

A REMARK ON THE UNCERTAINTY PRINCIPLE FOR HILBERTIAN BASIS

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1. INTRODUCTION.

We consider the one-dimensional case $L^2(\mathbb{R}, dx)$. Using the notations of [M₁], let $\psi \in L^2(\mathbb{R})$, $\|\psi\|_2 = \left(\int_{-\infty}^{\infty} |\psi(x)|^2 dx \right)^{1/2} = 1$. The Fourier transform is defined by $\hat{\psi}(\xi) = \int e^{-ix\xi} \psi(x) dx$. Assume moreover $\int x^2 |\psi(x)|^2 dx$ and $\int \xi^2 |\hat{\psi}(\xi)|^2 d\xi$ converging. Define then the mean values

$$\bar{x} = \int x |\psi(x)|^2 dx \quad \bar{\xi} = \frac{1}{2\pi} \int \xi |\hat{\psi}(\xi)|^2 d\xi$$

and the (quadratic) deviations

$$(\Delta x)_{\psi} = \left(\int |x - \bar{x}|^2 |\psi(x)|^2 dx \right)^{1/2} \quad (\Delta \xi)_{\psi} = \left(\frac{1}{2\pi} \int |\xi - \bar{\xi}|^2 |\hat{\psi}(\xi)|^2 d\xi \right)^{1/2}.$$

The uncertainty principle then tells us that

$$I = (\Delta x)_{\psi} (\Delta \xi)_{\psi} \geq \frac{1}{2}.$$

Note that the latter expression is invariant under rescaling of ψ . The minimum $\frac{1}{2}$ of I is obtained for the Gaussian (coherent states)

$$\psi(x) = (2\pi)^{-1/4} (\Delta x)^{-1/2} \exp \left[-\frac{1}{4} \frac{x^2}{(\Delta x)^2} \right]$$

The notion of basis for $L^2(\mathbb{R})$ has different meanings in litterature. Here it will mean a total orthonormal system. In [B], Balian studies basis obtained by translation of a fixed wave ψ over a lattice in phase space, i.e.

$$\psi_{\frac{\bar{x}}{a}, \frac{\bar{\xi}}{b}} = e^{ix\bar{\xi}} \psi(x - \bar{x})$$

where $\bar{x} = ma$, $\bar{\xi} = nb$ ($m, n \in \mathbb{Z}$) and a, b fixed positive numbers satisfying $ab = 2\pi$. It is proved in [B] that the constraints of completeness + orthogonality

forces the so-called strong uncertainty principle, i.e.

$$(\Delta x)_\psi (\Delta \xi)_\psi = \infty$$

In the same paper, Balian considers the more general problem for a non-periodic basis, i.e. the existence of a basis $\{\psi_i\}$ of $L^2(\mathbb{R})$ for which both quantities

$$\sup_i (\Delta x)_{\psi_i} ; \sup_i (\Delta \xi)_{\psi_i}$$

are finite. It is our purpose to construct such a system.

In [M,1], Meyer constructs a basis $\{\psi_i\}$ obtained by rescaling a same wave ψ , $\psi_i(x) = t_i \psi(t_i(x-a_i))$, where ψ satisfies $(\Delta x)_\psi (\Delta \xi)_\psi < \infty$. This basis and its variants turned out to be of considerable interest to various problems in analysis.

It has been shown by T. Steger that $L^2(\mathbb{R})$ does not admit a basis of the form $f_j(x) = e^{ib_j x} g_j(x-a_j)$ where g_j satisfies $\sup_j \|g_j\|_{A_\epsilon} < \infty$, defining

$$\|g\|_{A_\epsilon}^2 = \int (1+x^2)^{1+\epsilon} |g(x)|^2 dx + \int (1+\xi^2)^{1+\epsilon} |\hat{g}(\xi)|^2 d\xi.$$

Here $\epsilon > 0$ is any strictly positive number. His argument is based on the fact that the operations x (x -multiplication) and $\frac{d}{dx}$ in the latter basis would become "almost" diagonal operators, violating the non-commutation property $\left[\frac{d}{dx}, x\right] = I$. He also makes use of a density computation due to Y. Meyer [M,2] of the set Λ of pairs (a_j, b_j) in phase space. The condition $\epsilon > 0$ is important, in Steger's argument as well as for Meyer's distribution result to be valid.

The purpose of this note is to prove the following

THEOREM : Given $\rho > \frac{1}{\sqrt{2}}$, there is a basis $\{\psi_i\}$ for $L^2(\mathbb{R})$ satisfying

$$(\Delta x)_{\psi_i} < \rho \quad (1.1) \quad \text{and} \quad (\Delta \xi)_{\psi_i} < \rho \quad (1.2)$$

for each i .

Thus Balian's strong uncertainty principle does not hold for a non-periodic basis. The proof of the theorem is rather simple. It is obtained by a quantitative analysis of an abstract basis construction used previously in order to generate a uniformly bounded basis in various function spaces (for instance H^2 on the complex ball, see [R]). In fact, the basis elements turn out to be small perturbations of the coherent states.

I am grateful to Y. Meyer for some discussions on the subject.

2. CONSTRUCTION OF FINITE ORTHOGONAL SYSTEMS, WELL-LOCALIZED IN PHASE SPACE.

In the construction of the basis, we make use of some orthogonal systems of functions which we generate in this section. The letter C will denote various absolute constants. In order to simplify computations, approximate the gaussian $\psi(x) = \pi^{-1/4} e^{-x^2/2}$ by a compactly supported C^∞ -function φ . Thus let $\varepsilon > 0$ and assume φ to be supported by $\left[-\frac{K}{2}, \frac{K}{2}\right]$, $K = K(\varepsilon)$ satisfying

$$\int \varphi(x)^2 dx = 1 \quad (2.1)$$

$$\|\varphi' - \psi'\|_2 < \varepsilon \Rightarrow (\Delta \xi)_\varphi < \frac{1}{\sqrt{2}} + \varepsilon \quad (2.2)$$

$$\|(\hat{\varphi})' - (\hat{\psi})'\|_2 < \varepsilon \Rightarrow (\Delta x)_\varphi < \frac{1}{\sqrt{2}} + \varepsilon \quad (2.3)$$

Consider the Fourier transform

$$\lambda(\xi) = \int e^{-ix\xi} \varphi(x)^2 dx$$

that we may assume to satisfy

$$|\lambda(\xi)| < \frac{1}{1+C\xi^3} \quad (2.4)$$

Fix a large integer T and consider the lattice $((m+T)K, nK)$ ($0 \leq m, n < T$) as well as the functions

$$\varphi_{m,n}(x) = e^{inKx} \varphi(x - mK - TK) \quad (2.5)$$

Thus, by construction, $\varphi_{m,n}$ and $\varphi_{m',n'}$ are disjointly supported for $m \neq m'$.

Also

$$\|\sum_{m,n} a_{m,n} \varphi_{m,n}\|_2 \sim \left(\sum |a_{m,n}|^2 \right)^{1/2} \quad (2.6)$$

Denoting Φ the set of the $\varphi_{m,n}$ ($0 \leq m, n < T$) and $[\Phi]$ its linear span, the following inequalities are satisfied for $f \in [\Phi]$

$$\left(\int x^2 |f(x)|^2 dx \right)^{1/2} \leq 2TK \|f\|_2 \quad (2.7)$$

$$\left(\int \xi^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \sim \|f\|_2 \leq CT \|f\|_2 \quad (2.8)$$

(2.7) is obvious from the localization of f and the second inequality in (2.8) is easily deduced from (2.6).

Our purpose is to replace Φ by an orthogonal system using Gram-Schmidt orthogonalization on the $\varphi_{m,n}$. Thus we orthogonalize the $\varphi_{0,n}$ to a system τ_n and define then

$$\tau_{m,n}(x) = \tau_n(x - mK - TK) \quad (2.9)$$

Put $\tau_0 = \varphi_{0,0} = \varphi$. Write then for a fixed n

$$\tau_n = \varphi_{0,n} - \sum_{0 \leq j < n} a_j \varphi_{0,j}$$

and the orthogonality conditions $\tau_n \perp \tau_k$ ($0 \leq k < n$)

$$\lambda(K(n-k)) = \sum_{0 \leq j < n} a_j \lambda(K(j-k)) \quad (2.10)$$

Putting $\lambda(K(j-k)) = \delta_{jk} + H_{jk}$, the matrix H will be satisfied by (2.4)

$$H_{jj} = 0 \text{ and } H_{jk} < \frac{C}{K^3(j-k)^3} \quad (2.11)$$

Writing the inverse $S = (I+H)^{-1}$ as a Neumann series

$$S = \sum_{t=0}^{\infty} H^t$$

it is easily checked that S still fulfils

$$S_{jk} < \frac{C}{1+K^3(j-k)^3} \quad (2.12)$$

Since the solution $(a_j)_{0 \leq j < n}$ of (2.10) is given by

$$a_j = \sum_{0 \leq k < n} S_{jk} H_{n,k}$$

(2.11) and (2.12) imply

$$|a_j| < \frac{C}{1+K^3(n-j)^3} \quad (2.13)$$

Thus, by (2.13) and (2.6),

$$\|\tau_n - \varphi_{0,n}\|_2 < C(\sum a_j^2)^{1/2} < CK^{-3} \quad (2.14)$$

and

$$\|(e^{-inKx} \tau_n)' - \varphi'\|_2 \leq \sum |a_j| \|(e^{-inKx} \varphi_{0,j})'\|_2 \leq C \sum_{0 \leq j < n} \frac{(n-j)K}{1+K^3(n-j)^3} < CK^{-2} \quad (2.15)$$

By (2.3) and (2.14)

$$(\Delta x)_{\tau_{m,n}} = (\Delta x)_{\tau_n} \leq (\Delta x)_\varphi + CK^{-3} < \frac{1}{\sqrt{2}} + 2\epsilon \quad (2.16)$$

and by (2.1) and (2.15)

$$(\Delta \xi)_{\tau_{m,n}} = (\Delta \xi)_{\tau_n} \leq (\Delta \xi)_\varphi + CK^{-2} < \frac{1}{\sqrt{2}} + 2\epsilon \quad (2.17)$$

(for K large enough)

Denote S the system of L^2 -normalized functions $\tau_{m,n}$. If $s \in S$, then

$$|(\bar{x})_s| < 3TK ; |(\bar{\xi})_s| < 3TK \quad (2.18)$$

$$(\Delta x)_s < \frac{1}{\sqrt{2}} + 3\epsilon ; (\Delta \xi)_s < \frac{1}{\sqrt{2}} + 3\epsilon \quad (2.19)$$

$$s \text{ vanishes on the interval } \left[-\frac{T}{2}, \frac{T}{2}\right] \quad (2.20)$$

Note that (2.7) and (2.8) are valid for $f \in [S]$, since $[S] = [\emptyset]$.

3. CONSTRUCTION OF THE BASIS.

Let $\{f_k\}$ be a dense sequence in the unit sphere of $L^2(\mathbb{R})$ of compactly supported smooth functions. Fix $\epsilon > 0$. The basis β will be of the form

$$\beta = \bigcup_{k=1}^{\infty} \beta_k$$

where the β_k are finite families of orthogonal compactly supported smooth functions satisfying

$$(\Delta x)_b < \frac{1}{\sqrt{2}} + \varepsilon \quad (\Delta \xi)_b < \frac{1}{\sqrt{2}} + \varepsilon \quad (3.1)$$

$$\beta_k \perp \beta_{k'}, \quad \text{for } k \neq k' \quad (3.2)$$

$$\| P_{[\beta_1, \dots, \beta_k]} f_k \|_2 \geq v(\varepsilon) \quad (3.3)$$

where $v(\varepsilon) > 0$ and $P[\]$ stands for the orthogonal projection.

Condition (3.3) implies that β is total.

The construction of the β_k is done by induction on k , using the orthogonal families obtained in the previous section.

Assume that $\beta_1, \dots, \beta_{k-1}$ is obtained and let

$$f = f_k - P_{[\beta_1, \dots, \beta_{k-1}]} f_k$$

Choose T large enough such that in particular the members of $\beta_1 \cup \dots \cup \beta_{k-1}$ and f_k are supported by $\left[-\frac{T}{2}, \frac{T}{2}\right]$. We also let

$$\left(\int \xi^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} < T \quad (3.4)$$

Enumerate s_1, \dots, s_{T^2} the orthogonal system S built in Section 2. β_k will be contained in $[f, s]$ and hence orthogonal on $\beta_1, \dots, \beta_{k-1}$. Fix a constant $\theta > 0$ (to be specified later) and define the elements of β_k as follows

$$(3.5) \quad \left\{ \begin{array}{l} b_1 = \frac{\theta}{T} f + \gamma_1 s_1 \\ b_2 = \frac{\theta}{T} f + \sigma_1 s_1 + \gamma_2 s_2 \\ b_3 = \frac{\theta}{T} f + \sigma_1 s_1 + \sigma_2 s_2 + \gamma_3 s_3 \\ \vdots \\ b_{T^2} = \frac{\theta}{T} f + \sigma_1 s_1 + \dots + \sigma_{T^2-1} s_{T^2-1} + \gamma_{T^2} s_{T^2} \end{array} \right.$$

The σ 's are chosen to make the b 's orthogonal, i.e.

$$(3.6) \quad \frac{\theta^2}{T^2} \|f\|_2^2 + \sigma_1^2 + \dots + \sigma_{\ell-1}^2 + \gamma_\ell \sigma_\ell = 0 \quad (1 \leq \ell \leq T^2)$$

and the γ 's are normalization factors

$$(3.7) \quad \gamma_\ell^2 = 1 - \frac{\theta^2}{T^2} \|f\|_2^2 - \sigma_1^2 - \dots - \sigma_{\ell-1}^2 \quad (1 \leq \ell \leq T^2)$$

for $\theta < \frac{1}{2}$, (3.6) and (3.7) clearly imply

$$\sigma_\ell \leq \frac{\theta}{T^2} ; \quad 1 - \gamma_\ell < 2 \frac{\theta}{T} \quad (3.8)$$

hence

$$\|b_\ell - s_\ell\|_2 \leq 3 \frac{\theta}{T} \quad \text{for each } \ell \quad (3.9)$$

We verify the condition (3.1). Fix $\ell = 1, \dots, T^2$ and denote

$$\bar{x} = (\bar{x})_{s_\ell} ; \quad \bar{\xi} = (\bar{\xi})_{s_\ell}$$

It follows from (2.18) and (3.9) that

$$\left(\int |x - \bar{x}|^2 |b_\ell(x)|^2 dx \right)^{1/2} \leq \left(\int |x - \bar{x}|^2 |s_\ell(x)|^2 dx \right)^{1/2} + CKT \|b_\ell - s_\ell\|_2 \leq (\Delta x)_{s_\ell} + CK\theta$$

Hence, by (2.19) and for an appropriate choice of $\theta = \theta(\varepsilon)$, $(\Delta x)_{s_\ell} < \frac{1}{\sqrt{2}} + 4\varepsilon$.

Similarly

$$\begin{aligned} \left(\frac{1}{2\pi} \int |\xi - \bar{\xi}|^2 |\hat{b}_\ell(\xi)|^2 d\xi \right)^{1/2} &\leq (\Delta \xi)_{s_\ell} + \frac{\theta}{T} \left(\int |\xi - \bar{\xi}|^2 |\hat{f}(\xi)|^2 d\xi \right)^{1/2} \\ &\quad + \left(\int |\xi - \bar{\xi}|^2 |\hat{b}_\ell(\xi) - \hat{s}_\ell(\xi) - \frac{\theta}{T} \hat{f}(\xi)|^2 d\xi \right)^{1/2} \end{aligned}$$

By (3.4) and (2.18), the second term is bounded by $\theta + 3KT \frac{\theta}{T} \|f\|_2 < 4K\theta$.

Using (2.8) and (3.9), the last term is bounded by

$$3KT \|b_\ell - s_\ell - \frac{\theta}{T} f\|_2 + CKT \|b_\ell - s_\ell - \frac{\theta}{T} f\|_2 < CK\theta$$

and an appropriate choice of θ yields $(\Delta \xi)_{b_\ell} < \frac{1}{\sqrt{2}} + 4\epsilon$. It remains to check (3.3). Since $\|P_{[\beta_1, \dots, \beta_k]} f_k\|_2^2 = \|P_{[\beta_1, \dots, \beta_{k-1}]} f_k\|_2^2 + \|P_{[\beta_k]} f_k\|_2^2$

and by construction

$$\|P_{\beta_k} [f]\|_2 \geq |\langle f, \frac{1}{T} \Sigma b_\ell \rangle| = \theta \|f\|_2^2$$

we get

$$v(\epsilon) = \min_{0 \leq a \leq 1} \left[a + \theta^2 (1-a) \right]^{1/2} > \frac{\theta}{2}$$

This ends the proof.

4. REMARKS.

1. The previous construction yields a basis $(\psi_i)_{i=1,2,\dots}$ of $L^2(\mathbb{R})$ such that

$$\lim_{i \rightarrow \infty} (\Delta x)_{\psi_i} = \frac{1}{\sqrt{2}}, \quad \lim_{i \rightarrow \infty} (\Delta \xi)_{\psi_i} = \frac{1}{\sqrt{2}}$$

2. The higher dimensional generalization of the construction gives a basis (ψ_i) of $L^2(\mathbb{R}^d)$ such that

$$\left(\int_{\mathbb{R}^d} |x - \bar{x}|^{2d} |\psi_i(x)|^2 dx \right)^{1/2} + \left(\int_{\mathbb{R}^d} |\xi - \bar{\xi}|^{2d} |\hat{\psi}_i(\xi)|^2 d\xi \right)^{1/2} < c$$

R E F E R E N C E S

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