathcal{F} f]| + |\mathcal{F}^{-1}[\gamma_j - \Psi_{j,\overline{a}_\alpha})\mathcal{F} f]|]$$

(6)
$$\| \sup|\mathcal{F}^{-1}[\gamma_j \mathcal{F} f]| \|_2 \le (\sum_{\alpha \le A_\tau} \| \sup_{j \in S_\alpha} |\mathcal{F}^{-1}[\Psi_{j,\overline{a}_\alpha} \mathcal{F} f]\|_2^2)^{1/2} + \| \sup_{\alpha, j \in S_\alpha} |\mathcal{F}^{-1}[(\gamma_j - \Psi_{j,\overline{a}_\alpha})\mathcal{F} f]| \|_2.$$

For each $\alpha, \overline{a}_\alpha$ is a fixed element of ℓ^2_K which coordinates are δ-bounded. Therefore, one easily derives from (7.1) that

$$\| \sup_{j \in S_\alpha} |\mathcal{F}^{-1}[\Psi_{j,\overline{a}_\alpha} \mathcal{F} f]| \|_2 \le C\delta[\log \frac{1}{\delta} + (\log K)^2] \|f\|_2$$

and the first term in (6) contributes for

(7) $$\delta A_\tau^{1/2}(\log \frac{K}{\delta})^2 \|f\|_2.$$

Estimate by Hölder's inequality for $j \in S_\alpha$ (by (5))

$$|\mathcal{F}^{-1}[(\gamma_j - \Psi_{j,\overline{a}_\alpha})\mathcal{F}f]| \leq \sup_{|\overline{a}-\overline{b}|<\tau} \{\sum_k |\mathcal{F}^{-1}[(\Psi_{j,\overline{a}} - \Psi_{j,\overline{b}})\chi_{[\lambda_{k-1},\lambda_k[}\mathcal{F}f|\}$$
$$\leq 2\tau(\sum_k |\mathcal{F}^{-1}[\Psi_{j,\underline{1}}\chi_{[\lambda_{k-1},\lambda_k[}\mathcal{F}f]|^2)^{1/2}.$$

Thus the second term in (6) is bounded by

$$2\tau(\sum_k \|\sup_j|\mathcal{F}^{-1}[\Psi_{j,\underline{1}}\chi_{[\lambda_{k-1},\lambda_k[}\mathcal{F}f]|\|_2^2)^{1/2} \leq$$

(8) $$c\tau(\sum_k \|\mathcal{F}f|_{[\lambda_{k-1},\lambda_k]}\|_2^2)^{1/2} \sim \tau\|f\|_2.$$

Collecting estimates, (1) follows from (4), (7), (8).

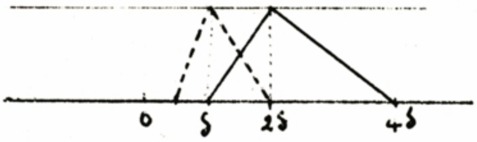
9. GENERALITIES ON THE PROOF OF MAIN INEQUALITY (5.1)

We now come back to section 5. Thus

$$p_c = |I_c|^{-1} \sum_{k \in I_c} \epsilon_k z^k \quad \text{where} \quad c \in \{0,1\}^m, m \leq n$$

and property (1), which remains to be proved.

Consider for diadic values of $0 < \delta < 1$, functions σ_δ



such that $\sum_{\substack{0<\delta<1 \\ \delta \text{ diadic}}} \sigma_\delta = 1$ on [0,1]. Thus

(1) $$p_c = \sum_\delta p_c \sigma_\delta(|p_c|).$$

We will introduce *localization functions* $0 \leq \eta_{c,\delta} \leq 1$, $|\eta'_{c,\delta}| \leq c|I_c|^{-1} \sim 2^{-m}N$. Denote

$$\overline{p}_{c,\delta} = p_c \sigma_\delta(|p_c|)\eta_{c,\delta} \quad \text{and} \quad \overline{p}_c = \sum_\delta \overline{p}_{c,\delta}. \tag{2}$$

For functions φ on Π, define

$$\|\varphi\|_A \equiv \|\varphi\|_{A(\mathbb{T})} = \sum_{k \in \mathbf{Z}} |\hat{\varphi}(k)|.$$

Since there is the pointwise inequality

$$|\mathcal{F}^{-1}[\varphi \mathcal{F} f]| \leq \|\varphi\|_A \cdot \|f\|_\infty$$

(applied here to 1-bounded functions f on $\mathbf{Z}$ supported by $[0, N]$) one may replace p_c by $\overline{p}_c$ in proving (5.1), provided

$$\|p_c - \overline{p}_c\|_A \leq \sum_\delta \|p_c \sigma_\delta(|p_c|) - \overline{p}_{c,\delta}\|_A < \frac{\tau}{2}. \tag{3}$$

The following lemma will be used in evaluating A-norms

Lemma. *If* $\|\varphi\|_2 \leq \kappa d^{-1/2}$, $\|\varphi'\|_2 \leq d^{1/2}$, *then* $\|\varphi\|_A \leq C\kappa^{1/2} (0 < \kappa < 1 < d)$

Proof: For (diadic) integers T, there are following estimates

$$\sum_{T \leq |k| \leq 2T} |\hat{\varphi}(k)| \leq T^{1/2} \|\varphi\|_2 \leq c \left(\frac{\kappa^2 T}{d}\right)^{1/2}$$

$$\sum_{T \leq |k| \leq 2T} |\hat{\varphi}(k)| \sim \sum_{T \leq |k| \leq 2T} |k|^{-1} |\hat{\varphi'}(k)| \leq T^{-1/2} \|\varphi'\|_2 \leq c\left(\frac{d}{T}\right)^{1/2}$$

from where

$$\|\varphi\|_A \leq c \sum_{T \text{ diadic}} \min\left(\frac{\kappa^2 T}{d}, \frac{d}{T}\right)^{1/2} \leq c\kappa^{1/2}.$$

Clearly, by construction, both

$$z^{-\sum_{j \leq m} c_j 2^{-j} N} p_c \sigma_\delta(|p_c|) \quad \text{and} \quad z^{-\sum_{j \leq m} c_j 2^{-j} N} \overline{p}_{c,\delta}.$$

have a gradient bounded by $C|I_c|^{-1}$. Hence, in order to fulfill (3), it suffices to have

$$\|p_c \sigma_\delta(|p_c|) - \overline{p}_{c,\delta}\|_2 \leq \kappa_\delta \, |I_c|^{1/2} \quad \text{where} \quad \kappa_\delta = \tau^2 \left(\log \frac{1}{\delta}\right)^{-4}. \tag{4}$$

For a family Φ of multipliers φ on Π , denote $D(\Phi)$ the best constant satisfying the maximal inequality

$$\|\sup_{\varphi\in\Phi}|\mathcal{F}^{-1}[\varphi\mathcal{F}f]|\|_2 \leq D(\Phi)\|f\|_2.$$

Obviously

$$D(\Phi_1\cup\Phi_2)^2 \leq D(\Phi_1)^2 + D(\Phi_2)^2. \tag{5}$$

We have to evaluate for fixed $\tau > 0$

$$\frac{1}{N}\#\{c\in\{0,1\}^n | B_\tau^{(N)}(\overline{p}_{c|j}|j=1,\ldots,n) > t\}. \tag{6}$$

Assume $|f|\leq 1$ and f supported by $\mathbf{Z}\cap[0,N]$, such that for some sequence of integers $1\leq j_1 < j_2 < \cdots < j_t \leq n$, one has

$$\|\sup_{j_{s-1}\leq j<j_s}|\mathcal{F}^{-1}[(\overline{p}_{c|j}-\overline{p}_{c|j_{s-1}})\mathcal{F}f]|\|_2 > \tau N^{1/2} \quad \text{for} \quad 1 < s \leq t. \tag{7}$$

Fixing some $\delta_o \leq \delta_o(\tau)$, it follows from (2) and triangle inequality that the left member of (7) is bounded by

$$\sum_{\delta>\delta_o}\|\sup_{j_{s-1}\leq j<j_s}|\mathcal{F}^{-1}[(\overline{p}_{c|j,\delta}-\overline{p}_{c|j_{s-1},\delta})\mathcal{F}f]|\|_2 + N^{1/2}\sum_{\delta\leq\delta_o} D(\overline{p}_{c|j,\delta}|1\leq j\leq n).$$

Hence, by (7), either for some $\delta > \delta_o$

$$B_{\tau'}^{(N)}(\overline{p}_{c|j,\delta}) > t' \quad \text{with } \tau' \sim (\log\frac{1}{\delta_o})^{-1}\tau, t' \sim (\log\frac{1}{\delta_o})^{-1}t \tag{8}$$

or for some $\delta < \delta_o$

$$D(\overline{p}_{c|j,\delta}) > (\log\frac{1}{\delta})^{-2}.$$

Consequently, in order to get a uniform estimate $\to 0$ on (6) for $t\to\infty$, it suffices to prove that
For fixed $\delta > \delta_o$ and t' sufficiently large

$$\frac{1}{N}\#\{c\in\{0,1\}^n | B_{\tau'}^{(N)}(\overline{p}_{c|j,\delta}) > t'\} \leq \varepsilon \tag{10}$$

and for δ sufficiently small

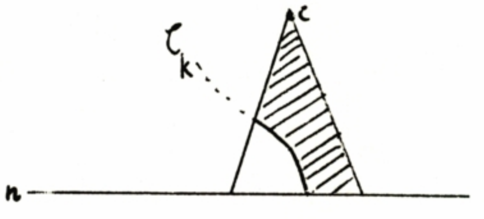
$$\frac{1}{N}\#\{c\in\{0,1\}^n\, | D(\overline{p}_{c|j,\delta}) > (\log\frac{1}{\delta})^{-2}\} < (\log\frac{1}{\delta})^{-2}. \tag{11}$$

We visualize $\bigcup_{m \le n} \{0,1\}^m$ as the nodes of a finite tree which branches are determined by the points in $\Omega \equiv \{0,1\}^n$.
Fixing $\delta > 0$, we will introduce an increasing sequence of stopping times $\tau_1 < \cdots < \tau_K \le n$, $K = K(\epsilon, \delta)$, defined on decreasing subsets $\Omega_1 \supset \Omega_2 \supset \cdots \supset \Omega_K$ of Ω. Extend τ_k to τ_k^* defined on the entire set Ω by letting $\tau_k^*(c) = n$ for $c \in \Omega \setminus \Omega_k$. Assume Ω_K such that

$$\frac{|\Omega_K|}{N} \le \min\left(\epsilon, (\log \frac{1}{\delta})^{-3}\right) \tag{12}$$

(Ω_K is a first contribution to the *exceptional set*). For complexes c, d, write $c < d$, provided that c is an initial segment of d.
Denote $\mathcal{C}_k = \{c|\tau_k(c)\,;\, c \in \Omega_k\}$. For $c \in \mathcal{C}_{k-1}$, denote $\mathcal{T}_c$ the *subtree*

$$\mathcal{T}_c = \{c' | c < c' \text{ and } c' \text{ has no predecessor in } \mathcal{C}_k\}$$



Denote $\mathcal{T}_c^*$ the maximal elements of $\mathcal{T}_c$. For $c \notin \Omega_K$, clearly

$$B_\tau(\overline{p}_{c|j,\delta}) \le K + \sum_{k \le K} B_\tau(\overline{p}_{c|j,\delta} | \tau_{k-1}^*(c) < j < \tau_k^*(c)) \tag{13}$$

and, by (5)

$$D(\overline{p}_{c|j,\delta}) \le \sum_{k \le K} D(\overline{p}_{c|j,\delta} | \tau_{k-1}^*(c) < j < \tau_k^*(c)). \tag{14}$$

By summation over the sub-trees, it will suffice to show that for some t one has for all k and $c_o \in \mathcal{C}_{k-1}$.

$$\sum \{2^{-|c|} | c \in \mathcal{T}_{c_o}^* \text{ and } B_\tau(\overline{p}_{c',\delta} | c_o < c' < c) > t\} < \epsilon' 2^{-|c_o|} \tag{15}$$

in order to obtain a bound $K(\epsilon, \delta)\epsilon' < \epsilon$ on the left member of (10). Here $\delta > \delta_o$. Similarly, using (14), (11) may be derived from

$$\sum \{2^{-|c|} | c \in \mathcal{T}_{c_o}^* \text{ and } D(\overline{p}_{c',\delta} | c_o < c' < c) > \delta_1\} < \delta_1 2^{-|c_o|} \tag{16}$$

where

$$\delta_1 = K(\delta)^{-1} (\log \frac{1}{\delta})^{-2} \tag{17}$$

or from the inequality

$$\sum_{c\in\mathcal{T}^*_{c_o}} 2^{-|c|}D(\overline{p}_{c',\delta}|c_o < c' < c)^2 \leq \delta_2 \quad 2^{-|c_o|} \tag{18}$$

with

$$\delta_2 = \delta_1^3. \tag{19}$$

The construction summarized above will be performed in the next section.

10. CONSTRUCTION OF THE STOPPING TIMES AND DISSECTION OF THE TREE.

Let $\delta > 0$ be fixed. We shall use the notations of the previous section. As before, $\mathbb{C}_1 = \{z \in \mathbb{C} ; |z| = 1\}$ will be frequently identified with $\pi = \mathbb{R}/\mathbb{Z}$ or $[0,1[$.

Consider numbers $0 < \kappa = \kappa(\delta) < 1$ and $R = R(\varepsilon,\delta)$, to be specified later in this section.

Define for each node $c \in \bigcup_{m \leq n} \{0,1\}^m$

$$A_c = \{z \in \mathbb{C}_1 \mid |p_c(z)| > \delta\} .$$

Next, define stopping times τ_k with domain $\Omega_k \subset \Omega = \{0,1\}^n$ as follows $\Omega_o = \Omega$, $\tau_o = 0$.

Let Ω_{k+1} be the set of those $\omega \in \Omega_k$ such that for some $\tau_k(\omega) < t \leq n$

$$\int_{A_{\omega|t} \setminus (B_k(\omega)+[-R\frac{2^t}{N}, R\frac{2^t}{N}])} |p_{\omega|t}|^2 > \kappa^2 \frac{2^t}{N} \tag{1}$$

where $B_k(\omega) = A_{\omega|\tau_1(\omega)} \cup \cdots \cup A_{\omega|\tau_k(\omega)}$. Let $\tau_{k+1}(\omega)$ be the smallest t for which (1) holds.

In this construction, we let $0 \leq k \leq K$, where K will be determined later. Let $C_k = \{\omega|\tau_k(\omega) \mid \omega \in \Omega_k\}$ and $C = \bigcup_{k \leq K} C_k$. It will be more convenient to work with the polynomials

$$P_c = |I_c| . p_c \text{ , thus } P_c(z) = \sum_{k \in I_c} \varepsilon_k z^k \text{ , } P_c = P_{c,o} + P_{c,1} .$$

For each $c \in C$, we will construct a polynomial Q_c satisfying in particular the following properties

$$
\begin{cases}
\text{(2)} \quad \operatorname{supp} \hat{Q}_c \subset I_c \\
\text{(3)} \quad \langle P_c, Q_c\rangle > \sigma |I_c| \\
\text{(4)} \quad \|Q_c\|_2^2 \le |I_c| \quad .
\end{cases}
$$

Write, using (3), (4)

$$
\text{(5)} \quad \int |P_c|^2 = \int |P_c - \sigma Q_c|^2 - \sigma^2 \int |Q_c|^2 + 2\sigma \langle P_c, Q_c\rangle > \int |P_c - \sigma Q_c|^2 + \sigma^2 |I_c|
$$

Notice that if $c, c' \in C$ are incomparable, $P_c \perp P_{c'}$, $Q_c \perp Q_{c'}$, $P_c \perp Q_{c'}$.
Thus, using (5), we may write

$$
\begin{aligned}
N &= \sum_{\omega \notin \Omega_K} \int |P_\omega|^2 + \sum_{c \in C_K} \int |P_c|^2 \\
\text{(6)} \quad &> \sum_{\omega \notin \Omega_K} \int |P_\omega|^2 + \sum_{c \in C_K} \int |P_c - \sigma Q_c|^2 + \sigma^2 |\Omega_K| \\
&= \sum_{\omega \notin \Omega_{K-1}} \int |P_\omega|^2 + \sum_{c \in C_{K-1}} \int |P_c - \sigma \sum_{\substack{c' > c \\ c' \in C_K}} Q_{c'}|^2 + \sigma^2 |\Omega_K|
\end{aligned}
$$

Considering the second term in (6), write similarly as in (5)

$$
\int |P_c - \sigma \Sigma Q_{c'}|^2 = \int |P_c - \sigma \Sigma Q_{c'} - \sigma Q_c|^2 - 2\sigma^2 \sum_{\substack{c' > c \\ c' \in C_K}} \langle Q_c, Q_{c'}\rangle + \sigma^2 |I_c|
$$

which substituted in (6) yield

$$
\begin{aligned}
N > \sum_{\omega \notin \Omega_{K-1}} \int |P_\omega|^2 + \sum_{c \in C_{K-1}} \int |P_c - \sigma Q_c - \sigma \sum_{\substack{c' > c \\ c' \in C_K}} Q_{c'}|^2 + \sigma^2 (|\Omega_{K-1}| + |\Omega_K|) - \\
- 2\sigma^2 \sum_{\substack{c \in C_{K-1} \\ c' \in C_K \\ c' > c}} |\langle Q_c, Q_{c'}\rangle|
\end{aligned}
$$

Continuing in this way leads to the inequality

$$(7) \qquad N > \sigma^2(|\Omega_1|+\cdots+|\Omega_K|) - 2\sigma^2 \sum_{\substack{c,c'\in\mathcal{C} \\ c'>c}} |\langle Q_c, Q_{c'}\rangle|$$

If the second term of (7) is smaller than $\frac{1}{2}$-first term, it follows

$$|\Omega_K| \leq 2\sigma^{-2} K^{-1} N$$

and condition (9.12) is fulfilled for

$$(8) \qquad K > C\kappa^{-4}\left(\frac{1}{\varepsilon} + \left(\log \frac{1}{\delta}\right)^3\right)$$

since $\sigma \sim \kappa^2$ in the construction of the Q_c .

It remains to define the polynomials Q_c .

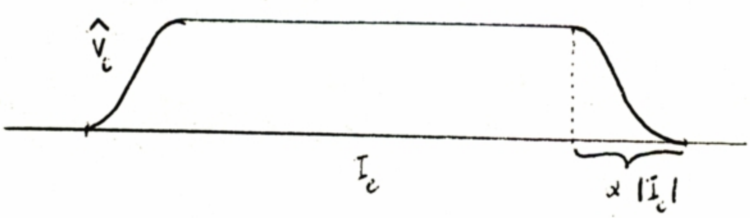
CONSTRUCTION OF THE POLYNOMIALS Q_c .

It follows from the definition of A_c that

$$(9) \qquad |A_c| \leq \delta^{-2} \|P_c\|_2^2 \leq \delta^{-2}|I_c|^{-1} \quad \text{and} \quad \|P_c \chi_{A_c}\|_1 \leq \delta^{-1}$$

Defining $\overline{A}_c = A_c \setminus \bigcup_{\substack{c'\in\mathcal{C} \\ c'<c \\ \neq}} \left(A_{c'} + \left[-\frac{R2^{|c|}}{N}, \frac{R2^{|c|}}{N}\right]\right)$, (1) gives

$\int_{\overline{A}_c} |P_c|^2 > \kappa^2 \frac{N}{2^{|c|}}$. For $\alpha \sim \kappa^4$, consider the de la Vallée Poussin type kernel based on I_c



Defining $Q_c = (P_c \chi_{\overline{A}_c})*V_c$, one gets thus

$$|\langle P_c, Q_c\rangle| \geq |\langle P_c, P_c\chi_{\overline{A}_c}| - |\langle P_c, P_c\chi_{\overline{A}_c} - Q_c\rangle| \geq \kappa^2 |I_c| - \|(P_c\chi_{\overline{A}_c})\hat{}(1-\hat{V}_c)\|_{\ell^1(I_c)}$$

$$\geq \kappa^2|I_c| - 2(\alpha|I_c|)^{1/2} \|P_c\chi_{\overline{A}_c}\|_2 \geq (\kappa^2 - 2\alpha^{1/2})|I_c| > \frac{1}{2}\kappa^2|I_c|$$

Hence (up to multiplication by a factor) Q_c fulfils (2), (3), (4) with $\sigma = \frac{1}{2}\kappa^2$. It is possible to choose $\hat{V}_c$ sufficiently smooth so that the decay property of $\hat{V}_c$ yields

(10)
$$\int_{\{|t| > \frac{R}{2|I_c|}\}} |V_c(t)| \leq C\kappa^{-20} R^{-8}$$

Define $Q_{c,1} = (P_c\chi_{\overline{A}_c})*(V_c \chi_{[-\frac{R}{2|I_c|}, \frac{R}{2|I_c|}]})$ which is supported by $\overline{A}_c + [-\frac{R}{2|I_c|}, \frac{R}{2|I_c|}]$. It also follows from (9), (10) that

(11)
$$\|Q_c - Q_{c,1}\|_1 \leq C\kappa^{-20}\delta^{-1}R^{-8} \equiv \theta$$

Let $c, c' \in \mathcal{C}$, $c' \underset{\neq}{<} c$. By definition of $\overline{A}_c$

(12)
$$\operatorname{dist}(\overline{A}_c, A_{c'} + [-\frac{R}{2|I_c|}, \frac{R}{2|I_c|}]) > \frac{R}{2|I_c|}$$

Estimate using (9), (11)

$$|\langle Q_c, Q_{c'}\rangle| \leq \|Q_c\|_\infty \|Q_{c'} - Q_{c',1}\|_1 + |\langle Q_c, Q_{c',1}\rangle|$$

(13)
$$\leq \theta|I_c| + \|Q_{c',1}\|_1 \|Q_c|_{\overline{A}_{c'} + [\frac{-R}{2|I_c|}, \frac{R}{2|I_c|}]}\|_\infty$$

$$\leq \theta|I_c| + \delta^{-1}|I_c| \, \|(\chi_{\overline{A}_c} * |V_c|)|_{A_{c'} + [\frac{-R}{2|I_c|}, \frac{R}{2|I_c|}]}\|_\infty$$

By (12), the second term in (13) is bounded by $\delta^{-1}|I_c| \int_{\{|t|>\frac{R}{2|I_c|}\}} |V_c(t)| < \theta|I_c|$, using (10). Consequently

$$|\langle Q_c, Q_{c'}\rangle| \leq C\kappa^{-20}\delta^{-1}R^{-8}|I_c| \quad \text{for} \quad c,c' \in \mathcal{C}, \quad c \gneq c' \tag{14}$$

and therefore, coming back to (7)

$$\begin{aligned}
\sum_{\substack{c,c'\in\mathcal{C}\\ c\gneq c'}} |\langle Q_c, Q_{c'}\rangle| &\leq C\kappa^{-20}\delta^{-1}R^{-8} \sum_{c'\in\mathcal{C}} \sum_{\substack{c\in\mathcal{C}\\ c\gneq c'}} |I_c| \\
&\leq C\kappa^{-20}\delta^{-1}R^{-8}K \sum_{c'\in\mathcal{C}} |I_{c'}| \\
&= C\kappa^{-20}\delta^{-1}R^{-8}K\,(|\Omega_1|+\dots+|\Omega_K|) \quad .
\end{aligned}$$

Thus in order that the second term in (7) should be smaller than the first

$$R^8 > C\kappa^{-20}K\delta^{-1} \quad . \tag{15}$$

This and (8) gives the estimate

$$R^8 \sim C\delta^{-1}\kappa^{-24}\left(\frac{1}{\varepsilon} + \left(\log\frac{1}{\delta}\right)^3\right) \quad . \tag{16}$$

<u>REMARK</u> : The dependence of R on δ in (15) may be avoided using a more delicate argumentation.

11. VERIFICATION OF INEQUALITY (9.18).

We will use the following lemma :

LEMMA : Let $p(z) = \frac{1}{d}\sum_{j=1}^{d} \varepsilon_j z^j$, $|\varepsilon_j| \leq 1$ and $\{z_k\} \subset \mathbb{C}_1$ a d^{-1}-separated set. Then

(1) $$\sum |p(z_k)|^2 \leq C$$

Proof : Let V be a kernel satisfying

(2) $$\begin{cases} \hat{V} = 1 \quad \text{on} \quad [0,d] \\ |V(t)| \leq Cd[1+(|t|d)^2]^{-1} \end{cases}$$

Writing $z_k = e^{2\pi i\theta_k}$, one has

$$|p(z_k)| \leq \int_0^1 |p(e^{2\pi i\psi})| \; |V(\theta_k-\psi)|d\psi$$

and hence, since by (2)

$$\sum |V(\theta_k-\psi)| \leq Cd \; \sum(1+d^2|\theta_k-\psi|^2)^{-1} < Cd \; ,$$

$$\Sigma_k|p(z_k)|^2 \leq \Sigma_k \int_0^1 |p(e^{2\pi i\psi})|^2 \; |V(\theta_k-\psi)|d\psi \leq Cd \; \|p\|_2^2 < C \; . \blacksquare$$

We verify (9.18) and estimate thus for $c_o \in C_{k-1}$ and T_{c_o} defined as in section 9

$$\sum_{c\in T^*_{c_o}} 2^{-|c'|} D(\overline{p}_{c',\delta} \mid c_o < c' < c)^2$$

where $\overline{p}_{c,\delta} = p_c\sigma_\delta(|p_c|)\eta_{c,\delta}$, the functions $\eta_{c,\delta}$ to be specified. It follows from the previous lemma that the entropy-number $E(A_c, \frac{2^{|c|}}{N}) \leq C\delta^{-2}$. Also, from the construction,

$$(3) \qquad \int_{A_c \setminus (B_{c_o} + [-R\frac{2^{|c|}}{N},\ R\frac{2^{|c|}}{N}])} |p_c|^2 \leq \kappa^2 \frac{2^{|c|}}{N} \quad \text{for} \quad c \in T_{c_o}$$

letting $B_{c_o} = \bigcup_{c \in C, c \leqq c_o} A_c$.

Since $E(B_{c_o}, \frac{2^{|c_o|}}{N}) \leq CK\delta^{-2}$, there is a set $P = P_o$,

$$(4) \qquad |P| \leq CK\delta^{-2}$$

such that for $c \in T_{c_o}$

$$(5) \qquad \int_{A_c \setminus (P + [-2R\frac{2^{|c|}}{N},\ 2R\frac{2^{|c|}}{N}])} |p_c|^2 \leq \kappa^2 \frac{2^{|c|}}{N} .$$

We will perform a further dissection of $T \equiv T_{c_o}$ according to the entropy numbers of the set P . Take ν satisfying

$$(6) \qquad \nu \leq \kappa^2 R^{-1} .$$

Let $n_o = |c_o|$. Perform the following construction : $n_1 > n_o$ is the largest integer such that $E(P, \frac{2^{n_1}}{N}) > E(P, \frac{2^{n_o}}{N}) - \nu\delta^{-2}$. Consider $P_1 \subset P$ a $\frac{2^{n_1}}{N}$ -separated set such that

$$P \subset P_1 + [-\frac{2^{n_1}}{N}, \frac{2^{n_1}}{N}]$$

$n_2 > n_1$ is the largest integer such that $E(P_1, \frac{2^{n_2}}{N}) > E(P_1, \frac{2^{n_1}}{N}) - \nu\delta^{-2}$. Consider $P_2 \subset P_1$ a $\frac{2^{n_2}}{N}$ -separated set such that

$$P_1 \subset P_2 + [-\frac{2^{n_2}}{N}, \frac{2^{n_2}}{N}]$$

etc...

Since $|P| = E(P_o , \frac{2^{n_o}}{N}) \geq E(P_o , \frac{2^{n_1+1}}{N}) + \nu\delta^{-2}$

$$\geq E(P_2 , \frac{2^{n_2}}{N}) + \nu\delta^{-2}$$

$$\geq E(P_2 , \frac{2^{n_3+1}}{N}) + 2\nu\delta^{-2} \geq \dots \geq \frac{v}{2}\, \nu\delta^{-2} ,$$

it follows from (4) that the construction must stop for

(7) $$v \leq CK\nu^{-1}$$

Tail $\mathcal{T}$ in the regions (sub-trees)

$$\mathcal{T}_u = \{c \in \mathcal{T} \mid n_{u-1} \leq |c| < n_u\} \qquad 1 \leq u < v .$$

It follows from (5), (6) and the construction that for $c \in \mathcal{T}_u$ and $Q = P_u$

(8) $$\int_{A_c \setminus (Q+[-3R \frac{2^{|c|}}{N} , 3R \frac{2^{|c|}}{N}])} (|p_c| \wedge \delta)^2 \leq \int_{A_c \setminus (P+[-2R \frac{2^{|c|}}{N} , 2R \frac{2^{|c|}}{N}])} |p_c|^2 + C\delta^2 R|P_{u-1} \setminus Q| \frac{2^{|c|}}{N} \leq (\kappa^2 + C R\nu) \frac{2^{|c|}}{N} < 2\kappa^2 \frac{2^{|c|}}{N}$$

since indeed

$$P \subset P_{u-1} + [-2 \frac{2^{n_{u-1}}}{N} , 2 \frac{2^{n_{u-1}}}{N}] ;$$

$$P+[-2R \frac{2^{|c|}}{N} , 2R \frac{2^{|c|}}{N}] \subset P_{u-1}+[-3R \frac{2^{|c|}}{N} , 3R \frac{2^{|c|}}{N}] .$$

For $c \in \mathcal{T}_u$, take $0 \leq \eta_{c,\delta} \leq 1$ satisfying

$$\begin{cases} \eta_{c,\delta} = 1 \quad \text{on} \quad 3R\,\frac{2^{|c|}}{N}\text{-neighborhood of } Q & (9) \\ \eta_{c,\delta} = 0 \quad \text{outside} \quad 4R\,\frac{2^{|c|}}{N}\text{-neighborhood of } Q & (10) \\ |\eta'_{c,\delta}| \le |I_c| & (11) \end{cases}$$

The functions $\overline{p}_{c,\delta} = p_c \sigma_\delta(|p_c|)\eta_{c,\delta}$ of section 9 satisfies then by (8)

$$\|\overline{p}_{c,\delta} - p_c\sigma_\delta(|p_c|)\|_2 \le C\kappa |I_c|^{-1/2}$$

and (9.4) is thus satisfied for $\kappa \sim \tau^2(\log \frac{1}{\delta})^{-4}$.

Consider the system $(p_c|_Q)_{c\in T_u}$ as an ℓ^2_Q-valued martingale. Since the points of $Q = P_u$ are $\frac{2^{n_u}}{N}$-separated and $n_u > |c|$ for $c \in T_u$, the above Lemma applied to

$$p = p_c\, z^{-\Sigma c_j 2^{-j} N}$$

gives that

$$\|p_c|_Q\|_{\ell^2_Q} \le C \tag{12}$$

It immediately follows from the scalar (or Hilbertian) variation estimate (6.1) with $p = 2$, that for $\lambda > 0$

$$\sum_{c\in T^*_{c_o}} 2^{-|c|}\, A_\lambda(p_{c|j}|_Q \;;\; n_{u-1} \le j < n_u) \le C\lambda^{-2}\, 2^{-|c_o|} \tag{13}$$

where A_λ refers to the λ-metrical entropy number.

For fixed $c \in T^*_{c_o}$, apply inequality (8.1) to the sequence of multipliers

$$(Z^{-\Sigma c_i 2^{-i} N} p_{c|j})_{n_{u-1} \leq j < n_u}$$

to get, by (9), (10), (11), for each $\lambda > 0$

$$D(\overline{p}_{c|j,\delta};\ n_{u-1} \leq j < n_u) \leq C(\lambda + \delta\ A_\lambda^{1/2}(\log \frac{|Q|R}{\delta})^2)\ .$$

Here $\Lambda = Q$ and $L = 1$. Hence, by (13) and (4), (10.16), (10.8)

$$\sum_{c \in T^*_{c_o}} 2^{-|c|}\ D(\overline{p}_{c|j,\delta};\ n_{u-1} \leq j < n_u)^2 \leq$$

$$C\lambda^2\ 2^{-|c_o|} + C\delta^2(\log \frac{KR}{\delta})^2 \sum_{c \in T^*_{c_o}} 2^{-|c|}\ A_\lambda(p_{c|j}\big|_Q;\ n_{u-1} \leq j < n_u) \leq$$

(14) $$C(\lambda^2 + \delta^2\lambda^{-2}(\log(\tau\varepsilon\delta)^{-1})^2)\ 2^{-|c_o|}$$

Summation over the different values $0 \leq u < v$ and applying (9.5), (7), (6), for an optimal choice of λ in (14)

$$\sum_{c \in T^*_{c_o}} 2^{-|c|}\ D(\overline{p}_{c',\delta}\ |\ c_o < c' < c)^2 \leq C\delta(\log(\tau\varepsilon\delta)^{-1}).v\ 2^{-|c_o|}$$

$$\leq c(\varepsilon\tau)^{-C}(\log \frac{1}{\delta})^C\ \delta^{7/8}\ 2^{-|c_o|}.$$

Thus, by (9.17), (9.19), (10.8), (9.18) will be fulfilled if $\delta^{7/8}(\log \frac{1}{\delta})^C > (\varepsilon\tau)^C$, which happens for δ small enough.

12. VERIFICATION OF INEQUALITY (9.15).

Let now $\delta > \delta_o$ and $c_o \in \mathcal{C}_{k-1}$ in the construction of Section 10, performed for this δ . For c in the subtree $\mathcal{T}_{c_o}$, the localization function $\eta_{c,\delta}$ is taken as follows :

(1) $\eta_{c,\delta} = 1$ on $2R\,\frac{2^{|c|}}{N}$ -neighborhood of P

(2) $\quad = 0$ outside $3R\,\frac{2^{|c|}}{N}$ " " "

(3) $|\eta'_{c,\delta}| \leq |I_c|$

Here P is the $\frac{2^{|c_o|}}{N}$ -net in B_{c_o} , introduced in the previous section, hence,

(4) $|P| \leq c(\varepsilon,\tau)$ (by (4), since $\delta > \delta_o(\tau)$)

(5) $\bar{p}_{c,\delta} = p_c\sigma_\delta(|p_c|)\eta_{c,\delta}$ satisfies condition (9.4) (by (10.5)).

It has to be shown that for t sufficiently large (depending on ε,δ_o) , one may achieve the estimate

(6) $$\Sigma\ \{2^{-|c|} \mid c \in \mathcal{T}^*_{c_o} \text{ and } B_\tau(\bar{p}_{c',\delta} \mid c_o < c' < c) > t\} < \varepsilon' 2^{-|c_o|} .$$

This is possible using the localization and tree-structure of the polynomials $(\bar{p}_{c,\delta})_{c\in\mathcal{T}_{c_o}}$ and $(p_c)_{c\in\mathcal{T}_{c_o}}$. Since P is a fixed set of bounded size and invoking the variation estimates for martingales discussed in Section 6, here applied to $(p_c|_P)_{c\in\mathcal{T}_{c_o}}$, (6) may easily be derived from the following

LEMMA : Let $(\varphi_j)_{j=1,2,\ldots}$ be a sequence of multipliers on $\mathbb{T}$, $\lambda_o \in \mathbb{T}$, such that

(7) $$|\varphi_j| \leq 1 \ , \quad |\varphi_j'| \leq 2^j$$

(8) The number of $c\tau^2$-jumps in the sequence $\{\varphi_j(\lambda_o)\}$ is at most r (where $c > 0$ is some constant).

Consider localizing functions $0 \leq \eta_j \leq 1$ satisfying

(9) $$\eta_j(\lambda_o) = 1 \ , \quad \eta_j \text{ vanishing outside a } 2^{-j}\text{-neighborhood of } \lambda_o$$

(10) $$|\eta_j'| \leq 2^j \ .$$

Then

(11) $$B_\tau(\varphi_j\eta_j \mid j = 1,2,\ldots) \leq r + C\tau^{-2} \log \frac{1}{\tau}$$

Proof : Define $\psi_j = \varphi_j\eta_j$. Let $f \in \ell^2(\mathbb{Z})$, $\|f\|_2 \leq 1$ and $j_1 < j_2 < \cdots < j_t$ integers such that

(12) $$|\varphi_j(\lambda_o) - \varphi_{j_{s-1}}(\lambda_o)| < C\tau^2 \quad \text{for} \quad j_{s-1} \leq j < j_s$$

(13) $$\Big\| \sup_{j_{s-1}\leq j<j_s} | F^{-1}(Ff(\psi_j-\psi_{j_{s-1}}))\Big\|_2 > \tau \ .$$

Fix $1 \leq s \leq t$. In order to avoid repetition of an argument appearing earlier in the paper, we make use of inequality (8.1) in Section 8. Take $a \sim \log \frac{1}{\tau}$ and define for $j_{s-1} \leq j < j_s$ the multiplier $\gamma_j = (\varphi_j-\varphi_{j_{s-1}})\eta_{j+a}$. Then by (12), (7), (9), for all λ

$$|\gamma_j(\lambda)| \leq (|\varphi_j(\lambda)-\varphi_j(\lambda_o)| + |\varphi_{j_{s-1}}(\lambda) - \varphi_{j_{s-1}}(\lambda_o)|)\eta_{j+a}(\lambda) + C\tau^2 < 2C\tau^2$$

while by (7), (10)

$$|\gamma_j'| \leq 2^a 2^j \quad .$$

Apply (8.1) to the system $\{\gamma_j \mid j_{s-1} \leq j < j_s\}$ with $\Lambda = \{\lambda_o\}$, $L = 2^a$, $\delta \sim \tau^2$.

It follows that

$$D(\gamma_j \eta_j \mid j_{s-1} \leq j < j_s) \leq C\tau^2 \log \frac{1}{\tau}$$

and therefore, by (13)

$$(14) \qquad \sup_{j_{s-1} \leq j < j_s} |F^{-1}[Ff(\psi_j - \psi_{j_{s-1}} - \gamma_j \eta_j)]| \|_2 > \frac{\tau}{2}$$

Write for $j_{s-1} \leq j < j_s$

$$(15) \qquad \psi_j - \psi_{j_{s-1}} - \gamma_j \eta_j = \varphi_{j_{s-1}}(\eta_j - \eta_{j_{s-1}}) + (\varphi_j - \varphi_{j_{s-1}})\eta_j(1-\eta_{j+a})$$

and

$$(16) \qquad \eta_j - \eta_{j_{s-1}} = (1-\eta_j)(\eta_{j_s} - \eta_{j_{s-1}}) + \eta_j(1-\eta_{j_{s-1}}) - \eta_{j_s}(1-\eta_j) \quad .$$

Substitution of (16) in (15) and using the square-function technique and Parseval's identity

$$\| \sup_{j_{s-1} \leq j < j_s} |F^{-1}[Ff(\psi_j - \psi_{j_{s-1}} - \gamma_j \eta_j)]| \|_2^2 \leq$$

$$(17) \qquad C \| \sup_{j_{s-1} \leq j < j_s} |F^{-1}[Ff(1-\eta_j)(\eta_{j_s} - \eta_{j_{s-1}})\varphi_{j_{s-1}}]| \|_2^2 +$$

$$C \int |\hat{f}(\lambda)|^2 \sum_{j_{s-1} \leq j < j_s} [|\eta_j(1-\eta_{j+a})|^2 + |\eta_j(1-\eta_{j_{s-1}})|^2 + |\eta_{j_s}(1-\eta_j)|^2] d\lambda$$

The first term of (17) is bounded by

$$(18) \qquad C \int |\hat{f}(\lambda)|^2 |\eta_{j_s} - \eta_{j_{s-1}}|^2 \, d\lambda \quad .$$

Summing of the second term of (17) and (18) requires to get a pointwise bound on

$$(19) \qquad \Sigma_j \eta_j (1-\eta_{j+a}) + \Sigma_s \sum_{j_{s-1} \le j < j_s} [(1-\eta_{j_{s-1}})\eta_j + (1-\eta_j)\eta_{j_s}] + \Sigma_s |\eta_{j_s} - \eta_{j_{s-1}}| .$$

From (9), (10) one easily derives that the first term is bounded by Ca and the next two by a constant. Hence we find by (14)

$$t \frac{\tau^2}{4} \le C \log \frac{1}{\tau} + C$$

implying (11).

This takes care of the condition (9.15), hence (9.10). This concludes the proof of the main inequality (5.1) and the theorem on the return-times stated in the first section.

APPENDIX 1 : APPLICATION OF MAXIMAL INEQUALITY TO POINTWISE ERGODIC THEOREMS FOR "ARITHMETIC SETS".

By arithmetic set, I mean sets such as
$\Lambda = \{n^2 \mid n=1,2,\ldots\}$
$\Lambda = \{n^3\}$
$\Lambda = \{p(n)\}$; $p(x)$ a polynomial with integer coefficients
$\Lambda = \{\text{prime numbers}\}$

for which the pointwise ergodic theorem was studied in [B1] (the L^2-theory) and [B2] (the L^p-theory). It turns out that the maximal inequality stated in section 7 permits to recover the results from [B1] (without using A. Weil's inequality) and improve the critical exponent in the L^p-theory relative to the method described in [B2]. Recall the following definition

DEFINITION : $\Lambda \subset \mathbb{Z}_+$ is a good sequence in L^p $(1 \le p \le \infty)$ for the pointwise ergodic theorem provided the averages

$$\frac{1}{|\Lambda_N|} \sum_{n \in \Lambda_N} T^n f$$

converge a.s, for any DS (Ω, μ, T) and $f \in L^p(\mu)$.

As before Λ_N stands for $\Lambda \cap [o,N]$.

The discussion in this section is based on [B1],[B2], which the reader may consult for further details, and inequality (7.1) of section 7 in this paper constitutes the new ingredient.

We consider the set $\Lambda = \{n^t\}$, $t \ge 2$ an integer. There are no essential differences in the treatment of the other sets listed above, except for the explicit description of the corresponding exponential sums. In the shift model $(\mathbb{Z}, S)$

$$\frac{1}{N} \sum_{n \le N} T^{(n^t)} f = f * K_N \text{ where } K_N = \frac{1}{N} \sum_{n \le N} \delta_{\{n^t\}}$$

and thus

$$\hat{K}_N(\alpha) = \frac{1}{N} \sum_{n\leq N} e^{2\pi i n^t \alpha} \quad ; \alpha \in \mathbb{T} \approx [o,1]$$

Based on inequality (7.1) in this paper, the estimate in Section 3 of [B1] may be replaced by the following

LEMMA: Let $o<\tau<1$ and $o\leq\varphi\leq 1$ vanish outside a τ-neighborhood of o a localizing function. Let R be τ-separated points in Π. Define for $f\in\ell^2(\mathbb{Z})$ and $T>o$

$$A_T f(x) = \sum_{\theta\in R} \int \hat{\chi}(T(\alpha-\theta))\hat{f}(\alpha)e^{2\pi i\alpha x}\varphi(\alpha-\theta)d\alpha \quad ; \chi = \chi_{[o,1]} \tag{1}$$

and the maximal operator

$$Mf = \sup_{j\in\mathbb{Z}} |A_{2^j} f|$$

The following inequality holds

$$\|Mf\|_2 \leq C(\log|R|)^2 \|f\|_2 \tag{2}$$

The details are routine (square function arguments as used earlier in this paper) and left to the reader. The main difference with the corresponding lemma in [B1] is that we do not assume R contained in a τ-separated arithmetic progression, provided a weak dependence on $|R|$ appears in (2).

Define for $1\leq a\leq q$, $(a,q) = 1$, the Weyl sum

$$S(q,a) = \frac{1}{q} \sum_{o\leq r<q} e^{-2\pi i r^t \frac{a}{q}} \tag{3}$$

and denote

$$K(x) = t^{-1} x^{\frac{1}{t}-1} \chi_{[o,1]}(x) \tag{4}$$

Consider the generations

$$R_s = \{\tfrac{a}{q} \mid 1<a<q,\ (a,q) = 1 \text{ and } 2^{s-1}\leq q<2^s\} \tag{5}$$

and the multipliers on Π

$$\psi_{s,N}(\alpha) = \sum_{\theta \in R_s} S(q,a)\ \hat{k}(N^t(\alpha-\theta))\ \varphi(4^s(\alpha-\theta)) \ ; \ \theta = \frac{a}{q} \tag{6}$$

φ = bumpfunction localizing on the interval $[-\frac{1}{2},\frac{1}{2}]$

The main problem consists then in obtaining an inequality for each $s = 1,2,...$

$$\| \sup_{N \text{ diadic}} |F^{-1}[\psi_{s,N} Ff]| \|_2 \le \delta_s \|f\|_2 \tag{7}$$

where

$$\Sigma \delta_s < \infty$$

Writing

$$k = \frac{1}{t}\chi - \int_o^1 yk'(y)\ \chi_y(x)\ dy$$

$$\chi \equiv \chi_{[o,1]} \quad \text{and} \quad \chi_y \equiv \frac{1}{y}\chi(\frac{x}{y}) = \frac{1}{y}\chi_{[o,y]}(x)$$

one deduces from previous lemma that

$$\| \sup_{N \text{ diadic}} \Big| \sum_{\theta \in R_s} \int \hat{k}(N^t(\alpha-\theta))\hat{f}(\alpha)e^{2\pi i\alpha x}\varphi(4^s(\alpha-\theta))d\alpha \Big| \|_2 \le C\ s^2 \|f\|_2$$

Here the lemma is applied with $R = R_s$, $\tau = 4^{-s}$, $\varphi = \varphi_s = \varphi(4^s(.))$. Thus $\log|R_s| \sim s$.

We will only need an estimate

$$|S(q,a)| \le C\ q^{-\varepsilon}, \quad \varepsilon = \varepsilon(t) > o \tag{8}$$

(obtained from H. Weyl's inequality).

Getting the $S(q,a)$-multiplications by introducing an additional multiplier uniformly bounded by $C\ 2^{-s\varepsilon}$ for $\frac{a}{q} \in R_s$, one indeed gets in (7) $\delta_s < C\ s^2\ 2^{-s\varepsilon}$, a summable sequence.

The previous argument applies to all examples of sets mentioned in the beginning of this section. It is likely that one may find other interesting examples.

Next, we consider the L^p-theory $p<2$ and refer the reader to [B2]. We consider the set $\Lambda = \{n^t\}$, as in [B2]. Our purpose is to lower the critical index $\frac{1}{2}(1+\sqrt{5})<p$ reached in [B2].

PROPOSITION: $\Lambda = \{n^t\}$ is a good sequence in L^p for $p>\frac{3}{2}$.
The method of [B2] is based on interpolation between L^2 and L^r, $r>1$. The improvement is due to better L^2-estimates from previous lemma. In order to get some L^r-controle for the different generations of major arcs, one has to partition the rationals in arithmetic progressions, taking into account the size of the exponential sums (3), and the finer results of A. Weil are of relevance for this purpose.

From A. Weil's result and the multiplicativity property, there is a bound

$$|S(q,a)| \leq \alpha(q) \quad (a,q) = 1$$

where $\alpha(q)$ satisfies

$$\Sigma_q \alpha(q)^\gamma < \infty \text{ for } \gamma>2 \tag{9}$$

(cf. Lemmas of [B2]). Thus if we let

$$Q_s = \{q \,|\, 2^{-s+1} \geq \alpha(q) > 2^{-s}\}.$$

then

$$|Q_s| < C_\gamma \, 2^{\gamma s}. \tag{10}$$

Denote $R(q) = \{1\leq a\leq q \,|\, (a,q) = 1\}$ and $R_s = \{\frac{a}{q} \,|\, q \in Q_s,\ a\in R(q)\}$.
Let $\psi_{s,N}$ be again given by

$$\psi_{s,N}(\alpha) = \sum_{\theta \in R_s} S(q,a)\hat{k}(N^t(\alpha-\theta))\varphi(M^s(\alpha-\theta)) \tag{11}$$

where $M = M(t)$ satisfies $|\theta-\theta'| > 10M^{-s}$ for $\theta \neq \theta'$ in R_s. This is possible by (8).

For $1<p\leq 2$, let $\delta_s(p)$ be the best constant satisfying the

inequality

$$\Big\| \sup_{N \text{ diadic}} | F^{-1}[\psi_{S,N} \, Ff] \Big\|_p \leq \delta_s(p) \| f \|_p \tag{12}$$

The condition

$$\Sigma_s \; \delta_s(p) < \infty \tag{13}$$

has to be satisfied.

Let $r > 1$ be fixed and $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{r}$. Interpolation gives

$$\delta_s(p) \leq \delta_s(2)^{1-\theta} \; \delta_s(r)^{\theta} \tag{14}$$

where, similarly as above, invoking (10)

$$\delta_s(2) \leq C \; 2^{-s} \; (\log|R_s|)^2 \leq C_t \; s^2 \; 2^{-s} \tag{15}$$

For $q \in Q_s$, denote $B_q(r)$ the best constant fulfilling

$$\Big\| \sup_{N \text{ diadic}} \Big| \sum_{a \in R(q)} S(q,a) \int \hat{k}(N^t(\alpha - \tfrac{a}{q}))\, \hat{f}(\alpha) e^{2\pi i \alpha x} \varphi(M^s(\alpha - \tfrac{a}{q}))\, d\alpha \Big| \Big\|_r \leq B_q(r) \| f \|_r$$

Thus, from the definition of R_s and (10),

$$\delta_s(r) \leq \sum_{q \in Q_s} B_q(r) \leq C \; 2^{\gamma s} \, . \sup_{q \in Q_s} B_q(r) \tag{16}$$

We now invoke [B2] (Lemma 7 and the estimates in section 6) to get for $q < c \, D$

$$\Big\| \sup_{N \text{ diadic}} \Big| \sum_{0 \leq a < q} \tilde{S}(\tfrac{a}{q}) \int \hat{k}(N^t(\alpha - \tfrac{a}{q}))\, \hat{f}(\alpha) e^{2\pi i \alpha x} \varphi(D(\alpha - \tfrac{a}{q}))\, d\alpha \Big| \Big\|_r \leq C_r \| f \|_r \tag{17}$$

where $\tilde{S}(\frac{a}{q}) \equiv S(q,a)$ for $(a,q) = 1$.

Denote $p_1, \ldots, p_m$ the different prime factors of $q \in Q_s$. Then writing

$$\sum_{a \in R(q)} = \sum_{I \subset \{1,\ldots,m\}} (-1)^{|I|} \sum_{1 \leq a' < q \prod_{\ell \in I} p_\ell^{-1}} \tag{18}$$

and using (17) to estimate the contribution of the I-sums in (18), it follows that

$$B_q(r) \le 2^m C_r < C_{r,\tau} 2^{s\tau} \quad (\tau > o) \tag{19}$$

Collecting estimates (14), (15) and (16), it follows that

$$\delta_s(p) \le C\ 2^{s\tau}\ 2^{-s(1-\theta)}\ 2^{\gamma s\theta} \quad (\tau > o, \gamma > 2) \tag{20}$$

where C depends on $r > 1$, τ and γ.

Clearly, for $p > \frac{3}{2}$, we may choose $r > 1$ such that $\theta < \frac{1}{3}$. Inequality (20) implies $\delta_s(p) < C\ 2^{-\tau' s}$ for some $\tau' > o$, hence condition (13).

REMARKS.

(1) We did not follow the notations of [B2] in what preceeds. In fact, the use of the new L^2-estimate leads to considerable simplifications and one does not have to introduce arithmetic progressions besides the natural ones.

(2) The proposition remains valid for the sets $\Lambda = \{p(n)\}$, where $p(x)$ is a polynomial with integer coefficients or taking integer-values for $x \in \mathbb{Z}$.

(3) Adaptation of previous argument for the primes shows that $\Lambda = \{\text{primes}\}$ is a good sequence in L^p for $p > \frac{4}{3}$ (cf. [B3])

We conclude this section with a discussion of the pointwise ergodic theorem for sets $\Lambda = \{[p(n)]; n = 1, 2, \ldots\}$, where $[x]$ stands for the integer part of $x \in \mathbb{R}$ and $p(x)$ is a polynomial with real coefficients.

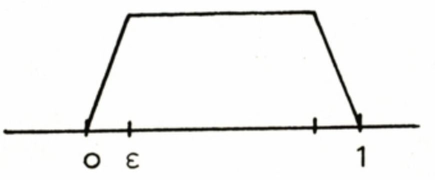
PROPOSITION: $\{[p(n)]\}$ satisfies the pointwise ergodic theorem for L^∞-functions.

It is easily seen that the case of polynomials with rational coeffficients reduces to the case of a polynomial with integer coefficients. Hence, we assume at least one of the coefficients of $p(x)$ irrational. Hence the sequence $p(n) - [p(n)]$ is uniformly distributed mod 1.

Let (Ω,μ,T) be a DS and $f \in L^\infty$. We have to show the a.s. convergence of

$$A_N f = \frac{1}{N} \sum_{1 \le n \le N} T^{[p(n)]} f \tag{21}$$

Fix $\varepsilon > 0$ and consider the function $\tau = \tau_\varepsilon$ on $\mathbb{R}$



Denote

$$\tilde{A}_N f = \frac{1}{N} \sum_{n=1}^{N} \sum_{m \in \mathbb{Z}} \tau(p(n)-m)\ T^m f \tag{22}$$

Clearly, invoking the uniform distribution property, there is the pointwise inequality

$$|A_N f - \tilde{A}_N f| \le \frac{\|f\|_\infty}{N} \#\{1 \le n \le N \mid \operatorname{dist}(p(n),\mathbb{Z}) < \varepsilon\} \le 2\varepsilon \|f\|_\infty$$

for N large enough.

Thus, it suffices to show the a.s. convergence of (22) (for a fixed $\varepsilon>0$), assuming $f \in L^2(\mu)$ (the hypothesis $f \in L^\infty$ is only of relevance when replacing $A_N f$ by $\tilde{A}_N f$).

The relevant exponential sums are $(z=e^{2\pi i\alpha})$

$$K_N(\alpha) = \frac{1}{N} \sum_{n\le N} \sum_{m\in\mathbb{Z}} \tau(p(n)-m)\, z^m$$

$$= \sum_{k\in\mathbb{Z}} \hat{\tau}(\alpha+k) \{\frac{1}{N} \sum_{n\le N} e^{2\pi i(\alpha+k)p(n)}\} \tag{23}$$

Denoting

$$\phi_N(\beta_1,\dots,\beta_t) = \frac{1}{N} \sum_{n\le N} e^{2\pi i(\beta_1 n+\beta_2 n^2+\dots+\beta_t n^t)}$$

and $p(x) = b_o+b_1x+\dots+b_t x^t$ $(b_t\neq o)$, (23) gives

$$\hat{K}_N(\alpha) = \sum_{k\in\mathbb{Z}} \hat{\tau}(\alpha+k)\, e^{2\pi i(\alpha+k)b_o}\, \phi_N(b_1(\alpha+k),\dots,b_t(\alpha+k)) \tag{24}$$

Observe also that

$$|\hat{\tau}(\lambda)| \le \frac{C}{1+\varepsilon^2\lambda^2} \tag{25}$$

The description of the behaviour of $\phi_N(\beta_1,\dots,\beta_t)$ (see [Vin],[B1]) permits then to find a suitable "major arc" type description of $\hat{K}_N$ in order to apply the approach described in [B1] and the beginning of this section. (Here, these major arcs are not necessarily centered around rational points). To be more precise, consider the particular case $p(x) = bx^t$, $b \in \mathbb{R}\setminus\mathbb{Q}$. For $s = o,1,2,\dots$ and $k \in \mathbb{Z}$, define the set of points

$$\mathcal{R}_{s,k} = \{\theta\in[o,1] \mid b(\theta+k) \equiv \frac{a}{q} \ (\text{mod}\,1) \text{ for some } 1\le a\le q,\ (a,q)=1,\ 2^{s-1}\le q<2^s\}.$$

For $\theta \in \mathcal{R}_{s,k}$, denote $S_\theta = S(q,a)$, where $b(\theta+k) = \frac{a}{q}$ (mod 1).

Define further

$$\psi_{s,k,N} = \sum_{\theta\in\mathcal{R}_{s,k}} S_\theta\, \hat{k}(N^t b(\alpha-\theta))\, \varphi(4^s b(\alpha-\theta)) ,$$

where the kernel k is given by (4),

$$\psi_{s,N}(\alpha) = \sum_{k\in\mathbb{Z}} \hat{\tau}(\alpha+k)\psi_{s,k,N}(\alpha)$$

Thus, by (24) and the description of the exponential sum $\frac{1}{N} \sum_{n \leq N} e^{2\pi i \beta n^t}$,

$$\hat{K}_N = \sum_{s=o}^{\infty} \psi_{s,N} + o(N^{-\varepsilon})$$

Since $|S_\theta| < C\, 2^{-\varepsilon s}$, $\varepsilon = \varepsilon(t) > o$, the same argument as above gives

$$\| \sup_{N \text{ diadic}} |F^{-1}[\psi_{s,k,N} Ff]| \|_2 \leq C\, s^2\, 2^{-\varepsilon s} \|f\|_2$$

and taking (25) into account, also

$$\| \sup_{N \text{ diadic}} |F^{-1}[\psi_{s,N} Ff]| \|_2 \leq C_\varepsilon\, s^2\, 2^{-\varepsilon s} \|f\|_2 \tag{26}$$

Thus the convergence problem reduces to the initial values of s. Again by (25), it suffices to consider individual sequences $F^{-1}[\psi_{s,k,N} Ff]$, N diadic, for which the following holds, $r>2$

$$\| \{F^{-1}[\psi_{s,k,N} Ff] \mid N \text{ diadic}\} \|_{\ell^2_{v_r}(\mathbb{Z})} \leq C_r(s) \|f\|_2$$

where v_r are the variationspaces defined in section 6 of this paper.

REMARK. It is possible to prove that $\{[p(n)]\}$ satisfies the pointwise ergodic theorem on L^p, $p > \frac{3}{2}$. This can be achieved by interpolation methods based on the approximation $|A_N f - A_{N,\varepsilon} f|$, $A_{N,\varepsilon} = \tilde{A}_N$ for $f \in L^\infty$ and the actual bound $\|\hat{\tau}_\varepsilon\|_{L^1(\mathbb{R})} \leq C \log \frac{1}{\varepsilon}$. In the case $p(x) = bx^t$ for instance, one has to take the rational approximation properties of b into account.

APPENDIX 2 : BASES PROPERTIES FOR RETURN TIME SEQUENCES.

In this section, we will consider the return time sequences for weakly mixing DS's (Ω,β,μ,T), meaning that T is ergodic and has no non-trivial point-spectrum as a unitary operator. These sequences have the following interesting structural property

PROPOSITION : Let T be weakly mixing, $\mu(A)>o$ and $\Lambda_\omega=\{n \in \mathbb{Z}_+ | T^n\omega \in A\}$. Then, for almost all ω

(1) $\Lambda_\omega + \Lambda_\omega + \Lambda_\omega$ contains any sufficiently large integer (i.e. Λ_ω is an asymptotic basesof order 3

(2) For any positive interger t, the set $\{n^t | n \in \Lambda_\omega\}$ is an asymptotic bases.

The proofs are rather straightforward combinations of the ideas around the Wiener-Wintner theorem and the Hardy-Littlewood circle method. Following fact will be used

LEMMA : Let $g \in L^\infty(\Omega,\mu)$, $\int g d\mu = o$. Then a.s. in ω

$$\lim_{N\to\infty} \sup_{|z|=1} \frac{1}{N} \Big| \sum_{n\leq N} T^n g(\omega) z^{(n^t)} \Big| = o \tag{1}$$

PROOF. Eliminate the z-dependence by successive applications of van der Corput's lemma. Thus define inductively

$$g_{j_1} = T^{j_1} g.\overline{g}$$

$$g_{j_1 j_2} = T^{j_2} g_{j_1}.\overline{g_{j_1}}$$

$$\vdots$$

$$g_{j_1\dots j_t} = T^{j_t} g_{j_1\dots j_{t-1}}.\overline{g_{j_1\dots j_{t-1}}}$$

(1) may then be derived from the following fact

$$(2)\quad \lim_{k_1\to\infty} \dots \lim_{k_t\to\infty} \lim_{N\to\infty} \frac{1}{k_1} \sum_{j_1\le k_1} \left(\frac{1}{k_2} \sum_{j_2\le k_2} \left(\dots \frac{1}{k_t} \sum_{j_t\le k_t} \left| \frac{1}{N} \sum_{n\le N} T^n g_{j_1\dots j_t} \right| \right)\right) = 0$$

a.s. in ω .

Since T is ergodic, a.s. in ω

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n\le N} T^n g_{j_1\dots j_t}(\omega) = \int g_{j_1,\dots,j_t}\, d\mu \ .$$

Hence, it remains to verify that

$$(3)\quad \lim_{k_1\to\infty} \dots \lim_{k_t\to\infty} \frac{1}{k_1} \sum_{j_1\le k_1} \left(\dots \frac{1}{k_t} \sum_{j_t\le k_t} \left| \langle T^{j_t} g_{j_1\dots j_{t-1}}, g_{j_1\dots j_{t-1}} \rangle \right| \right) = 0$$

Since T is weakly mixing

$$\frac{1}{k} \sum_{j\le k} |\langle T^j h, h\rangle| \to |\int h| \,.\, |\int h| \qquad (4)$$

Iterated applications of (4) leads to property (3), since $\int g d\mu = 0$.

PROOF OF PROPOSITION (1) : Applying the circle method, examine the expressions

$$\int_0^1 \left(\sum_{n\le N} T^n \chi_A(\omega)\ e^{2\pi i n\theta} \right)^3 e^{-2\pi i k\theta}\, d\theta \qquad (5)$$

Applying the previous lemma with $t=1$ to $g = \chi_A - \mu(A)$, one gets $\Omega' \subset \Omega$ of full measure, such that for given $\omega \in \Omega'$ and N_0 large enough

$$\left\| \sum_{n\le N} T^n g(\omega)\ e^{2\pi i n\theta} \right\|_\infty \le \frac{1}{100} \mu(A)^3 N \quad \text{if} \quad N > N_0 \qquad (6)$$

Fix $k > N_0$ and consider (5) with $N=k$. Writing

$$\sum_{n\le k} T^n \chi_A(\omega) e^{2\pi i n\theta} = \mu(A) \sum_{n\le k} e^{2\pi i n\theta} + \sum_{n\le k} T^n g(\omega)\ e^{2\pi i n\theta}.$$

it follows from (6) and the estimate

$$|\int_{\Pi} g_1 g_2 g_3| \leq \| g_1 \|_{\infty} \| g_2 \|_2 \| g_3 \|_2$$

that

$$\left| (5) - \mu(A)^3 \int_o^1 \left(\sum_{n \leq k} e^{2\pi i n \theta} \right)^3 e^{-2\pi i k \theta} \, d\theta \right| < \frac{1}{10} \mu(A)^3 k^2 \tag{7}$$

Since the integral in (7) equals $\frac{1}{2} k(k+1)$, (5) does not vanish. Thus k belongs to $\Lambda_\omega + \Lambda_\omega + \Lambda_\omega$.

The proof of the second part of the proposition uses the following observation

LEMMA: For fixed t and r sufficiently large, there is the inequality

$$\int_o^1 \left| \sum_{n \leq N} c_n e^{2\pi i n^t \theta} \right|^r d\theta \leq c N^{r-t} \tag{8}$$

if $|c_n| \leq 1$

PROOF. If $c_n = 1$, then the statement follows from the description of the Weyl sums (see [Vaug] for instance). By Hölder's inequality, writing $\frac{1}{r} = \frac{1-\theta}{r-1} + \frac{\theta}{r+1}$, we may assume r even, in which case

$$\int_o^1 \left| \sum_{n \leq N} c_n e^{2\pi i n^t \theta} \right|^r \leq \int_o^1 \left| \sum e^{2\pi i n^t \theta} \right|^r .$$

PROOF OF PROPOSITION (2). Consider the expression

$$\int_o^1 \left(\sum_{n \leq N} T^n \chi_A(\omega) \, e^{2\pi i n^t \theta} \right)^{r+1} e^{-2\pi i k \theta} \, d\theta \tag{9}$$

Fix $\varepsilon > o$ (to be specified later) and apply (1) to $g = \chi_A - \mu(A)$ to get $\Omega' \subset \Omega$ of full measure, such that for each $\omega \in \Omega'$ there is N_o fulfilling

$$\| \sum_{n\leq N} T^n g(\omega) e^{2\pi i n^t\theta} \|_\infty < \varepsilon N \quad \text{for} \quad N>N_o \tag{10}$$

Let $k>(N_o)^t$ and consider $N = [k^{1/t}]$ in (9). Write again

$$\sum_{n\leq N} T^n \chi_A(\omega)\, e^{2\pi i n^t\theta} = \mu(A) \sum_{n\leq N} e^{2\pi i n^t\theta} + \sum_{n\leq N} T^n g(\omega)\, e^{2\pi i n^t\theta}$$

From Hölder's inequality, (8) and (10), it easily follows that

$$\left| (9) - \mu(A)^{r+1} \int_o^1 \left(\sum_{n\leq N} e^{2\pi i n^t\theta}\right)^{r+1} e^{-2\pi i k\theta} \right| \leq C\,\varepsilon\, N\, N^{r-t} \tag{11}$$

From the solution of Waring's problem by the circle method, for sufficiently large r, the integral in (11) exceeds $C\,N^{r+1-t}$. Letting ε be small enough, (11) implies that $(9) \neq o$, hence k belongs to the $(r+1)$-fold sum-set of $\{n^t | n \in \Lambda_\omega\}$.

This completes the proof.

APPENDIX 3 : AN EXAMPLE.

Our purpose is to construct a sequence of positive density satisfying the mean ergodic theorem but not the pointwise ergodic theorem. Thus the pointwise result on the return times is not a formal consequence of the Wiener-Wintner ergodic theorem. The ideas behind the example are not unrelated to the harmonic analysis of sequences exploited to prove positive results.

Denote $D = \{z \in \mathbb{C}; |z| \leq 1\}$ the unit disc and $\mathbb{C}_1 = \{z \in \mathbb{C}; |z| = 1\}$ the unit circle. We will construct a sequence $(\lambda_n)_{n=o,1,2,\ldots}$ in D satisfying

(1) $$\frac{1}{N} \sup_{|z|=1} \Big| \sum_{n \leq N} \lambda_n z^n \Big| \to o \quad \text{for} \quad N \to \infty$$

(2) (λ_n) is not a "good weight" for the pointwise ergodic theorem, i.e. there exists a DS (Ω,μ,T) , μ a probability measure, and $f \in L^\infty(\Omega,\mu)$, such that

$$\int \{\overline{\lim_N} | \frac{1}{N} \sum_{n \leq N} \lambda_n T^n f |\} \, d\mu > o \tag{3}$$

Splitting the sequence $\overline{\lambda} = (\lambda_n)$ in its real and imaginary part, we may assume $\overline{\lambda}$ ranging in the interval $[-\frac{1}{2},\frac{1}{2}]$. Use then the probabilistic balayage technique to represent

$$\lambda_n + \frac{1}{2} = \sigma_n + \eta_n$$

where the sequence $\overline{\sigma} = (\sigma_n)$ is o,1-valued and $\overline{\eta} = (\eta_n)$ satisfies

$$\frac{1}{N} \sup_{|z|=1} \Big| \sum_{n \leq N} \eta_n z^n \Big| \leq C \left(\frac{\log N}{N} \right)^{1/2} \tag{4}$$

Hence

$$\frac{1}{N} \sum_{n \leq N} \eta_n T^n g \to o \quad \text{almost surely} \tag{5}$$

We refer the reader to [B1], section 8 for details.

Define $\Lambda = \{n \in \mathbb{Z}_+ | \sigma_n = 1\}$ and $g = f - \int f d\mu$. It follows from (1), (4) that Λ satisfies the mean ergodic theorem (Λ is an ergodic sequence). Since from (3), (4) and Birkhoff's theorem

$$\int \{ \overline{\lim_N} | \frac{1}{N} \sum_{n \leq N} \sigma_n T^n g | \} d\mu > o ,$$

Λ fails the pointwise ergodic theorem. Observe that $d(\Lambda) = \frac{1}{2}$. Next, we describe the dynamical system. Denote Ω the product space $D^{\mathbb{Z}}$ endowed with the product topology and T the shift. Assume given an invariant mean L on $\mathbb{Z}$ and a sequence $\overline{a} = (a_n)_{n \in \mathbb{Z}}$, $a_n \in D$. This sequence $\overline{a}$ will be specified later. The measure μ on Ω is defined as follows on cylinders $\tilde{G} = \ldots D \times D \times G \times D \times D \ldots$, where G is an open subset of $D_{-k} \times \ldots \times D_o \times \ldots \times D_k$

$$\mu(\tilde{G}) = L(n \in \mathbb{Z} | (a_n, a_{n+1}, \ldots, a_{n+2k}) \in G) \tag{6}$$

Since L is an invariant mean, μ is T-invariant. Let us mention that some additional work permits to replace the space $D^{\mathbb{Z}}$ by $\{o,1\}^{\mathbb{Z}}$. The previous construction is then the same as in Furstenberg's reduction of Sremeredi's theorem on arithmetical progressions to a problem about dynamical systems. We will introduce a sequence $\overline{\lambda} = (\lambda_n)$ satisfying (1) and rapidly increasing integers

$$\ldots << N_{k,1} << N_{k,2} << \ldots << N_{k,j(k)} << N_{k+1,1} <<$$

such that for each k

$$\sup_{1 \leq j \leq j(k)} N_{k,j}^{-1} | \sum_{n \leq N_{k,j}} a_{m+n} \lambda_n | > \rho > o \tag{7}$$

for all sufficiently large (depending on k) m taken in a set A of positive upper density. Let $f = \pi_o$ be the o-coordinate projection. Since (7) means that

$$\sup_{1\leq j\leq j(k)} N_{k,j}^{-1} \left| \sum_{n\leq N_{k,j}} \lambda_n T^n f(T^m \bar{a}) \right| > \rho$$

it follows from the definition of μ that

$$\mu\{\omega \in \Omega \mid \sup_{1\leq j\leq j(k)} N_{k,j}^{-1} \left| \sum_{n\leq N_{k,j}} \lambda_n T^n f(\omega) \right| > \frac{\rho}{2} \} \geq L(A)$$

which for appropriate choice of L will be non-zero. Hence (3) will be fulfilled.

We consider a system $\{z_c \mid c \in \{o,1\}^r , r = 1,2,\ldots\}$ in $\mathbb{C}_1$ satisfying in particular the condition

$$z_{c_1} z_{c_2} \bar{z}_{c_3} \bar{z}_{c_4} \neq 1 \text{ unless } \{c_1,c_2\} = \{c_3,c_4\} \tag{8}$$

hence ($z^n = n^{th}$ power of z)

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left|\Sigma' b_c z_c^n\right|^4 \sim (\Sigma' |b_c|^2)^2 \tag{9}$$

Denote $\theta(z) = \frac{\delta z}{1+\delta|z|}$, $z \in \mathbb{C}$. Thus θ ranges in D and

$$|\theta(z) - \delta z| < \delta^2 |z|^2 \tag{10}$$

For $2^{r-1} \leq |n| < 2^r$, define

$$a_n = \theta(2^{-r/2} \sum_{c\in\{o,1\}^r} \varepsilon_c z_c^{-n}) \tag{11}$$

where the sequence $\varepsilon_c = \pm 1$ is chosen such that

$$\# B_r > \alpha 2^r \tag{12}$$

where B_r is the set of those $2^{r-1} \le |n| < 2^r$ satisfying

$$\max_{r' \le r} \sum_{c' \in \{o,1\}^{r'}} \Big| \sum_{\substack{c \in \{o,1\}^r \\ c' < c}} \varepsilon_c \, z_c^n \Big|^2 < \frac{1}{\alpha} 2^r$$

$$\min_{r' \le r} \sum_{c' \in \{o,1\}^{r'}} \Big| \sum_{\substack{c \in \{o,1\}^r \\ c' < c}} \varepsilon_c \, z_c^n \Big|^2 > \alpha 2^r$$

Here $\alpha > o$ is some constant (that this is possible is an easy verification left to the reader).

Define $A = \bigcup_r \left(B_r \cap \{2^{r-1} + 2^{r/2} \le |n| \le 2^r - 2^{r/2}\} \right)$, where B_r is the set considered in (12).

For each k, denote $\xi^1, \xi^2, \ldots, \xi^{j(k)}$ an enumeration of a $\frac{1}{10}$-net in the unit ball of $\ell^2_{2^k}$, the 2^k-dimensional complex Hilbert space. For $N_{k,j-1} \le n < N_{k,j}$, put

$$\lambda_n = \theta \Big(\sum_{|c|=k} \xi_c^j \, z_c^n \Big) \tag{13}$$

where $\ell^2_{2^k}$ is identified with $\ell^2(\{o,1\}^k)$.

Assume the system (z_c) such that for $N \ge N_{k,1}$

$$\left(\frac{1}{N} \sum_{n=1}^{N} \Big| \sum_{|c|=k} b_c \, z_c^n \Big|^4 \right)^{1/4} \sim \left(\frac{1}{N} \sum_{n=1}^{N} \Big| \sum_{|c|=k} b_c \, z_c^n \Big|^2 \right)^{1/2} \approx \left(\Sigma |b_c|^2 \right)^{1/2} \tag{14}$$

and

$$|z_c - z_{c'}| << \frac{1}{N_{k,j(k)}} \text{ for } c \in \{o,1\}^k, c' > c \tag{15}$$

(the orthogonality relations needed to fulfil (14) follow from (8)).

Compute (7) using the definition of the sequences (11),(13). Assume $m \in A$ and $2^{r-1} \le m \le 2^r$, where $2^r >> N^2_{k,j(k)}$.

For $1 \le j \le j(k)$ by construction,

$$N_{k,j}^{-1} \sum_{n \le N_{k,j}} a_{m+n} \lambda_n \approx$$

$$N_{k,j}^{-1} \sum_{N_{k,j-1} < n < N_{k,j}} a_{m+n} \lambda_n \approx$$

$$N_{k,j}^{-1} \sum_{n \le N_{k,j}} \theta\left(\sum_{|c|=k} \xi_c^j z_c^n \right) \theta\left(2^{-k/2} \sum_{|c|=k} \left(2^{-\frac{r-k}{2}} \sum_{\substack{|c'|=r \\ c'>c}} \varepsilon_{c'} z_{c'}^{-m} \right) z_c^{-n} \right) \quad (16)$$

invoking (15). Defining

$$\zeta_c = 2^{-\frac{r-k}{2}} \sum_{\substack{|c'|=r \\ c<c'}} \varepsilon_{c'} z_{c'}^{-m} \quad (17)$$

the hypothesis $m \in A$ gives

$$\frac{1}{\alpha} 2^k > \sum_{|c|=k} |\zeta_c|^2 > \alpha 2^k \quad (18)$$

Estimate by (10) and Hölder's inequality

$$|(16)| \ge \delta^2 N_{k,j}^{-1} \left| \sum_{n \le N_{k,j}} \left(\sum_{|c| \le k} \xi_c^j z_c^n \right) \left(2^{-k/2} \sum_{|c| \le k} \zeta_c z_c^{-n} \right) \right| \quad (19)$$

$$(20) \qquad - \delta^3 N_{k,j}^{-1} \left(\sum_{n \le N_{k,j}} \left| \sum_{|c| \le k} \xi_c^j z_c^n \right|^4 \right)^{1/2} \left(\sum_{n \le N_{k,j}} \left| 2^{-k/2} \sum_{|c| \le k} \zeta_c z_c^{-n} \right|^2 \right)^{1/2}$$

$$(21) \qquad - \delta^3 N_{k,j}^{-1} \left(\sum_{n \le N_{k,j}} \left| \sum_{|c| \le k} \xi_c^j z_c^n \right|^2 \right)^{1/2} \left(\sum_{n \le N_{k,j}} \left| 2^{-k/2} \sum_{|c| \le k} \zeta_c z_c^{-n} \right|^4 \right)^{1/2}$$

It now follows from the properties of the system (z_c), in particular (14)

$$(19) \approx \delta^2 \Big|\sum_{|c|\leq k} 2^{-k/2}\, \xi_c^j\, \zeta_c\Big| \tag{22}$$

$$(20) \sim \delta^3\, 2^{-k/2} \left(\sum_{|c|\leq k} |\xi_c^j|^2\right) \left(\sum_{|c|\leq k} |\zeta_c|^2\right)^{1/2} \tag{23}$$

$$(21) \sim \delta^3\, 2^{-k} \left(\sum_{|c|\leq k} |\xi_c^j|^2\right)^{1/2} \left(\sum_{|c|\leq k} |\zeta_c|^2\right) \tag{24}$$

Invoking (18), $(23)+(24) \leq c\,.\delta^3$. Therefore

$$(7) \geq \frac{1}{2}\, \delta^2\, 2^{-k/2} \sup_{j\leq j(k)} \Big|\sum_{|c|\leq k} \xi_c^j\, \zeta_c\Big| - C\delta^3 \tag{25}$$

Since by construction $\{\xi^j \mid 1\leq j\leq j(k)\}$ is $\frac{1}{10}$-dense in the unit-ball of $\ell^2(\{o,1\}^k)$, (25) and the lower estimate in (18) yield

$$(7) \geq \frac{1}{3}\, \delta^2\, 2^{-k/2} \left(\sum_{|c|\leq k} |\zeta_c|^2\right)^{1/2} - C\,\delta^3 > \frac{\sqrt{\alpha}}{3}\,\delta^2 - C\,\delta^3 > \rho$$

taking δ small enough in the definition of θ . Here $\rho > o$ is some positive number and this completes the construction.

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