CIRCLE MAPPINGS

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These lectures are devoted to the theory of a particularly simple kind of dynamical system—continuous invertible mappings of the circle to itself. We begin by reviewing the classical theory of these systems; then take up very recent developments using renormalization group ideas.

1. Circle mappings and rotation numbers

Our first step will be to pass from mappings of the circle to itself to mappings of the real line to itself. We will represent the circle as the quotient $\mathbf{R}/\mathbf{Z}$ of the real line by the integers. It is easy to see that any continuous mapping of the circle to itself can be "unrolled" to a continuous mapping of the line to itself. More precisely, any continuous mapping of the circle to itself is induced, through passage to quotients, by a continuous mapping f of $\mathbf{R}$ to itself. This f is unique up to adding an integer, and it has a periodicity property: There is an integer k such that $f(x+1) = f(x) + k$ for all x. Conversely, any continuous f with this property does induce a continuous mapping of the circle to itself. An f which induces in this way a given continuous mapping of the circle is called a *lift* of that mapping.

The induced mapping on the circle is invertible if and only if either:

- f is strictly decreasing and $k = -1$, in which case we say the induced map is *orientation reversing*, or
- f is strictly increasing and $k = +1$, in which case we say the induced map is *orientation reversing*.

From the point of view of dynamical systems, the orientation reversing case is easily reduced to the orientation preserving one, since any even iterate of an orientation reversing mapping is orientation preserving. We will therefore consider only lifts of orientation preserving mappings of the circle to itself, i.e., the objects we will be studying are continuous strictly increasing mappings f of $\mathbf{R}$ to itself satisfying

$$f(x+1) = f(x) + 1 \qquad \text{for all } x. \tag{1.1}$$

For brevity, we will refer to such an f as a *circle mapping*, and the condition (1.1) as the *circle mapping identity*. We note that the circle mapping identity is equivalent to the condition that $f(x) - x$ is periodic with period one, and also to the condition that f commutes with the unit translation $x \mapsto x + 1$.

Proposition 1.1. *Let f be a circle mapping. Then*

$$\lim_{n\to\infty} \left(f^n(x) - x\right)/n$$

exists for all x and is independent of x.

The x-independent value of this limit is called the *rotation number* of f; we will denote it by $\rho(f)$. We collect a few of the elementary properties of the rotation number in the following proposition:

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PROPOSITION 1.2. *1. If f_n is a sequence of circle mappings converging uniformly to f on $[0,1]$, then $\rho(f_n)$ converges to $\rho(f)$. In other words $f \mapsto \rho(f)$ is continuous in the topology of uniform convergence. (Note that, for circle mappings, uniform convergence on $[0,1]$ is equivalent to uniform convergence on all of* $\mathbf{R}$.)

2. If f_1, f_2 are circle mappings, and if $f_1(x) \leq f_2(x)$ for all x, then $\rho(f_1) \leq \rho(f_2)$.

3. Let f be a circle mapping p, q integers with $q > 0$. Then $p/q > \rho(f)$ if and only if $x + p > f^q(x)$ for all x; $p/q < \rho(f)$ if and only if $x + p < f^q(x)$ for all x; and $p/q = \rho(f)$ if and only if there is at least one x_0 such that $f^q(x_0) = x_0 + p$.

Note that, in the notation of 3., the image of x_0 in $\mathbf{R}/\mathbf{Z}$ is a periodic point for the mapping of which f is a lift, and that its period is a divisor of q. It is easy to see that its period is in fact q itself, provided that p and q are relatively prime. One consequence of 3. is that a circle mapping f has rational rotation number if and only if the mapping it induces on the circle has a periodic orbit. In fact we have:

PROPOSITION 1.3. *Let f be a circle mapping with rational rotation number p/q, where we take p, q to be relatively prime integers with $q > 0$. Then* **every** *orbit of the mapping induced by f on $\mathbf{R}/\mathbf{Z}$ is asymptotic to a periodic orbit with period q.*

This proposition finishes, for all practical purposes, the theory of the dynamics of circle mappings with rational rotation numbers. The situation for irrational rotation number is quite different. The general principle is that, modulo some regularity conditions, a circle mapping with irrational rotation number differs from a translation ("rigid rotation") only by a change of coordinates. More precisely: We say that two circle mappings f_1 and f_2 are *topologically conjugate* if there exists a circle mapping h such that $f_2 = h^{-1} f_1 h$. It can then be shown that a circle mapping f with irrational rotation number is necessarily topologically conjugate to $x + \rho(f)$, provided that either

- f is continuously differentiable; $f'(x) > 0$ everywhere; and $f'(x)$ is a function of bounded variation.
- f is infinitely differentiable and $f'(x)$ has no zero of infinite order.

(The sufficiency of the first condition is a classical result of Denjoy; that of the second a recent result of Yoccoz.) Thus, if one of these conditions holds, those dynamical properties of f (or of the mapping which it induces on $\mathbf{R}/\mathbf{Z}$) which are invariant under topological changes of coordinates are identical to those of the corresponding rotation. We can sum this up by saying that, as in the case of rational rotation number, the dynamics of a sufficiently regular circle mapping with irrational rotation number are essentially determined by the rotation number. It is perhaps worth noting that this kind of complete classification of the dynamics generated by *all* continuous and continuously invertible mappings has not been accomplished for any space more complicated than the circle, and, indeed, it seems unrealistically optimistic to hope to arrive at such a complete picture in any higher-dimensional situation. Even in the case of the circle or the interval, things become much more complicated if the condition of invertibility is dropped.

If f is a circle mapping with irrational rotation number ρ, which is topologically conjugate to a $x + \rho$, then the conjugator h is essentially unique: Any other conjugator has the form $h(x + \alpha)$ for some constant α. Thus, the question of whether or not h is smooth is well-posed. This is a much more delicate question than those discussed above; important progress has been made on it by V. I. Arnol'd, J. Moser, and M. Herman. The general character of results in this area is that h can be shown to smooth provided both that f is sufficiently smooth and that the rotation number has "sufficiently good number-theoretic properties", i.e., cannot be approximated too well by rational numbers with small denominators.

As already noted, the rotation number of f varies continuously with f. There is, nevertheless, a surprise in how the rotation number varies. Consider a one parameter family of circle mappings f_μ, and suppose for concreteness that $\rho(f_\mu)$ is non-decreasing in μ. It then turns of that, unless the family f_μ is "exceptional", each rational value of $\rho(f_\mu)$ which is taken on at all is taken on **on an interval of non-zero length.** This is expressed by saying that the graph of $\rho(f_\mu)$ as a function of μ is a *devil's staircase.*

The above formulation is deliberately vague. To give a precise statement which is general enough to be usable in many cases of interest, we introduce some special terminology. A circle mapping f will be said to be *strictly periodic* if there are integers p, q, with $q > 0$, such that $f^q(x) = x + p$ for all x. (This is to be contrasted with the condition that $f^q(x) = x + p$ for *some* x, which we have seen to be equivalent to $\rho(f) = p/q$.) It is intuitively clear that only exceptional circle mappings are strictly periodic. One class of circle mappings which are easily seen not to be strictly periodic are ones which are non-linear entire functions: If f is entire and $f^q(x) = x + p$ on $\mathbf{R}$, then this identity extends to all of $\mathbf{C}$. Thus, in particular, f^q is one-to-one on $\mathbf{C}$, so f is one-to-one on $\mathbf{C}$, and a one-to-one entire function is linear.

PROPOSITION 1.4. *Let f_μ be a continuous one parameter family of circle mappings, defined, say, for $\mu \in [0,1]$. Assume that $\rho(f_\mu)$ is non-decreasing in μ, and that* **no** *f_μ is strictly periodic. Then for each rational number r strictly between $\rho(f_0)$ and $\rho(f_1)$, the set of parameter values μ where $\rho(f_\mu) = r$ is a closed interval of non-zero length.*

If f_μ is any continuous one parameter family of circle mappings, we will refer to any interval on which the rotation number of f_μ takes on a constant rational value as a *phase locking interval.*

Again, the situation for irrational rotation numbers is quite different. The following proposition shows in particular that, if f_μ is a one parameter family such that $f_\mu(x)$ is strictly increasing in μ for each fixed x, the $\rho(f_\mu)$ takes on irrational values only once.

PROPOSITION 1.5. *Let f_1, f_2 be circle mappings with $f_1(x) < f_2(x)$ for all x. If at least one of $\rho(f_1)$, $\rho(f_2)$ is irrational, then $\rho(f_1) < \rho(f_2)$.*

Let up see what these general results say about a concrete example. For any $k \in [-1,1]$ and $\mu \in \mathbf{R}$, $f_\mu(x) = x + \mu - (k/2\pi)sin(2\pi x)$ is a circle mapping. We fix a $k \neq 0$ and consider the behavior of this family as μ runs from 0 to 1.

- Since $f_\mu(x)$ is jointly continuous in (x,μ), $\rho(f_\mu)$ varies continuously with μ.
- Since $f_0(0) = 0$, $\rho(f_0) = 0$, and since $f_1(0) = 1$, $\rho(f_1) = 1$.
- Since $f_\mu(x)$ is strictly increasing in μ for each x, $\rho(f_\mu)$ is a non-decreasing function of x which is strictly increasing at each point where it takes on an irrational value.
- Since f_μ is a non-linear entire function for each μ, it is not strictly periodic for any μ; hence, $\rho(f_\mu)$ takes on each rational value between 0 and 1 on an interval of non-zero length.

One consequence of the above is that any open parameter interval not contained in a phase locking interval contains infinitely many subintervals where $\rho(f_\mu)$ takes different constant values. It is remarkable that such a simple example can produce such thoroughly non-analytic behavior.

There is another lesson to be learned from this example. No matter how small $|k|$ is, provided only that it is not exactly zero, the above analysis shows that the set of parameter values where f_μ has rational rotation number contains an open dense set. Thus, rationality of the rotation number is, in the technical topological sense, a *generic* property. On the other hand, Arnol'd has proved a theorem which implies that, in this family, the Lebesgue measure of the set of parameters where the rotation number is rational goes to zero with $|k|$. There is thus a clear contradiction between the topological and measure theoretic indications as to whether irrational rotation number ought to be expected to occur "frequently". In this case, at least, the measure theoretic indication seems to correspond better to what one actually sees.

Bibliographic note.

The material sketched in this section is standard. Two useful sources are Chapter 3 of Cornfeld, Fomin, and Sinai [1] and Herman [3]. The former gives a succinct exposition of the main points; the latter is more systematic (and covers much more than has been discussed here.)

2. Dependence of rotation number on parameter in families of critical circle mappings

The dependence of rotation number on parameter in one parameter families of analytic circle mappings with critical points exhibits some remarkable "universal rates" which can be explained on the basis of some hypotheses about the action of an explicit concrete nonlinear operator on—roughly—the space of critical circle mappings. In this section, we will describe some of these phenomena, and in the subsequent sections we will develop the theory to account for them. The theory will be organized around the use of the continued fraction representation of real numbers, so we will start with a digression in which we review some standard facts about continued fractions.

Continued fractions.

For $\rho \in (0,1)$, we define recursively a finite or infinite sequence $r_1, r_2, \ldots$ of strictly positive integers and a corresponding sequence $\rho_1, \rho_2, \ldots$ in $[0,1)$ as follows: If $\rho_j = 0$, the sequences terminate with the jth term; otherwise, we put r_{j+1} equal to the integer part of $1/\rho_j$ (i.e., the largest integer not greater than $1/\rho_j$) and ρ_{j+1} equal to the fractional part of $1/\rho_j$ (i.e., $1/\rho_j - r_{j+1}$). To start the construction we put ρ_0 equal to ρ. The recursion step may be summarized by the formula

$$\rho_{j-1} = \frac{1}{r_j + \rho_j}.$$

It is easy to see that the sequences continue forever if and only if ρ is irrational.

With this notation, we get relations like

$$\rho = \cfrac{1}{r_1 + \cfrac{1}{r_2 + \cfrac{1}{r_3 + \cfrac{1}{r_4 + \rho_4}}}}.$$

Since writing these relations in this way is typographically inconvenient, we introduce a notation $[s_1, s_2, \ldots, s_n]$ for finite continued fractions. This notation may be formally defined by

$$[s_1, \ldots, s_n] = \frac{1}{s_1 + [s_2, \ldots, s_n]} \quad \text{with} \quad [s_1] = \frac{1}{s_1},$$

and is quite general; the s_i have no need to be positive integers or even real numbers but can be elements of an arbitrary field.

The relation between ρ, $r_1, \ldots, r_n$, and ρ_n can be reorganized in a useful way as follows:

Proposition 2.1. *Suppose the r_j, ρ_j are defined for j up to n. Let p_j, q_j, $j = 2, \dots, n$, be defined recursively by*

$$p_j = r_j p_{j-1} + p_{j-2}, \qquad q_j = r_j q_{j-1} + q_{j-2}, \tag{2.1}$$

with $p_0 = 0$, $p_1 = 1$, $q_0 = 1$, $q_1 = r_1$. Then, for $1 \le j \le n$,

$$\rho = \frac{p_j + p_{j-1}\rho_j}{q_j + q_{j-1}\rho_j} \tag{2.2}$$

$$\rho_j = -\frac{\rho q_j - p_j}{\rho q_{j-1} - p_{j-1}} \tag{2.3}$$

$$q_j p_{j-1} - p_j q_{j-1} = (-1)^j. \tag{2.4}$$

The proof of this proposition is completely straightforward; (2.2) and (2.4) are proved by induction and (2.3) by solving (2.2) for ρ_j. The proof, incidentally, uses only the relations $\rho_{j-1} = 1/(r_j + \rho_j)$ and not the fact that the r_j are integers and that $\rho_n \in [0,1)$. If we put in the extra information we see that p_j and q_j are integers; that they are relatively prime (from (2.4)); that $p_j/q_j = [r_1, \dots, r_j]$; that p_j/q_j is greater than ρ for j odd and less than ρ for j even (from (2.3) and the fact that $\rho \ge 1/r_1$); and hence that

$$|p_j/q_j - \rho| \le |p_j/q_j - p_{j+1}/q_{j+1}| = 1/(q_j q_{j+1})$$

(by dividing (2.4), with j replaced by $j+1$, by $q_j q_{j+1}$.)

A further fact about the continued fraction expansion which is important for motivating what follows but which is not actually used in the formal development is that the ratios p_j/q_j are the best rational approximations to ρ with denominators of a given size, in the sense that the smallest positive integer q such that the distance from $q\rho$ to the integers is smaller than the distance from $q_j\rho$ to the integers is exactly q_{j+1}.

For the remainder of these notes we will find it convenient to use the following convention: When we write $\rho = [r_1, \dots, r_n, \dots]$, we mean that $\rho \in (0,1)$ has continued fraction expansion beginning $r_1, \dots, r_n$ **and which has at least one more term**, i.e., that ρ has the form $[r_1, \dots, r_n + \rho_n]$ with $0 < \rho_n < 1$.

Phenomenology in one parameter families.

The sorts of one-parameter families $f_\mu(x)$ of circle mappings we want to consider are ones such that $f_\mu(x)$ is analytic in x and at least continuously differentiable in μ; such that the derivative with respect to x of each $f_\mu(x)$ vanishes somewhere; but such that $(f_\mu)'''(x) > 0$ wherever $(f_\mu)'(x)$ vanishes (i.e., the zeroes of the $(f_\mu)'(x)$ are all of third order, the smallest order consistent with the condition that $f_\mu(x)$ is increasing in x.) Furthermore, to fix ideas, we will suppose that $\rho(f_\mu)$ is non-decreasing in μ and runs from 0 to 1 as μ runs from 0 to 1. A standard example of such a family is the "sine family"

$$f_\mu(x) = x + \mu - \frac{1}{2\pi}\sin(2\pi x).$$

It then appears that there is a particular number $\delta_1 = -2.83361\dots$ such that, for a large class of families of the above type, the length of the phase locking interval with rotation number

$[1, \ldots, 1]$ (n 1's) is asymptotic to a constant multiple of $|\delta_1|^{-n}$. What is surprising is that the exponential rate of decrease of the length of the phase locking interval with n appears to be independent of what family is examined. The constant multiplying $|\delta_1|^{-n}$ is not independent of family; indeed, it can be changed simply by making a smooth reparametrization of a given family. It moreover appears that, for any fixed finite sequence $r_1, \ldots, r_j$, the length of the phase locking interval with rotation number $[r_1, \ldots, r_j, 1, 1, \ldots, 1]$ (n 1's after r_j) is again asymptotic to a constant multiple of $|\delta_1|^{-n}$.

More generally, for each $r = 1, 2, \ldots$, there is a δ_r such that, again for a large class of families of the above type, the length of the phase locking interval with rotation number $[r, \ldots, r]$ (n r's) is asymptotic to a constant times $|\delta_r|^{-n}$; again, the same rate appears for rotation numbers whose continued fractions are generated by prepending an arbitrary fixed $r_1, \ldots, r_j$ to a sequence of n r's. There doesn't appear to be any simple relation among the δ_r's.

To get away from rotation numbers of the relatively simple types considered above, we need a more complicated formulation: What seems to be the case is that there exist two "structure functions" $\gamma(r_0|r_1, r_2, \ldots)$ and $\sigma(r_0|r_1, r_2, \ldots)$. These are functions defined on the space of all sequences $r_0, r_1, \ldots$ of strictly positive integers, taking values in $(0, 1)$, which depend exponentially little on the r_j for large j. Their relation to the length of phase locking intervals is that, for a large class of families of the above type, the length of a *general* phase locking interval with rotation number $[r_1, \ldots, r_n]$ is given approximately by

$$(2.5) \quad \sigma(r_n|r_{n-1}, r_{n-2}, \ldots)\gamma(r_{n-1}|r_{n-2}, r_{n-3}, \ldots)\gamma(r_{n-2}|r_{n-3}, r_{n-4}, \ldots) \ldots \gamma(r_1|r_0, r_{-1}, \ldots).$$

Here, the approximation is in the sense that the ratio of the length of the phase locking interval with rotation number $[r_1, \ldots, r_n]$ to the quantity (2.5) is bounded uniformly in $n, r_1, \ldots, r_n$. The quantity (2.5) depends on an infinite sequence of extra indices $r_0, r_{-1}, \ldots$. Because, however, of the exponential decrease of the dependence of γ and σ on their successive arguments, these extra indices can be chosen in any convenient way—e.g., all equal to 1—without affecting the validity of the approximation. The simpler asymptotic behavior discussed earlier for phase locking intervals with rotation numbers $[r, \ldots, r]$ fits naturally into this more general framework; the connection is that $\gamma(r|r, r, \ldots) = |\delta_r|^{-1}$. Note that, because the quantity (2.5) is a product of $n-1$ γ's and only one σ, γ plays a more important role than σ in determining the asymptotic behavior of the length of phase locking intervals as n goes to infinity. We cannot leave σ out altogether, however, because $\sigma(r_0|r_1, \ldots)$ goes to zero as r_0 goes to infinity.

The status of our understanding of the above behavior is as follows: The assertion about phase locking intervals with rotation numbers $[1, \ldots, 1]$ is a *theorem*. It follows by a standard renormalization group analysis from the existence of a hyperbolic fixed point with one expanding direction for an explicit operator (which we will construct in Section 3.) A formal analysis deducing universal rates from the existence and properties of a fixed point was given by Ostlund, Rand, Sethna, and Siggia [7] and—in another formalism—by Feigenbaum, Kadanoff, and Shenker [2]. The existence and relevant properties of the fixed point were proved by Mestel [6] and (in part) by Lanford and de la Llave [5] using detailed estimates verified rigorously by computer. It should be noted that the analysis does not show that the rate δ_1 is "universal" in the straightforward sense. What it shows is that there is a non-empty open set of one parameter families to which this rate applies. This set consists of those families which cross the stable manifold of the fixed point transversally, and is known as the *universality class* of the fixed point. Although in principle there might be many such fixed points and hence many universality classes, or even open sets of one parameter families not in any universality class, it is a fact of experience that all one parameter families arising in practice, provided that they have the general formal properties described in the first few paragraphs of this subsection, seem to be in one standard universality class.

The formal analysis of Ostlund et. al. referred to in the preceding paragraph generalizes in a straightforward way to account for the assertions about the lengths of phase locking intervals with rotation numbers $[r, \ldots, r]$: One constructs an operator for each r, and the asserted behavior follows—on the appropriate universality class—if this operator is shown to admit a hyperbolic fixed point with only one expanding direction. The operators for different r's are formally similar, but their fixed point problems seem to be mathematically independent. It certainly appears that such a fixed point does exist for each r, and that the machinery used to prove this for the case $r = 1$ could be applied in a mechanical way to any other r, but to my knowledge this has not been done. Numerical studies indicate that some simplifications appear in the asymptotic behavior for large r, but this limiting regime is also not yet completely understood.

The assertion about the lengths of general phase locking intervals is more speculative. We are going to sketch in Section 4 an argument showing that, if a particular concrete operator admits a hyperbolic invariant Cantor set with one expanding direction, then the assertion follows. That is: We will show how to deduce the formula (2.5) from a generalized renormalization group picture where the "universal object" is an invariant Cantor set rather than a fixed point. The existence of this invariant Cantor set appears quite likely, but is far from being proved. Since its existence implies, among many other things, the existence of all the fixed points referred to above, the only approach to proving its existence which appears promising at this time is via the same sort of detailed numerical estimates verified rigorously by computer which were used in proving the existence of the fixed point for $r = 1$.

3. The renormalization operator

The renormalization group analysis which we are going to describe applies to smooth circle mappings which have critical points, i.e., places where their derivatives vanish. What we do is to study the behavior of particular iterates of the mapping, restricted to small neighborhoods of a critical point. It will therefore be convenient to suppose that the critical point on which we are concentrating is at 0; this can always be arranged by a shift of the origin. Thus, for our purposes, a *critical* circle mapping will mean one whose first derivative vanishes at 0.

Special iterates of circle mappings.

Let f denote a smooth critical circle mapping with rotation number $\rho = [r_1, \ldots, r_n, \ldots]$, and write, as in Proposition 2.1, $[r_1, \ldots, r_j]$ as p_j/q_j, $j = 1, \ldots, n$. Intuitively, since the $q_j\rho$ are particularly close to integers, we expect that the orbit of 0 under the action induced on the circle by f to come back particularly close to itself at the (discrete) times q_j. We thus define

$$f_j(x) = f^{q_j}(x) - p_j \quad \text{and } x_j = f_j(0) \quad \text{for } j = 1, \ldots, n. \tag{3.1}$$

The idea will be to study the behavior of these iterates on appropriately chosen small neighborhoods of 0. To do this, it is convenient to introduce rescaled functions

$$\eta_j^f(y) = (1/x_{j-1}) f_j(x_{j-1}y) \quad \text{and} \quad \xi_j^f(y) = (1/x_{j-1}) f_{j-1}(x_{j-1}y). \tag{3.2}$$

The first part of the picture one should have of these objects is that, asymptotically for large j, the ξ_j^f, η_j^f depend only on $r_1, \ldots, r_j$, i.e., on the rotation number of f, and not on any more detailed properties of f.

Note that the ξ_j^f, η_j^f are not circle mappings—they are obtained by rescaling circle mappings by a factor which goes to zero as j goes to infinity. On the other hand, they have some simple formal properties:

(1) $\xi_j^f(x) > x$ for all x and $\xi_j^f(0) = 1$.

(2) $\eta_j^f(x) < x$, and $\xi_j^f \eta_j^f(x) > x$, for all x.

(3) ξ_j^f and η_j^f commute.

Furthermore, there is a simple formula for constructing ξ_{j+1}^f, η_{j+1}^f given ξ_j^f, η_j^f: From the recursion relations

$$p_{j+1} = r_{j+1}p_j + p_{j-1}, \qquad q_{j+1} = r_{j+1}q_j + q_{j-1},$$

it follows at once that

$$f_{j+1} = (f_j)^{r_{j+1}} f_{j-1}.$$

Rescaling this equation appropriately gives

$$\text{(3.3a)} \qquad \eta_{j+1}^f(x) = \left(1/\lambda_j^f\right) \left(\eta_j^f\right)^{r_{j+1}} \xi_j^f(\lambda_j^f x), \qquad \text{where } \lambda_j^f = \eta_j^f(0),$$

and we also have

$$\text{(3.3b)} \qquad \xi_{j+1}^f = \left(1/\lambda_j^f\right) \eta_j(\lambda_j^f x).$$

The thing to notice about these formulas is that f does not appear in them except through r_{j+1} which depends only on the rotation number. We can thus take the right hand sides of (3.3ab) as defining an operator $\mathcal{T}$ acting on pairs like $\left(\xi_j^f, \eta_j^f\right)$; then by studying the iteration of that operator we can get information about the asymptotic behavior of ξ_j^f, η_j^f for large j.

Weakly commuting pairs.

In the next few subsections, we develop a formalism for implementing the program sketched above. That is: We define a space of pairs containing in particular the pairs ξ_j^f, η_j^f constructed as above from circle mappings and we show that the right hand side of (3.3) defines an operator on this space with reasonable formal properties. The formalism we develop will deviate in two respects from the setup sketched above:

- The set of strictly commuting pairs may not form a manifold in any natural way. To avoid such difficulties, we work with pairs satisfying only an approximate commutativity condition.
- The analysis leading to the propagation formula (3.3) used implicitly the fact that ξ_j^f, η_j^f are defined on the whole real axis. While this is no restriction for pairs coming from circle mappings, we do not want to assume it for all the pairs we consider. Instead, we make a space of pairs defined on judiciously chosen finite intervals and argue that the formula (3.3) applied to such a pair gives another one with an appropriate domain.

Moreover, although we are really interested in pairs of smooth functions, it costs no more to develop the basic formalism for pairs which are no more than continuous.

By a *weakly commuting pair* we will mean a pair (ξ, η) where

(P1) η is a continuous strictly increasing real valued function defined of $[0,1]$, with $\eta(x) < x$ for all x in $[0,1]$.We write λ for $\eta(0)$; note that $\lambda < 0$.

(P2) ξ is a continuous strictly increasing real valued function defined on $[\lambda, 0]$, with $\xi(0) = 1$ and $\xi(x) > x$ for all x in $[\lambda, 0]$.

(P3) $\xi\eta(x) > x$ where the left hand side is defined, i.e., on $[0, \eta^{-1}(0)]$.

(P4) ξ and η commute at 0, i.e.

$$\xi(\lambda) = \xi(\eta(0)) = \eta(\xi(0)) = \eta(1). \tag{3.4}$$

Note that $0 < \xi(\lambda) = \eta(1) < 1$.

We sometimes need a stronger condition than commutation at the single point zero, and accordingly we say that a weakly commuting pair (ξ, η) *commutes to order* k if ξ and η are of class C^k on their respective domains and if

$$\lim_{x \to 0^+} (\xi\eta)^{(j)} = \lim_{x \to 0^-} (\eta\xi)^{(j)} \qquad \text{for } j = 1, 2, \ldots, k.$$

An *analytic (strictly) commuting pair* will mean an weakly commuting pair such that ξ and η are real analytic (and so have analytic extensions to complex neighborhoods of their respective domains) and if their analytic continuations commute exactly on a neighborhood of 0.

If f is a circle mapping with rotation number strictly between 0 and 1, then $\xi_0^f(x) = x + 1$, $\eta_0^f(x) = -f(-x)$ (each restricted to the appropriate interval) is easily verified to be a weakly commuting pair which indeed commutes to order k if f is of class C^k and strictly if f is real analytic. We will see shortly that this notation is consistent with the definitions given above for ξ_j^f, η_j^f which make sense only for $j > 0$.

Generalized rotation number.

The next step is to note that the notion of rotation number can be extended to weakly commuting pairs. To motivate the definition we are going to give, consider a circle mapping f with rotation number between 0 and 1. The rotation number of f can be determined from ξ_0^f, η_0^f as follows: Pick any x, and for each $n = 1, 2, \ldots$, let m_n denote the number of iterations of $x \mapsto x + 1 = \xi_0^f(x)$ necessary to move $\left(\eta_0^f\right)^n(x) = -f^n(-x)$ into $(0, 1]$, i.e., m_n is the least integer strictly greater than $f^n(-x)$. From this second description of m_n, it is evident that m_n/n approaches $\rho(f)$ as n goes to infinity. It is easy to express this construction in a way that applies to an arbitrary weakly commuting pair (ξ, η): Pick any point x in $[0, 1]$, and begin applying η to it. After some number of iterations, the point obtained will be to the left of 0 and thus outside the domain $[0, 1]$ of η. When that happens, apply ξ to push it back into $[0, 1]$ and continue. Let m_n denote the number of times ξ gets applied in the course of applying η n times.

PROPOSITION 3.2. *With the notation as above,* $\lim_{n \to \infty} m_n/n$ *exists, is independent of* x, *and lies in* $(0, 1)$.

We call the limiting ratio the *rotation number* of (ξ, η), and denote it by $\rho(\xi, \eta)$. It is not difficult to see that this extended definition of rotation number shares many of the elementary properties of the ordinary rotation number, e.g. that $\rho(\xi, \eta) = p/q$ if and only if a product of q η's and p ξ's, in some order, has a fixed point.

The renormalization operator.

We can now investigate the operator defined by formulas (3.3).

Proposition 3.3. *Let r be a strictly positive integer and let ξ, η be a weakly commuting pair with rotation number strictly between $1/(r+1)$ and $1/r$. Define $\hat{\xi}$ and $\hat{\eta}$ by*

$$\hat{\xi}(x) = (1/\lambda)\,\eta(\lambda x); \quad \hat{\eta}(x) = (1/\lambda)\,\eta^r\xi(\lambda x) \quad (\text{where } \lambda = \eta(0),) \tag{3.5}$$

wherever the right hand sides are defined. Then

1. *The domain of $\hat{\eta}$ is $[0,1]$, and the domain of $\hat{\xi}$ contains $[\hat{\eta}(0), 0]$.*
2. *$\hat{\xi}$ (appropriately restricted) and $\hat{\eta}$ form a weakly commuting pair.*
3. *If ξ, η commute to order k, then so do $\hat{\xi}$, $\hat{\eta}$. If ξ, η are analytic and strictly commuting, then so are $\hat{\xi}$, $\hat{\eta}$.*
4. *If ρ denotes the rotation number of (ξ, η), and $\hat{\rho}$ that of $(\hat{\xi}, \hat{\eta})$, then $\rho = 1/(r + \hat{\rho})$, i.e., $\hat{\rho}$ is the image of ρ under the Gauss map.*

We thus define, for each strictly positive integer r, an operator $\mathcal{T}_r$ mapping ξ, η to $\hat{\xi}$, $\hat{\eta}$ given by (3.4); the domain $\mathcal{D}_r$ of $\mathcal{T}_r$ is the set of all weakly commuting pairs with rotation number strictly between $1/(r+1)$ and $1/r$. Since the $\mathcal{D}_r$ are pairwise disjoint, we can define a single operator $\mathcal{T}$ whose domain $\mathcal{D}$ is the union of the $\mathcal{D}_r$ and which is given by $\mathcal{T}_r$ on each $\mathcal{D}_r$. Note that $\mathcal{D}$ is just the set of those weakly commuting pairs whose rotation numbers are not reciprocals of integers.

With this notation it is easy to check

Proposition 3.3. *For any sequence $r_1, \ldots, r_n$ of strictly positive integers*

1. The domain of $\mathcal{T}_{r_n} \ldots \mathcal{T}_{r_1}$ is the set of weakly commuting pairs (ξ, η) with $\rho(\xi, \eta) = [r_1, \ldots, r_n, \ldots]$, and, for such (ξ, η), $\rho(\mathcal{T}^n(\xi, \eta)))$ is $[r_{n+1}, \ldots]$.

2. If f is a circle mapping with rotation number $[r_1, \ldots, r_n, \ldots]$, then $\left(\xi_n^f, \eta_n^f\right)$ (restricted to the appropriate subintervals) is $\mathcal{T}_{r_n} \ldots \mathcal{T}_{r_1}\left(\xi_0^f, \eta_0^f\right)$

For historical reasons, we refer to the operator $\mathcal{T}$, or to any one of the $\mathcal{T}_r$'s, as the *renormalization operator.* The reader is cautioned against reading too much into the use of the word "renormalization" in this context; the connection with the original use of this term in quantum field theory is remote.

Bibliographic note.

The formalism developed in this section is essentially that of Ostlund, Rand, Sethna, and Siggia [7]

4. The global hyperbolicity hypothesis

We will now describe a set of hypotheses about the action of the renormalization operator $\mathcal{T}$ which permit us to explain the "universal formula" (2.5) for lengths of all phase locking intervals. The formulation we will give is intended only to show what the idea is; it will not be entirely precise. We do this not only to keep the exposition from becoming too cumbersome but also because it is not yet clear exactly what technical formulation—if any—is correct.

The general idea is that the renormalization operator acting on a large region in the space of analytic commuting pairs is expansive in one direction and contractive in all others. To try to clarify what this means, we offer the following preliminary formulation which, while having the advantage of relative simplicity, is likely to be too restrictive. (It is also not completely specific, as we will not say what Banach space of commuting or approximately commuting pairs is to be used for the analysis.) The formulation is as follows: We suppose that there exist

(1) An open set $\mathcal{V}$ in the space of commuting pairs which is invariant under the action of $\mathcal{T}$ in the sense that, if ζ is in both $\mathcal{V}$ and the domain of $\mathcal{T}$, then $\mathcal{T}\zeta$ is again in $\mathcal{V}$. (We of course also want to require that $\mathcal{V} \cap \mathcal{D}(\mathcal{T})$ be non-empty.)

(2) A set of coordinates (x, y) for $\mathcal{V}$, where x runs over the unit ball in some infinite-dimensional Banach space and y over the open unit interval.

(3) A real number $\kappa < 1$

such that, if we write the action of $\mathcal{T}$ in the coordinates (x, y) as $(x, y) \longmapsto (X(x, y), Y(x, y))$ then X, Y are smooth and

$$\|X(x, y)\| \leq \kappa \tag{4.1}$$

$$\|\partial X/\partial x\| \leq \kappa \tag{4.2}$$

$$|\partial Y/\partial y| \geq \kappa^{-1} \tag{4.3}$$

$$\|\partial X/\partial y\| \leq \epsilon \quad \text{and} \quad \|\partial Y/\partial x\| \leq \epsilon \quad \text{for some } \epsilon < \sqrt{\kappa + \kappa^{-1} - 2}. \tag{4.4}$$

Condition (4.1) says that $\mathcal{V}$ is mapped horizontally (i.e., in the x direction) well within itself; condition (4.2) says that the horizontal derivative of the horizontal component of $\mathcal{T}$ is uniformly contractive; condition (4.3) says that the vertical derivative of the vertical component of $\mathcal{T}$ is uniformly expansive; and (4.4) turns out to be the right condition to ensure that the off-diagonal derivatives are not big enough to spoil things. We will also want to assume that, for fixed x, the rotation number of the pair corresponding to (x, y) is non-decreasing in y and strictly increasing where irrational, and that this rotation number approaches 0 as y approaches 0 and 1 as y approaches 1.

If we assume that something like the above holds, then for each r the set $\mathcal{D}_r$ of pairs with rotation number between $1/(r+1)$ and $1/r$ intersects $\mathcal{V}$ in a set which, in the coordinates (x, y) is a short cylinder. The image of $\mathcal{D}_r$ under $\mathcal{T}_r$ is stretched out vertically so that it runs from $y = 0$ to $y = 1$, but is squeezed horizontally; that is, it is a thin cylinder of full height. If we pick any r_1, r_2, the image of $\mathcal{T}_{r_1}\mathcal{T}_{r_2}$, i.e., the image under $\mathcal{T}_{r_1}$ of $\mathcal{D}_{r_1} \bigcap \mathcal{T}_{r_2}\mathcal{D}_{r_2}$ again runs from $y = 0$ to $y = 1$, but is still thinner horizontally. If we take any sequence $r_1, r_2, \ldots$, then the images of the products $\mathcal{T}_{r_1} \ldots \mathcal{T}_{r_n}$ are a decreasing sequence of thinner and thinner full-height cylinders. Their intersection has vanishing horizontal diameter and in fact it follows easily from standard ideas in invariant manifold theory that this intersection is in fact a smooth curve. This curve depends on the sequence $r_1, r_2, \ldots$; we will call it *a branch of the unstable manifold* and denote it by $W^u(r_1, r_2, \ldots)$ (For the purposes of this summary, we don't actually need to describe the invariant Cantor set of which these curves are the unstable manifolds; the unstable manifolds themselves contain, as we shall see, the essential information.)

So far, we have described the unstable manifolds only as point sets. We next want to argue that they carry a natural parametrization. The idea is as follows: For each $r_0, r_1, \ldots$, the curve $W^u(r_1, \ldots)$ crosses the domain of $\mathcal{T}_{r_0}$, and the image under $\mathcal{T}_{r_0}$ of the part of the curve in its domain is easily seen—directly from the definitions—to be exactly $W^u(r_0, r_1, \ldots)$. It turns out that there is an essentially unique way of parametrizing simultaneously all the $W^u(r_1, r_2, \ldots)$'s which is smooth along each curve and continuous in $r_1, r_2, \ldots$, and such that the mapping $\mathcal{T}_{r_0}$ from the piece of $W^u(r_1, r_2, \ldots)$ on which it is defined to all of $W^u(r_0, r_1, \ldots)$ is *linear* in the respective parametrizations. It is clear that this description cannot completely fix the parametrization, since any parametrization obtained from one having this property by linearly reparametrizing each curve, in a way which depends continuously on the curve but is otherwise arbitrary, will also have the specified property. It becomes unique if we impose an appropriate normalization condition, which we will do by requiring that the parameter run from 0 to 1 as the rotation number runs from 0 to 1. From now on, when we refer to $W^u_\mu(r_1, r_2, \ldots)$, we mean the unique parametrization satisfying the above conditions and normalization.

We will restrict ourselves from now on to one-parameter families ζ_μ of commuting (or approximately commuting) pairs which satisfy the normalization condition that the rotation number runs from 0 to 1 as the parameter runs over the same range, and which are "everywhere sufficiently vertical" in the coordinates (x, y) introduced above. For each $r = 1, 2, \ldots$, we define an operator $\mathcal{T}^*_r$ acting on these one-parameter families as follows: Take the family ζ_μ, restrict μ to the subinterval on which $\zeta_\mu \in \mathcal{D}_r$; apply $\mathcal{T}_r$; then make a linear change of parameter to restore the normalization. More formally, let μ^0_r be the supremum of the set of μ's where the rotation number is (strictly) less than $1/r$, and let μ^1_r be the infimum of the set of μ's where the rotation number is greater than $1/(r+1)$; then put

$$(\mathcal{T}^*_r \zeta)_\mu = \mathcal{T}_r \zeta_{\mu^0_r + \mu(\mu^1_r - \mu^0_r)}.$$

The heart of the renormalization group analysis for general rotation number is the observation that, *for any sequence $r_1, r_2, \ldots$, and any one-parameter family ζ_μ of the sort considered above, the sequence $\mathcal{T}^*_{r_1}\mathcal{T}^*_{r_2} \cdots \mathcal{T}^*_{r_n}\zeta$ of one-parameter families converges to the one-parameter family $W^u(r_1, r_2, \ldots)$, and that the convergence is exponentially fast in n, uniformly in $r_1, r_2, \ldots$.*

Here is how to extract information about the lengths of phase locking intervals from this observation: Let $\sigma(r_0|r_1, r_2, \ldots)$ (respectively $\gamma(r_0|r_1, r_2, \ldots)$) denote the length of the parameter intervals where $W^u_\mu(r_1, r_2, \ldots)$ has rotation number $1/r$ (respectively between $1/(r+1)$ and $1/r$). If ζ_μ is a one-parameter family, it is a simple consequence of the definitions that the ratio of the length of the parameter interval where $\rho(\zeta_\mu) = [r_1, r_2, \ldots, r_n, \ldots]$ to the length of the larger parameter interval where $\rho(\zeta_\mu) = [r_1, \ldots, r_{n-1}, \ldots]$ is the length of the parameter interval where $\mathcal{T}^*_{r_{n-1}}\mathcal{T}^*_{r_{n-2}} \cdots \mathcal{T}^*_{r_1}\zeta$ has rotation number $[r_n, \ldots]$, and, for large n, this is exponentially near to $\gamma(r_n|r_{n-1} \ldots)$. Similarly, the ratio of the length of the parameter interval where $\rho(\zeta_\mu) = [r_1, \ldots, r_n]$ to the length of the interval where $\rho(\zeta_\mu) = [r_1, \ldots, r_{n-1}, \ldots]$ is, for large n, very near to $\sigma(r_n|r_{n-1}, \ldots)$. Applying these observations repeatedly gives the "universal approximate formula" (2.5) for lengths of phase locking intervals.

Bibliographic note.

For a more complete exposition of the above analysis, see Lanford [4].

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