A REMARK ON THE UNCERTAINTY PRINCIPLE FOR HILBERTIAN BASIS

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1. INTRODUCTION.

We consider the one-dimensional case $L^2(\mathbb{R},dx)$. Using the notations of $[M_1]$, let $\psi \in L^2(\mathbb{R})$, $\|\psi\|_2 = \left(\int_{-\infty}^{\infty} |\psi(x)|^2 dx\right)^{1/2} = 1$. The Fourier transform is defined by $\hat{\psi}(\xi) = \int e^{-ix\xi}\psi(x)\,dx$. Assume moreover $\int x^2|\psi(x)|^2dx$ and $\int \xi^2|\hat{\psi}(\xi)|^2d\xi$ converging. Define then the mean values

$$\bar{x} = \int x|\psi(x)|^2dx \qquad \bar{\xi} = \frac{1}{2\pi}\int \xi|\hat{\psi}(\xi)|^2d\xi$$

and the (quadratic) deviations

$$(\Delta x)_\psi = \left(\int |x-\bar{x}|^2 \quad |\psi(x)|^2dx\right)^{1/2} \qquad (\Delta\xi)_\psi = \left(\frac{1}{2\pi}\int |\xi - \bar{\xi}|^2 \quad |\hat{\psi}(\xi)|^2d\xi\right)^{1/2} .$$

The uncertainty principle then tells us that

$$I = (\Delta x)_\psi \ (\Delta\xi)_\psi \geq \frac{1}{2} .$$

Note that the latter expression is invariant under rescaling of ψ . The minimum $\frac{1}{2}$ of I is obtained for the Gaussian (coherent states)

$$\psi(x) = (2\pi)^{-1/4} \ (\Delta x)^{-1/2} \exp\left[-\frac{1}{4}\frac{x^2}{(\Delta x)^2}\right]$$

The notion of basis for $L^2(\mathbb{R})$ has different meanings in litterature. Here it will mean a total orthonormal system. In [B], Balian studies basis obtained by translation of a fixed wave ψ over a lattice in phase space, i.e.

$$\psi_{\bar{x},\bar{\xi}} = e^{ix\bar{\xi}}\ \psi(x-\bar{x})$$

where $\bar{x} = ma$, $\bar{\xi} = nb$ $(m,n \in \mathbb{Z})$ and a,b fixed positive numbers satisfying $ab = 2\pi$. It is proved in [B] that the constraints of completeness + orthogonality forces the so-called strong uncertainty principle, i.e.

$$(\Delta x)_\psi \, (\Delta \xi)_\psi = \infty$$

In the same paper, Balian considers the more general problem for a non-periodic basis, i.e. the existence of a basis $\{\psi_i\}$ of $L^2(\mathbb{R})$ for which both quantities

$$\sup_i (\Delta x)_{\psi_i} \ ; \ \sup_i (\Delta \xi)_{\psi_i}$$

are finite. It is our purpose to construct such a system.

In [M,1], Meyer constructs a basis $\{\psi_i\}$ obtained by rescaling a same wave ψ, $\psi_i(x) = t_i \psi(t_i(x-a_i))$, where ψ satisfies $(\Delta x)_\psi (\Delta \xi)_\psi < \infty$. This basis and its variants turned out to be of considerable interest to various problems in analysis.

It has been shown by T. Steger that $L^2(\mathbb{R})$ does not admit a basis of the form $f_j(x) = e^{ib_j x} g_j(x-a_j)$ where g_j satisfies $\sup_j \| g_j \|_{A_\varepsilon} < \infty$, defining

$$\| g \|^2_{A_\varepsilon} = \int (1+x^2)^{1+\varepsilon} |g(x)|^2 dx + \int (1+\xi^2)^{1+\varepsilon} |\hat{g}(\xi)|^2 d\xi \ .$$

Here $\varepsilon > 0$ is any strictly positive number. His argument is based on the fact that the operations x (x-multiplication) and $\frac{d}{dx}$ in the latter basis would become "almost" diagonal operators, violating the non-commutation property $\left[\frac{d}{dx}, x\right] = I$. He also makes use of a density computation due to Y. Meyer [M,2] of the set Λ of pairs (a_j, b_j) in phase space. The condition $\varepsilon > 0$ is important, in Steger's argument as well as for Meyer's distribution result to be valid.

The purpose of this note is to prove the following

<u>THEOREM</u> : <u>Given</u> $\rho > \frac{1}{\sqrt{2}}$, <u>there is a basis</u> $\{\psi_i\}$ <u>for</u> $L^2(\mathbb{R})$ <u>satisfying</u>

$$(\Delta x)_{\psi_i} < \rho \quad (1.1) \quad \underline{\text{and}} \quad (\Delta\xi)_{\psi_i} < \rho \quad (1.2)$$

<u>for each</u> i .

Thus Balian's strong uncertainty principle does not hold for a non-periodic basis. The proof of the theorem is rather simple. It is obtained by a quantitative analysis of an abstract basis construction used previously in order to generate a uniformly bounded basis in various function spaces (for instance H^2 on the complex ball, see [R]). In fact, the basis elements turn out to be small perturbations of the coherent states.

I am grateful to Y. Meyer for some discussions on the subject.

2. CONSTRUCTION OF FINITE ORTHOGONAL SYSTEMS, WELL-LOCALIZED IN PHASE SPACE.

In the construction of the basis, we make use of some orthogonal systems of functions which we generate in this section. The letter C will denote various absolute constants. In order to simplify computations, approximate the gaussian $\psi(x) = \pi^{-1/4} e^{-x^2/2}$ by a compactly supported C^∞-function φ . Thus let $\varepsilon > 0$ and assume φ to be supported by $\left[-\frac{K}{2}, \frac{K}{2}\right]$, $K = K(\varepsilon)$ satisfying

$$\int \varphi(x)^2 dx = 1 \tag{2.1}$$

$$\|\varphi' - \psi'\|_2 < \varepsilon \Rightarrow (\Delta\xi)_\varphi < \frac{1}{\sqrt{2}} + \varepsilon \tag{2.2}$$

$$\|(\hat{\varphi})' - (\hat{\psi})'\|_2 < \varepsilon \Rightarrow (\Delta x)_\varphi < \frac{1}{\sqrt{2}} + \varepsilon \tag{2.3}$$

Consider the Fourier transform

$$\lambda(\xi) = \int e^{-ix\xi} \varphi(x)^2 dx$$

that we may assume to satisfy

$$|\lambda(\xi)| < \frac{1}{1+C\xi^3} \tag{2.4}$$

Fix a large integer T and consider the lattice $((m+T)K, nK)$ $(o \leq m, n < T)$ as well as the functions

$$\varphi_{m,n}(x) = e^{inKx} \quad \varphi(x - mK - TK) \tag{2.5}$$

Thus, by construction, $\varphi_{m,n}$ and $\varphi_{m',n'}$ are disjointly supported for $m \neq m'$. Also

$$\|\Sigma\, a_{m,n}\, \varphi_{m,n}\|_2 \sim \Sigma\, |a_{m,n}|^{2\ 1/2} \tag{2.6}$$

Denoting Φ the set of the $\varphi_{m,n}(o \leq m,n < T)$ and $[\Phi]$ its linear span, the following inequalities are satisfied for $f \in [\Phi]$

$$\left(\int x^2 |f(x)|^2 dx\right)^{1/2} \leq 2\,TK\|f\|_2 \tag{2.7}$$

$$\left(\int \xi^2 |\hat{f}(\xi)|^2 d\xi\right)^{1/2} \sim \|f'\|_2 \leq CTK\|f\|_2 \tag{2.8}$$

(2.7) is obvious from the localization of f and the second inequality in (2.8) is easily deduced from (2.6).

Our purpose is to replace Φ by an orthogonal system using Gram-Schmidt orthogonalization on the $\varphi_{m,n}$. Thus we orthogonatize the $\varphi_{o,n}$ to a system τ_n and define then

$$\tau_{m,n}(x) = \tau_n(x - mK - TK) \tag{2.9}$$

Put $\tau_o = \varphi_{o,a} = \varphi$. Write then for a fixed n

$$\tau_n = \varphi_{o,n} - \sum_{o \leq j < n} a_j\, \varphi_{o,j}$$

and the orthogonality conditions $\tau_n \perp \tau_k$ $(o \leq k < n)$

$$\lambda(K(n-k) = \sum_{o \leq j < n} a_j \, \lambda(K(j-k)) \tag{2.10}$$

Putting $\lambda(K(j-k)) = \delta_{jk} + H_{jk}$, the matrix H will be satisfied by (2.4)

$$H_{jj} = o \text{ and } H_{jk} < \frac{C}{K^3 (j-k)^3} \tag{2.11}$$

Writing the inverse $S = (I + H)^{-1}$ as a Neumann series

$$S = \sum_{t=o}^{\infty} H^t$$

it is easily checked that S still fulfils

$$S_{jk} < \frac{C}{1+K^3 (j-k)^3} \tag{2.12}$$

Since the solution $(a_j)_{o \leq j < n}$ of (2.10) is given by

$$a_j = \sum_{o \leq k < n} S_{jk} \, H_{n,k}$$

(2.11) and (2.12) imply

$$|a_j| < \frac{C}{1+K^3 (n-j)^3} \tag{2.13}$$

Thus, by (2.13) and (2.6),

$$\| \tau_n - \varphi_{o,n} \|_2 < C(\Sigma a_j^2)^{1/2} < CK^{-3} \tag{2.14}$$

and

$$\| (e^{-inKx} \tau_n)' - \varphi' \|_2 \leq \Sigma |a_j| \, \| (e^{-inKx} \varphi_{o,j})' \|_2 \leq C \sum_{o \leq j < n} \frac{(n-j)K}{1+K^3 (n-j)^3} < CK^{-2} \tag{2.15}$$

By (2.3) and (2.14)

$$(\Delta x)_{\tau_{m,n}} = (\Delta x)_{\tau_n} \leq (\Delta x)_{\varphi} + CK^{-3} < \frac{1}{\sqrt{2}} + 2\varepsilon \tag{2.16}$$

and by (2.1) and (2.15)

$$(\Delta\xi)_{\tau_{m,n}} = (\Delta\xi)_{\tau_n} \leq (\Delta\xi)_{\varphi} + CK^{-2} < \frac{1}{\sqrt{2}} + 2\varepsilon \tag{2.17}$$

(for K large enough)

Denote S the system of L^2-normalized fonctions $\tau_{m,n}$. If $s \in S$, then

$$|(\bar{x})_s| < 3TK \ ; \ |(\bar{\xi})_s| < 3TK \tag{2.18}$$

$$(\Delta x)_s < \frac{1}{\sqrt{2}} + 3\varepsilon \ ; \ (\Delta\xi)_s < \frac{1}{\sqrt{2}} + 3\varepsilon \tag{2.19}$$

$$s \text{ vanishes on the interval } \left[-\frac{T}{2}, \frac{T}{2}\right] \tag{2.20}$$

Note that (2.7) and (2.8) are valid for $f \in [S]$, since $[S] = [\Phi]$.

3. CONSTRUCTION OF THE BASIS.

Let $\{f_k\}$ be a dense sequence in the unit sphere of $L^2(\mathbb{R})$ of compactly supported smooth functions. Fix $\varepsilon > 0$. The basis β will be of the form

$$\beta = \bigcup_{k=1}^{\infty} \beta_k$$

where the β_k are finite families of orthogonal compactly supported smooth functions satisfying

$$(\Delta x)_b < \frac{1}{\sqrt{2}} + \varepsilon \quad (\Delta\xi)_b < \frac{1}{\sqrt{2}} + \varepsilon \tag{3.1}$$

$$\beta_k \perp \beta_{k'} \quad \text{for} \quad k \neq k' \tag{3.2}$$

$$\| P_{[\beta_1,\ldots,\beta_k]} f_k \|_2 \geq \nu(\varepsilon) \tag{3.3}$$

where $\nu(\varepsilon) > o$ and $P_{[\ ]}$ stands for the orthogonal projection. Condition (3.3) implies that β is total.

The construction of the β_k is done by induction on k, using the orthogonal families obtained in the previous section.

Assume that $\beta_1,\ldots,\beta_{k-1}$ is obtained and let

$$f = f_k - P_{[\beta_1,\ldots,\beta_{k-1}]} f_k$$

Choose T large enough such that in particular the members of $\beta_1 \cup\ldots\cup \beta_{k-1}$ and f_k are supported by $\left[-\frac{T}{2}, \frac{T}{2}\right]$. We also let

$$\left(\int \xi^2 |\hat{f}(\xi)|^2 \, d\xi\right)^{1/2} < T \tag{3.4}$$

Enumerate $s_1,\ldots,s_{T^2}$ the orthogonal system S built in Section 2. β_k will be contained in $[f,s]$ and hence orthogonal on $\beta_1,\ldots,\beta_{k-1}$. Fix a constant $\theta > o$ (to be specified later) and define the elements of β_k as follows

$$(3.5) \quad \begin{cases} b_1 = \frac{\theta}{T} f + \gamma_1 s_1 \\ b_2 = \frac{\theta}{T} f + \sigma_1 s_1 + \gamma_2 s_2 \\ b_3 = \frac{\theta}{T} f + \sigma_1 s_1 + \sigma_2 s_2 + \gamma_3 s_3 \\ \vdots \\ b_{T^2} = \frac{\theta}{T} f + \sigma_1 s_1 + \ldots + \sigma_{T^2-1} s_{T^2-1} + \gamma_{T^2} s_{T^2} \end{cases}$$

The σ's are chosen to make the b's orthogonal, i.e.

$$(3.6) \qquad \frac{\theta^2}{T^2}\|f\|_2^2 + \sigma_1^2 + \ldots + \sigma_{\ell-1}^2 + \gamma_\ell \sigma_\ell = o \quad (1 \leq \ell < T^2)$$

and the γ's are normalization factors

$$(3.7) \qquad \gamma_\ell^2 = 1 - \frac{\theta^2}{T^2}\|f\|_2^2 - \sigma_1^2 - \ldots - \sigma_{\ell-1}^2 \quad (1 \leq \ell \leq T^2)$$

for $\theta < \frac{1}{2}$, (3.6) and (3.7) clearly imply

$$\sigma_\ell \leq \frac{\theta}{T^2} \quad ; \quad 1 - \gamma_\ell < 2\frac{\theta}{T} \tag{3.8}$$

hence

$$\|b_\ell - s_\ell\|_2 \leq 3\frac{\theta}{T} \quad \text{for each } \ell \tag{3.9}$$

We verify the condition (3.1). Fix $\ell = 1,\ldots,T^2$ and denote

$$\overline{x} = (\overline{x})_{s_\ell} \quad ; \quad \overline{\xi} = (\overline{\xi})_{s_\ell}$$

It follows from (2.18) and (3.9) that

$$\left(\int |x-\overline{x}|^2 \, |b_\ell(x)|^2 dx\right)^{1/2} \leq \left(\int |x-\overline{x}|^2 \, |s_\ell(x)|^2\right)^{1/2} + CKT\|b_\ell - s_\ell\|_2 \leq (\Delta x)_{s_\ell} + CK\theta$$

Hence, by (2.19) and for an appropriate choice of $\theta = \theta(\varepsilon)$, $(\Delta x)_{s_\ell} < \frac{1}{\sqrt{2}} + 4\varepsilon$.

Similarly

$$\left(\frac{1}{2\pi}\int|\xi-\overline{\xi}|^2|\hat{b}_\ell(\xi)|^2 d\xi\right)^{1/2} \leq (\Delta\xi)_{s_\ell} + \frac{\theta}{T}\left(\int|\xi-\overline{\xi}|^2|\hat{f}(\xi)|^2\right)^{1/2}$$

$$+ \left(\int|\xi-\overline{\xi}|^2 \, |\hat{b}_\ell(\xi) - \hat{s}_\ell(\xi) - \frac{\theta}{T}\hat{f}(\xi)|^2\right)^{1/2}$$

By (3.4) and (2.18), the second term is bounded by $\theta + 3KT\frac{\theta}{T}\|f\|_2 < 4K\theta$.

Using (2.8) and (3.9), the last term is bounded by

$$3KT\| b_\ell - s_\ell - \frac{\theta}{T} f\|_2 + CKT\| b_\ell - s_\ell - \frac{\theta}{T} f\|_2 < CK\theta$$

and an appropriate choice of θ yields $(\Delta\xi)_{b_\ell} < \frac{1}{\sqrt{2}} + 4\varepsilon$. It remains to check (3.3). Since $\|P_{[\beta_1,\ldots,\beta_k]}f_k\|_2^2 = \|P_{[\beta_1,\ldots,\beta_{k-1}]}f_k\|_2^2 + \|P_{[\beta_k]}f\|_2^2$

and by construction

$$\|P_{\beta_k}[f]\|_2 \geq |< f, \frac{1}{T}\Sigma b_\ell >| = \theta \|f\|_2^2$$

we get

$$\nu(\varepsilon) = \min_{o\leq a\leq 1} \left[a + \theta^2(1-a)^2\right]^{1/2} > \frac{\theta}{2}$$

This ends the proof.

4. REMARKS.

1. The previous construction yields a basis $(\psi_i)_{i=1,2,\ldots}$ of $L^2(\mathbb{R})$ such that

$$\lim_{i\to\infty} (\Delta x)_{\psi_i} = \frac{1}{\sqrt{2}} \quad , \quad \lim_{i\to\infty} (\Delta\xi)_{\psi_i} = \frac{1}{\sqrt{2}}$$

2. The higher dimensional generalization of the construction gives a basis (ψ_i) of $L^2(\mathbb{R}^d)$ such that

$$\left(\int_{\mathbb{R}^d} |x-\bar{x}|^{2d} |\psi_i(x)|^2 \, dx\right)^{1/2} + \left(\int_{\mathbb{R}^d} |\xi-\bar{\xi}|^{2d} |\hat{\psi}_i(\xi)|^2 d\xi\right)^{1/2} < C$$

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