A REMARK ON THE MAXIMAL FUNCTION ASSOCIATED TO AN ANALYTIC VECTOR FIELD IN THE PLANE

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1. Introduction

We consider the planar case. Let thus $\Omega \subset \mathbb{R}^2$ be a bounded open set and $v: \Omega' \to S^1$ or, more generally, $v: \Omega' \to \mathbb{R}^2$ be a vectorfield defined on a neighborhood Ω' of the closure $\overline{\Omega}$ of Ω.

For $\epsilon < \epsilon_0$ (taken small enough), consider the averages (along v)

$$A_\epsilon f = \frac{1}{\epsilon} \int_0^\epsilon f(x+tv(x))dt \tag{1.1}$$

where f is a priori a bounded measurable function

The differentiation problem $f = \lim_{\epsilon \to 0} A_\epsilon f$ almost sure (a.s.) leads naturally to estimating the corresponding maximal function

$$M_v f = Mf = \sup_{\epsilon < \epsilon_0} |A_\epsilon f| ,$$

although our interest in this object here will be rather the purely harmonic analysis aspects. Our purpose is to prove the L^2-inequality in the real-analytic case (see Theorem 2 below).

Already the boundedness of $A_\epsilon f$ (as an operator on L^2 say) requires hypothesis on v, as shown by the example of the Nikodým set (see [G]). The differentiation problem has been solved affirmatively in the analytic case but is still open for C^∞ vector fields. More precisely, estimates on v were obtained when v has non-vanishing curvature (see [N-S-W]), say

$$\inf_{x \in \Omega} \det[Dv(x)v(x), v(x)] > 0 . \tag{1.2}$$

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Assume v normalized and $\det[Dv(x)v(x), v(x)] > 0$ on Ω. Then an L^2-bound on M_v may in fact be formulated in terms of the quantity

$$\frac{\sup\limits_{\Omega} \det[Dv(x)v(x), v(x)]}{\inf\limits_{\Omega} \det[Dv(x)v(x), v(x)]} \tag{1.3}$$

thus the ratio between maximum and minimum curvature of the integral curves of v. In the context of "non-vanishing curvature", this theory has been generalized and developed in several directions (for instance, the higher dimensional case and maximal functions associated to certain Radon-transforms, cf [Chr], [Ph-St.]).

It has been observed that for a constant curve $\Gamma_x = \Gamma$, the maximal function

$$M_\Gamma f(x) = \sup_{\epsilon > 0} \frac{1}{\epsilon} \int_0^\epsilon |f(x + \Gamma(t))| dt \tag{1.4}$$

requires geometric conditions on Γ in order to have non-trivial boundedness properties. However, these restrictions on Γ did not indicate so far necessary conditions on the integral curves in the vectorfield case. Possibly M_v is bounded as soon as v is $\mathcal{C}^1$.

2. Statement of a Condition and Verification in the Analytic Case

The letters $c, C > 0$ will stand for various constants.

The result proved here is obtained using known ideas and techniques. As such, a condition on v will be imposed, expressing "how well v turns". This condition will permit us to prove the L^2-boundedness of M_v and will apply also in certain flat cases, even including straight lines among the integral curves of v.

For $x \in \Omega$ and t small enough, define the function

$$\omega_x(t) = \omega(t) = \left|\det[v(x + tv(x)), v(x)]\right| . \tag{2.1}$$

We require a uniform estimate of the type

$$\operatorname{mes}\{t \in [-\epsilon,\epsilon]: \omega(t) < \tau \sup_{-\epsilon \leq t \leq \epsilon} \omega(t)\} \leq C\tau^c \epsilon \tag{2.2}$$

valid for all $0 < \tau < 1$, $0 < \epsilon < \epsilon_0$, where $0 < c, C < \infty$ are constants independent of the point $x \in \Omega$.

<u>Theorem 1</u>: If v is C^1 and satisfies a condition (2.2), then M_v is bounded on $L^2(\Omega)$.

Combining the argument with some additional interpolation, the analogue statement for $p > 1$ may be obtained. We do not carry the details out in this paper. It is also likely that similar ideas permit us to prove boundedness results on the Hilbert transform along v

$$H_v f(x) = \mathrm{Pv} \int_{-\epsilon_0}^{\epsilon_0} \frac{f(x + tv(x))}{t} \, dt .$$

In case of positive curvature, (2.2) holds with $c = 1$ and for C the expression (1.3). The main additional application is the following strengthening of the differentiation result for real analytic vectorfields.

<u>Theorem 2</u>: Let v be real analytic on Ω'. Then for $\epsilon_0 > 0$ chosen small enough, M_v is bounded on $L^2(\Omega)$.

In order to verify (2.2), consider first the polynomial case. Clearly, for fixed x,

$$p(t) = \det[v(x + tv(x)), v(x)]$$

is a polynomial in t of degree bounded independently of x.

Now on polynomials $p = p(t)$ of a given degree d, there is a uniform estimate (2.2) In fact, one has for $p \neq 0$,

$$\{\frac{1}{\epsilon}\int_{-\epsilon}^{\epsilon} |p(t)|^{-\rho}\, dt\}^{1/\rho}\, \{\sup_{|\tau| < \epsilon} |p(t)|\} < C(\rho,d)\ , \quad \rho < \frac{1}{d} \tag{2.3}$$

which is easily checked by factorization of p and using Hölder's inequality.

Assume now $v = v(x_1,x_2)$ real analytic in a neighborhood of the closure $\overline{\Omega}$ of Ω. If each point $x_0 \in \overline{\Omega}$ has a neighborhood on which (2.2) holds, then (2.2) will be valid on $\overline{\Omega}$, hence Ω, by compactness.

Since v is real analytic, we may write for $|x-x_0| < \delta$

$$F(x,t) = \sum_{k \geq 1} f_k(x-x_0)t^k$$

where for some constant C,

$$|\hat{f}_k(\alpha)| \leq C^{k+|\alpha|}\ , \alpha \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}\ . \tag{2.4}$$

We use the following

<u>Lemma 2.3</u>: There is $\nu > 0$ such that for $|x| < \nu$,

$$|f_k(x)| \leq C \cdot C^k \max_{k \leq k_0} |f_k(x)| \tag{2.5}$$

holds, where k_0 is some integer.

It will then clearly suffice to have a uniform estimate (2.2) for real analytic functions $\omega(t) = \sum_{k > 0} \hat{\omega}(k) t^k$ satisfying

$$\sup_{k > 0} |\hat{\omega}(k)| = \sup_{k \leq k_0} |\hat{\omega}(k)| = 1 . \tag{2.6}$$

Such ω may be written on a (fixed) neighborhood of 0 as a product $p \cdot \omega_1$ where p is a polynomial of bounded degree (depending on k_0) and $|\hat{\omega}_1(0)| > \delta = \delta(k_0)$. Verification of (2.2) then reduces to p, i.e., the polynomial case.

Proof of Lemma 2.3: We will invoke a quantitative version of the division theorem for convergent power series, which may be found in [Br] (see Th. II). Take thus $\ell_1, \ell_2 > 0$ linearly independent over $\mathbb{Z}$ and consider the Banach algebra $K[\rho]$ of formal power series f in x_1, x_2 such that

$$\|f\| = \sum_{j_1, j_2 \geq 0} \rho_1^{j_1} \rho_2^{j_2} |\hat{f}(j_1, j_2)| < \infty$$

denoting $\rho = (\rho_1, \rho_2)$, $\rho_1 = \tau\eta^{\ell_1}$, $\rho_2 = \tau\eta^{\ell_2}$ and where $0 < \tau$, $\eta < 1$ will be suitably chosen. Given pairs $E_1, \dots, E_p$ in $\mathbb{N}^2$, denote $\Delta = \bigcup_{1 \leq i \leq p} (E_i + \mathbb{N}^2)$ and $R(\Delta, \rho)$ the subspace of elements $f = \sum_\alpha \hat{f}(\alpha) x^\alpha$ in $K[\rho]$ such that $\hat{f}(\alpha) = 0$ for $\alpha \in \Delta$. For appropriate $\tau > 0$, by (2.5), the sequence $\{f_k\}$ is contained in $K[\rho]$ and $\|f_k\| \leq C^k$. Let $\mathfrak{a}$ be the ideal generated by $\{f_k\}$ in $K[\rho]$.

It is proved in [Br] (see Th. II) that there this $\tau, \eta > 0$, a finite generating sequence $F_1, \dots, F_p$ in $\mathfrak{a}$ and a set $\Delta \subset \mathbb{N}^2$ as above, such that the map φ given by

$$\varphi(g_1,\ldots,g_p,h) = \sum_{i=1}^{p} g_i F_i + h$$

maps $H[\rho]$ onto $K[\rho]$, denoting

$$H[\rho] = K[\rho]^p \oplus R(\Delta,\rho)$$

the Banach space with norm

$$\|(g_1,\ldots,g_p,h)\| = \sum_{i=1}^{p} \rho^{E_i}\|g_i\| + \|h\| \ .$$

Moreover, every element f in $K[\rho]$ is equivalent up to an element of $\mathcal{A}$ with a unique element of $R(\Delta,\rho)$. Thus for $f \in \mathcal{A}$, $\|f\| \leq 1$, there is a representation $f = \sum_{1 \leq i \leq p} g_i F_i$ for some $g_1,\ldots,g_p \in K[\rho]$, $\|g_i\| \leq B$ $(1 \leq i \leq p)$. In particular, for some $\nu > 0$,

$$|f(x)| \leq C \max|F_i(x)| \quad \text{if} \quad |x| < \nu \ . \tag{2.7}$$

Since the $F_i \in \mathcal{A}$, there is an integer k_0 such that the right member of (2.7) is bounded by $C \max_{k \leq k_0} |f_k(x)|$. Applying (1.11) to the functions $f = C^{-k} f_k$ completes the proof.

<u>Remark</u>: The author is grateful to P. Milman for pointing out the reference [Br] to him, simplifying an earlier argument.

We now come back to the proof of Theorem 1. The basic idea is the following. To the vectorfield v, we associate a "natural" system of rectangles, say $\mathcal{C}$, for which the corresponding maximal operator $M_{\mathcal{C}}$ satisfies a weak-type estimate and hence is bounded on L^2. The difference between M_v and $M_{\mathcal{C}}$ is then taken care of by "local" estimates, possible because of condition (2.2).

In the next section, a general estimate is proved which is later exploited using (2.2).

3. A Local Estimate on the A_ε-Averages

Assume $|\nabla v| < B$ and choose $\varepsilon < \varepsilon_0 < \frac{1}{100B}$. It then easily follows from a change of variable that

$$\|A_\varepsilon f\|_2 \leq 2\|f\|_2 \tag{3.1}$$

where, for convenience, $A_\varepsilon f$ is redefined as

$$A_\varepsilon f(x) = \int f(x + \varepsilon t v(x))\alpha(t)dt \tag{3.2}$$

taking for α a fixed, positive C^∞-function supported by $[-1,1]$, $\alpha(-t) = \alpha(t)$, $\int \alpha(t)dt = 1$.

Since M_v is a positive operator, we may as well put

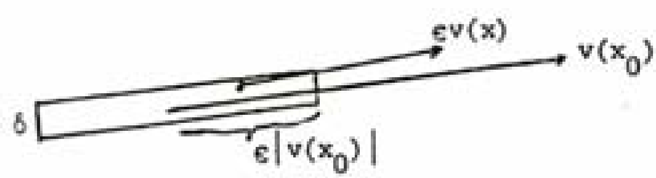
$$M_v f(x) = \sup_{\varepsilon < \varepsilon_0} |A_\varepsilon f|$$

where $A_\varepsilon f$ is defined by (3.2).

Take a point $x_0 \in \Omega$ and assume $v(x_0) \neq 0$. Let R be the rectangle with center x_0, orientation $v(x_0)$, length $\varepsilon|v(x_0)|$ (in direction $v(x_0)$) and width

$$\delta = \varepsilon \cdot \sup_{|t| < \varepsilon} \left|\det\left[v(x_0 + tv(x_0)), \frac{v(x_0)}{|v(x_0)|}\right]\right| \tag{3.3}$$

which we assume non-zero.



Since $\epsilon < \epsilon_0 < \frac{1}{100B}$, $x + \epsilon v(x)$ will clearly lie in a 2δ-neighborhood of $x_0 + \mathbb{R}v(x_0)$, hence in the doubled rectangle R_1 of R, for each $x \in R$.

Let T be a positive number (practically $T > \delta^{-1}$) and assume $\operatorname{supp} \hat{f} \subset B(0,2T)\backslash B(0,T)$. Let φ, ψ be a pair of positive C^∞-functions on $\mathbb{R}^2$ satisfying

$$\int \varphi(x)\,dx = 1 = \int \psi(x)\,dx \tag{3.4}$$

$$\varphi, \hat{\psi} \text{ are supported by the disc } B(0,1)\ . \tag{3.5}$$

Denote as usual $\varphi_\delta(x) = \delta^{-2}\varphi(\delta^{-1}x)$, hence supported by $B(0,\delta)$. Thus, denoting R_3 a doubling of R_2, $\chi_{R_3} * \varphi_\delta = 1$ on R_2 and by previous considerations, also

$$A_\epsilon f|_R = A_\epsilon f_1|_R \quad \text{where} \quad f_1 = f(\chi_{R_3} * \varphi_\delta)^2\ . \tag{3.6}$$

The function

$$g = f(\chi_{R_3} * \varphi_\delta * \psi_{\frac{10}{T}})^2$$

satisfies the following properties, by (3.5)

$$|g| \leq |f| \cdot (\chi_{R_4} * \psi_{\frac{10}{T}}) \tag{3.7}$$

$$\operatorname{supp} \hat{g} \subset B(0,3T)\backslash B(0, \tfrac{T}{2})\ . \tag{3.8}$$

Also

$$\|f_1 - g\|_2 \leq 2\|f \cdot (\chi_{R_4} * \psi_{\frac{10}{T}})\|_2 \cdot \|\chi_{R_3} * \varphi_\delta * (\delta_0 - \psi_{\frac{10}{T}})\|_\infty$$

where δ_0 stands for Dirac measure at 0. Further

$$\|\chi_{R_3} * \varphi_\delta * (\delta_0 - \psi_{\frac{10}{T}})\|_\infty \le \|\varphi * (\delta_0 - \psi_{\frac{10}{\delta T}})\|_1 \le \frac{c}{\delta T}$$

and thus, by (3.1), (3.6),

$$\|A_\varepsilon f|_R - A_\varepsilon g|_R\|_2 \le 2\|f_1 - g\|_2 < \frac{c}{\delta T}\ \|f(\chi_{R_4} * \psi_{\frac{10}{T}})\|_2 \ . \tag{3.9}$$

Our purpose now is to evaluate $\|A_\varepsilon g|_R\|_2$. By (3.8), we may write

$$A_\varepsilon g(x) = \int \hat{g}(\xi) e^{i\langle x,\xi\rangle}\, \hat{\alpha}(\varepsilon\langle v(x),\xi\rangle)\beta(T^{-1}|\xi|)d\xi \tag{3.10}$$

where β is a C^∞-function, $\beta = 1$ on $[\frac{1}{2},3]$ and vanishing outside $[\frac{1}{4},4]$. Estimate by duality

$$\|A_\varepsilon g|_R\|_2 = \int \hat{g}(\xi)[\int h(x)e^{i\langle x,\xi\rangle}\,\hat{\alpha}(\varepsilon\langle v(x),\xi\rangle)dx]\beta(T^{-1}|\xi|)d\xi$$

$$\le \|g\|_2 |\iint h(x)\overline{h(x')}[\int e^{i\langle x-x',\xi\rangle}\,\hat{\alpha}(\varepsilon\langle v(x),\xi\rangle)\hat{\alpha}(\varepsilon\langle v(x'),\xi\rangle)\beta(T^{-1}|\xi|d\xi]dxdx']| .$$

We simply appeal to Shur's inequality to estimate the integral above. Thus by (3.7), (3.9), this gives the following bound

$$\|A_\varepsilon f|_R\|_2 \le c[\frac{1}{\delta T} + S^{1/2}]\|f(\chi_{R_4} * \psi_{\frac{10}{T}})\|_2 \tag{3.11}$$

denoting

$$S = \sup_{x \in R} \int_R |\int e^{i\langle x-x',\xi\rangle}\,\hat{\alpha}(\varepsilon\langle v(x),\xi\rangle)\hat{\alpha}(\varepsilon\langle v(x'),\xi\rangle)\beta(T^{-1}|\xi|)d\xi|dx' \tag{3.12}$$

and which remains to be evaluated.

Put

$x-x' = |x-x'|e^{i\phi}$, $\xi = r\,e^{i\theta}$, $v(x) = |v(x)|e^{i\nu}$, $v(x') = |v(x')|e^{i\nu'}$

and rewrite in polar coordinates

$$\int e^{i\langle x-x',\xi\rangle}\hat{\alpha}(\ldots)\hat{\alpha}(\ldots)\beta(T^{-1}|\xi|)d\xi$$

$$= \iint e^{i|x-x'|r\cos(\theta-\phi)}\hat{\alpha}(\varepsilon r|v(x)|\cos(\theta-\nu))$$

$$\times\,\hat{\alpha}(\varepsilon r|v(x')|\cos(\theta-\nu'))\beta(T^{-1}r)\,r\,dr\,d\theta\ . \qquad (3.13)$$

Put $r = tT$, $\frac{1}{4} \le t \le 4$, replace θ by $\bar{\theta} = \frac{\pi}{2} + \theta$ and estimate (3.13) as

$$T^2\int\Big|\int e^{i|x-x'|tT\sin(\bar{\theta}-\phi)}\hat{\alpha}(\varepsilon tT|v(x)|\sin(\bar{\theta}-\nu))$$

$$\times\,\hat{\alpha}(\varepsilon tT|v(x')|\sin(\bar{\theta}-\nu'))\beta(t)\,t\,dt\Big|d\bar{\theta}\ . \qquad (3.14)$$

Evaluate the t-integral in (3.14). First, by partial integration and the fact that $\hat{\alpha} \in \mathcal{S}$, there is a bound

$$C(1 + |x-x'|T|\sin(\bar{\theta}-\phi)|)^{-6}\ . \qquad (3.15)$$

Secondly, using just the decay of $\hat{\alpha}$, there is also the estimate

$$C(1 + \varepsilon T|v(x)||\sin(\bar{\theta}-\nu)| + \varepsilon T|v(x')||\sin(\bar{\theta}-\nu')|)^{-6}\ . \qquad (3.16)$$

Note that for $x \in R$, $|v(x) - v(x_0)| \le B|x-x_0| \le 2B\varepsilon|v(x_0)|$, hence $|v(x)| \sim |v(x_0)|$. Also $|x-x'| < 3\varepsilon|v(x_0)|$ for $x, x' \in R$.

Recombining the terms of (3.15), (3.16), one gets therefore

$$[1 + \varepsilon T|v(x_0)||\sin(\bar{\theta}-\nu)| + \varepsilon T|v(x_0)||\sin(\nu-\nu')| + T|x-x'||\sin(\phi-\nu)|]^{-6}$$

bounded by

$$[1 + \varepsilon T|v(x_0)||\sin(\bar{\theta}-\nu)|]^{-2}$$

$$\times\,[1 + \frac{T}{|v(x_0)|}(|\det[x-x',v(x)]| + \varepsilon|\det[v(x),v(x')]|)]^{-4}\ .$$

Integrating in $\overline{\theta}$

$$(3.14) \leq \frac{T}{\epsilon|v(x_0)|} [1 + \frac{T}{|v(x_0)|} (|\det[x-x',v(x)]|$$

$$+ \epsilon|\det[v(x),v(x')]|)]^{-4} . \qquad (3.17)$$

We now come back to (3.12) and integrate on R. For fixed $x \in R$, write $v(x)^{\perp}$ for the normalized direction perpendicular on $v(x)$ and for $x' \in R$

$$x'-x = \alpha v(x) + \beta v(x)^{\perp} \quad (|\alpha| \leq \epsilon, |\beta| \leq \delta) .$$

Since

$$|\det[x-x',v(x)]| \sim |\beta||v(x_0)|$$

and

$$|\det[v(x),v(x')]| \geq |\det[v(x),v(x + \alpha v(x))]| - B|v(x)||x'-x-\alpha v(x)|,$$

integration of (3.17) gives after change of coordinates

$$\frac{T}{\epsilon} \iint_{\substack{|\alpha| < \epsilon \\ |\beta| < \delta}} [1 + T|\beta| + \frac{\epsilon T}{|v(x_0)|} |\det[v(x),v(x+\alpha v(x))]|]^{-4} \, d\alpha \, d\beta$$

$$\leq c \, \epsilon^{-1} \int_{|\alpha| < \epsilon} [1 + \epsilon T|v(x_0)|^{-1} \omega_x(\alpha)]^{-2} d\alpha \quad . \qquad (3.18)$$

Substitution of (3.18) in (3.12), (3.11) gives finally

$$\|A_\epsilon f|_R\|_2 \leq C\|f(\chi_R, * \phi_{\frac{1}{T}})\|_2 \cdot \{\frac{1}{\delta T} +$$

$$\sup_{x \in R} \|(1 + \epsilon T|v(x_0)|^{-1}\omega_x)^{-1}\|_{L^2[-\epsilon,\epsilon]}\} \qquad (3.19)$$

where R' is some multiple of R, supp $\hat{f} \subset B(0,2T)\backslash B(0,T)$ and where $L^2[-\epsilon,\epsilon]$ is L^2 on $[-\epsilon,\epsilon]$ with normalized measure.

We will specify (3.19) under hypothesis (2.2).

Observe first that (2.2) easily implies a doubling estimate

$$\sup_{|t| < 2\epsilon} \omega_x(t) \leq C \sup_{|t| < \epsilon} \omega_x(t) . \tag{3.20}$$

We continue to use the letter B for the Lipschitz constant of v.

<u>Lemma 3.21</u>: Assume x,x' two points in Ω, which are "close" in the sense that $x' = x + \alpha v(x) + y$ where $|\alpha| < C\epsilon$, $|y| < C\epsilon|v(x)|^{-1} \sup_{|t| < \epsilon} \omega_x(t)$. Then (provided $\epsilon < \epsilon_0$ assumed small enough),

$$\sup_{|t| < \epsilon} \omega_x(t) \sim \sup_{|t| < \epsilon} \omega_{x'}(t) . \tag{3.22}$$

<u>Proof</u>: We show the inequality

$$\sup_{|t| < \epsilon} \omega_x(t) \leq C \sup_{|t| < \epsilon} \omega_{x'}(t) \tag{3.23}$$

(the other inequality is similar, in fact simpler).

Notice first that since for $|t| < \epsilon$

$$\omega_x(t) \leq |v(x)||v(x)-v(x + tv(\alpha))| \leq B\epsilon|v(x)|^2 ,$$

one has

$$|v(x)-v(x')| \leq B|x-x'| \leq CB(\epsilon|v(x)| + C\epsilon B\epsilon|v(x)|) \leq \frac{1}{2} |v(x)| ,$$

hence

$$|v(x)| \sim |v(x')| . \tag{3.24}$$

Thus the length is essentially preserved on the neighborhood described above.

Next estiate

$$|\det[v(x),v(x+tv(x))]| \leq C|\det[v(x'),v(x'-\alpha v(x)-y)]|$$

$$+ C|\det[v(x'),v(x'+(t-\alpha)v(x)-y)]| \tag{3.25}$$

and the first term by

$$|v(x')|B|y| + |\det[v(x'),v(x'-\beta v(x'))]|$$

$$+ |v(x')||v(x'-\beta v(x')) - v(x'-\alpha v(x)-y)| \tag{3.26}$$

where $\beta \sim \alpha$ will be introduced later.

First term in (3.26) is at most $CB\epsilon\|\omega_x\|_{L^\infty(-\epsilon,\epsilon)}$.

Second term in (3.26) is bounded by

$$\|\omega_{x'}\|_{L^\infty(-C\epsilon,C\epsilon)} \sim \|\omega_{x'}\|_{L^\infty(-\epsilon,\epsilon)} \text{ by (3.20) .}$$

Third term in (3.26) may be for an appropriate choice of β estimated by

$$B|v(x)||-\beta v(x') + \alpha v(x) - y|$$

$$\leq B|v(x)||y| + B|v(x)| \leq B|y| + B|v(x)||\alpha v(x) - \beta v(x + \alpha v(x))|$$

$$\leq CB\epsilon\|\omega_x\|_{L^\infty(-\epsilon,\epsilon)} + B\epsilon|\det[v(x),v(x + \alpha v(x)]| .$$

The second term in (3.25) is bounded similarly to the first (replacing α by α-t). Collecting previous estimates, (3.25) is bounded by

$$C\|\omega_{x'}\|_{L^\infty(-\epsilon,\epsilon)} + BC\epsilon_0\|\omega_x\|_{L^\infty(-\epsilon,\epsilon)}$$

hence

$$\|\omega_x\|_{L^\infty(-\epsilon,\epsilon)} \le C\|\omega_{x'}\|_{L^\infty(-\epsilon,\epsilon)} + \frac{1}{2}\|\omega_x\|_{L^\infty(-\epsilon,\epsilon)} \text{ implying (3.23).}$$

This proves the lemma.

We now come back to (3.19). By previous lemma, $\|\omega_x\|_{L^\infty(-\epsilon,\epsilon)} \sim \|\omega_{x_0}\|_{L^{-\infty}(-\epsilon,\epsilon)}$ for $x \in R$. Moreover, for fixed $x \in R$, it follows from (2.2)

$$\int_{-\epsilon}^{\epsilon} [1 + \epsilon T|v(x_0)|^{-1}\omega_x(t)]^{-2}\, dt$$

$$\le \text{mes}\{|t| < \epsilon \mid \omega_x(t) < \tau\|\omega_x\|_{L^\infty(-\epsilon,\epsilon)}\}$$

$$+ \epsilon[\epsilon T|v(x_0)|^{-1}\tau\|\omega_x\|_{L^\infty(-\epsilon,\epsilon)}]^{-2}$$

$$\le C\tau^c\epsilon + \epsilon(\tau T\delta)^{-2} \tag{3.27}$$

where $1 > \tau > 0$ may be arbitrarily chosen. Therefore, for an appropriate choice of τ in (3.27)

<u>Lemma 3.28</u>: Under hypothesis (2.2) and with R,R' and f as in (3.19),

$$\|A_\epsilon f|_R\|_2 \le C(T\delta)^{-c}\, \|f(x_{R'} * \phi_{\frac{1}{T}})\|_2 \tag{3.29}$$

for some constants $0 < c,C < \infty$ depending on (2.2).

4. Geometrical Properties of Associated Rectangles

We use the notation R', R'', etc. for dilates of a given rectangle R. For $x \in \Omega$, $\varepsilon < \varepsilon_0$, define $R_{x,\varepsilon}$ as the rectangle considered in previous section, i.e., with center x, length $\varepsilon|v(x)|$ and width $\delta(R_{x,\varepsilon}) = \varepsilon|v(x)|^{-1}\|\omega_x\|_{L^\infty(-\varepsilon,\varepsilon)}$. As a consequence of Lemma 3.21 and its proof, there is the following property.

Lemma 4.1: Let $x' \in R'_{x,\varepsilon}$, then

$$|v(x)| \sim |v(x')| \tag{4.2}$$

$$\|\omega_x\|_{L^\infty(-\varepsilon,\varepsilon)} \sim \|\omega_{x'}\|_{L^\infty(-\varepsilon,\varepsilon)} \tag{4.3}$$

$$\delta(R_{x,\varepsilon}) \sim \delta(R_{x',\varepsilon}) \tag{4.4}$$

$R_{x,\varepsilon}$ is contained in a multiple of $R_{x',\varepsilon}$ and vice versa . (4.5)

There is the following corollary.

Lemma 4.6: Assume $R'_{x,\varepsilon} \cap R'_{x_1,\varepsilon_1} \neq \phi$ and $\varepsilon_1 < C\varepsilon$. Then

$$R_{x_1,\varepsilon_1} \subset R''_{x,\varepsilon} \quad \text{and} \quad \varepsilon^{-1}\delta(R_{x,\varepsilon}) \geq c\varepsilon_1^{-1}\delta(R_{x_1,\varepsilon_1}) .$$

Proof: Take $x_2 \in R'_{x,\varepsilon} \cap R'_{x_1,\varepsilon_1}$. Then by Lemma 4.1, since also $x_2 \in R'_{x_1,\varepsilon}$,

$$R_{x_1,\varepsilon_1} \subset R'_{x_1,\varepsilon} \subset R''_{x_2,\varepsilon} \subset R'''_{x,\varepsilon} .$$

Also, since

$$|v(x)| \sim |v(x_2)| \sim |v(x_1)| : \|\omega_x\|_{L^\infty(-\varepsilon,\varepsilon)} \sim \|\omega_{x_2}\|_{L^\infty(-\varepsilon,\varepsilon)} \sim \|\omega_{x_1}\|_{L^\infty(-\varepsilon,\varepsilon)}$$

$$\varepsilon_1^{-1}\delta(R_{x_1,\varepsilon_1}) = |v(x_1)|^{-1}\|\omega_{x_1}\|_{L^\infty(-\varepsilon_1,\varepsilon_1)} \le C|v(x)|^{-1}\|\omega_x\|_{L^\infty(-\varepsilon,\varepsilon)} = C\varepsilon^{-1}\delta(R_{x,\varepsilon}) .$$

Let $\mathcal{B}$ stand for the collection of rectangles $R_{x,\varepsilon}$ where $x \in \Omega$, $\varepsilon_o > \varepsilon > 0$.

Lemma 4.7: Let $\{R_j = R_{x_j,\varepsilon_j}\}$ be a sequence in $\mathcal{B}$ and $\delta > 0$ such that

(i) $\delta(R_j) \sim \delta$

(ii) x_{j+1} does not belong to $R_1 \cup \ldots \cup R_j$

Then

$$\| \Sigma \chi_{R_j'} \|_\infty < C . \tag{4.8}$$

Proof: Suppose $R'_{x_j,\varepsilon_j} \cap R'_{x_k,\varepsilon_k} \neq \phi$ and $\varepsilon_j \le \varepsilon_k$ By (4.6) and (4.7)(i), it follows that

$$\varepsilon_k^{-1}\delta > c\varepsilon_j^{-1}\delta \Rightarrow \varepsilon_j \sim \varepsilon_k \quad \text{and} \quad R_k \subset R_j' .$$

From this observation, it is easily seen that having a fixed point in too many R_j'-rectangles must contradict (ii).

The next lemma is a Besicovitch-type covering property.

Lemma 4.9: Every subset $\mathcal{B}_0$ of $\mathcal{B}$ has a further subset $\mathcal{B}_1 \subset \mathcal{B}_0$ satisfying

(i) $\left| \bigcup_{R \in \mathcal{B}_0} R' \right| \le C \sum_{R \in \mathcal{B}_1} |R|$.

(ii) $\|\sum_{R \in \mathcal{S}_1} \chi_{R'}\|_\infty \leq C$.

Proof: For $s > 0$, define $\mathcal{B}_s = \{R \in \mathcal{S}_0 | 2^{-s-1} \leq \delta(R) < 2^{-s}\}$.
Construct $\overline{\mathcal{B}}_s \subset \mathcal{B}_s$ satisfying condition (4.7)(ii) and also

$$R_{x,\epsilon} \in \mathcal{B}_s \Rightarrow x \in \bigcup_{R \in \overline{\mathcal{B}}_s} R .$$

Hence we have by (4.5), (4.8),

$$\bigcup_{R \in \mathcal{B}_s} R' \subset \bigcup_{R \in \overline{\mathcal{B}}_s} R'' \tag{4.10}$$

$$\|\sum_{R \in \overline{\mathcal{B}}_s} \chi_{R'}\|_\infty \leq C . \tag{4.11}$$

Construct now by induction systems $\mathcal{S}_s \subset \overline{\mathcal{B}}_s$ as follows:

$$\mathcal{S}_0 = \overline{\mathcal{B}}_0$$

$$\mathcal{S}_{s+1} = \{R \in \overline{\mathcal{B}}_{s+1} | R' \text{ does not intersect } R_0'$$
$$\text{for any } R_0 \in \mathcal{S}_1 \cup \ldots \cup \mathcal{S}_s\} .$$

By construction and (4.11), denoting $\mathcal{S}_1 = \bigcup \mathcal{S}_s$,

$$\|\sum_{R \in \mathcal{S}_1} \chi_{R'}\|_\infty = \sup_s \|\sum_{R \in \mathcal{S}_s} \chi_{R'}\|_\infty \leq C \text{ , i.e. (ii)}$$

Assume $R \in \overline{\mathcal{B}}_{s+1} \backslash \mathcal{S}_{s+1}$. Then R' intersects R_0' for some $R_0 \in \mathcal{S}_{s'}$, $s' \leq s$. If $R = R_{x,\epsilon}$, $R_0 = R_{x_0,\epsilon_0}$, it is easily deduced from (4.6) that $\epsilon \leq C\epsilon_0$, since the other alternative leads to a contradiction. Applying again (4.6) we then get $R \subset R_0'$. Consequently

$$\bigcup_{R \in \overline{\mathcal{B}}_{s+1}} R' \subset \bigcup_{s' \leq s+1} \bigcup_{R \in \mathcal{S}_{s'}} R'' .$$

Using (4.10), it follows that

$$\bigcup_{R \in \mathcal{S}_0} R' = \bigcup_{s} \bigcup_{R \in \mathcal{S}_s} R' \subset \bigcup_{s'} \bigcup_{R \in \mathcal{S}_{s'}} R''' = \bigcup_{R \in \mathcal{S}_1} R'''$$

$$\Big| \bigcup_{R \in \mathcal{S}_0} R' \Big| \leq C \sum_{R \in \mathcal{S}_1} |R| \text{ , i.e. (i)}$$

Lemma 4.9 has the following corollary.

Lemma 4.12: The maximal function

$$M_{\mathcal{S}} f(x) = \sup_{R \in \mathcal{S},\ x \in R} \Big\{ \frac{1}{|R|} \int_R |f| \Big\}$$

has a weak-type estimate and hence is L^2-bounded.

5. Estimation of the Maximal Function

Define for $0 < \varepsilon < \varepsilon_0$, $s > 0$,

$$\Omega_{\varepsilon,s} = \{x \in \Omega | 2^{-s-1} \leq \varepsilon |v(x)|^{-1} \|\omega_x\|_{L^\infty(-\varepsilon,\varepsilon)} = \delta(R_{x,\varepsilon}) < 2^{-s}\} . \tag{5.1}$$

For real analytic vectorfields, either the integral curves of the vectorfield are straight lines (in which case the problem is 1-dimensional) or for each $\varepsilon > 0$, the complement of the set $\bigcup_s \Omega_{\varepsilon,s}$ has measure 0. It then suffices to prove the L^2-inequality on the maximal function $M_v f = \sup_{\varepsilon > 0} |A_\varepsilon f|$ restricted to $\bigcap_{\varepsilon > 0} \bigcup_{\delta > 0} \Omega_{\varepsilon,s}$. Hence, it is no restriction to assume for each $\varepsilon > 0$ that

$$\Omega = \bigcup_{s > 0} \Omega_{\varepsilon,s}$$

(since, a priori, only finitely many ϵ's have to be considered).

In the general case of v satisfying (2.2), it obviously suffices to prove the L^2-bound on $\Omega^* = \{x \in \Omega | v(x) \neq 0\}$. Since it suffices to prove the inequality for functions with finitely supported Fourier transform, the rectangles $R_{x,\epsilon}$, $x \in \Omega^*$, may always be redefined such that $\delta(R_{x,\epsilon}) \geq \tau\epsilon|v(x)|$ for some $\tau > 0$. In particular, the rectangles will be non-degenerated and again $\Omega^* = \bigcup_{s>0} \Omega^*_{\epsilon,s}$ for each $\epsilon > 0$.

For $\epsilon < \epsilon_0$, we only consider dyadic values 2^{-j}, $j > j_0$. Clearly, for fixed s, the $\Omega_{\epsilon,s}$ are essentially disjoint. From (4.4), if $x \in \Omega_{\epsilon,s}$ and $x' \in R'_{x,\epsilon}$, then $\delta(R_{x',\epsilon}) \sim 2^{-s}$, hence $x' \in \bigcup_{|s-s'|<C} \Omega_{\epsilon,s'}$. Hence

$$\Omega'_{\epsilon,s} \equiv \bigcup_{x \in \Omega_{\epsilon,s}} R'_{x,\epsilon} \subset \bigcup_{|s-s'|<C} \Omega_{\epsilon,s'}$$

and therefore

$$\Big\| \sum_{\epsilon} \chi_{\Omega'_{\epsilon,s}} \Big\|_\infty < C \, . \tag{5.2}$$

Let $f \in L^2$ and $f = \sum_{T \text{ dyadic}} f_T$ be a Littlewood-Paley decomposition where $\operatorname{supp} \hat{f} \subset B(0,2T) \backslash B(0,T)$. Thus $A_\epsilon f = \sum A_\epsilon f_T$ and for $x \in \Omega_{\epsilon,s}$ evaluated as

$$|A_\epsilon f(x)| \leq \Big| \sum_{T \leq 2^s} A_\epsilon f_T(x) \Big| + \sum_{T > 2^s} |A_\epsilon f_T(x)| \tag{5.3}$$

where the second term in (5.3) is bounded by

$$\sum_{s>0} \sum_{T>2^s} |A_\epsilon f_T| \chi_{\Omega_{\epsilon,s}} = \sum_{j>0} \Big[\max_\epsilon \Big(\sum_{s>0} |A_\epsilon (f_{2^{s+j}})|^2 \chi_{\Omega_{\epsilon,s}} \Big)^{1/2} \Big] \tag{5.4}$$

an expression independent of ϵ. Evaluate in L^2-norm replacing the ϵ-supremum by the square function to get

$$\|(5.4)\|_2 \le \sum_j \{ \sum_{\epsilon,s} \|A_\epsilon(f_{2^{s+j}})\chi_{\Omega_{\epsilon,s}}\|_2^2 \}^{1/2} . \tag{5.5}$$

To evaluate the inner L^2-norms, use (3.29). First cover $\Omega_{\epsilon,s}$ with rectangles $R_\ell \in \mathcal{B}$ such that condition (ii) of (4.7) holds, hence $\|\sum_\ell \chi_{R'_\ell}\|_\infty \le C$ and therefore

$$\sum_\ell \chi_{R'_\ell} \le C\,\chi_{\Omega'_{\epsilon,s}} . \tag{5.6}$$

Write, using (3.9) with $T = 2^{s+j}$, $\delta = 2^{-s}$

$$\|A_\epsilon(f_{2^{s+j}})\chi_{\Omega_{\epsilon,s}}\|_2^2 \le \sum_\ell \|A_\epsilon(f_{2^{s+j}})\chi_{R_\ell}\|_2^2 \le 2^{-cj} \sum_\ell \|f_{2^{s+j}}(\chi_{R'_\ell} * \psi_{\frac{1}{2^{s+j}}})\|_2^2 .$$

Thus, by (5.6),

$$\|A_\epsilon(f_{2^{s+j}})\chi_{\Omega_{\epsilon,s}}\|_2 \le C2^{-cj}\|f_{2^{s+j}}(\chi_{\Omega'_{\epsilon,s}} * \psi_{2^{-s-j}})\|_2 .$$

Hence, substituting in (5.5) and exploiting (5.2),

$$\|(5.4)\|_2^2 \le C \sum_j 2^{-cj}\Big(\sum_{\epsilon,s} \|f_{2^{s+j}}(\chi_{\Omega'_{\epsilon,s}} * \psi_{2^{-s-j}})\|_2^2\Big)$$

$$\le \sum_j 2^{-cj} \sum_s \|f_{2^{s+j}}\|_2^2 \le C\|f\|_2^2 .$$

It remains to take care of the contribution of the first term in (5.3). The idea is to replace M_v by $M_{\mathcal{B}}$.

<u>Lemma 5.7</u>: If $\operatorname{supp} \hat{g} \subset B(0,T)$ and $|x-x'| < \frac{1}{T}$, then $|g(x)| \le Cg^*(x')$, where g^* refers to the usual Hardy-Littlewood maximal function.

<u>Proof</u>: Assume $x = 0$ and φ satisfying $\hat{\varphi} = 1$ on $B(0,1)$, $|\varphi(x)| < C(1 + |x|^3)^{-1}$. Since $\hat{\varphi}_{\frac{1}{T}} = 1$ on supp $\hat{f}$, we have

$$|g(0)| = |\int \hat{g}(\xi)d\xi| = |\langle g, \varphi_{\frac{1}{T}}\rangle| \leq C \sum_{k \geq 0} T^2 8^{-k} \int_{B(0, \frac{2^k}{T})} |g(x)|dx \leq Cg^*(x') .$$

Taking for g the function $\sum_{T < 2^s} f_T$ and $x \in \Omega_{\epsilon,s}$, by (5.7)

$$|A_\epsilon g(x)| = |\int g(x + \epsilon t v(x))\alpha(t)dt|$$

$$\leq c \int 4^s \{\int_{B(x+\epsilon t v(x), 2^{-s})} g^*(y)dy\} \alpha(t)dt$$

$$\leq C \frac{2^s}{\epsilon|v(x)|} \int_{R_{x,\epsilon}} g^* \leq CM_{\mathcal{B}}(g^*)(x) \leq CM_{\mathcal{B}}(f^{**})(x)$$

since we may assume $|\sum_{T < 2^s} f_T| \leq Cf^*$, for each s.

Invoking (4.12), it follows that the L^2-contribution of the first term in (5.3) is bounded by

$$\|M_{\mathcal{B}} f^{**}\|_2 \leq C\|f^{**}\|_2 \leq C\|f\|_2$$

which completes the proof of Theorem 1.

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