ON SETS OF RECURRENCE

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SUMMARY: The existence is shown of a positive non-continuous measure μ on the circle which Fourier transform vanishes on a set of recurrence, i.e. $S = \{n \in \mathbb{Z}; \hat{\mu}(n) = 0\}$ is a set of recurrence but not a Van der Corput set. The method is constructive and involves some combinatorial considerations. In fact, we prove that the generic density condition for both properties are the same.

1. DEFINITIONS AND PRELIMINARIES.

Given a subset S of $\mathbb{Z}$, let

$$D^*(S) = \overline{\lim_{N\to\infty}} \frac{|S \cap [1,N]|}{N}$$

be the upper density of S.

Recall following defintions for a subset A of $\mathbb{Z}$

(1) A is a set of recurrence or Poincaré set (P) iff

$$A \cap (S - S) \neq \phi, \text{whenever } S \subset \mathbb{Z},\ D^*(S) > 0.$$

(2) A is a Van der Corput set (vdC) iff

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$$\mu \in M_+(\mathbb{T}),\ \hat{\mu}(n) = 0 \ \text{ if } \ n \in \Lambda \Rightarrow \mu \ \text{ continuous.}$$

(3) Λ is an FC^+-set (forcing continuity for positive measures) iff

$$\mu \in M_+(\mathbb{T}),\ \hat{\mu}(n) \to 0 \ \text{ on } \ \Lambda \Rightarrow \mu \ \text{ continuous.}$$

It is obvious that (3)⇒(2) and well-known that (2)⇒(1). I intend to prove that (1) does not imply (2), thus the existence of a positive non-continuous measure vanishing on a (P)-set. A first step, in fact containing the main idea, consists in disproving the implication (1)⇒(3). The actual vanishing property for the Fourier transfrom is then achieved by modifying the previous construction, modifications mainly of technical nature.

The definition of (vdC)-set given above is equivalent with the following one: "If (u_n) is a sequence of reals, then its uniform distribution (mod 1) is implied by the uniform distribution of the difference sequences $(u_{n+h} - u_n)$, whenever $h \in \Lambda$".

The equivalence of those definitions follows from [K-M] and the work of Y. Peres [P]. It also explains the terminology.

The following observation (cf. [K-M]) will be useful in the next section.

<u>LEMMA 1</u>: Let $\Lambda \subset \mathbb{Z}$, for which there exists a sequence $(\varphi_n)_{n=1,2,\ldots}$ in $A(\mathbb{T})$ satisfying the conditions

(i) $\hat{\varphi}_n$ is supported by Λ

(ii) $\sup \|\varphi_n\|_{A(\mathbb{T})} < \infty$

(iii) $\varphi_n(0) = 1$

(iv) $\varphi_n(x) \to 0$ for $x \in \mathbb{T},\ x \neq 0$.

Then Λ is an FC^+-set.

Proof: Let $\mu \in M_+(\Pi)$. Since (φ_n) is uniformly bounded, (iv) implies

$$\lim_{n\to\infty} \int_{\Pi} \varphi_n(x)\mu(dx) = \mu(\{0\}). \tag{1}$$

Since $\lim_{n\to\infty} \hat{\varphi}_n(k) = 0$ for each k, it follows from (ii)

$$\overline{\lim_{n}} \left|\sum \hat{\varphi}_n(k)\hat{\mu}(k)\right| \leq c \overline{\lim_{k\in A}} |\hat{\mu}(k)|. \tag{2}$$

It follows that if $\hat{\mu} \to 0$ on A, then $\mu(\{0\}) = 0$.

2. THEN GENERIC CASE

Notice that negation of the properties (P), (vdC), FC^+) leads to classes of sets stable under finite union.

The following remark is a variant of a result of Y. Katznelson [k] related to density in the Bohr compactification for a random subset of $\mathbb{Z}$ with prescribed density.

Lemma 2: Let $\mathbb{N} = \cup I_k$ be a partition of the integers in intervals, say $I_k = [2^{2^k}, 2^{2^{k+1}}]$. Coose for each k a random subset A_k of I_k, $|A_k| = N_k$, assigning to each element of I_k the same probability δ_k. Let $A = \cup_{k=1}^{\infty} A_k$. Then almost surely

(1) If $\overline{\lim_{k}}\, 2^{-k}N_k < \infty$, then A is not a recurrent set

(2) If $\overline{\lim_{k}}\, 2^{-k}N_k = \infty$, then A is an FC^+-set.

Proof.

(1) It is well-known (cf. [Ka]) that if $\overline{\lim}\, 2^{-k}N_k < \infty$, then almost surely Λ is not dense in the Bohr compactification (in fact Λ will be a Helson set). Hence, there are finitely many poitns $t_1, \dots, t_J$ in Π such that $\inf_{n\in\Lambda} \sum_{j=1}^{J} |1 - e^{2\pi i n t_j}| > 0$. Clearly such Λ is not recurrent.

(2) Assume $\overline{\lim\limits_k}\, 2^{-k}N_k = \infty$. Fix k and consider independent $(0,1)$-valued selectors $(\xi_n)_{n\in I_k}$ of mean δ, $\delta\cdot|I_k| = N_k$. Define the random function

$$\varphi_\omega(x) = \frac{1}{N_k}\sum_{n\in I_k} \xi_n(\omega)e^{inx}.$$

Thus

$$\int \|\varphi_\omega\|_\Lambda d\omega = 1 = \int \varphi_\omega(0)d\omega. \tag{3}$$

Also

$$|\varphi_\omega(x)| \le \frac{\delta}{N_k}\Big|\sum_{n\in I_k} e^{inx}\Big| + \frac{1}{N_k}\Big|\sum\nolimits_{I_k}(\xi_n - \delta)e^{inx}\Big|. \tag{4}$$

The first term is bounded by $|I_k|^{-1}|1 - e^{ix}|^{-1}$, which is small for x not to close to 0. For the second term in (4), we apply standard probabilistic estimates to get

$$\int\Big\|\sum\nolimits_{I_k}(\xi_n(\omega) - \delta)e^{inx}\Big\|_\infty d\omega \le c(\log|I_k|)^{1/2}\int\Big(\sum\nolimits_{I_k}(\xi_n(\omega)^2\Big)^{1/2}d\omega \le c(2^kN_k)^{1/2}.$$

This will be $o(N_k)$ provided $2^{-k}N_k \to \infty$. Thus it results from (4) that given $\tau > 0$, for an appropriate k

$$\int \sup_{|x|>\tau} |\varphi_\omega(x)| d\omega < \tau. \tag{5}$$

Using (3), (5) it is now easy to satisfy the hypothesis of Lemma 1. This completes the proof.

The previous result shows that generically for subsets of $\mathbb{Z}$ with given density, conditions (P), (vdC), (FC^+) are equivalent (namely the condition for density in the Bohr compactification).

3. CONSTRUCTION OF A RECURRENT SET WHICH IS NOT (FC^+).

For $t \in \Pi$, let δ_t be the Dirac measure.

PROPOSITION 3: Define the infinite convolution

$$\nu = \mathop{*}_{j=1}^{\infty} \left[\frac{1}{2}\left(\delta_{\frac{2\pi}{j!}} + \delta_{-\frac{2\pi}{j!}}\right)\right]$$

and let $\mu = \delta_0 + \nu$ Let $\alpha : \mathbb{N} \to \mathbb{R}_+$ satisfy $\lim_{n\to\infty} \alpha(n) = \infty$. Then the set

$$A = \bigcup_N \{1 \le n \le N! ;\ |\hat{\mu}(n)| < \frac{\alpha(N)}{N}\}$$

is recurrent.

Obviously A is not (FC^+) provided $\lim_{n\to\infty} \frac{\alpha(n)}{n} = 0$.

Proposition 3 is clearly a consequence of

LEMMA 4: Let A be a subset of $\{1,2,\dots,J!\}$, $|A| > cJ!$ Then

$$\min_{n\in A-A} |\hat{\mu}(n)| < \frac{M(c)}{J}. \tag{6}$$

Denote $\Omega_j = \{0,1,\dots,j\}$ and $\Omega = \Omega_1 \times \Omega_2 \times \cdots \Omega_{J-1}$. The representation

$$n = \sum_{j=1}^{J-1} q_j \, j! \qquad (0 \le q_j \le j) \qquad (7)$$

defines a one-to-one map from $\{1,2,\dots,J! - 1\}$ to Ω. Denote by υ the normalized counting measure on Ω (= product measure). Let A be a subset of $\{1,\dots,J! - 1\}$, $|A| > cJ!$ and $\tilde{A} \subset \Omega$ the image of A under the mapping considered above. Thus $\upsilon(\tilde{A}) > c$. Following combinatorial lemma will be used.

<u>LEMMA 5</u>: Let $B \subset \Omega$, $\upsilon(B) > c$. Then there is a pair of points x, x' in B and an integer $cJ < s < J$ such that

$$x_1 = x'_1, \dots, x_{s-1} = x'_{s-1}$$
$$x_s = 0,\ x'_s = [\tfrac{s}{2}]$$
$$|x_{s+1} - x'_{s+1}| \le 2, \dots, |x_{J-1} - x'_{J-1}| \le 2.$$

<u>Proof</u>. Perform following construction

$B_J = B$

$B_{J-1} = \{t \in \Omega \mid \text{there is } t' \in B_J \text{ with } t_1 = t'_1, \dots, t_{J-2} = t'_{J-2},\ |t_{J-1} - t'_{J-1}| \le 1\}$

$B_{J-2} = \{t \in \Omega \mid \text{there is } t' \in B_{J-1} \text{ with } t_1 = t'_1, \dots, t_{J-3} = t'_{J-3},\ |t_{J-2} - t'_{J-2}| \le 1, t_{J-1} = t'_{J-1}\}$

⋮

$B_s = \{t \in \Omega \mid \text{there is } t' \in B_{s+1} \text{ with } t_1 = t'_1, \dots, t_{s-1} = t'_{s-1},\ |t_s - t'_s| \le 1, t_{s+1} = t'_{s+1}, \dots\}$

⋮

Thus each element of B_s can be perturbed in the s-th coordinate by at most one unit to become an element of B_{s+1}. Perturbing the $J - s$ last coordinates, an element of B is obtained.

Denote by $\pi_{\check{s}}$ the projection on the coordinates $1,\dots,s-1,s+1,\dots,J-1$. If for each $x \in \pi_{\check{s}}(B_{s+1})$ there is $t \in \Omega\backslash B_{s+1}$ with $\pi_{\check{s}}(t) = x$, then clearly

$$\nu(B_s\backslash B_{s+1}) \geq \frac{1}{s}\,\nu_{\check{s}}(\pi_{\check{s}}(B_{s+1})) \geq \frac{1}{s}\,\nu(B_{s+1})$$

$$\nu(B_s) \geq (1 + \tfrac{1}{s})\nu(B_{s+1}). \tag{8}$$

Fixing $1 < \bar{J} < J$, (8) and the fact that $\prod_{s=\bar{J}}^{J} (1 + \frac{1}{s}) = \frac{J}{\bar{J}}$, imply the existence of some $s > \nu(B)\cdot J$ and a point $x \in \pi_{\check{s}}(B_{s+1})$ satisfying

$$t \in \Omega,\ \pi_{\check{s}}(t) = x \Rightarrow t \in B_{s+1}.$$

Thus, by construction, one may find for each $p \in \{0,1,\dots,s\}$ an element

$$(x_1,\dots,x_{s-1},p,x'_{s+1},\dots,x'_{J-1})$$

in B, where $|x'_{s+1} - x_{s+1}| \leq 1,\dots,|x'_{J-1} - x_{J-1}| \leq 1$. The lemma follows.

<u>Proof of Lemma 4</u>: Applying Lemma 5 to the set $\tilde{\Lambda}$, a pair of elements

$$n = \sum_{j=1}^{J-1} x_j\, j! \qquad\qquad n' = \sum_{j=1}^{J-1} x'_j\, j!$$

in A is obtained, where (x_j), (x'_j) fulfill the condition of Lemma 5. Thus $m \equiv n' - n = [\frac{s}{2}]\, s! + (x'_{s+1} - x_{s+1})(s+1)! + \cdots + (x'_{J-1} - x_{J-1})(J-1)!$ is in the difference set $A - A$. By definition of μ

$$\hat{\mu}(m) = 1 + \prod_{j=1}^{\infty} \cos 2\pi \frac{m}{j!} = 1 + \prod_{j=s+1}^{\infty} \cos 2\pi \frac{m}{j!} .$$

Hence

$$\begin{aligned} |\hat{\mu}(m)| &\le \left|1 + \cos 2\pi \frac{m}{(s+1)!}\right| + \sum_{j>s} \left|1 - \cos 2\pi \frac{m}{j!}\right| \\ &= 2\cos^2 \pi \frac{m}{(s+1)!} + 2 \sum_{j>s} \sin^2 \pi \frac{m}{j!} \qquad (9) \end{aligned}$$

where

$$\left|\cos \pi \frac{m}{(s+1)!}\right| = \left|\cos \pi [\tfrac{s}{2}] \frac{1}{s+1}\right| = O(\tfrac{1}{s})$$

and since $|x_{s+1} - x'_{s+1}| \le 2, \ldots, |x_{J-1} - x'_{J-1}| \le 2$, for $j \ge s+2$

$$\left|\sin \pi \frac{m}{j!}\right| = \left|\sin \pi \left([\tfrac{s}{2}] \frac{s!}{j!} + (x'_{s+1} - x_{s+1}) \frac{(s+1)!}{j!} + \cdots + (x'_{j-1} - x_{j-1}) \frac{(j-1)!}{j!}\right)\right| = O(\tfrac{1}{j}).$$

Substitution in (9) yield thus

$$|\hat{\mu}(m)| \le \text{const}\left(\frac{1}{s^2} + \sum_{j>s} \frac{1}{j^2}\right) \sim \frac{1}{s} < \frac{1}{cJ}$$

using the lower estimate on s given by Lemma 5. This completes the proof of Lemma 4.

The purpose of the next sections is to modify previous construction in order to obtain a (P)-set on which $\hat{\mu}$ actually vanishes.

4. REDUCTION TO A LOCAL PROBLEM.

Assume for each j positive integers $n_j < \frac{1}{10} N_j$ given and a trigonometric polynomial p_j satisfying following conditions

$$p_j \geq 0, \quad \hat{p}_j(0) = 1 \tag{10}$$

$$\operatorname{supp} \hat{p}_j \subset [-\tfrac{1}{4} N_j, \tfrac{1}{4} N_j]. \tag{11}$$

$$\text{If } \ A \subset [0,n_j], \ |A| > \tfrac{1}{j} n_j \ \text{ then } \ \hat{p}_j(m) = -\tfrac{1}{2} \text{ for some } m \in A - A. \tag{12}$$

LEMMA 6: Under the conditions (10), (11), (12), there is a positive measure μ, $\mu(\{0\}) = 1$, such that $\hat{\mu}$ vanishes on some set of recurrence.

Proof: Define

$$q_j(t) = p_1(t)p_2(N_1 t)p_3(N_1 N_2 t)\cdots p_j(N_1\cdots N_{j-1} t)$$

for which

$$\operatorname{supp} \hat{q}_j \subset [-\tfrac{1}{3} N_1\cdots N_j, \tfrac{1}{3} N_1\cdots N_j].$$

Thus since

$$q_j(t) = q_{j-1}(t)p_j(N_1\cdots N_{j-1} t)$$

we have

$$\int q_j(t)dt = (\int q_{j-1})(\int p_j) = 1. \tag{13}$$

Also for $|n| < \frac{1}{2} N_1 \cdots N_{j-1}$.

$$\hat{q}_j(n) = \hat{q}_{j-1}(n). \tag{14}$$

If $n = N_1 \cdots N_{j-1} m$, then

$$\hat{q}_j(n) = \hat{p}_j(m). \tag{15}$$

Let υ be the weak*-limit of the sequence $\{q_j\}$ in $M(\pi)$. By (13, (14). (15)

$$\|\upsilon\| = 1 \tag{16}$$

$$\hat{\upsilon}(n) = \hat{p}_j(m) \quad \text{if} \quad n = N_1 \cdots N_{j-1} m \quad \text{if} \quad m < \frac{1}{2} N_j. \tag{17}$$

Define

$$\mu = \delta_0 + 2\gamma. \tag{18}$$

Take $S \subset \mathbb{Z}$, $D^*(S) > \varepsilon > \frac{1}{j}$. Since the class $\{A - A | A \subset S,\ A \text{ finite}\}$ is homogeneous in the sense of [R], there is a subset A_1 of $[0, N_1 \cdots N_{j-1} n_j]$ for which

$$A_1 - A_1 \subset S - S \tag{19}$$

$$|A_1| > \varepsilon N_1 \cdots N_{j-1} n_j. \tag{20}$$

Hence there is $A \subset \{1,\dots,n_j\}$ satisfying

$$(A - A)N_1 \cdots N_{j-1} \subset A_1 - A_1 \tag{21}$$

$$|A| > \varepsilon n_j.$$

By (12), $\hat{p}_j(m) = -\frac{1}{2}$ for some $m \in A - A$. Thus if $n = N_1 \cdots N_{j-1} m$, then $n \in S - S$ by (21), (19) and since $|m| < \frac{1}{2} N_j$, by (17)

$$\hat{\mu}(n) = 1 - 2\hat{\rho}(n) = 1 + 2\hat{p}_j(m) = 0.$$

This proves the lemma.

Fixing an integer n and $\varepsilon > 0$, our purpose will be to construct a positive measure μ, $\|\mu\| = \frac{3}{4}$, satisfying

$$\hat{\mu}(m) = -\frac{1}{2} \text{ for some } m \in A - A$$

whenever $A \subset [0,n]$, $|A| > \varepsilon n$. $\qquad (*)$

Given μ, let for some N

$$p_1 = (\mu * F_N) + [\mu - (\mu * F_N)] * D_n$$

where

$F_N = N$ - Féjer kernel and $D_n = n$ - Dirichlet kernel. For N large enough, we may ensure that

$$\|[\mu - (\mu * F_N)] * D_n\|_\infty < \frac{1}{4}$$

Thus $p = p_1 + \frac{1}{4}$ is a positive polynomial and

$$\hat{p}(m) = \hat{p}_1(m) = \hat{\mu}(m) \quad \text{for} \quad |m| \leq n,\ m \neq 0.$$

It is now clear how to get from (*) a sequence (p_j) satisfying the conditions of Lemma 6.

5. CONSTRUCTION OF CERTAIN MEASURES.

Fix an integer N and consider the basic measure

$$\sigma = \frac{1}{2}(\delta_{\frac{2\pi}{N}} + \delta_{-\frac{2\pi}{N}})$$

with transform

$$\hat{\sigma}(n) = \cos 2\pi\frac{n}{N}\ .$$

In this section we construct a perturbation σ_1 of σ satisfying the following conditions.

LEMMA 7: Given R and a number L, there is a positive measure σ_1 such that σ_1 is supported by the N-th roots of unity and

$$\hat{\sigma}_1(n) = 1 \quad \text{if} \quad L \leq |n| \leq RL \tag{22}$$

$$\hat{\sigma}_1(n) = -1 \quad \text{if} \quad |\frac{N}{2} - n| \leq RL \tag{23}$$

$$\|\sigma - \sigma_1\|_{M(\Pi)} \leq c(R)(\frac{L}{N})^2. \tag{24}$$

<u>Proof</u>: We assume $RL \ll N$, N even. Consider the polynomial

$$q_1(t) = \sum_{|n| \leq RL} [1 - \cos \frac{2\pi}{N} (RL - |n|)] e^{int}.$$

Since the function

$$\begin{cases} 1 - \cos \frac{2\pi}{N} (RL - n) & \text{if } 0 \leq n \leq RL \\ 0 & \text{if } n > RL \end{cases}$$

is nonnegative, decreasing and convex, q_1 is positive. Hence

$$\|q_1\|_1 = 1 - \cos \frac{2\pi}{N} RL < 10(\frac{RL}{N})^2. \tag{25}$$

Define

$$q_2 = q_1 * F_L \quad ; \quad F_L(t) = \sum_{|n| \leq L} \frac{L - |n|}{L} e^{int} = \text{Féjer kernel}.$$

Then

$$\text{supp } \hat{q}_2 \subset [-L, L] \tag{26}$$

$$q_2 \geq 0, \ \|q_2\|_1 \leq 10(\frac{RL}{N})^2 \tag{27}$$

$$q_1 \leq 10 R \, q_2. \tag{28}$$

Next, define the polynomial

$$q_3(t) = 40\,R\,q_2(t) + [2\cos RLt - \cos(\tfrac{N}{2} - RL)t - \cos(\tfrac{N}{2} + RL)t]q_1(t).$$

By (28), (27)

$$q_3 \geq 0,\ \|q_3\|_1 \leq 500\,R(\tfrac{RL}{N})^2. \tag{29}$$

If $L < n < RL$, then

$$\hat{q}_3(n) = \hat{q}_1(n - RL) = 1 - \cos\frac{2\pi}{N}n. \tag{30}$$

If $|\frac{N}{2} - n| < RL$, then

$$2\hat{q}_3(n) = -\hat{q}_1(n - \tfrac{N}{2} + RL) - \hat{q}_1(n - \tfrac{N}{2} - RL) = -\hat{q}_1(RL - |n - \tfrac{N}{2}|) = -1 - \cos\frac{2\pi}{N}n. \tag{31}$$

Finally consider the positive measure

$$\sigma_1 = \sigma + \frac{1}{N}q_3 \cdot \sum_{k=0}^{N-1} \delta_{2\pi\frac{k}{N}}$$

for which by (29)

$$\|\sigma_1 - \sigma\|_{M(\Pi)} \leq \frac{1}{N}\sum_{k=0}^{N-1} |q_3(2\pi\tfrac{k}{N})| \leq 2\|q_3\|_1 < 2000\,R^3(\tfrac{L}{N})^2$$

while

$$\hat{\sigma}_1(n) = \hat{\sigma}(n) + \frac{1}{N} \sum_{k=0}^{N-1} (q_3 e^{-int})|_{t=\frac{2\pi k}{N}} = \cos \frac{2\pi}{N} n + \sum (\hat{q}_3(m) | m - n \in N\mathbb{Z})$$

and (22), resp. (23) follow from (30), resp. (31), as easily verified.

6. PROOF OF EXISTENCE OF A (P)-SET WHICH IS NOT (V.d.C.).

Our aim is to satisfy (*) in Section 4. We will use arguments similar to those of Section 3 and the measures constructed in the previous section. Take n of the form Q^P, Q even. Fix an integer R. Use the representation

$$m = \sum_{j=0}^{P-1} q_j Q^j \qquad (0 \leq q_j < Q)$$

to get a one-to-one map from $[0, n-1]$ into $\Omega \equiv \{0, 1, \ldots, Q-1\}^P$. Denote ν the normalized counting measure on Ω. Identifying $\{0, 1, \ldots, Q-1\}$ with the cyclic group $\mathbb{Z}/Q\mathbb{Z}$, denote θ the coordinate wise shift acting on Ω. By Lemma 7, we get for each j a positive measure σ_j on Π satisfying the conditions

$$\hat{\sigma}_j(m) = 1 \quad \text{if} \quad Q^j \leq |m| \leq RQ^j \tag{32}$$

$$\hat{\sigma}_j(m) = -1 \quad \text{if} \quad \left|\frac{Q^{j+1}}{2} - m\right| \leq RQ^j \tag{33}$$

$(L = Q^j, N = Q^{j+1})$. Moreover

$$\|\sigma_j\|_{M(\Pi)} \leq 1 + C(R) \frac{1}{Q^2} \tag{34}$$

and σ_j is supported by the Q^{j+1}-roots of unity, implying Q^{j+1}-periodicity of $\hat{\sigma}_j$. Define

$$\nu = \sigma_0 * \sigma_1 * \cdots * \sigma_{P-1}.$$

Hence by (34)

$$\|\nu\|_{M(\Pi)} \leq 1 + C(R)\,\frac{P}{Q^2}\,. \tag{35}$$

Let A be a subset of $[0,n-1]$, $|A| > \varepsilon n$. Let $\tilde{A} \subset \Omega$ be its image under the correspondence mentioned earlier. Thus $\nu(\tilde{A}) > \varepsilon$. Consider next the sets $\tilde{A}, \theta(\tilde{A}), \ldots, \theta^R(\tilde{A})$. It is easily seen that for some $4 \leq r \leq R-2$ say, the set

$$B = \tilde{A} \cap \theta^{-r}(\tilde{A})$$

will satisfy $\nu(B) > \frac{\varepsilon^2}{10}$, provided we choose $R > \frac{10}{\varepsilon}$. (This is the recurrence principle).

Assuming now

$$(1 + \tfrac{1}{Q})^P > 10\varepsilon^{-2} \tag{36}$$

the same combinatorial argument as described in Lemma 5 gives a pair of points x, x' in B

$$x = (x_1, \ldots, x_{s-1}, 0, x_{s+1}, \ldots, x_p) \quad x' = (x_1, \ldots, x_{s-1}, \tfrac{Q}{2}, x'_{s+1}, \ldots, x'_p)$$

where

$$|x_j - x'_j| \leq 2 \quad \text{if} \quad s < j \leq P.$$

Thus, x, $\theta^r(x')$ are both in $\tilde{\Lambda}$ corresponding to a pair of points a, a' in $\dot{\Lambda}$

$$a = \sum_{j=0}^{P-1} q_j Q^j \;;\; a' = \sum_{j=0}^{P-1} q'_j Q^j$$

where

$$q'_s - q_s \in \{\tfrac{Q}{2} - R + 1, \ldots, \tfrac{Q}{2} + R - 1\} + Q\mathbb{Z} \tag{37}$$

$$q'_j - q_j \in \{-R + 1, \ldots, -2, 2, \ldots, R - 1\} + Q\mathbb{Z}. \tag{38}$$

Let $m = a' - a$. Thus $m \in \Lambda - \Lambda$ and we claim that $\hat{\upsilon}(m) = -1$. Since $\hat{\upsilon}(m) = \Pi\hat{\sigma}_j(m)$, in view of (32), (33), it suffices to show that

$$m \in [\tfrac{1}{2} Q^{s+1} - RQ^s, \tfrac{1}{2} Q^{s+1} + RQ^s] + Q^{s+1}\mathbb{Z} \tag{39}$$

and

$$m \in ([-RQ^j, RQ^j] \setminus [-Q^j, Q^j]) + Q^{j+1}\mathbb{Z} \quad \text{if} \quad j \neq s. \tag{40}$$

Clearly, for a fixed j

$$m \in (q'_j - q_j)Q^j + [-Q^j, Q^j] + Q^{j+1}\mathbb{Z}.$$

Therefore (39), resp. (40) follows from (37), resp. (38). To satisfy (36), take $P = c(\varepsilon)Q$. Then (35) implies together with the condition on R

$$\|\nu\|_{M(\Pi)} \leq 1 + c'(\varepsilon)\frac{1}{Q} < 2$$

for Q large enough. Take $\mu = \frac{1}{2}\nu + (1 - \frac{1}{2}\hat{\nu}(0))$ which will fulfill (*). This completes the proof.

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