LYAPUNOV EXPONENTS AND HODGE THEORY

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ABSTRACT. Claude Itzykson was fascinated (among other things) by the mathematics of *integrable* billiards (see [AI]). This paper is devoted to new results about the *chaotic* regime. It is an extended version of the talk of one of us (M.K.) on the conference "The Mathematical Beauty of Physics", Saclay, June 1996, dedicated to the memory of C. Itzykson.

We started from computer experiments with simple one-dimensional ergodic dynamical systems, and quite unexpectedly ended with topological string theory. The result is a formula connecting fractal dimensions in one dimensional "conformal field theory" and explicit integrals over certain moduli spaces. Also a new analogy arose between ergodic theory and complex algebraic geometry.

We will finish the preface with a brief summary of what is left behind the scene. Our moduli spaces are close relatives of those arising in Seiberg-Witten approach to the supersymmetric Yang-Mills theory. The integrals in the main formula can also be considered as correlators in a topological string theory with $c = 1$. Probably, there is way to calculate them in terms of a matrix model and an integrable hierarchy. In the derivation we use some identity in Kähler geometry which looks like a use of $N = 2$ supersymmetry.

1. INTERVAL EXCHANGE TRANSFORMATIONS

Let us consider a classical mechanical system with the action of a quasi-periodic external force. Mathematically such a system can be described as a symplectic manifold (X, ω) and a closed 1-form α on it. The Hamiltonian of the system is a multivalued function H such that $dH = \alpha$. Branches of H differ from each other by additive constants. One can write the equations of motions $dF(x(t))/dt = \{F, H\}(x(t))$, $F \in C^\infty(X)$, as usual. In contrast with the case of globally defined Hamiltonians, the system does not have first integrals. More precisely, one still can make a reduction to codimension 1 near local minima or maxima of H. Nevertheless, the dynamics on an open part of X is expected to be ergodic.

Many physical systems produce after averaging multivalued Hamiltonians. Examples include celestial mechanics, magnetic surfaces, motions of charged particles on Fermi surfaces in crystals, etc. (see the survey of S.P.Novikov [Nov]).

We consider the simplest case of 2-dimensional phase space. Thus we have a closed oriented surface Σ with an area element $\omega \in \Omega^2(\Sigma)$ and an area-preserving vector field ξ:

$$i_\xi \omega = \alpha \in \Omega^1(\Sigma), \quad d\alpha = 0, \quad Lie_\xi(\omega) = 0$$

The main feature of the 2-dimensional case is that the system depends essentially on a finite number of parameters. Generically the surface splits into a finite number of components filled with periodic trajectories, and finite number of *minimal* components, where every trajectory is dense. We can associate with every minimal component a so-called *interval exchange transformation* T (see [CFS]). First of all, we choose an interval I on Σ transversal to the vector field ξ. The transformation T is defined as the first return map (the Poincaré map) from I to itself. The form α defines a measure dx and an orientation on I. The map T preserves both dx and the orientation. Also, it is easy to see that generically T has a finite number of discontinuity points $a_1, \ldots a_{k-1}$ where k is the number of intervals of continuity of T. Thus we can identify I with an interval in $\mathbb{R}$ and write T as follows:

$$I = [0, a] \subset \mathbb{R}, \quad 0 = a_0 < a_1 < a_2 \cdots < a_{k-1} < a = a_k$$

$$T(x) = x + b_i \quad for \quad a_i < x < a_{i+1}$$

where $b_i \in \mathbb{R}$, $i = 0, \ldots, k-1$, are some constants. Moreover, intervals (a_0, a_1), $(a_1, a_2), \ldots, (a_{k-1}, a_k)$ after the application of this map will be situated on I in an order described by a permutation $\sigma \in S_k$ and without overlapping. Thus numbers b_i can be reconstructed uniquely from the numbers a_i and the permutation σ.

We did not use the area element on the surface Σ in this construction. Everything is defined in terms of a closed 1-form α and an orientation on Σ. Thus it is enough to have an oriented foliation with a transversal measure (and with finitely many singularities) on an oriented surface. It is easy to go back from interval exchange transformations to oriented surfaces with measured foliations. The systems which we will get by the inverse construction correspond to multivalued Hamiltonians H without local minima and maxima.

Also we can consider possibly non-orientable foliations with transversal measures on possibly non-orientable surfaces. This leads to the consideration of mechanical systems with various additional symmetries. The first return map is defined on an interval in an appropriate ramified double covering of the surface.

The permutation σ is called *irreducible* if, for any j, $1 \leq j \leq k-1$, one has

$$\sigma(\{1, 2, \ldots, j\}) \neq \{1, 2, \ldots, j\}$$

It is called *nondegenerate* if in addition it obeys some extra conditions (see 3.1-3.3 in [M] or equivalent conditions 5.1-5.5 in [V1]). Morally these conditions mean that the permutation does not determine any fake zeros of the form α — zeros of order 0. In particular $\sigma(j + 1) \neq \sigma(j) + 1$ $j = 1, \ldots, k-1$. First return maps for ergodic flows give irreducible permutations; appropriate choice of transversal gives nondegenerate permutations.

Theorem . (H. Masur [M]; W. Veech [V1]). *Let us consider the interval exchange map T for an irreducible permutation σ and generic values of continuous parameters a_i (generic with respect to the Lebesgue measure on the parameter space $\mathbb{R}^k_+ = \{(a_1, \ldots, a_k)\}$). Then the map T is ergodic with respect to the Lebesgue measure dx.*

The entropy of the map T is 0.

An analogous result is true for non-orientable measured foliations on *orientable* surfaces. The case of non-orientable surfaces is always degenerate. In such case foliations almost always have non-trivial families of closed leaves (see [N]). Presumably, the interesting (ergodic) part is always reduced to measured foliations on

orientable surfaces. In order to simplify the exposition we will mainly consider here the case when both the surface and the foliation are orientable.

2. ERROR TERMS: FIRST RESULTS AND COMPUTER EXPERIMENTS

Let x be a generic point on I and let (y_1, y_2) be a generic subinterval of I. Since the map T is ergodic (for generic values of lengths of subintervals) we have the following equality

$$\#\{i : 0 \leq i \leq N - 1, \ T^{(i)}(x) \in (y_1, y_2)\} = (y_2 - y_1)N + o(N)$$

as $N \longrightarrow +\infty$. It was first observed in computer experiments (see [Z1]) that this error term (denoted above by $o(N)$) typically has the growth of a power of N,

$$error \ \ term \sim O(N^\lambda) \ .$$

Here $\lambda < 1$ is an universal exponent depending only on the permutation σ (see [Z2] for the proof of the related statement).

In the case of 2 or 3 intervals, one has $\lambda = 0$. In these cases the genus of the surface is 1 and the transformation itself is equivalent to the generic irrational rotation of a circle.

In the case of 4 intervals for all nondegenerate permutations one has

$$\lambda = 0.33333+ \sim (10^{-6})$$

In the case of 5 intervals for all nondegenerate permutations one has

$$\lambda = 0.50000+ \sim (10^{-6})$$

These two cases correspond to surfaces of genus 2.

If we have 6 intervals (surfaces of genus 3), then the number λ depends on the permutation:

$$\lambda = 0.6156\ldots \ \text{or} \ \ 0.7173\ldots \ .$$

These two numbers are probably irrational.

Also computer experiments show (see [Z1]) that a generic closed 1-form on a surface Σ defines a filtration on $H_1(\Sigma, \mathbb{R})$ ("fractal Hodge structure") by subspaces

$$H_1(\Sigma, \mathbb{R}) \supset F^{\lambda_g} \supset \cdots \supset F^{\lambda_2} \supset F^{\lambda_1} \supset 0, \ \ g = \text{genus of } \Sigma, \ \ \dim(F^{\lambda_j}) = j$$

where $1 = \lambda_1 > \lambda_2 > \cdots > \lambda_g > 0$ are some universal constants depending only on the permutation. The number λ which gives the error term in the ergodic theorem is the second exponent λ_2. The highest term of the filtration F^{λ_g} is a Lagrangian subspace of $H_1(\Sigma, \mathbb{R})$.

One can see numbers λ_i geometrically. Let us consider a generic trajectory of the area-preserving vector field ξ on Σ. We consider a sequence of pieces of this trajectory $x(t)$ of lengths $l_j \longrightarrow +\infty$, $j = 1, 2, \ldots$ such that $x(l_j)$ is close to the starting point $x(0)$. We connect two ends of these pieces by short intervals and get a sequence of closed oriented curves C_j on Σ. Homology classes of curves C_j are elements $\mathbf{v}_j = [C_j]$ in the group $H_1(\Sigma, \mathbb{Z})$.

Vectors $\mathbf{v}_j$ at the first approximation are close to a one-dimensional space,

$$\mathbf{v}_j = \mathbf{u}\, l_j + o(l_j)$$

where $\mathbf{u}$ is a non-zero element of $H_1(\Sigma, \mathbb{R})$. Homology class $\mathbf{u}$ is Poincaré dual to the cohomology class $[\alpha]$ of the 1-form α. The lowest non-trivial term in the filtration is

$$F^{\lambda_1} = \mathbb{R}\mathbf{u} \ .$$

After the projection to the quotient space $H_1(\Sigma, \mathbb{R}) \longrightarrow H_1(\Sigma, \mathbb{R})/\mathbb{R}\mathbf{u}$ we get again a sequence of vectors. It turns out that for large j these vectors are again close to a 1-dimensional subspace $\mathcal{L}$. Also these vectors mostly will have size $(l_j)^{\lambda_2 + o(1)}$. We define 2-dimensional space $F^{\lambda_2} \subset H_1(\Sigma, \mathbb{R})$ as the inverse image of the 1-dimensional space $\mathcal{L}$. We can repeat the procedure g times. On the last step we get a chaotic sequence of vectors of bounded length in the g-dimensional quotient space $H_1(\Sigma, \mathbb{R})/F^{\lambda_g}$ (see also [Z3]).

There is, presumably, an equivalent way to describe numbers λ_i. Namely, let ϕ be a smooth function on Σ. Assume for simplicity that the multi-valued Hamiltonian has only non-degenerate (Morse) singularities. Then, for generic trajectory $x(t)$, we expect that the number $\int_0^T \phi(x(t))dt$ for large T with high probability has size $T^{\lambda_i + o(T)}$ for some $i \in \{1, \ldots, g\}$. Exponent λ_1 appears for all functions with non-zero average value,

$$\int_\Sigma \phi\omega \neq 0 \ .$$

The next exponent, λ_2, should work for functions in a codimension 1 subspace in $C^\infty(\Sigma)$ etc.

We discovered in computer experiments (more than 100 cases) that the sum of numbers λ_j is rational,

$$\lambda_1 + \cdots + \lambda_g \in \mathbb{Q} \ .$$

For example, if genus of Σ is 3 and we have 4 simple saddle points for the foliation, then

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 + 0.5517\cdots + 0.3411\cdots = 53/28 \ .$$

Also, our observation explains why the case of genus 2 is exceptional. If we have two numbers first of which is equal to 1 and the sum is rational, then the second number is rational too.

3. Moduli spaces

We want to study the renormalization procedure for interval exchange maps. For example, we can define a map from the space of parameters

$$\{(a_1, \ldots, a_k; \sigma)\} = \mathbb{R}_+^k \times S_k$$

to itself considering the first return map of the half $[0, a/2]$ of the original interval $I = [0, a]$. There are also other ways, but the most elegant is the one which we describe at the end of this section. In order to do it we introduce, following H. Masur and W. Veech certain moduli space.

The space $\Omega^1_{\text{closed}}(\Sigma)/\text{Diff}(\Sigma)$ of equivalence classes of closed 1-forms on a surface Σ is non-Hausdorf. In order to cure it we consider a "doubling" of this space consisting of the space of closed complex-valued 1-forms $\alpha_{\mathbb{C}}$ satisfying the condition

$$\text{Re}\,\alpha \wedge \text{Im}\,\alpha_{|_x} > 0$$

for almost all points $x \in \Sigma$. The notion of positivity here is well-defined because the surface Σ is oriented.

The real-valued 1-form α whose leaves we considered before is the real part, $\text{Re}\,\alpha_{\mathbb{C}}$, of the complex-valued form $\alpha_{\mathbb{C}}$. First of all, we should be sure that we did not restrict ourselves. It follows from results of E. Calabi (see [C]), or from results of A. Katok (see [K]) that, for any closed real 1-form α giving a minimal (everywhere dense) foliation, there exists at least one closed 1-form $\alpha' \neq 0$ such that $\alpha \wedge \alpha' \geq 0$

everywhere except points where α vanishes. Thus we have a complex valued closed 1-form $\alpha_{\mathbb{C}} = \alpha + i\alpha'$.

Any such complex-valued 1-form defines a complex structure on Σ. Locally outside of zeroes of $\alpha_{\mathbb{C}}$ there is a complex-valued coordinate $z : \Sigma \longrightarrow \mathbb{C}$ such that $dz = \alpha_{\mathbb{C}}$. Holomorphic functions are defined as continuous functions holomorphic in coordinate z. Also, there is a canonical flat metric $(\operatorname{Re}\alpha_{\mathbb{C}})^2 + (\operatorname{Im}\alpha_{\mathbb{C}})^2$ on Σ with singularities at zeroes of $\alpha_{\mathbb{C}}$.

Let us fix a sequence of non-negative integers $\mathbf{d} = (d_1, \ldots, d_n)$ such that $\sum_i d_i = 2g - 2$ where $g \geq 2$ is the genus of the surface. We denote by $\mathcal{M}_{\mathbf{d}}$ the moduli space of triples $(C; p_1, \ldots, p_n; \alpha_{\mathbb{C}})$ where C is a smooth complex curve of genus g, p_i are pairwise distinct points of C, and $\alpha_{\mathbb{C}}$ is a holomorphic 1-form on C which vanishes up to order d_i at p_i and is non-zero at all other points of C. From this definition it is clear that $\mathcal{M}_{\mathbf{d}}$ is a Hausdorf complex analytic (and algebraic) space (see [V3]).

First of all, $\mathcal{M}_{\mathbf{d}}$ is a complex orbifold of dimension $2g - 1 + n$. Let us consider the period map from a neighborhood of a point $(C; p_1, \ldots, p_n; \alpha_{\mathbb{C}})$ of $\mathcal{M}_{\mathbf{d}}$ into the cohomology group $H^1(C, \{p_1, \ldots, p_n\}; \mathbb{C})$. Closed form $\alpha_{\mathbb{C}}$ defines an element of the relative cohomology group $H^1(C, \{p_1, \ldots, p_n\}; \mathbb{C})$ by integration along paths connecting points p_i. In a neighborhood of any point $(C; p_1, \ldots, p_n; \alpha_{\mathbb{C}})$ of $\mathcal{M}_{\mathbf{d}}$, we can identify cohomology groups $H^1(C', \{p'_1, \ldots, p'_n\}; \mathbb{C})$ with $H^1(C, \{p_1, \ldots, p_n\}; \mathbb{C})$ using the Gauss-Manin connection.

Thus we get a map (the period map) from this neighborhood into a vector space. An easy calculation shows that the deformation theory is not obstructed and we get locally a one-to-one correspondence between $\mathcal{M}_{\mathbf{d}}$ and an open domain in the vector space $H^1(C, \{p_1, \ldots, p_n\}; \mathbb{C})$.

We claim that $\mathcal{M}_{\mathbf{d}}$ has structures 1), 2), 3), 4) listed below.

1. *a holomorphic affine structure on $\mathcal{M}_{\mathbf{d}}$ modelled on the vector space $H^1(C, \{p_1, \ldots, p_n\}; \mathbb{C})$,*
2. *a smooth measure μ on $\mathcal{M}_{\mathbf{d}}$,*
3. *a locally quadratic non-holomorphic function $A : \mathcal{M}_{\mathbf{d}} \longrightarrow \mathbb{R}_+$,*
4. *a non-holomorphic action of the group $GL_+(2, \mathbb{R})$ on $\mathcal{M}_{\mathbf{d}}$.*

The first structure we already defined using the period map.

The tangent space to $\mathcal{M}_{\mathbf{d}}$ at each point contains a lattice,

$$H^1(C, \{p_1, \ldots, p_n\}; \mathbb{C}) \doteq H^1(C, \{p_1, \ldots, p_n\}; \mathbb{R} \oplus i\mathbb{R}) \supset$$
$$\supset H^1(C, \{p_1, \ldots, p_n\}; \mathbb{Z}) \oplus i \cdot H^1(C, \{p_1, \ldots, p_n\}; \mathbb{Z}) \,.$$

The Lebesgue measure (= the Haar measure) on the tangent space to $\mathcal{M}_{\mathbf{d}}$ can be uniquely normalized by the condition that the volume of the quotient torus is equal to 1. Thus we defined the density of a measure μ at each point of $\mathcal{M}_{\mathbf{d}}$.

We define the function $A : \mathcal{M}_{\mathbf{d}} \longrightarrow \mathbb{R}_+$ by the formula $A(C, \alpha_{\mathbb{C}}) = \frac{i}{2} \int_C \alpha_{\mathbb{C}} \wedge \overline{\alpha_{\mathbb{C}}}$. In other terms, it is the area of C for the flat metric associated with $\alpha_{\mathbb{C}}$.

The group $GL_+(2, \mathbb{R})$ of 2×2-matrices with positive determinant acts by linear transformations with constant coefficients on the pair of real-valued 1-forms $(\operatorname{Re}(\alpha_{\mathbb{C}}), \operatorname{Im}(\alpha_{\mathbb{C}}))$. In the local affine coordinates, this action is the action of $GL_+(2, \mathbb{R})$ on the vector space

$$H^1(C, \{p_1, \ldots, p_n\}; \mathbb{C}) \simeq \mathbb{C} \otimes H^1(C, \{p_1, \ldots, p_n\}; \mathbb{R}) \simeq \mathbb{R}^2 \otimes H^1(C, \{p_1, \ldots, p_n\}; \mathbb{R})$$

through the first factor in the tensor product. From this description it is clear that the subgroup $SL(2, \mathbb{R})$ preserves the measure μ and the function A.

On the hypersurface $\mathcal{M}_{\mathbf{d}}^{(1)} = A^{-1}(1)$ (the level set of the function A) we define the induced measure by the formula

$$\mu^{(1)} = \frac{\mu}{dA}$$

The group $SL(2, \mathbb{R})$ acts on $\mathcal{M}_{\mathbf{d}}^{(1)}$ preserving $\mu^{(1)}$.

Theorem (H. Masur; W. Veech). *The total volume of $\mathcal{M}_{\mathbf{d}}^{(1)}$ with respect to the measure $\mu^{(1)}$ is finite.*

Let us denote by $\mathcal{M}$ any connected component of $\mathcal{M}_{\mathbf{d}}$ and by $\mathcal{M}^{(1)}$ its intersection with $\mathcal{M}_{\mathbf{d}}^{(1)}$.

Theorem (H. Masur; W. Veech). *The action of the one-parameter group $\{diag(e^t, e^{-t})\} \subset SL(2, \mathbb{R})$ on $(\mathcal{M}^{(1)}, \mu^{(1)})$ is ergodic.*

The action in this theorem is in fact the renormalization group flow for interval exchange maps. Another name for this flow is the "Teichmüller geodesic flow", because it gives the Euler-Lagrange equations for geodesics for the Teichmüller metric on the moduli space of complex curves. Notice that this metric is not a Riemannian metric, but only a Finsler metric.

The intuitive explanation of the ergodicity is that the group $\{diag(e^t, e^{-t})\}$ expands leaves of the foliation by affine subspaces parallel to $H^1(C, \{p_1, \ldots, p_n\}, \mathbb{R})$ and contracts leaves of the foliation by subspaces parallel to $H^1(C, \{p_1, \ldots, p_n\}, i\mathbb{R})$.

4. TOPOLOGY OF THE MODULI SPACE

In the last theorem from the previous section we consider *connected components* of the moduli space $\mathcal{M}_{\mathbf{d}}$. From the first glance it seems to be not necessary because normally moduli spaces are connected. It is not true in our case. W. Veech and P. Arnoux discovered by direct calculations in terms of permutations that there are several connected components. The set of irreducible permutations is decomposed into certain equivalence classes called extended Rauzy classes. These classes correspond to connected components of spaces $\mathcal{M}_{\mathbf{d}}$. For a long time the geometric origin of non-connectedness was not clear.

Recently we have obtained the complete classification of connected components. First of all, there are two series of connected components of $\mathcal{M}_{\mathbf{d}}$ consisting of hyperelliptic curves such that the set of singular points is invariant under the hyperelliptic involution. The first series corresponds to curves with one singular point, $\mathbf{d} = (2g - 2)$ for $g \geq 2$. The second series corresponds to curves with two singular points, $\mathbf{d} = (g - 1, g - 1)$.

If all orders of zeros are even numbers, we have a spin structure on C given by a half of the canonical divisor

$$S = \sum_i \frac{d_i}{2}[p_i] \in Pic(C) .$$

It is well-known that spin structures have a topological characteristic (parity) which does not change under continuous deformations (see [At]). The *parity of a spin structure* is the parity of the dimension of the space of global sections of the corresponding line bundle.

Classification Theorem . *There are hyperelliptic and non-hyperelliptic connected components of the moduli spaces of holomorphic 1-forms. For non-hyperelliptic components there are two cases: the vector* **d** *is divisible by 2, or not. If* **d** *is divisible by 2 then there are two components corresponding to even and odd spin structures. There are exceptional cases when we get an empty set: 1) for $g = 2$: all non-hyperelliptic strata; 2) for $g = 3$: non-hyperelliptic strata with* **d** *divisible by 2 and even spin structure.*

We have analogous results for the moduli space of quadratic differentials. At the moment we do not know anything about the topology of connected components except for the hyperelliptic locus.

Conjecture . *Each connected component $\mathcal{M}$ of $\mathcal{M}_{\mathbf{d}}$ has homotopy type $K(\pi, 1)$, where π is a group commensurable with some mapping class group.*

5. LYAPUNOV EXPONENTS

We recall here the famous multiplicative ergodic theorem (see also the other famous theorem on the related matter — theorem of H. Furstenberg [F] for a product of random matrices).

Theorem . **(V. Oseledets [O]).** *Let $T_t : (X, \mu) \longrightarrow (X, \mu)$, $t \in \mathbb{R}_+$, be an ergodic flow on a space X with finite measure μ; let V be an $\mathbb{R}_+$-equivariant measurable finite-dimensional real vector bundle. We also assume that a (non-equivariant) norm $\| \ \|$ on V is chosen such that, for all $t \in \mathbb{R}_+$,*

$$\int_X \log \left(1 + \|T_t : V_x \longrightarrow V_{T_t(x)}\|\right) \mu < +\infty .$$

Then there are real constants $\lambda_1 > \lambda_2 > \cdots > \lambda_k$ and an equivariant filtration of the vector bundle V

$$V = V_{\lambda_1} \supset \cdots \supset V_{\lambda_k} \supset 0$$

such that, for almost all $x \in X$ and all $v \in V_x \setminus \{0\}$, one has

$$\|T_t(v)\| = \exp(\lambda_j t + o(t)), \quad t \longrightarrow +\infty$$

where j is the maximal value for which $v \in (V_{\lambda_j})_x$. The filtration V_{λ_j} and numbers λ_j do not change if we replace norm $\| \ \|$ by another norm $\| \ \|'$ such that

$$\int_X \log \left(\max_{v \in V_x \setminus \{0\}} \left(\max \left(\frac{\|v\|}{\|v\|'}, \frac{\|v\|'}{\|v\|} \right) \right) \right) \mu < +\infty .$$

Analogous statement is true for systems in discrete time $\mathbb{Z}_+$.

Numbers λ_j are called Lyapunov exponents of the equivariant vector bundle V. Usually people formulate this theorem using language of matrix-valued 1-cocycles instead of equivariant vector bundles. This is equivalent to the formulation above because any vector bundle on a measurable space can be trivialized on the complement to a subset of measure zero.

If our system is reversible, we can change the positive direction of the time. Lyapunov exponents will be replaced by negative Lyapunov exponents. A new filtration will appear. This new filtration is opposite to the previous one, and they together define an equivariant splitting of V into the direct sum of subbundles.

Lyapunov exponents are, in general, very hard to evaluate other than numerically. We are aware only about two examples of explicit formulas. One example is the geodesic flow on a locally symmetric domain and V being a homogeneous vector

bundle. In this case one can explicitly construct the splitting of V. The second example is the multiplication of random independent matrices whose entries are independent equally distributed Gaussian random variables. In this case one can calculate Lyapunov exponents using rotational invariance and the Markov property.

Our calculation seems to be the first calculation of Lyapunov exponents in a non-homogeneous situation. As the reader will see later, our proof uses a replacement of a deterministic system by a Markov process.

Let us define a vector bundle H^1 over $\mathcal{M}$ by saying that its fiber at a point $(C; p_1, \ldots, p_n; \alpha_{\mathbb{C}})$ is the cohomology group $H^1(C, \mathbb{R})$. We apply the multiplicative ergodic theorem to the action of $\{diag(e^t, e^{-t})\}$ on $\mathcal{M}^{(1)}$ and to the bundle H^1. The action of the group on this bundle is defined by the lift using the natural flat connection (Gauss-Manin connection). We will not specify for a moment the norm on H^1 because all natural choices are equivalent in the sense specified in our formulation of the multiplicative ergodic theorem.

The structure group of the bundle H^1 is reduced to $Sp(2g, \mathbb{R}) \subset GL(2g, \mathbb{R})$. One can see easily that in this case Lyapunov exponents form a symmetric subset of $\mathbb{R}$. Also, in all experiments we have simple spectrum of Lyapunov exponents, i.e. the picture is

$$\lambda_1 > \lambda_2 > \cdots > \lambda_g > \lambda_{g+1} = -\lambda_g > \cdots > \lambda_{2g} = -\lambda_1$$

Theorem . *1) The lowest Lyapunov exponent $\lambda_{2g} = -\lambda_1$ is equal to -1 and has multiplicity one. The corresponding one-dimensional subbundle is $\mathrm{Re}(\alpha)\mathbb{R} \subset H^1$. 2) The second Lyapunov exponent λ_2 governs the error term in the ergodic theorem for interval exchange maps. 3) The filtration on H^1 related with the positive time dynamics depends locally only on the cohomology class $[\mathrm{Re}\,\alpha] \in H^1(\Sigma, \{p_1, \ldots, p_n\}; \mathbb{R})$.*

The first part is quite easy. At least, the growth of the norm for the 1-dimensional bundle $\mathrm{Re}(\alpha)\mathbb{R} \subset H^1$ is exponential with the rate 1.

The second part looks more mysterious. We compare two different dynamical systems, the original flow on the surface and the renormalization group flow on the moduli space. The time in one system is morally an exponent of the time in another system. The technical tool here is a mixture of the ordinary (additive) and the multiplicative ergodic theorem for an action of the group $\mathrm{Aff}(\mathbb{R}^1)$ of affine transformation of line. We are planning to write in a future a detailed proof; a rather technical proof of a related statement can be found in [Z2].

The third part is not hard, but surprising. In fact, the positive-time filtration on H^1 coincides with the filtration for *real-valued* closed 1-forms described in section 2. Thus it is independent on the choice of the imaginary part.

In computer experiments we observed that the spectrum of Lyapunov exponents is simple. In the rest of the paper wee will assume for simplicity that the non-degeneracy holds always. The general reason to believe in it is that there is no additional symmetry in the system which can force the Lyapunov spectrum to be degenerate.

6. Analogy with the Hodge theory

We see that our moduli space locally is decomposed into the product of two manifolds

$$H^1(\ldots; \mathbb{R}) \times H^1(\ldots; i\mathbb{R}) \ .$$

More precisely, we have two complementary subbundles in the tangent bundle satisfying the Frobenius integrability condition. This is quite analogous the geometry

of a complex manifold. If N is an almost complex manifold, then we have two complementary subbundles $T^{1,0}$ and $T^{0,1}$ in the *complexified* tangent bundle $T_N \otimes \mathbb{C}$. The integrability condition of the almost-complex structure is equivalent to the formal integrability of distributions $T^{1,0}$ and $T^{0,1}$.

Also, if we have a family of complex manifolds X_b, $b \in B$, parametrized holomorphically by a complex manifold B, then for every integer k we have a holomorphic vector bundle over B with the fiber $H^k(X_b; \mathbb{C})$. This bundle carries a natural flat connection and a holomorphic filtration by subbundles coming from the standard spectral sequence.

This picture (variations of Hodge structures, see [G]) is parallel to the situation in the multiplicative ergodic theorem applied to a smooth dynamical system. Let M denote the underlying manifold of the system. The tangent bundle T_M is an equivariant bundle. Thus, in the case of ergodicity and convergence of certain integrals we get a canonical measurable splitting of T_M into the direct sum of subbundles indexed by Lyapunov exponents. It is well known in many cases (and is expected in general) that these subbundles, and also all terms of both filtrations are integrable, i.e. they are tangent to leaves of (non-smooth) foliations on M. Two most important foliations (expanding and contracting foliations) correspond to terms of filtrations associated with all positive or all negative exponents. It is known that if the invariant measure is smooth then the sum of positive exponents is equal to the entropy of the system (Pesin formula).

7. FORMULA FOR THE SUM OF EXPONENTS

The main result of our work is an explicit formula for the sum of positive Lyapunov exponents $\lambda_1 + \cdots + \lambda_g$ for the equivariant bundle H^1 over the connected component $\mathcal{M}$ of moduli spaces of curves with holomorphic 1-forms.

We want to warn the reader that this equivariant bundle is not the whole tangent bundle $T_{\mathcal{M}}$. Lyapunov exponents for $T_{\mathcal{M}}$ are calculated in terms of the numbers λ_j in the following way:

$$2 > (1 + \lambda_2) > (1 + \lambda_3) > \cdots > (1 + \lambda_g) > \underbrace{1 = \cdots = 1}_{n-1} >$$

$$> (1 - \lambda_g) > \cdots > (1 - \lambda_2) > 0 = 0 > -(1 - \lambda_2) > \cdots > -(1 - \lambda_g) >$$

$$> \underbrace{-1 = \cdots = -1}_{n-1} > -(1 + \lambda_g) > -(1 + \lambda_{g-1}) > \cdots > -(1 + \lambda_2) > -2$$

Here n is the number of zeros $p_1, \ldots, p_n$ of a corresponding holomorphic 1-form. The entropy of the Teichmüller geodesic flow is equal by the Pesin formula to the complex dimension of $\mathcal{M}$. In short, what we are computing here is more delicate information than the entropy of the system.

Hypersurface $\mathcal{M}^{(1)}$ is isomorphic to the quotient space $\mathcal{M}/\mathbb{R}_+^*$, where $\mathbb{R}_+^*$ is identified with subgroup $\{diag(e^t, e^t)\}$ of $GL_+(2, \mathbb{R})$. We denote by $\mathcal{M}^{(2)}$ the quotient space

$$\mathcal{M}^{(1)}/SO(2, \mathbb{R}) \simeq \mathcal{M}/\mathbb{C}^* .$$

This space is a complex algebraic orbifold.

Orbits of the group $GL_+(2, \mathbb{R})$ define a 4-dimensional foliation on $\mathcal{M}$. It induces a 3-dimensional foliation on $\mathcal{M}^{(1)}$ by orbits of $SL(2, \mathbb{R})$, and a 2-dimensional foliation

$\mathcal{F}$ on $\mathcal{M}^{(2)}$. Leaves of $\mathcal{F}$ are complex curves in $\mathcal{M}^{(2)}$, but the foliation itself is not holomorphic.

3-dimensional foliation on $\mathcal{M}^{(1)}$ carries a natural transversal measure. This measure is the quotient of $\mu^{(1)}$ by the Haar measure on $SL(2,\mathbb{R})$. The transversal measure on $\mathcal{M}^{(1)}$ induces a transversal measure on $\mathcal{M}^{(2)}$. We have natural orientations on $\mathcal{M}^{(2)}$ and on leaves of $\mathcal{F}$ arising from complex structures. Thus we can construct differential form β such that

$$\beta \in \Omega^{\dim_{\mathbb{R}} \mathcal{M}^{(2)} - 2}(\mathcal{M}^{(2)}), \;\; d\beta = 0, \;\; Ker\,\beta = \mathcal{F} \; .$$

The natural projection $\mathcal{M} \longrightarrow \mathcal{M}^{(2)}$ is a holomorphic $\mathbb{C}^*$-bundle with a Hermitean metric given by the function A. Thus we have a natural curvature form $\gamma_1 \in \Omega^2(\mathcal{M}^{(2)})$, $d\gamma_1 = 0$ representing the first Chern class $c_1(\mathcal{M} \longrightarrow \mathcal{M}^{(2)})$. This form is given locally by the formula

$$\gamma_1 = \frac{1}{2\pi i} \partial\bar{\partial} \log{(A(s))}$$

where s is a non-zero holomorphic section of the line bundle $\mathcal{M} \longrightarrow \mathcal{M}^{(2)}$.

We also have another holomorphic vector bundle on $\mathcal{M}^{(2)}$. The fiber of this bundle (denoted by $H^{(1,0)}$) is equal to $H^0(C, \Omega_C^1)$, the term of the Hodge filtration in $H^1 \otimes \mathbb{C}$. This holomorphic bundle carries a natural hermitean metric coming from the polarization in Hodge theory. The formula for the metric is

$$\|\omega\|^2 := \frac{1}{2\pi i} \int_C \omega \wedge \bar{\omega}, \;\; \omega \in \Gamma(C, \Omega_{hol}^1)$$

This metric defines again a canonical closed 2-form γ_2 representing the characteristic class $c_1(H^{(1,0)})$.

Main Theorem .

$$\lambda_1 + \cdots + \lambda_g = \frac{\int_{\mathcal{M}^{(2)}} \beta \wedge \gamma_2}{\int_{\mathcal{M}^{(2)}} \beta \wedge \gamma_1}$$

In this formula we can not go directly to the cohomology, because the orbifold $\mathcal{M}^{(2)}$ is not compact. In order to overcome this difficulty, we constructed a compactification $\overline{\mathcal{M}}^{(2)}$ of $\mathcal{M}^{(2)}$ with toroidal singularities. All three differential forms $\beta, \gamma_1, \gamma_2$ in the formula seem to be smooth on $\overline{\mathcal{M}}^{(2)}$. Both γ_1 and γ_2 represent classes in $H^2(\overline{\mathcal{M}}^{(2)}, \mathbb{Q})$. It seems that β also represents a rational cohomology class although we don't have a proof yet. In the case of *one* critical point of 1-form α it is true because, by invariance reasons, the form β is proportional to a power of γ_1. Another possible explanation of rationality is that $[\beta]$ is *proportional* to a rational class because the part of $H^{dim-2}(\overline{\mathcal{M}}^{(2)}, \mathbb{R})$ consisting of classes vanishing on boundary divisors, can be one-dimensional. In any case, we almost explained the rationality of $\sum_{j=1}^g \lambda_j$ observed in experiments.

8. PROOF OF THE FORMULA

Any leaf of the foliation $\mathcal{F}$ carries a natural hyperbolic metric. The generic leaf is a copy of the upper half-plane $SL(2,\mathbb{R})/SO(2,\mathbb{R})$. We are studying the behavior of the monodromy of the Gauss-Manin connection in H^1 along *a long geodesic going in a random direction on a generic leaf of $\mathcal{F}$*. It was an old idea of Dennis Sullivan to replace the walk along random geodesic by a random walk on the hyperbolic

plane (the Brownian motion). The trajectory of the random walk goes to infinity in a random direction with approximately constant speed.

The meaning of the sum $\lambda_1 + \cdots + \lambda_g$ is the following. We move using the parallel transport a generic Lagrangian subspace L in the fiber of H^1 and calculate the average growth of the volume element L associated with the Riemannian metric on L induced from the natural metric (polarization) on H^1.

As we discuss above, we can replace the geodesic flow by the Brownian motion. We will approximate the random walk by a sequence of small jumps of a fixed length in random uniformly distributed directions on the hyperbolic plane.

Identity. *Fix $x \in \mathcal{M}^{(2)}$ and identify the leaf $\mathcal{F}_x$ of $\mathcal{F}$ passing through x with the model of the Lobachevsky plane in unit disc $\{z \in \mathbb{C} | \cdot |z| < 1\}$ in such a way that $x \mapsto z = 0$. Also, we trivialize the vector bundle H^1 over $\mathcal{F}_x$ using the Gauss-Manin connection. Then, for any Lagrangian subspace $L \subset H_0^1$, and for any ϵ, $0 < \epsilon < 1$ the following identity holds:*

$$\frac{1}{2\pi} \int\limits_0^{2\pi} d\theta \, \log \left(\frac{volume \ \ on \ \ L, \ \ for \ \ metric \ \ in \ \ H_{\epsilon e^{i\theta}}^1}{volume \ \ on \ \ L, \ \ for \ \ metric \ \ in \ \ H_0^1} \right) = \int\limits_{disc \ |z| \le \epsilon} \log \left(\frac{\epsilon^2}{|z|^2} \right) \gamma_2 \, .$$

The idea of the proof of this identity follows. Let us choose a locally constant basis $l_1, \ldots, l_g$ of L and a basis $v_1(z), \ldots, v_g(z)$ in $H_z^{1,0}$ depending holomorphically on $z \in \mathcal{F}_x$. Then we have

$$\|l_1 \wedge \cdots \wedge l_g\|_z^2 = \frac{(l_1 \wedge \cdots \wedge l_g \wedge v_1 \wedge \cdots \wedge v_g) \otimes (l_1 \wedge \cdots \wedge l_g \wedge \overline{v}_1 \wedge \cdots \wedge \overline{v}_g)}{(v_1 \wedge \cdots \wedge v_g \wedge \overline{v}_1 \wedge \cdots \wedge \overline{v}_g) \otimes (v_1 \wedge \cdots \wedge v_g \wedge \overline{v}_1 \wedge \cdots \wedge \overline{v}_g)}$$

where the numerator and the denominator are considered as elements of the one dimensional complex vector space $(\wedge^{2g}(H_0^1 \otimes \mathbb{C}))^{\otimes 2}$.

If we apply the Laplace-Beltrami operator $\Delta = (1/2\pi i) \times \partial_z \overline{\partial_z}$ to the logarithms of both sides of the formula from above, we get that

$$\Delta(\log(\|l_1 \wedge \cdots \wedge l_g\|_z^2)) =$$
$$= \Delta(holomorphic \ \ function) + \Delta(antiholomorphic \ \ function) + \gamma_2|_{\mathcal{F}_x}$$

It implies that the average growth of the volume on L depends not on L but only on the position of the point $x \in \mathcal{M}^{(2)}$. Because of ergodicity we can average over the invariant probability measure $Z^{-1} \times \mu^{(2)}$, where $Z = \int_{\mathcal{M}^{(2)}} \mu^{(2)}$ is the total volume of $\mathcal{M}^{(2)}$. The invariant measure $\mu^{(2)}$ is proportional to $\beta \wedge \gamma_1$. This explains the denominator in the formula for $\lambda_1 + \cdots + \lambda_g$.

9. GENERALIZATIONS

In our proof we treat the higher-dimensional moduli space $\mathcal{M}^{(2)}$ as a "curve with hyperbolic metric". In general, in many situations ergodic foliations with transversal measures and certain differential-geometric structures along leaves can be considered as virtual manifolds with the same type of geometric structure. Also, ergodic actions of groups can be considered as virtual discrete subgroups (Mackey's philosophy) etc.

Our proof works literally in a different situation. Let C be a complex curve of genus $g > 0$ parametrizing polarized abelian varieties A_x, $x \in C$ of complex dimension G.

We endow C with the canonical hyperbolic metric and consider the geodesic flow on it. It gives us an ergodic dynamical system. For the equivariant bundle we will take the symplectic local system H^1 over C with fibers $H^1(A_x, \mathbb{R})$. Again, the sum of positive Lyapunov exponents is rational:

$$\lambda_1 + \cdots + \lambda_G = \frac{deg(H^{1,0})}{2g - 2}$$

10. CONJECTURES ON THE VALUES OF LYAPUNOV EXPONENTS

Fix a collection of integers $\mathbf{d} = (d_1, \ldots, d_n)$, such that each d_i, $i = 1, \ldots, n$, is either positive, or equals -1. Assume that $\sum_i d_i = 4g - 4$. We denote by $\mathcal{Q}_\mathbf{d}$ the moduli space of triples $(C; p_1, \ldots, p_n; q)$ where C is a smooth complex curve of genus g, p_i are pairwise distinct points of C, and q is a meromorphic quadratic differential on C with the following properties. It has zero of order d_i at p_i if $d_i > 0$, it has a simple pole at p_i if $d_i = -1$ and it does not have any other zeros or poles on C. We also require that quadratic differential q is not a square of a holomorphic differential.

H.Masur and J.Smillie showed in [MS] that any singularity data $\mathbf{d}$ satisfying conditions above can be realized by a meromorphic quadratic differential with the following four exceptions:

$$\mathbf{d} \neq \quad (\ ), \quad (-1, 1), \quad (1, 3), \quad (4)$$

$\mathcal{Q}_\mathbf{d}$ is a Hausdorf complex analytic (and algebraic) space (orbifold) (see [V3]).

A meromorphic quadratic differential q on a complex curve C determines a two-sheet cover (or a ramified two-sheet cover) $\pi : \tilde{C} \to C$ such that $\pi^* q$ becomes a square of a holomorphic differential on $\tilde{C}$. Genus $\tilde{g}$ of the minimal cover and the singularity data $\tilde{\mathbf{d}}$ of the quadratic differential $p^* q$ is the same for all quadratic differentials from $\mathcal{Q}_\mathbf{d}$. By *effective genus* we will call the difference $g_{eff} = \tilde{g} - g$.

Conjecture . *For any moduli space $\mathcal{Q}_\mathbf{d}$ of meromorphic quadratic differentials on $\mathbb{CP}^1$ the sum of the Lyapunov exponents $\lambda_1 + \cdots + \lambda_{g_{eff}}$ equals*

$$\lambda_1 + \cdots + \lambda_{g_{eff}} = 1 + \sum_{i=1}^n \frac{\left[\frac{d_i+1}{2}\right]^2}{d_i + 2} = \frac{1}{4} \cdot \sum_{\substack{j \text{ such that} \\ d_j \text{ is odd}}} \frac{1}{d_j + 2}$$

By definition of the hyperelliptic components $\mathcal{M}^{\mathcal{H}}_{(2g-2)}$ and $\mathcal{M}^{\mathcal{H}}_{(g-1,g-1)}$ of the moduli spaces $\mathcal{M}_{(2g-2)}$ and $\mathcal{M}_{(g-1,g-1)}$ there are canonical isomorphisms:

$$\mathcal{M}^{\mathcal{H}}_{(2g-2)} = \mathcal{Q}_{(\underbrace{-1,\ldots,-1}_{2g+1},2g-3)} \qquad \mathcal{M}^{\mathcal{H}}_{(g-1,g-1)} = \mathcal{Q}_{(\underbrace{-1,\ldots,-1}_{2g+2},2g-2)}$$

Thus Conjecture above gives hypothetical value for the sum of the Lyapunov exponents for all hyperelliptic components of the moduli spaces of holomorphic differentials:

$$\lambda_1 + \cdots + \lambda_g \stackrel{?}{=} \frac{g^2}{2g - 1} \quad \text{for} \quad \mathcal{M}^{\mathcal{H}}_{(2g-2)}$$

$$\lambda_1 + \cdots + \lambda_g \stackrel{?}{=} \frac{g + 1}{2} \quad \text{for} \quad \mathcal{M}^{\mathcal{H}}_{(g-1,g-1)}$$

As we already mention above, the second Lyapunov exponent λ_2 is responsible for the deviation from the average for interval exchange transformations and for

related dynamical systems. Below we present the approximate values for λ_i for small genera. One can see in particular that λ_2 ranges considerably already for these few moduli spaces. Still we wish to believe in the following conjecture.

Conjecture . *For the hyperelliptic components* $\mathcal{M}^{\mathcal{H}}_{(2g-2)}$ *and* $\mathcal{M}^{\mathcal{H}}_{(g-1,g-1)}$

$$\lim_{g\to\infty} \lambda_2 = 1$$

For all other components and other moduli spaces of holomorphic differentials

$$\lim_{g\to\infty} \lambda_2 = \frac{1}{2}$$

APPENDIX A. APPROXIMATE VALUES OF THE LYAPUNOV EXPONENTS FOR SMALL GENERA

In our experiments we used averaging over several pseudotrajectories of the *fast Rauzy induction* — the discrete analog of the Teichmüller geodesic flow. The maximal length of trajectories available to us ($\sim 10^{10}$ iterations) presumably allows to compute the numbers λ_i with approximately 5 digits of precision. This precision is still insufficient to determine corresponding rational numbers when their denominator is about $\sim 10^3$ or more.

TABLE 1. Genus $g = 3$

Types of zeros d	Hyperelliptic or spin structure	Lyapunov exponents		
		λ_2	λ_3	$\sum\limits_{j=1}^{g} \lambda_j$
(4)	*hyperelliptic*	0.6156	0.1844	9/5
(4)	*odd*	0.4179	0.1821	8/5
(1,3)	−	0.5202	0.2298	7/4
(2,2)	*hyperelliptic*	0.6883	0.3117	4/2
(2,2)	*odd*	0.4218	0.2449	5/3
(1,1,2)	−	0.5397	0.2936	11/6
(1,1,1,1)	−	0.5517	0.3411	53/28

TABLE 2. Genus $g = 4$

Types of zeros **d**	Hyperelliptic or spin structure	Lyapunov exponents			
		λ_2	λ_3	λ_4	$\sum\limits_{j=1}^{g} \lambda_j$
(6)	*hyperelliptic*	0.7375	0.4284	0.1198	16/7
(6)	*even*	0.5965	0.2924	0.1107	14/7
(6)	*odd*	0.4733	0.2755	0.1084	13/7
(1, 5)	—	0.5459	0.3246	0.1297	2
(2, 4)	*even*	0.6310	0.3496	0.1527	32/15
(2, 4)	*odd*	0.4789	0.3134	0.1412	29/15
(3, 3)	*hyperelliptic*	0.7726	0.5182	0.2097	5/2
(3, 3)	—	0.5380	0.3124	0.1500	2
(1, 2, 3)	—	0.5558	0.3557	0.1718	25/12
(1, 1, 4)	—	0.55419	0.35858	0.15450	?
(2, 2, 2)	*even*	0.6420	0.3785	0.1928	166/75
(2, 2, 2)	*odd*	0.4826	0.3423	0.1749	2
(1, 1, 1, 3)	—	0.5600	0.3843	0.1849	66/31
(1, 1, 2, 2)	—	0.5604	0.3809	0.1982	?
(1, 1, 1, 1, 2)	—	0.5632	0.4032	0.2168	?
(1, 1, 1, 1, 1, 1)	—	0.5652	0.4198	0.2403	?

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