A COMPUTER-ASSISTED PROOF OF THE FEIGENBAUM CONJECTURES

by

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1. Introduction.

Let M denote the space of continuously differentiable even mappings ψ of the interval $[-1,1]$ into itself such that

M1. $\psi(0) = 1$

M2. $x\psi'(x) < 0$ for $x \neq 0$

M2 says that ψ is strictly increasing on $[-1,0)$ and strictly decreasing on $(0,1]$, so M is a space of mappings which are unimodal in a strict sense.

Condition M1 says that the unique critical point 0 is mapped to 1. We want to consider ψ's which map 1 slightly - but not too far - to the left of 0. It may then be possible to find non-overlapping intervals I_o about 0 and I_1 near 1 which are exchanged by ψ. Technically, we proceed as follows : Write a for $-\psi(1) = -\psi^2(0)$ and b for $\psi(a)$; we suppress from the notation the dependence of a and b on ψ. Define $\mathcal{D}(T)$ to be the set of all ψ's in M such that :

D1. $a > 0$

D2. $b > a$

D3. $\psi(b) \leq a$.

The two intervals $I_o = [-a,a]$ and $I_1 = [b,1]$ are then non-overlapping and ψ maps I_o into I_1 and vice versa. If $\psi \in \mathcal{D}(T)$, then $\psi \circ \psi|_{I_o}$ has a single critical point, which is a minimum. By making the change of variables $x \to -ax$, we replace I_o by $[-1,1]$ and the minimum by a maximum, i.e., if we define

$$T\psi(x) = -\frac{1}{a}\,\psi \circ \psi(-ax) \quad \text{for} \quad x \in [-1,1]$$

then $T\psi$ is again in M . Thus, T defines a mapping of $\mathcal{D}(T)$ into M . (In general, $T\psi$ need not lie in $\mathcal{D}(T)$. If a is small, then $T\psi(1)$ will be approximately 1 so $T\psi$ will not satisfy D1 . On the other hand, if $T\psi(b)$ is near a , then $T\psi(1)$ will be near -1 from which it follows that $T\psi$ does not satisfy D2) .

M. Feigenbaum [4] has proposed an explanation for some universal features displayed by infinite sequences of period doubling bifurcations based on some conjectures about T . We will not review his argument here; a version with due regard for mathematical technicalities may be found in Collet and Eckmann [2] , Collet, Eckmann and Lanford [3] , or in Lanford [6]. The purpose of this note is to announce a proof of essentially all of these conjectures and to indicate the kind of analysis used.

2. Statement of results.

Theorem 1. There exists a function g , analytic and even on $\{Z \in \mathbb{C} : |Z| < \sqrt{8}\}$ whose restriction to $[-1,1]$ is a fixed point for T . The Schwarzian derivative of g is negative on $[-1,1]$.

Let Ω denote $\{Z \in \mathbb{C} : |Z^2 - 1| < 2.5\}$ and write

$\mathfrak{H}$ for the Banach space of even functions bounded and analytic on Ω , real on real points, equipped with the supremum norm.

$\mathfrak{H}_0$ for the subspace of $\mathfrak{H}$ consisting of those functions vanishing (to second order) at 0 .

$\mathfrak{H}_1$ for $\mathfrak{H}_0 + 1$.

Proposition 2. There is an open neighborhood V of g in $\mathfrak{H}_1$ such that

. Every $\psi \in V$ is in $\mathcal{D}(T)$ (i.e., its restriction to $[-1,1]$ is)

. If $\psi \in V$, $T\psi \in \mathfrak{h}_1$

. T is infinitely differentiable as a mapping from V into $\mathfrak{h}_1$,

. The derivative $DT(\psi)$ is compact operator on $\mathfrak{h}_o$ for each $\psi \in V$.

<u>Theorem</u> 3. $DT(g)$ is hyperbolic on $\mathfrak{h}_o$ with one-dimensional expanding subspace; the expanding eigenvalue δ is positive.

In other words : The spectrum of $DT(g)$ does not intersect the unit circle, and the part of the spectrum outside the unit circle consists of a single simple positive eigenvalue δ .

It then follows from invariant manifold theory that T admits locally invariant local stable and local unstable manifolds, of codimension one and dimension one respectively. Because of the non-invertibility of T , we do notconstruct global stable and unstable manifolds; we will let W_s and W_u denote respectively some particular local stable and local unstable manifolds.

Let Σ_o denote the bifurcation surface for the simple period-doubling bifurcation. By this we mean the following : Any ψ in M has exactly one fixed point x_o in $[0,1]$; Σ_o then denotes $\{\psi \in M : \psi'(x_o) = -1 ; (\psi \circ \psi)'''(x_o) < 0\}$. As a one-parameter family of ψ's crosses Σ_o (in the appropriate direction) the fixed point x_o loses stability in favor of an attracting orbit of period 2.

<u>Theorem 4</u>. There is a positive integer j and an element g_j^* of W_u such that $T^j g_j^* \in \Sigma_o$.

Except for the difficulties in defining a global unstable manifold, we could formulate this theorem by saying that the unstable manifold crosses Σ_o . We would like to know more, viz., that the crossing is transversal. This - properly formulated - is almost certainly true, but we have not proved it.

Let $\psi_\mu^{(o)}(x)$ denote the quadratic mapping $1-\mu x^2$.

Theorem 5. There is a positive integer j and a parameter value μ_∞ (between 1.4011550 and 1.411554) such that $\psi_\mu^{(o)}$ is in $\mathcal{D}(T^j)$ for μ sufficiently near to μ_∞ and such that the curve $T^j\psi_\mu^{(o)}$ crosses W_s transversally at $\mu = \mu_\infty$.

Except for technicalities, this says that $\psi_\mu^{(o)}$ crosses the stable manifold transversally at μ_∞ .

3. Remarks on the method of proof.

The heart of the proof is a set of complicated numerical estimates proved rigorously with the aid of a computer. To formulate these estimates, we have first to establish some notations. We will work, initially, not in $\mathfrak{H}_1$ but in a subspace equipped with a stronger norm. The idea is that we want to write ψ as

$$\psi(x) = 1-x^2h(x^2)$$

and to use the ℓ^1 norm for the Taylor coefficients of h at 1 . Formally, given an element (u,v) of $\mathbb{R}\oplus\ell^1$, we associate with it an element ψ of $\mathfrak{H}_1$ by

$$\psi(x) = 1-x^2\{10.u+\sum_{n=1}^{\infty} v_n(\frac{(x^2-1)}{2.5})^n\} . \tag{3.1}$$

We denote the set of ψ's obtained in this way by A, and we equip A with the norm $|u| + \Sigma\, |v_n|$. Note that A contains any element of $\mathcal{H}_1$ which is analytic on the closure of Ω. (Of course, $\mathbb{R} \oplus \ell^1$ could have been identified with ℓ^1, but we have singled out the u component - and introduced the factor of 10 in the formula (3.1) for $\psi(x)$ - for convenience later on). For the remainder of this section, the norm of an element of A will always mean the norm of ℓ^1 type just introduced.

The first step is to choose an explicit polynomial ψ_o which will turn out to be a good approximate fixed point. We will take ψ_o to be the polynomial of degree 10 defined by the first ten terms of the series given in Table 1. below. It can be checked without difficulty that

. For any $\psi \in A$ with $\| \psi - \psi_o \| < .01$, $T\psi \in A$

. T is infinitely differentiable from $\{ \|\psi - \psi_o\| < .01 \}$ to A.

. For any ψ in this ball, $DT(\psi)$ is a compact operator on A.

Identifying A with $\mathbb{R} \oplus \ell^1$, we can represent $DT(\psi)$ as a matrix

$$\begin{pmatrix} \alpha(\psi) & \beta(\psi) \\ \gamma(\psi) & \delta(\psi) \end{pmatrix}$$

with $\alpha \in \mathbb{R}$; $\beta \in (\ell^1)^*$; $\gamma \in \ell^1$; $\delta \in L(\ell^1, \ell^1)$. In this notation, we can formulate :

<u>Estimate 1</u>. If $\|\psi - \psi_o\| < .01$, then

$$|\alpha - 4.669| < .148 \; ; \; \|\beta\| < .560 \; ; \; \|\gamma\| < .756 \; ; \|\delta\| < .719$$

These bounds imply that the inequality

$$[\alpha(\psi) - 1][1 - \|\delta(\psi)\|] > \|\beta(\psi)\| . \|\gamma(\psi)\| \tag{3.2}$$

holds uniformly on the ball of radius .01 about ψ_o. If T has a fixed point g in this ball, then hyperbolicity of $DT(g)$ acting on A follows readily from (3.2).

To prove the existence of a fixed point, we use a variant of Newton's method. Instead of studying

$$\psi \longmapsto \psi - (DT(\psi) - \mathbb{1})^{-1} [T\psi - \psi],$$

we replace $(DT(\psi) - \mathbb{1})^{-1}$ by the approximation

$$J = \begin{pmatrix} \frac{1}{3.669} & 0 \\ 0 & -\mathbb{1} \end{pmatrix},$$

and we apply the contraction mapping principle to the mapping

$$\psi \longmapsto \Phi(\psi) = \psi - J\cdot[T\psi - \psi]$$

which has the same fixed points as T.

A simple calculation using Estimate 1 shows that

$$\|D\Phi(\psi)\| < .9 \text{ for } \|\psi - \psi_o\| < .01 .$$

It will then follow from the contraction mapping theorem that Φ has a fixed point in this ball provided that

$$\frac{\|\Phi(\psi_o) - \psi_o\|}{1 - .9} < .01$$

For this we have :

Estimate 2.

$$\|\Phi(\psi_o) - \psi_o\| < 4 \times 10^{-6}$$

Thus T has a fixed point in A , and DT at the fixed point, acting on A , has the hyperbolicity properties stated in Theorem 3. Domains of analyticity may be enlarged using the functional equation for g , and in this way we arrive at Theorem 1 and 3 as formulated.

Furthermore, Estimate 1 makes it possible to establish the existence of a system of expanding and contracting cones for T on $\{\psi : \|\psi - \psi_o\| < .01\}$, which in turn makes it possible to construct local stable and unstable manifolds which are not too small. This facilitates the proofs of Theorem 4 and 5.

The proofs of Estimates 1. and 2. are completely straightforward, if long. Consider, for example, Estimate 1. Since A is essentially ℓ^1 , we can think of $DT(\psi)$ as an infinite matrix. Norms of matrices acting on ℓ^1 are easy to compute in terms of the matrix elements. Any matrix element can be expressed in terms of ψ . All but finitely many of these matrix elements are estimated analytically. For the remainder, strict upper and lower bounds are computed numerically from bounds on the Taylor coefficients for ψ . The arithmetic operations are performed in finite precision floating point arithmetic; the methods of interval arithmetic are used to control the effect of round-off error.

4. Supplementary remarks.

1. The results described here are descendants of (and improvements on) the results announced in [5]. Since that announcement, a completely different proof for the existence of g has been given by Campanino, Epstein, and Ruelle [1].

2. The approach to proving Theorem 1 outlined in the preceding section produces strict bounds on the difference between an approximate fixed point and the exact one. These estimates can be applied to higher precision calculations. Let

$$g_{40}^{(o)}(x) = 1 + \sum_{n=1}^{40} g_n^{(o)} x^{2n}$$

where the $g_n^{(o)}$ are given by the following table :

n	$g_n^{(o)}$
1	-1.52763 29970 363014 54035 8903 E+00
2	1.04815 19478 730373 32167 4261 E-01
3	2.67056 70525 193354 03265 2095 E-02
4	-3.52740 96609 087091 70234 1908 E-03
5	8.16009 66547 531745 17219 0486 E-05
6	2.52850 84233 963536 17626 2552 E-05
7	-2.55631 71662 784938 46353 2541 E-06
8	-9.65127 15508 912032 16372 5768 E-08
9	2.81934 63974 504091 37075 6629 E-08
10	-2.77305 11607 990117 24373 1657 E-10
11	-3.02842 70221 305663 29838 6443 E-10
12	2.67058 92807 480755 53960 2441 E-11
13	9.96229 16410 284823 10598 0831 E-13
14	-3.62420 29829 041560 84558 1742 E-13
15	2.17965 77448 270704 77019 8956 E-14
16	1.52923 28994 809626 05606 8735 E-15
17	-3.18472 87899 527757 99377 9066 E-16
18	1.13467 21062 118714 17757 0937 E-17
19	1.88167 60568 254399 33331 1887 E-18
20	-2.27561 25646 32121 77152 62473 E-19
21	-9.82244 76294 21762 12269 93846 E-22
22	2.06412 97560 04508 72566 18024 E-21
23	-1.24932 00592 43689 29083 39227 E-22
24	-1.07706 12046 93546 96384 01973 E-23
25	1.87274 68082 18657 19228 25376 E-24
26	-2.57770 82101 03665 08803 84193 E-26
27	-1.55419 04560 84084 34313 83993 E-26
28	1.28044 34650 13446 65240 71115 E-27
29	5.58505 87986 18652 30574 68715 E-29
30	-1.52783 46925 29937 97901 86234 E-29
31	5.04174 26639 85470 06223 96180 E-31
32	1.01653 68070 20109 67909 77329 E-31
33	-1.00690 21392 64415 27335 99472 E-32
34	-5.24253 64364 14750 06805 77950 E-34
35	1.72437 64381 50161 48350 91165 E-34
36	-1.31439 18669 87224 62488 66269 E-35
37	-1.85830 45136 26377 53427 02339 E-38
38	8.05506 40492 58197 17747 92644 E-38
39	-6.26717 53968 04057 97900 56349 E-39
40	1.76882 81623 81787 65332 72841 E-40

Table 1.

We then have strict bounds

$$|g(Z) - g_{40}^{(o)}(Z)| \leq \begin{cases} 1.5\times 10^{-23} & \text{for } |Z|^2 \leq 1.5 \\ 5.5\times 10^{-13} & \text{for } |Z| \leq 2 \\ 5\times 10^{-7} & \text{for } |Z|^2 \leq 6 \\ 1.7\times 10^{-2} & \text{for } |Z| \leq 8 \,. \end{cases}$$

These bounds are probably very conservative.

3. The domain Ω used in the statements of Proposition 2 and Theorem 3 was chosen for convenience. Many other domains, including arbitrarily small open neighborhoods of $[-1,1]$, could have been used instead. The hyperbolicity statement of Theorem 3 is formally stronger for small domains than for large ones. (For $\Omega_1 \subset \Omega_2$, any eigenfunction for $D\mathcal{T}(g)$ on Ω_2 is also an eigenfunction on Ω_1) . It can be shown, however, that any function analytic on a neighborhood of $[-1,1]$ and satisfying there the formal functional equation for an eigenvector of $D\mathcal{T}(g)$ is actually analytic and bounded on the domain Ω .

4. It follows easily from the functional equation for g that g is analytic on a neighborhood of the whole real axis. H. Epstein (private communication) has observed that a similar argument shows that it is analytic on a neighborhood of the imaginary axis. On the other hand, it is essentially certain that g is not entire. Indeed, it appears - but has not been proved - that the singularities of g nearest to the origin occur at a set of 4 periodic points of period 2 for $Z \longmapsto g(-\lambda Z)$, $\lambda = -g(1)$, located approximately at :

$$Z^2 = -3.8428 \pm i\ 9.8215$$

5. Proposition 2 and Theorem 3 remain true if the requirement that ψ be even is dropped. In other words : No new expanding eigenvectors are introduced if we let $D\mathcal{T}(g)$ act on functions which are not necessarily even (but which vanish to second order at 0).

6. Theorem 4 can be extended considerably. To formulate the extension, we need the theory of kneading sequences for unimodal mappings as developed, for example, in Chapter III. 1 of Collet-Eckmann [2]. Let $\underline{K}$ be a finite kneading sequence. Except for the simple case $\underline{K}$ = RC , there are associated with $\underline{K}$ three hypersurfaces in $\mathcal{M}$:

. The set of superstable ψ's with kneading sequence $\underline{K}$.

. The saddle-node bifurcation surface where the attracting periodic orbit passing through the critical point on the preceding surface appears.

. The period-doubling bifurcation surface where that periodic orbit becomes unstable.

It can be shown that, intuitively, the unstable manifold crosses these three surfaces for each $\underline{K}$; a precise version of this statement must be formulated with the same circumspection as Theorem 4. There is no reason to doubt that these crossings are all transverse.

A simple argument using the apparatus developed in [ ] reduces the proof of Theorem 4 and the above extension to establishing the existence, on the local unstable manifold, of one point whose kneading sequence strictly precedes, and one whose kneading sequence strictly follows, that of g (in the combinatorial ordering for kneading sequences). The proof proceeds by finding with sufficient precision two points on the unstable manifold and computing initial segments of their kneading sequences.

7. Although done by computer, the computations involved in proving the results stated are just on the boundary of what it is feasible to verify by hand. I estimate that a carefully chosen minimal set of estimates sufficient to prove Theorems 1. and 3. could be carried out, with the aid only of a non-programmable calculator, in a few days.

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REFERENCES

[1] M. Campanino, H. Epstein, and D. Ruelle, "On Feigenbaum's functional equation", IHES preprint P/80/32 (1980), to appear in Topology.

M. Campanino and H. Epstein, "On the existence of Feigenbaum's fixed point", IHES preprint P/80/35 (1980), Commun. Math. Phys. 79, 261-302 (1981).

[2] P. Collet and J.P. Eckmann, Iterated Maps of the Interval As Dynamical Systems, Birkhäuser, Boston-Basel-Stuttgart, 1980.

[3] P. Collet, J.P. Eckmann, and O.E. Lanford, "Universal properties of maps on an interval", Commun. Math. Phys. 76 (1980) 211-254.

[4] M. Feigenbaum, "Quantitative universality for a class of non-linear transformations", J. Stat. Phys. 19 (1978) 25-52.

M. Feigenbaum, "The universal metric properties of non-linear transformations", J. Stat. Phys. 21 (1979) 669-706.

[5] O.E. Lanford, "Remarks on the accumulation of period-doubling bifurcations", in Mathematical Problems in Theoretical Physics, Springer Lecture Notes in Physics 116, (1980) 340-342.

[6] O.E. Lanford, "Smooth transformations of intervals", Séminaire Bourbaki, 1980/81, N° 563.