A SZEMEREDI TYPE THEOREM FOR SETS OF POSITIVE DENSITY IN $\mathbb{R}^k$

J. BOURGAIN

Institut des Hautes Etudes Scientifiques
35, route de Chartres
91440 Bures-sur-Yvette (France)

July 1985

IHES/M/85/46

A SZEMEREDI TYPE THEOREM FOR SETS OF POSITIVE DENSITY IN $\mathbb{R}^k$

J. BOURGAIN(*)

SUMMARY : Let $k \geq 2$ and A a subset of $\mathbb{R}^k$ of positive upper density. Let V be the set of vertices of a (non-degenerated) $(k-1)$-dimensional simplex. It is shown that there exists $\ell = \ell(A,V)$ such that A contains an isometric image of $\ell'.V$ whenever $\ell' > \ell$. The case $k=2$ yields a new proof of a result of Katznelson and Weiss [4]. Using related ideas, a proof is given of Roth's theorem on the existence of arithmetic progressions of length 3 in sets of positive density.

1. INTRODUCTION

The following result has been obtained by Katznelson and Weiss [4].

THEOREM 1 : Whenever A is a subset of $\mathbb{R}^2$ with positive upper density, then a number $\ell = \ell(A)$ such that $|x-y| = \ell'$ for some $x,y \in A$, fixing any $\ell' > \ell$.

Recall that $A \subset \mathbb{R}^k$ has positive upper density provided

$$\delta(A) \equiv \overline{\lim_R} \frac{|B(o,R) \cap A|}{|B(o,R)|} > 0$$

where $B(o,R) = \{x \in \mathbb{R}^k ; |x| < R\}$.

Their argument combines ergodic theory and measure theory. In the next section, a short proof will be given based on elementary harmonic analysis. This proof can be elaborated in order to get the result mentioned above, thus

THEOREM 2 : Assume $A \subset \mathbb{R}^k, \delta(A) > 0$ and V a set of k points spanning a $(k-1)$-dimensional hyperplane. There exists some number ℓ such that A contains an isometric copy of $\ell'.V$ whenever $\ell' > \ell$.

(*) I.H.E.S.

REMARKS

(a) Theorem 2 is of the same nature as the generalizations of Széméredy's theorem [7] obtained in [3] (see also [2]). More precisely, the dilations are replaced by rotations. Although the method presented here requires an increasing dimension, the exact rôle of the dimension k does not seem well understood yet.

(b) The following simple example clarifies the necessity of the non-degeneracy hypothesis on the set V. Let $V = \{-1,0,1\}$ and $A = \{x \in \mathbb{R}^k ; |x|^2 \in [0,\frac{1}{10}] + \mathbb{Z}_+\}$. Clearly $\delta(A) > 0$. Assume now $x \in A$ and $y \in \mathbb{R}^k$, $|y| = t$ satisfying $x + y \in A$ and $x - y \in A$. Then

$$2t^2 = 2|y|^2 = |x + y|^2 + |x - y|^2 - 2|x|^2 \in [0,\tfrac{1}{5}] + \mathbb{Z}_+$$

implying the existence of some $k \in \mathbb{Z}_+$ s.t.

$$|t - \sqrt{\tfrac{k}{2}}| < \frac{1}{5\sqrt{k}} .$$

Consequently, there are arbitrary large values of ℓ such that A does not contain an isometric copy of $\ell.V$.

This example permits several variations.

It is easily seen that Thms 1 and 2 result from the following "compact" version.

PROPOSITION 3 : Let V be as in Th. 2, diam $V < 1$. Let $A \subset [0,1]^k$, $|A| > \varepsilon$ and $0 < t_j < 1$ a sequence satisfying $t_{j+1} < \frac{1}{2} t_j$. Then there exists $j \leq J(\varepsilon, V)$ such that A contains an isometric image of $t_j.V$. In fact, for $t = t_j$.

$$(1) \quad \int_{\mathbb{R}^k} \int_{SO(k)} f(x) f(x + tOa_1) \ldots f(x + tOa_{k-1})\, dx\, dO > \frac{1}{2}\, \varepsilon^k$$

where $f = \chi_A$, $V = \{0, a_1, \ldots, a_{k-1}\}$ and dO refers to the normalized invariant measure on the orthogonal group $SO(k)$.

For the sake of clarity, the case $k = 2$ will be handled separately. The complete proof of Prop. 3 is given in section 3 of this paper. The last section is an appendix in which it is shown how a new proof of Roth's theorem (see [5]) can be obtained using similar ideas. The letters $0 < c, C < \infty$ denote numerical constants.

2. A PROOF OF THE KATZNELSON-WEISS THEOREM

As usual $\hat{F}(\xi) = \int_{\mathbb{R}^k} F(x) e^{-2\pi i\langle x,\xi\rangle}\, dx$ stands for the Fourier transform. In case $k = 2$, the left member of (2) becomes

$$\iint f(x) f(x+ty)\, dx\, \sigma(dy) = \int \hat{f}(\xi)\hat{f}(-\xi)\hat{\sigma}(t\xi)\, d\xi = \int |\hat{f}(\xi)|^2 \hat{\sigma}(t|\xi|)\, d\xi \tag{2}$$

where σ denotes the normalized arc-length measure of the unit circle. Thus

$$|\hat{\sigma}(\xi)| \le C|\xi|^{-\frac{1}{2}} \quad \text{and} \quad |1-\hat{\sigma}(\xi)| < C|\xi| \tag{3}$$

Also, by definition of f

$$|\hat{f}(\xi)-\hat{f}(o)| \le 2\pi \int_A |\langle x,\xi\rangle|\, dx \ ; \ |\hat{f}(\xi) - |A|\ | < C\ |\xi|\,|A| \ .$$

Hence, for $\delta > t$ to be specified later, as consequence of (3)

$$\int |\hat{f}(\xi)|^2 \hat{\sigma}(t|\xi|)\, d\xi = \Big\{\int_{[|\xi| \le \delta t^{-1}]} + \int_{[\delta t^{-1} < |\xi| < \delta^{-1} t^{-1}]} + \int_{[|\xi| > \delta^{-1} t^{-1}]}\Big\} |\hat{f}(\xi)|^2 \hat{\sigma}(t\xi)\, d\xi$$

$$\ge \frac{1}{2} \int_{[|\xi| \le \delta t^{-1}]} |\hat{f}(\xi)|^2 d\xi - \int_{[\delta t^{-1} < |\xi| < \delta^{-1} t^{-1}]} |\hat{f}(\xi)|^2 d\xi - C\delta^{1/2} \int |\hat{f}(\xi)|^2 d\xi$$

$$\ge c_1 |A|^2 - C\delta^{1/2} |A| - \int_{[\delta t^{-1} < |\xi| < \delta^{-1} t^{-1}]} |\hat{f}(\xi)|^2 d\xi \ .$$

Assume $\delta \ll |A|^2$. It is clear that there exists some $j \le C(\log\frac{1}{\delta})\varepsilon^{-1} \sim \frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ satisfying

$$\int_{[\delta t_j^{-1} < |\xi| < \delta^{-1} t_j^{-1}]} |\hat{f}(\xi)|^2 d\xi < \frac{c_1}{3}\, \varepsilon^2$$

and therefore

$$\iint f(x)\, f(x+t_j y)\, dx\sigma(dy) \geq \frac{C_1}{2} |A|^2 \qquad \text{QED.}$$

REMARK : Combined with the results on the spherical maximal function in the plane, Th. 1 can be improved as follows :

THEOREM 1' : If $A \subset \mathbb{R}^2$, $\delta(A) > 0$, there exists $\ell = \ell(A)$ such that whenever $\ell_1 > \ell$ there is a point $x \in A$ fulfilling the condition

$$\{|x-y| \ ; \ y \in A\} \supset [\ell, \ell_1] \ .$$

Denote P_t the Poisson-semigroup kernel on $\mathbb{R}^k$. Thus $\hat{P}_t(\xi) = e^{-t|\xi|}$. In general, let $K_t(x) = t^{-k}K(t^{-1}x)$ satisfying $\hat{K}_t(\xi) = \hat{K}(t\xi)$.
The key estimate of [1] related to the planar spherical maximal operator can be formulated as follows :

PROPOSITION 1 : For $p > 2$, there are constants $C(p) < \infty$ and $\alpha(p) > 0$ satisfying

$$\| \max_{s \geq t_0} |[f-(f*P_t)]*\sigma_s| \|_p \leq C(p)\left(\frac{t}{t_0}\right)^{\alpha(p)} \|f\|_p \ , \ t_0 > t \qquad (4)$$

Similarly as in proving Th 1, the negation of Th 1' leads to a subset A of $[0,1]^2$, $|A| > \varepsilon$ and a sequence of positive numbers

$$s_1 > t_1 > s_2 > t_2 > \dots \qquad > s_J > t_J$$

where J can be taken arbitrarily large, satisfying the properties

$$s_{j+1} < \frac{1}{2} t_j \qquad (5)$$

and

$$x \in A \cap [s_j, 1-s_j]^2 \Rightarrow \sup_{s_j > t > t_j} [(1_R - f)*\sigma_t] = 1 \quad (f = 1_A \text{ and } R = [0,1]^2)$$

Hence, we may write for a fixed $\tau > 0$ and choosing $j < J$ large enough

$$\int f \sup_{s_j>t>t_j} [(1_R - f)*\sigma_t] < (1-\tau)\int f \tag{6}$$

Fix $\delta > 0$. As a consequence of (4), we may write

$$\| \sup_{s_j>t>t_j} [(1_R-f)*\sigma_t] - \sup_{s_j>t>t_j} [(1_R-f)*P_{\delta t_j}*\sigma_t]\|_1 \leq \| \sup_{t>t_j} |[f - f*P_{\delta t_j}]*\sigma_t\|_p + \tau < C\delta^\alpha + \tau \tag{7}$$

For $t < s_j$, also

$$|[(1_R-f)*P_{\delta^{-1}s_j}*\sigma_t](x) - [(1_R-f)*P_{\delta^{-1}s_j}](x)| \leq \| P_{\delta^{-1}s_j} - (P_{\delta^{-1}s_j}*\sigma_t)\|_1 < C\delta \tag{8}$$

Thus, again by (4), using (6), (7), (8)

$$C\|[(1_R-f)*P_{\delta^{-1}s_j}] - [(1_R-f)*P_{\delta t_j}]\|_p \geq \int f \sup_{s_j>t>t_j} \{[(1_R-f)*P_{\delta t_j}*\sigma_t] - [(1_R-f)*P_{\delta^{-1}s_j}*\sigma_t]\}$$

$$\geq (1-\tau)\int f - \int f[(1_R-f)*P_{\delta^{-1}s_j}] - C\delta^\alpha - \tau$$

$$\geq -2\tau\int f + (\int f)^2 - C\delta^\alpha - \tau$$

$$\geq (\varepsilon - 2\tau)\varepsilon - C\delta^\alpha - \tau$$

Taking τ, δ small enough and J sufficiently large, a contradiction follows. Indeed, if $t_{j+1} < \frac{1}{2}t_j$, then for $2 \leq p \leq \infty$

$$\{\Sigma_j \| (f*P_{t_{j+1}}) - (f*P_{t_j}) \|_p^p\}^{\frac{1}{p}} \leq C\|f\|_p$$

This completes the proof of Th.1'.

3. PROOF OF THEOREM 2 IN GENERAL CASE

Let $V = \{0, a_1, a_2, \ldots a_{k-1}\}$ be non-degenrated. Simple invariance arguments show that the left member of (1) may be rewritten as

$$\int f(x) f(x+ty_1) f(x+ty_2) \ldots f(x+ty_{k-1}) \sigma^{(k-1)}(dy_1) \sigma^{(k-2)}_{y_1}(dy_2) \ldots \sigma^{(1)}_{y_1,\ldots,y_{k-2}}(dy_{k-1}) \tag{9}$$

where $\sigma^{(j)}_{y_1,\dots,y_{k-j-1}}$ is the average on a (j)-dimensional sphere in $\mathbb{R}^k$ dependent on the points $y_1,\dots,y_{k-j-1}$ already fixed and on V. We will use the estimate

$$| \widehat{\sigma^{(j)}_{y_1,\dots,y_{k-j-1}}}(\xi) | \leqslant C_V [1+\mathrm{dist}(\xi,[y_1,\dots,y_{k-j-1}])]^{-\frac{j}{2}} \tag{10}$$

which is a consequence of the decay at infinity of the Fourier transform of the j-sphere in R^{j+1}. Denote $G_{k,m}(m<k)$ the Grassmannian of m-dimensional subspaces of $\mathbb{R}^k$ endowed with the normalized Haar-measure.

<u>LEMMA 1</u> : For $m < k$

$$\int_{\mathbb{R}^k}\int_{G_{k,m}} [\mathrm{dist}(\xi,F)+1]^{-\rho} |\hat{f}(\xi)|^2 (1-e^{-\delta|\xi|})^2 \, d\xi dF < C_k(\delta+\delta^{\frac{\rho}{2}}) \|f\|_2^2 \tag{11}$$

<u>Proof</u> : Estimate the left member of (11) as

$$C\delta \|f\|_2^2 + C \{ \sup_{|\xi| > \delta^{-1/2}} \int_{G_{k,m}} [\mathrm{dist}(\xi,F)+1]^{-\rho} dF \} \|f\|_2^2$$

<u>Proof of Theorem 2</u> : Denote for simplicity

$$d\Omega_j(y_1,\dots,y_j) = \sigma^{(k-1)}(dy_1)\sigma^{(k-2)}_{y_1}(dy_2)\dots\sigma^{(k-j)}_{y_1,\dots,y_{j-1}}(dy_j)$$

Fix $\delta > 0$ and compare the expressions

$$\int f(x)f(x+ty_1)\dots f(x+ty_{k-2})f(x+ty_{k-1})dx d\Omega_{k-1}(y_1,\dots,y_{k-1}) \tag{12}$$

and

$$\int f(x)f(x+ty_1)\dots f(x+ty_{k-2})(f*P_{\delta t})(x+ty_{k-1})dx \, d\Omega_{k-1}(y_1,\dots,y_{k-1}) \tag{13}$$

which difference can be estimated as

$$|(12)-(13)| \leq \int \| [f-(f*P_{\delta t})]*[\sigma^{(1)}_{y_1,\dots,y_{k-2}}]_t \|_2 d\Omega_{k-2}(y_1,\dots y_{k-2})$$

or by Parseval's identity, using (10), (11), as

$$C_V\{\int_{\mathbb{R}^k}\int_{G_{k,k-2}} |\hat{f}(\xi)|^2 [1-e^{-\delta t|\xi|}]^2 \{1+\mathrm{dist}(t\xi,F)\}^{-1/2} d\xi dF\}^{1/2} \leq C_V\delta^{1/4} \|f\|_2 \tag{14}$$

Next we compare the expressions

$$\int f(x)(f*P_{\delta^{-1}t})(x)f(x+ty_1)\dots f(x+ty_{k-2})dx\, d\Omega_{k-1}(y_1,\dots y_{k-1}) \tag{15}$$

and

$$\int f(x)f(x+ty_1)\dots f(x+ty_{k-2})(f*P_{\delta^{-1}t})(x+ty_{k-1})dx\, d\Omega_{k-1}(y_1,\dots,y_{k-1}) , \tag{16}$$

which difference is simply majorated by

$$\sup_{|y|<1} \|(f*P_{\delta^{-1}t})(x) - (f*P_{\delta^{-1}t})(x+ty)\|_{L^2(dx)} \leq$$

$$\sup_{|y|<1} \{\int|\hat{f}(\xi)|^2|1-e^{2\pi i<ty,\xi>}|^2 e^{-\delta^{-1}t|\xi|}d\xi\}^{1/2} < C\delta\,\|f\|_2 \tag{17}$$

Collecting estimates, it now follows

$$|(12)-(15)| \leq |(12)-(13)|+|(15)-(16)|+|(13)-(16)| \leq C_V\delta^{1/4}\|f\|_2+\|(f*P_{\delta^{-1}t})-(f*P_{\delta t})\|_2$$

In the expression (15)

$$(15) = \int f(x)(f*P_{\delta^{-1}t})(x)f(x+ty_1)\dots f(x+ty_{k-2})dx d\Omega_{k-2}(y_1,\dots,y_{k-2})$$

the variable y_{k-1} does not appear any more. We treat (15) the same way as (12) where y_{k-2} plays the rôle of y_{k-1}. Thus defining

$$(17) = \int f(x)(f*P_{\delta^{-1}t})^2(x)f(x+ty_1)\dots f(x+ty_{k-3})dx d\Omega_{k-3}(y_1,\dots,y_{k-3})$$

Similar computations give

$$|(15-(17)| \leq C_V(\delta+\delta^{1/2})\|f\|_2 + \|(f*P_{\delta^{-1}t})-(f*P_{\delta t})\|_2$$

Iteration of the procedure yields that

$$|(12)-\int f(x)(f*P_{\delta^{-1}t})^{k-1}(x)dx| \leq C_V(k\delta+\sum_{\delta=1}^{k-1}\delta^{r/4})\,\|f\|_2+k\|(f*P_{\delta^{-1}t})-(f*P_{\delta t})\|_2 \tag{18}$$

Further

$$\varepsilon^k \leq \int(f*P_{\delta^{-1}t})^k$$

$$\leq \int(f*P_{\delta^{-1}t})^{k-1}(f*P_{\delta t}) + \|(f*P_{\delta t}) - (f*P_{\delta^{-1}t})\|_2$$

$$\leq \int f.(f* P_{\delta^{-1}t})^{k-1} + \|(f* P_{\delta^{-1}t})^{k-1} - [P_{\delta t}*(f* P_{\delta^{-1}t})^{k-1}]\|_2 + \|(f*P_{\delta t})-(f* P_{\delta^{-1}t})\|_2$$

where the second term is dominated by

$$\sqrt{2}\{\int(f* P_{\delta^{-1}t})^{2(k-1)} - \int[P_{\frac{\delta}{2}t}*(f* P_{\delta^{-1}t})^{k-1}]^2\}^{1/2} \leq$$

$$\sqrt{2}\{\int(f* P_{\delta^{-1}t})^{2(k-1)} - \int(f* P_{\delta^{-1}t}*P_{\frac{\delta}{2}t})^{2(k-1)}\}^{1/2} \leq$$

$$Ck\,\|(f* P_{\delta^{-1}t})-(f* P_{\delta^{-1}t}*P_{\frac{\delta}{2}t})\|_2 \leq Ck\delta^2\|f\|_2 \quad .$$

Therefore, as a consequence of (18) and previous computation

$$(12) \geq \varepsilon^k \quad C_V k\delta^{1/4}\|f\|_2 - \|(f*P_{\delta t}) - (f* P_{\delta^{-1}t})\|_2 \ (k+1)$$

Taking suitable $t \in \{t_1 > t_2 > \ldots > t_J\}$ $(t_{j+1} < \frac{1}{2}t_j)$, we may dominate

$$\|\ (f*P_{\delta t})-(f* P_{\delta^{-1}t})\ \|_2 \leq \frac{C}{J}(\log\tfrac{1}{\delta})\ \|f\|_2$$

so that

$$(12) \geq \varepsilon^k - C_V k(\delta^{1/4} + J^{-1}(\log\tfrac{1}{\delta}))\quad \sqrt{\varepsilon} \quad > \tfrac{1}{2}\varepsilon^k$$

for an appropriate choice of δ and J . This completes the proof.

4. APPENDIX :

A PROOF OF ROTH'S THEOREM ON ARITHMETIC PROGRESSIONS OF LENGTH 3

Let G be a compact Abelian group and $\Gamma = \hat{G}$ the dual group.

THEOREM 3 : Given $\varepsilon > 0$, there exists $\varepsilon' = \varepsilon'(\varepsilon)$ such that whenever f is a function on G , $0 \leq f \leq 1$ and $\int_G f(x)dx > \varepsilon$, then

$$\iint_{G\times G} f(x)f(x+y)f(x+2y)dxdy > \varepsilon' \ . \qquad (1)$$

Applying the result to a finite cyclic group $G = \mathbb{Z}/_{N\mathbb{Z}}$ (taking N large enough) and $f = \chi_S$ $(S \subset G, |S| > \varepsilon)$ yields Roth's theorem ([5]) .

The proof is based on two lemmas :

<u>LEMMA 2</u> : $|\iint f_1(x)f_2(x+y)f_3(x+2y)K(y)dxdy| \leq \|K\|_{A(G)} \prod_{i=1}^{3} \|\hat{f}_i\|_\infty^{1/3} \|f_i\|_2^{2/3}$

<u>Proof</u> : $|\langle f_1, \int f_2(\cdot+y)f_3(\cdot+2y)K(y)dy\rangle| \leq \|\hat{f}_1\|_\infty \|\int f_2(\cdot+y)f_3(\cdot+2y)K(y)dy\|_{A(G)}$

and the second factor is dominated by $\|K\|_{A(G)} \|f_2\|_2 \|f_3\|_2$. Reversing the rôle of f_1, f_2 and making the product gives the estimate.

<u>LEMMA 3</u> : (Bozejko-Pelczynski theorem on invariant approximation, cf. [8]).

Given a finite subset Λ of Γ and $\tau>0$, there exists a kernel K satisfying

(i) $K \geq 0$, $\hat{K} \geq 0$ and $\hat{K}(0) = 1$

(ii) $|\hat{K}(\gamma)-1| < \tau$ for $\gamma \in \Lambda$

(iii) $|\text{supp } \hat{K}| < N(|\Lambda|, \tau)$

<u>Proof of Th. 3</u> : Let f be as in Th. 1. Combining lemmas (1),(2), it follows that given a kernel K with $\hat{K}$ finitely supported, there exists K' satisfying (i) of Lemma 2 and

(2) $|K - (K*K')| < \tau$

(3) $|\iint f(x)f(x+y)f(x+2y)K(y)dxdy - \iint (f*K')(x)(f*K')(x+y)(f*K')(x+2y)K(y)dxdy| < \tau$

(4) $|\text{supp } \hat{K}'| < N'(|\text{supp } \hat{K}|, \|\hat{K}\|_\infty, \tau)$.

Take $K_0 = 1$. Previous considerations and an inductive construction lead to a sequence $\{K_i\}_{0 \leq i < I}$ satisfying (i) of Lemma 2 (I is a positive integer of size $\sim\varepsilon^{-3}$) .

Denote $f_i = f * K_i$. By (2), $|f_i - (f_i * K_{i+1})| < \tau$. Thus

$$\|f_{i+1}-f_i\|_2^2 = \|f_{i+1}\|_2^2 + \|f_i\|_2^2 - 2\langle f_i, f_{i+1}\rangle < \|f_{i+1}\|_2^2 + \|f_i\|_2^2 - 2\langle f_i, f\rangle + 2\tau \leq \|f_{i+1}\|_2^2 - \|f_i\|_2^2 + 2\tau$$

and summation shows the existence of some $1 \leq i \leq I$ fulfilling

$$\|f_{i+1} - f_{i-1}\|_1 < 4\tau + 2I^{-1}$$

and hence

(5) $|\iint f_{i+1}(x)f_{i+1}(x+y)f_{i+1}(x+2y)K_i(y)dxdy - \iint f_{i-1}(x)f_{i-1}(x+y)f_{i-1}(x+2y)K_i(y)dxdy| < 12\tau+6I^{-1}$

Assume (1) does not hold. From (3) and the construction $(K = K_i, K' = K_{i+1})$ it now follows from (5)

(7) $|\iint f_{i-1}(x)f_{i-1}(x+y)f_{i-1}(x+2y)K_i(y)dxdy| < 13\tau+6I^{-1}+\varepsilon'\|K_i\|_\infty$

Also, for $\gamma = 1,2$

(8) $\iint |f_{i-1}(x+\gamma y)-f_{i-1}(x+(\gamma-1)y)|K_i(y)dxdy \leq$

$$\left(\iint |f_{i-1}(x+y)-f_{i-1}(x)|^2 K_i(y)dxdy\right)^{1/2} = \sqrt{2}\left(\|f_{i-1}\|_2^2 - \langle f_{i-1}, f_{i-1} * K_i\rangle\right)^{1/2} < 4\sqrt{\tau}$$

which permits us to replace in the left member of (7) $f_{i-1}(x+y)$, $f_{i-1}(x+2y)$ by $f_{i-1}(x)$. Hence

$$\varepsilon^3 \leq \left(\int_G f\right)^3 \leq \iint f_{i-1}(x)^3 K_i(y)dxdy < 16\tau+6I^{-1}+\varepsilon'\|K_i\|_\infty$$

giving a lower bound on ε'.

<u>REMARK</u> : It follows for instance from the construction of Salem and Spencer (see [6],p) that $\varepsilon'(\varepsilon)$ is not a polynomial function of ε in Theorem 3. However there exist known methods providing better bounds than results from the previous argument.

REFERENCES

[1] J. BOURGAIN : On the spherical maximal function in the plane, preprint IHES

[2] H. FURSTENBERG : Ergodic behavior of diagonal measures and a theorem of Szemerédi on Arithmetic progressions, J. Analyse Math. 31 (1977), 204-256.

[3] H. FURSTENBERG, Y. KATZNELSON : An ergodic Szemerédi theorem for commuting transformations, J. Analyse Math. 34 (1978), 275-291.

[4] Y. KATZNELSON, B. WEISS : preprint

[5] K. ROTH : 1952

[6] R. SALEM : Oeuvres Choisies

[7] E. SZEMEREDI : 1975

[8] Séminaire d'Analyse Fonctionnelle, Ecole Polytechnique, 78-79, Exp 9