HOMOLOGICAL MIRROR SYMMETRY AND TORUS FIBRATIONS

Maxim KONTSEVICH and Yan SOIBELMAN

Institut des Hautes Études Scientifiques
35, route de Chartres
91440 – Bures-sur-Yvette (France)

Janvier 2001

IHES/M/01/01

Homological mirror symmetry and torus fibrations

Maxim Kontsevich and Yan Soibelman

November 24, 2000

1 Introduction

1.1 Homological mirror symmetry and degenerations

Mathematically mirror symmetry can be interpreted in many ways. In this paper we will make a bridge between two approaches, the homological mirror symmetry ([Ko]) and the duality between torus fibrations (a version of Strominger-Yau-Zaslow conjecture, see [SYZ]).

The mirror symmetry is a duality between Calabi-Yau manifolds, i.e. complex manifolds with Kähler metrics with vanishing Ricci curvarure. In fact, it is rather duality not between individual manifolds, but between manifolds in certain "degenerating" families ("large complex structure limit" and "large symplectic structure limit"). In this paper we propose some conjectures of differential-geometric nature about the degenerations. In particular, we assume that in the limit both dual manifolds X and $X^\vee$ become fiber bundles with toroidal fibers over the *same* base Y (see Section 3). The manifold Y is a (real) Riemannian manifold whose dimension is half of the dimension of X and $X^\vee$. Also, manifold Y carries a rich geometric structure which contains as a part a "combinatorial" data, so-called integral affine structure. This picture is partially motivated by the classical theory of collapsing of general Riemannian manifolds (see [CG]). Another origin of our geometric conjectures is the [SYZ] version of mirror duality. We recast it in somewhat different terms in Section 2, devoted to the moduli space of conformal field theories and its natural compactification. In a recent preprint [GW] similar differential-geometric conjectures were proposed and verified in the case of degenerating K3-surfaces.

The homological mirror symmetry conjecture (proposed in [Ko]) is a statement about an equivalence between two A_∞-categories: the (derived) category of coherent sheaves on a Calabi-Yau manifold X and the Fukaya category of the dual Calabi-Yau manifold $X^\vee$. The former is defined in holomorphic (or algebraic) terms, the latter is defined in terms of symplectic geometry.

We apply the geometric picture of limits of Calabi-Yau metrics to the homological mirror conjecture. The Fukaya category $F(X^\vee, \omega^\vee)$ of a symplectic manifold $(X^\vee, \omega^\vee)$, with $[\omega^\vee] \in H^2(X^\vee, \mathbf{Z})$, and its degeneration are defined as A_∞-categories over the field of Laurent formal power series $\mathbf{C}((q))$. The parameter q enters in the story when one writes higher compositions, which have expressions $q^{\int_\beta \omega^\vee}$ as coefficients, $\beta \in H_2(X^\vee, \mathbf{Z})$. We can set $q = exp(-1/\varepsilon), \varepsilon \to 0$, where the parameter ε corresponds to the rescaling of the symplectic form: $\omega^\vee \mapsto \omega^\vee/\varepsilon$. If $[\omega^\vee]$ does not belong to $H^2(X, \mathbf{Z})$, one can work over the field $\mathbf{C}_\varepsilon := \{\sum_{i\geq 0} a_i e^{-\lambda_i/\varepsilon} | \, a_i \in \mathbf{C}, \lambda_i \in \mathbf{R}, \lambda_i \to +\infty\}$.

In the picture of torus fibration, a full subcategory of the limiting Fukaya category can be described in terms of the Morse theory on the base of the torus fibration. The higher products giving the A_∞-structure can be written as sums over sets of planar trees. In the case of cotangent bundle (instead of the torus filtration) this description was proposed earlier by Fukaya and Oh (see [FO]).

On the holomorphic side of mirror symmetry, the degeneration of the dual family X_q is described in non-archimedean terms: we have a Calabi-Yau manifold $\mathcal{X}_{mer}$ over the field $\mathbf{C}_q^{mer}$ of germs at $q = 0$ of meromorphic functions. Changing scalars, we get a Calabi-Yau manifold $\mathcal{X}_{form}$ over the local field of Laurent series $\mathbf{C}((q))$. Let us call this degeneration picture *analytic*. There is a description of some algebraic Calabi-Yau manifolds over arbitrary local fields (complete with respect to discrete valuations) in terms of real C^∞-manifolds with integral affine structures. We expect that differential-geometric and (non-archimedean) analytic pictures of the degeneration are equivalent. In this paper we discuss the relationship between integral affine manifolds and varieties over non-archimedean fields only in the simplest case of flat tori and abelian varieties. The general case will be described elsewhere.

The homological mirror conjecture says that the Fukaya category $F(X^\vee, \omega^\vee)$ is equivalent (as an A_∞-category over $\mathbf{C}((q))$) to the derived category of coherent sheaves $D^b(\mathcal{X}_{form})$. Apparently, it implies well-known numerical predictions for the numbers of rational curves on a Calabi-Yau manifold (genus zero Gromov-Witten invariants).

Using the assumptions about the collapse, we offer in the paper a general approach to the proof of Homological Mirror Conjecture and apply it in the case when the torus fibration has no singularities. This happens in the case of abelian varieties. Also, we deal not with all objects in the Fukaya category, but with a certain subclass. In general, one should investigate the input of singularities of the base of torus fibration.

It should be clear from the above discussion that the non-archimedean analysis plays an important role in the formulation and proof of the main result. Analytic picture of the degeneration seems to be related to the theory of rigid analytic spaces in the version of Berkovich (see [Be]). In particular, there is a striking similarity between the base of torus fibration and a certain canonically defined subset (see 3.3) of the skeleton of an analytic space introduced in [Be]. This subject definitely deserves further investigation.

1.2 Content of the paper

In Section 2 we discuss motivations from the Conformal Field Theory. In Section 3 we formulate the conjectures about analytic and geometric pictures of the large complex structure limit. In Section 4 we describe a general framework of A_∞-pre-categories adapted to the transversality problem in the definition of the Fukaya category. Section 5 is devoted to the Fukaya category and its degeneration. The reader will notice an advantage of working over the field of Laurent power series: one can consider *all* local systems over Lagrangian submanifolds, while in the conventional approach unitarity of the holonomy is required. Section 6 is devoted to the A_∞-category of smooth functions introduced by Fukaya (and then studied by Fukaya and Oh in [FuO]). We prove that this A_∞-category has very simple de Rham model. This part of the paper can be read independently of the rest. On the other hand, the technique of the proof will be used later in the paper. One important technical tool is an explicit A_∞-structure on a subcomplex of a differential-graded algebra (see [GS], [Me]). We restate the formulas from [Me] in term of sums over a set of planar trees. The proof of the equivalence of Morse and de Rham A_∞-categories uses the technique of [HL]. Section 7 is devoted to the analytic side of the homological mirror conjecture. We give a construction of mirror symmetry functor for torus fibrations in terms of the non-archimedean geometry. The use of non-archimedean analysis allows us to avoid problems with convergence of series in the definition of the Fukaya category. In Section 8 we construct an A_∞-pre-category which is equivalent to a full A_∞-subcategory of the derived category coherent sheaves on the Calabi-Yau manifold over $\mathbf{C}_\varepsilon$. Similarly to the comparison of Morse and de Rham pictures, we will prove that this category is also equivalent to an A_∞-subcategory of the Fukaya category of the mirror dual torus fibration. In Appendix (Section 9) we describe the analogs of our constructions in the case of complex geometry.

2 Degenerations of unitary Conformal Field Theories

In this section we will explain physical motivations for our picture of mirror symmetry. We assume that the reader is familiar to some extent with the basic notions of Conformal Field Theory. For example, the lectures [Gaw] contain most of what we need.

Unitary Conformal Field Theory (abbreviated by CFT below) is well-defined mathematically. It is described by the following data:

1) A real number $c \geq 0$ called central charge.

2) A bi-graded pre-Hilbert *space of states* $H = \oplus_{p,q \in \mathbf{R}_{\geq 0}} H^{p,q}, p - q \in \mathbf{Z}$ such that $dim(\oplus_{p+q \leq E} H^{p,q})$ is finite for every $E \in \mathbf{R}_{\geq 0}$. Equivalently, there is an action of the Lie group $\mathbf{C}^*$ on H, so that $z \in \mathbf{C}^*$ acts on $H^{p,q}$ as $z^p \bar{z}^q := (z\bar{z})^p \bar{z}^{q-p}$.

3) An action of the product of Virasoro and anti-Virasoro Lie algebras $Vir \times \overline{Vir}$ (with the same central charge c) on H, so that the space $H^{p,q}$ is an eigenspace of the generator L_0 (resp. $\overline{L}_0$) with the eigenvalue p (resp. q).

4) The space H carries some additional structures derived from the operator product expansion (OPE). The OPE is described by a linear map $H \otimes H \to H \widehat{\otimes} \mathbf{C}\{z, \bar{z}\}$. Here $\mathbf{C}\{z, \bar{z}\}$ is the topological ring of formal power series $f = \sum_{p,q} c_{p,q} z^p \bar{z}^q$ where $c_{p,q} \in \mathbf{C}$, $p, q \to +\infty$, $p, q \in \mathbf{R}$, $p - q \in \mathbf{Z}$. The OPE satisfies a list axioms, which we are not going to recall here (see [Gaw]).

Let $\phi \in H^{p,q}$. Then the number $p + q$ is called the *conformal dimension* of ϕ (or the *energy*), and $p - q$ is called the *spin* of ϕ. Notice that, since the spin of ϕ is an integer number, the condition $p + q < 1$ implies $p = q$.

The central charge c can be described by the formula $dim(\oplus_{p+q \leq E} H^{p,q}) =$

$exp(\sqrt{4/3\pi^2 cE(1+o(1))})$ as $E \to +\infty$. It is expected that all possible central charges form a countable well-ordered subset of $\mathbf{Q}_{\geq 0} \subset \mathbf{R}_{\geq 0}$. If $H^{0,0}$ is a one-dimensional vector space, the corresponding CFT is called irreducible. A general CFT is a sum of irreducible ones. The *trivial* CFT has $H = H^{0,0} = \mathbf{C}$ and it is the unique irreducible unitary CFT with $c = 0$.

Remark 1 *Geometric considerations of this paper are related to $N = 2$ Superconformal Field Theories (SCFT). There is a version of the above data and axioms for SCFT. In particular, each $H^{p,q}$ is a hermitian super vector space. There is an action of the super extension of the product of Virasoro and anti-Virasoro algebra on H. In the discussion of the moduli spaces below we will not distinguish between CFTs and SCFTs, because except of some minor details, main conclusions are true in both cases.*

2.1 Moduli space of Conformal Field Theories

For a given CFT one can consider its group of symmetries (i.e. automorphisms of the space $H = \oplus_{p,q} H^{p,q}$ preserving all the structures). It is expected that the group of symmetries is a compact Lie group of dimension less or equal than $dim\, H^{1,0}$.

Let us fix $c_0 \geq 0$ and $E_{min} > 0$, and consider the moduli space $\mathcal{M}_{c\leq c_0}^{E_{min}}$ of all irreducible CFTs with the central charge $c \leq c_0$ and

$$min\{p+q>0 | H^{p,q} \neq 0\} \geq E_{min}$$

It is expected that $\mathcal{M}_{c\leq c_0}^{E_{min}}$ is a compact real analytic stack of finite local dimension. The dimension of the base of the minimal versal deformation of a given CFT is less or equal than $dim\, H^{1,1}$. We define $\mathcal{M}_{c\leq c_0} = \cup_{E_{min}>0} \mathcal{M}_{c\leq c_0}^{E_{min}}$. We would like to compactify this stack by adding boundary components corresponding to certain asymptotic descriptions of the theories with $E_{min} \to 0$. The compactified space is expected to be a compact stack $\overline{\mathcal{M}}_{c\leq c_0}$. In what follows we will loosely use the word "space" instead of the word "stack".

Remark 2 *There are basically only two classes of rigorously defined CFTs: the rational theories (RCFT) and the lattice CFTs. Considerations of this paper correspond to the case of sigma models which produce neither of these. The description of sigma models as path integrals corresponding to certain Lagrangians did not give yet a mathematically satisfactory construction. As we will explain below, there is an alternative way to speak about sigma models in terms of degenerations of CFTs.*

2.2 Physical picture of a simple collapse

In order to compactify $\mathcal{M}_{c\leq c_0}$ we consider degenerations of CFTs as $E_{min} \to 0$. A degeneration is given by a one-parameter (discrete or continuous) family $H_\varepsilon, \varepsilon \to 0$ of bi-graded spaces as above, where $(p,q) = (p(\varepsilon), q(\varepsilon))$. These spaces are equipped with OPEs. The subspace of fields with conformal dimensions vanishing as $\varepsilon \to 0$ gives rise to a commutative algebra $H^{small} = \oplus_{p(\varepsilon)\ll 1} H_\varepsilon^{p(\varepsilon),p(\varepsilon)}$ (the algebra structure is given by the leading terms in OPEs). The spectrum X of H^{small} is expected to be a compact space ("manifold with singularities") such that $dim\, X \leq c_0$. It follows from the conformal invariance and the OPE, that the grading of H^{small} (rescaled as $\varepsilon \to 0$) is given by the eigenvalues of a second order differential operator defined on the smooth part of X. The operator has positive eigenvalues and is determined up to multiplication by a scalar. This implies that the smooth part of X carries a metric g_X, which is also defined up to multiplication by a scalar. Other terms in OPEs give rise to additional differential-geometric structures on X.

Thus, as a first approximation to the real picture, we assume the following description of a "simple collapse" of a family of CFTs. The degeneration of the family is described by the point of the boundary of $\overline{\mathcal{M}}_{c\leq c_0}$ which is a triple $(X, \mathbf{R}_+^* \cdot g_X, \phi_X)$, where the metric g_X is defined up to a positive scalar factor, and $\phi_X : X \to \mathcal{M}_{c\leq c_0 - dim\, X}$ is a map. One can have some extra conditions on the data. For example, the metric g_X can satisfy the Einstein equation.

Although the scalar factor for the metric is arbitrary, one should imagine that the curvature of g_X is "small", and the injectivity radius of g_X is "large". The map ϕ_X appears naturally from the point of view of the simple collapse of CFTs described above. Indeed, in the limit $\varepsilon \to 0$, the space H_ε becomes an H^{small}-module. It can be thought of as a space of sections of an infinite-dimensional vector bundle $W \to X$. One can argue that fibers of W generically are spaces of states of CFTs with central charges less or equal than $c_0 - dim\, X$. This is encoded in the map ϕ_X. In the case when CFTs from $\phi_X(X)$ have non-trivial symmetry groups, one expects a kind of a gauge theory on X as well.

Purely bosonic sigma-models correspond the case when $c_0 = c(\varepsilon) = dim\, X$ and the residual theories (CFTs in the image of ϕ_X) are all trivial. The target space X in this case should carry a Ricci flat metric. In the supersymmetric case the target space X is a Calabi-Yau manifold, and the residual bundle of CFTs is a bundle of free fermion theories.

Remark 3 *We expect that all compact Ricci flat manifolds (with the metric defined up to a constant scalar factor) appear as target spaces of degenerating CFTs. Thus, the construction of the compactification of the moduli space of CFTs should include as a part a compactification of the moduli spaces of Einstein manifolds. Notice that in differential geometry there is a fundamental result of Gromov (see [G]) about the precompactness of the moduli space of pointed connected complete Riemannian manifolds of a given dimension, with the Ricci curvature bounded from below. One can speculate about the relationship between the compactification of the moduli space of CFTs and the Gromov's compactification. For example, is it true that all target spaces appearing as limits of CFTs have non-negative Ricci curvature?*

2.3 Multiple collapse and the structure of the boundary

In terms of the Virasoro operator L_0 the collapse is described by a subset (cluster) S_1 in the set of eigenvalues of L_0 which approach to zero "with the same speed", as $E_{min} \to 0$. The next level of the collapse is described by another subset S_2 of eigenvalues of L_0. Elements of S_2 approach to zero "modulo the first collapse" (i.e. at the same speed, but "much slower" than elements of S_1). One can continue to build a tower of degenerations. It leads to an hierarchy of boundary strata. Namely, if there are further degenerations of CFTs parametrized by X, one gets a fiber bundle over the space of triples $(X, \mathbf{R}_+^* \cdot g_X, \phi_X)$ with the fiber which is the space of triples of similar sort. Finally, we obtain the following qualitative geometric picture of the boundary $\partial\overline{\mathcal{M}}_{c\leq c_0}$.

A boundary point is given by the following data:

1) A finite tower of maps of compact topological spaces $p_i : \overline{X}_i \to \overline{X}_{i-1}, 0 \leq i \leq k$, $\overline{X}_0 = \{pt\}$.

2) A sequence of smooth manifolds $(X_i, g_{X_i}), 0 \leq i \leq k$, such that X_i is a dense subspace of $\overline{X}_i$, and $dim\, X_i > dim\, X_{i-1}$, and p_i defines a fiber bundle $p_i : X_i \to X_{i-1}$.

3) Riemannian metrics on the fibers of the restrictions of p_i to X_i, such that the diameter of each fiber is finite. In particular the diameter of X_1 is finite, because it is the only fiber of the map $p_1 : X_1 \to \{pt\}$.

4) A map $X_k \to \mathcal{M}_{c\leq c_0 - dim\, X_k}$.

The data above are considered up to the natural action of the group $(\mathbf{R}_+^*)^k$ (it rescales the metrics on fibers).

There are some additional data, like non-linear connections on the bundles $p_i : X_i \to X_{i-1}$. The set of data should satisfy some conditions, like differential equations on the metrics. We cannot formulate this portion of data more precisely in general case. It will be done below in the case of $N=2$ SCFTs corresponding to sigma models with Calabi-Yau target spaces.

2.4 Example: Toroidal models

Non-supersymmetric toroidal model is described by the so-called Narain lattice, endowed with some additional data. More precisely, let us fix the central charge $c=n$ which is a positive integer number. What physicists call the Narain lattice $\Gamma^{n,n}$ is a unique unimodular lattice of rank $2n$ and the signature (n,n). It can be described as $\mathbf{Z}^{2n}$ equipped with the quadratic form $Q(x_1,...,x_n,y_1,...,y_n)=\sum_i x_i y_i$. The moduli space of toroidal CFTs is

$$\mathcal{M}_{c=n}^{tor} = O(n,n,\mathbf{Z})\backslash O(n,n,\mathbf{R})/O(n,\mathbf{R})\times O(n,\mathbf{R}).$$

Equivalently, it is a quotient of the open part of the Grassmannian $\{V_+ \subset \mathbf{R}^{n,n} | \, dim\, V_+ = n, Q_{|V}>0\}$ by the action of $O(n,n,\mathbf{Z}) = Aut(\Gamma^{n,n},Q)$. Let V_- be the orthogonal complement to V_+. Then every vector of $\Gamma^{n,n}$ can be uniquely written as $\gamma=\gamma_+ + \gamma_-$, where $\gamma_\pm \in V_\pm$. For the corresponding CFT one has

$$\sum_{p,q} dim(H^{p,q}) z^p \bar{z}^q = \left|\prod_{k\geq 1}(1-z^k)\right|^{-2n} \sum_{\gamma\in\Gamma^{n,n}} z^{Q(\gamma_+)}\bar{z}^{-Q(\gamma_-)}$$

Let us try to compactify the moduli space $\mathcal{M}_{c=n}^{tor}$. Suppose that we have a one-parameter family of toroidal theories such that $E_{min}(\varepsilon)$ approaches zero. Then for corresponding vectors in H_ε one gets $p(\varepsilon)=q(\varepsilon)\to 0$. It implies that $Q(\gamma(\varepsilon))=0, Q(\gamma_+(\varepsilon))\ll 1$. It is easy to see that one can add vectors $\gamma(\varepsilon)$ satisfying these conditions. Thus one gets a (part of) lattice of the rank less or equal than n. In the case of "maximal" simple collapse the rank will be equal to n. One can see that the corresponding points of the boundary give rise to the following data: $(X, \mathbf{R}_+^* \cdot g_X, \phi_X^{triv}; B)$, where (X,g_X) is a flat n-dimensional torus, $B\in H^2(X,\mathbf{R}/\mathbf{Z})$ and ϕ_X^{triv} is the constant map form X to the trivial theory point in the moduli space of CFTs. These data in turn give rise to a toroidal CFT, which can be realized as a sigma model with the target space (X, g_X) and given B-field B. The residual bundle of CFTs on X is trivial.

Let us consider a 1-parameter family of CFTs defined by the family $(X, \lambda g_X, \phi_X^{triv}; B = 0)$, where $\lambda \in (0, +\infty)$. There are two degenerations of this family, which define two points of the boundary $\partial\overline{\mathcal{M}}^{tor}_{c=n}$. As $\lambda \to +\infty$, we get a toroidal CFT defined by $(X, \mathbf{R}_+^* \cdot g_X, \phi_X^{triv}; B = 0)$. As $\lambda \to 0$ we get $(X^\vee, \mathbf{R}_+^* \cdot g_{X^\vee}, \phi_X^{triv}; B = 0)$, where $(X^\vee, g_{X^\vee})$ is the dual flat torus.

There might be further degenerations of the lattice. Thus one obtains a stratification of the compactified moduli space of lattices (and hence CFTs). Points of the compactification are described by flags of vector spaces $0 = V_0 \subset V_1 \subset V_2 \subset ... \subset V_k \subset \mathbf{R}^n$. In addition one has a lattice $\Gamma_{i+1} \subset V_{i+1}/V_i$, considered up to a scalar factor. These data give rise to a tower of torus bundles $X_k \to X_{k-1} \to ... \to X_1 \to \{pt\}$ over tori with fibers $(V_{i+1}/V_i)/\Gamma_{i+1}$. If $V_k \simeq \mathbf{R}^{n-l}, l \geq 1$, then one has also a map from the total space X_k of the last torus bundle to the point $[H_k]$ in the moduli space of toroidal theories of smaller central charge: $\phi_n : X_k \to \mathcal{M}^{tor}_{c=l}$, $\phi_k(X_k) = [H_k]$.

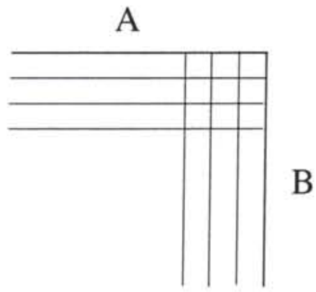
2.5 Example: WZW model for $SU(2)$

In this case we have a discrete family with $c = \frac{3k}{k+2}$, where $k \geq 1$ is an integer number called *level*. In the limit $k \to +\infty$ one gets $X = SU(2) = S^3$ equipped with the standard metric. The corresponding bundle is the trivial bundle of trivial CFTs (with $c = 0$ and $H = H^{0,0} = \mathbf{C}$). Analogous picture holds for an arbitrary compact simply connected simple group G.

2.6 A-model and B-model of $N = 2$ SCFT as boundary strata

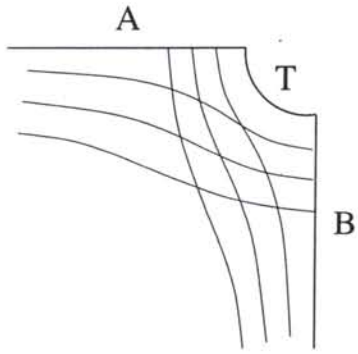
The boundary of the compactified moduli space $\overline{\mathcal{M}}^{N=2}$ of $N = 2$ SCFTs with a given central charge contains an open stratum given by sigma models with Calabi-Yau targets. Each stratum is parametrized by the classes of equivalence of quadruples $(X, J_X, \mathbf{R}_+^* \cdot g_X, B)$ where X is a compact real manifold, J_X a complex structure, g_X is a Calabi-Yau metric, and $B \in H^2(X, i\mathbf{R}/\mathbf{Z})$ is a B-field. The residual bundle of CFTs is a bundle of free fermion theories.

As a consequence of supersymmetry, the moduli space $\mathcal{M}^{N=2}$ of superconformal field theories is a complex manifold which is locally isomorphic to the product of two complex manifolds. [1] It is believed that this decomposition (up to certain corrections) is global. Also, there are two types of sigma models with Calabi-Yau targets: A-models and B-models. Hence, the traditional picture of the compactified moduli space looks as follows:



Here the boundary consists of two open strata (A-stratum and B-stratum) and a mysterious meeting point. This point corresponds, in general, to a submanifold of codimension one in the closure of A-stratum and of B-stratum.

We argue that this picture should be modified. There is another open stratum of $\partial\overline{\mathcal{M}}^{N=2}$ (we call it T-stratum). It consists of toroidal models (i.e. CFTs associated with Narain lattices), parametrized by a manifold Y with a Riemannian metric defined up to a scalar factor. This subvariety meets both A and B strata along the codimension one stratum corresponding to the double collapse. Therefore the "true" picture is obtained from the traditional one by the real blow-up at the corner:



---

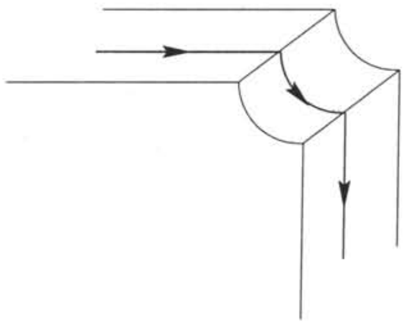
[1] Strictly speaking, one should exclude models with chiral fields of conformal dimension $(2,0)$, e.g. sigma models on hyperkähler manifolds, see [AM].

2.7 Mirror symmetry and the collapse

Mirror symmetry is related to the existence of two different strata of the boundary $\partial\overline{\mathcal{M}}^{N=2}$ which we called A-stratum and B-stratum. As a corollary, same quantities admit different geometric descriptions near different strata. In the traditional picture, one can introduce natural coordinates in a small neighborhood of a boundary point corresponding to $(X, J_X, \mathbf{R}_+^* \cdot g_X, B)$. Skipping X from the notation, one can say that the coordinates are (J, g, B) (complex structure, Calabi-Yau metric and the B-field). Geometrically, the pairs (g, B) belong to the preimage of the Kähler cone under the natural map $Re : H^2(X, \mathbf{C}) \to H^2(X, \mathbf{R})$ (more precisely, one should consider B as an element of $H^2(X, i\mathbf{R}/\mathbf{Z})$). It is usually said, that one considers an open domain in the complexified Kähler cone with the property that with the class of metric $[g]$ it contains also the ray $t[g], t \gg 1$. The mirror symmetry gives rise to an identification of neighborhoods of $(X, J_X, \mathbf{R}_+^* \cdot g_X, B_X)$ and $(X^\vee, J_{X^\vee}, \mathbf{R}_+^* \cdot g_{X^\vee}, B_{X^\vee})$ such that J_X is interchanged with $[g_{X^\vee}] + iB_{X^\vee})$ and vice versa.

We can describe this picture in a different way. Using the identification of complex and Kähler moduli, one can choose $([g_X], B_X, [g_{X^\vee}], B_{X^\vee})$ as local coordinates near the meeting point of A-stratum and B-stratum. There is an action of the additive semigroup $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$ in this neighborhood. It is given explicitly by the formula $([g_X], B_X, [g_{X^\vee}], B_{X^\vee}) \mapsto (e^{t_1}[g_X], B_X, e^{t_2}[g_{X^\vee}], B_{X^\vee})$ where $(t_1, t_2) \in \mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$. As $t_1 \to +\infty$, a point of the moduli space approaches the B-stratum, where the metric is defined up to a positive scalar only. The action of the second semigroup $\mathbf{R}_{\geq 0}$ extends by continuity to the non-trivial action on the B-stratum. Similarly, in the limit $t_2 \to +\infty$ the flow retracts the point to the A-stratum.

This picture should be modified, if one makes a real blow-up at the corner, as discussed before. Again, the action of the semigroup $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$ extends continuously to the boundary. Contractions to A-stratum and B-stratum carry non-trivial actions of the corresponding semigroups isomorphic to $\mathbf{R}_{\geq 0}$. Now, let us choose a point in, say, A-stratum. Then the semigroup flow takes it along the boundary to the new stratum, corresponding to the double collapse. The semigroup $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$ acts trivially on this stratum. A point of the double collapse is also a limiting point of a 1-dimensional orbit of $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$ acting on the T-stratum. Explicitly, the element (t_1, t_2) changes the size of the tori defined by the Narain lattices, rescaling them with the coefficient $e^{t_1 - t_2}$. This flow carries the point of T-stratum to another point of the double collapse, which can be moved then inside of the B-stratum. The whole path, which is the intersection of $\partial\overline{\mathcal{M}}^{N=2}$ and the $\mathbf{R}_{\geq 0} \times \mathbf{R}_{\geq 0}$-orbit, connects an A-model with the corresponding B-model through the stratum of toroidal models. We can depict it as follows:



The T-portion of the path (we call it T-path) connects dual torus fibrations over the same Riemannian base. This is mirror symmetry in our picture.

This description is inspired by [SYZ]. The reader notices however, that in our picture, the mirror symmetry phenomenon is explained entirely in terms of the *boundary* of the compactified moduli space. In order to explain the mirror symmetry phenomenon it is not necessary to build full SCFTs. It is sufficient to work with simple toroidal models on the boundary of the compactified moduli space $\overline{\mathcal{M}}^{N=2}$. Also, in contrast with [SYZ], we do not use supersymmetric cycles (D-branes) in our description.

3 Calabi-Yau manifolds in the large complex structure limit

3.1 Maximal degenerations of Calabi-Yau manifolds

Let $\mathbf{C}_q^{mer} = \{f = \sum_{n \geq n_0} a_n q^n\}$ be the field of germs at $q = 0$ of meromorphic functions in one variable.

Let $\mathcal{X}_{mer}$ be an algebraic n-dimensional Calabi-Yau manifold over $\mathbf{C}_q^{mer}$ (i.e. $\mathcal{X}_{mer}$ is a smooth projective manifold over $\mathbf{C}_q^{mer}$ with the trivial canonical class: $K_{\mathcal{X}} = 0$). We fix an algebraic non-vanishing volume element $vol \in \Gamma(\mathcal{X}_{mer}, K_{\mathcal{X}})$. The pair $(\mathcal{X}_{mer}, vol)$ defines a 1-parameter analytic family of complex Calabi-Yau manifolds $(X_q, vol_q), 0 < |q| < r_0$, for some $r_0 > 0$.

Let $[\omega] \in H^2_{DR}(\mathcal{X}_{mer})$ be the cohomology class in the ample cone. Then for every q, such that $0 < |q| < r_0$ it defines a Kähler class ω_q on X_q. By the Yau theorem, there exists a unique Calabi-Yau metric g_{X_q} on X_q with the Kähler class $[\omega_q]$.

It follows from the resolution of singularities, that as $q \to 0$ one has the following formula:

$$\int_{X_q} vol_q \wedge \overline{vol_q} = C(log|q|)^m |q|^k (1 + o(1))$$

for some $C \in \mathbf{C}^*, k \in \mathbf{Z}, 0 \le m \le n = dim\,(\mathcal{X}_{mer})$.

Definition 1 *We say that $\mathcal{X}_{mer}$ has maximal degeneration at $q = 0$ if in the formula above we have $m = n$.*

One can show easily that this definition is equivalent to the usual one, given in terms of variations of Hodge structures (see [Mo], [LTY]).

Lemma 1 *$\mathcal{X}_{mer}$ has maximal degeneration iff there exists a vector $v \in H^n(X_q, \mathbf{C})$ such that $(T - Id)^{n+1}v = 0$ and $(T - id)^n v \neq 0$ where T is the monodromy operator.*

In fact, the vector v in the lemma can be chosen to be proportional to the cohomology class of vol_q for any given q. Notice that in [De] a slightly stronger condition was imposed: the weight filtration on $H^*(X_q, \mathbf{C})$ associated with the monodromy operator should be complementary to the Hodge filtration.

Let us recall the definition of the Gromov-Hausdorff metric ρ_{GH}. It is a metric on the space of isometry classes of metric spaces of finite diameter. We say that two metric spaces M_1 and M_2 are ε-close in ρ_{GH} if there exists a metric space M containing both M_1 and M_2 as metric subspaces, such that M_1 belongs to the ε-neighborhood of M_2 and vice versa.

Let us rescale the Calabi-Yau metric: $g^{new}_{X_q} = g_{X_q}/diam(X_q, g_{X_q})^{1/2}$. Thus we obtain a 1-parameter family of Riemannian manifolds $X^{new}_q = (X_q, g^{new}_{X_q})$ of the diameter 1.

Conjecture 1 *If $\mathcal{X}_{mer}$ has maximal degeneration at $q = 0$ then there is a limit $(\overline{Y}, g_{\overline{Y}})$ of X^{new}_q in the Gromov-Hausdorff metric, such that:*

a) $(\overline{Y}, g_{\overline{Y}})$ *is a compact metric space, which contains a smooth oriented Riemannian manifold* (Y, g_Y) *of dimension* n *as a dense open metric subspace. The Hausdorff dimension of* $Y^{sing} = \overline{Y} \setminus Y$ *is less or equal than* $n-2$.

b) Y *carries an integral affine structure. This means that it carries a torsion-free flat connection* ∇ *with the holonomy contained in* $SL(n, \mathbf{Z})$.

c) The metric g_Y *has a potential. This means that it is locally given in affine coordinates by a symmetric matrix* $(g_{ij}) = (\partial^2 K/\partial x_i \partial x_j)$, *where* K *is a smooth function (defined modulo adding an affine function, i.e. the sum of a linear function and a constant).*

d) In affine coordinates the metric volume element is constant, $det(g_{ij}) = det(\partial^2 K/\partial x_i \partial x_j) = const$ *(real Monge-Ampère equation).*

At the end of this section we propose a non-rigorous explanation of our conjecture based on differential-geometric considerations.

Remark 4 *1) Since the matrix* (g_{ij}) *defined by the metric* g_Y *is positive, the function* K *is convex. In particular, there is locally well-defined Legendre transform of* K. *This fact will be used later, when we will discuss the duality of Monge-Ampère manifolds.*

2) It seems plausible that in the case when all X_q *are simply-connected, and* $h^{k,0}(X_q) = 0$ *for* $0 < k < n$, *the metric space* $\overline{Y}$ *is a homological sphere of dimension* n. *In all examples it is in fact homeomorphic to* S^n.

The conjecture opens the way for compactification of the moduli space of Calabi-Yau metrics on a given Calabi-Yau manifold M, by adding as a boundary component the set of pairs $(\overline{Y}, \mathbf{R}_+^* \cdot g_{\overline{Y}})$ for all 1-parameter maximal degenerations X_q, such that $X_{q'} = M$ for some q'. This corresponds to a choice of a "cusp" in the moduli space of Calabi-Yau manifolds. This choice is usually described in terms of certain algebro-geometric data: the action of the monodromy operator, variation of Hodge structures, mixed Hodge structure of the special fiber, etc. The previous conjecture offers a pure "metric" description of a cusp.

It follows from part b) of the conjecture that one can choose a ∇-covariant lattice $T_{Y,y}^{\mathbf{Z}} \subset T_{Y,y}, y \in Y$. Suppose we are given a triple (Y, g_Y, ∇), satisfying the properties a)-c) of the conjecture, and we have fixed a covariant lattice $T_Y^{\mathbf{Z}}$ in the tangent bundle T_Y. Then we can construct a 1-parameter family of *non-compact* complex Calabi-Yau manifolds, endowed with Ricci flat Kähler metrics. Namely, let X^ε be the total space of the torus bundle $p_\varepsilon : X^\varepsilon \to$

Y with fibers $T_{Y,y}/\varepsilon T^{\mathbf{Z}}_{Y,y}, y \in Y, 0 < \varepsilon \leq \varepsilon_0$. The total space TY of the tangent bundle T_Y carries a canonical complex structure coming from the isomorphism $T_{TY} \simeq \pi^* T_Y \oplus \pi^* T_Y \simeq \pi^* T_Y \otimes \mathbf{C}$ where $\pi : TY \longrightarrow Y$ is the canonical projection (here we use the affine structure on Y). Using the same identification, we introduce a metric on TY, namely $g_{TY} = \pi^* g_Y \oplus \pi^* g_Y$. It is easy to see, that g_{TY} is a Kähler metric with the potential $\pi^* K$. It follows from the Monge-Ampère equation that the metric g_{TY} is Ricci flat. Passing to the quotient, we obtain on X^ε a complex structure J_{X^ε} and a Ricci flat Kähler metric g_{X^ε}.

Let $U \subset Y$ be an open simply-connected subset. Then there is an action of the torus $T^n \simeq T_{Y,y}/T^{\mathbf{Z}}_{Y,y}$ on $p_\varepsilon^{-1}(U), y \in U$ (different tori are identified for different points $y \in U$ by means of the connection ∇). It implies that for any $t \in H^1(Y, (T_Y/T_Y^{\mathbf{Z}})^{discr})$ (cohomology with coefficients in the local system of tori considered as abstract groups) one can define a twisted manifold $X^{\varepsilon,t}$, which is the total space of the torus fibration $p_{\varepsilon,t} : X^{\varepsilon,t} \to Y$.

Roughly speaking, the next conjecture says that the "leading asymptotic term" of the family of Calabi-Yau manifolds $X_q^{new}, q = e^{-1/\varepsilon}$ near the point of maximal degeneration $\varepsilon = 0$, is isomorphic up to a twist to the family $(X^\varepsilon, J_{X^\varepsilon})$ associated with the torus bundle described above.

More precisely, we formulate it as follows.

Conjecture 2 *Let $(\mathcal{X}_{mer}, vol) = (X_q, vol_q)$ be a 1-parameter family of maximally degenerate Calabi-Yau manifolds, and X_q^{new} be the family with rescaled metrics, as before. There exist a constant $C > 0$ and a function $t(q)$ such that Kähler manifolds X_q^{new} and $X^{\varepsilon(q),t(q)}$ with $\varepsilon(q) = C(log|q|)^{-1}$ are close to each other (as $q \to 0$) in the following sense:*

for any $\delta > 0$ there exist a decomposition $X_q = X_q^{sm} \sqcup X_q^{sing}$ and an embedding of smooth manifolds $j_q : X_q \to p^{-1}_{\varepsilon(q),t(q)}(Y \setminus (Y^{sing})^\delta)$, where $(Y^{sing})^\delta$ is a δ-neighborhood of Y^{sing}, such that:

a) (X_q, X_q^{sing}) converges in the Gromov-Hausdorff metric to the pair $(\overline{Y}, Y^{sing})$.

b) j_q identifies up to $o(1)$ terms, uniformly in $x \in X_q^{sm}$, the scalar products and complex structures on the tangent spaces $T_x X_q$ and $T_{j_q(x)} X^{\varepsilon(q),t(q)}$.

There is the following motivation for the Conjectures 1 and 2. In general, for a degenerating family of Riemannian metrics with non-negative Ricci curvature, one expects a description in terms of a tower of fibrations (collapses) with singularities (compare with 2.3). [2] In the case of Kähler manifolds there are two basic pictures of a simple collapse. The first case is when both the base and the fiber are Kähler manifolds. In the second case fibers are flat totally real tori of dimension m and the base looks locally as a product of a domain in $\mathbf{R}^m$ with a Kähler manifold. The logarithmic factor in the asymptotic behavior of the volume should come only from torus fibers. Thus, the largest possible power of the logarithm can appear only when we have a tower of purely torus fibrations. It seems that the fixing (up to a scalar) of the Kähler class forbids the multiple collapse. These considerations give an intuitive "explanation" of our conjectures.

Remark 5 *During the preparation of this text we learned that conjectures similar to ours were proposed independently by M. Gross and P. Wilson (see [GW]). A remarkable achievement in [GW] consists of the verification of conjectures in the case of degenerating $K3$ surfaces, together with a precise description of the behavior of metrics near singular fibers. Also, in a recent preprint [Le] mirror symmetry was discussed from a similar point of view. In the main body of the present paper we will consider degenerations of complex abelian varieties. In this case the conjectures obviously hold.*

3.2 Monge-Ampère manifolds and duality of torus fibrations

In this section we propose a mathematical language for the geometric mirror symmetry, understood as a duality of torus fibrations.

Definition 2 *A Monge-Ampère manifold is a triple (Y, g, ∇), where (Y, g) is a smooth Riemannian manifold with the metric g, and ∇ is a flat connection on T_Y such that:*

a) ∇ defines an affine structure on Y.

b) Locally in affine coordinates $(x_1, ..., x_n)$ the matrix $((g_{ij}))$ of g is given by $((g_{ij})) = ((\partial^2 K/\partial x_i \partial x_j))$ for some smooth real-valued function K.

c) The Monge-Ampère equation $det((\partial^2 K/\partial x_i \partial x_j)) = const$ is satisfied.

Monge-Ampère manifolds were studied (under a different name) in [CY] where is was proven that if Y is *compact* then its finite cover is a torus.

[2] Some steps in the program of compactification of the space of metrics are accomplished now (see e.g. [CC]), but still there are many non-clarified issues.

Let us consider a (non-compact) example motivated by the mirror symmetry for K3 surfaces (see also [GW]). Let S be a complex surface endowed with a holomorphic non-vanishing volume form vol_S, and $\pi : S \to C$ be a holomorphic fibration over a complex curve C, such that fibers of π are non-singular elliptic curves.

We define a metric g_C on C as the Kähler metric associated with the $(1,1)$-form $\pi_*(vol_S \wedge \overline{vol_S})$. Let us choose (locally on C) a basis (γ_1, γ_2) in $H_1(\pi^{-1}(x), \mathbf{Z}), x \in C$. We define two closed 1-forms on C by the formulas

$$\alpha_i = Re(\int_{\gamma_i} vol_S), i = 1, 2.$$

It follows that $\alpha_i = dx_i$ for some functions $x_i, i = 1, 2$. We define an affine structure on C, and the corresponding connection ∇, by saying that (x_1, x_2) are affine coordinates. One can check directly that (C, g_C, ∇) is a Monge-Ampère manifold. In a typical example of elliptic fibration of a K3 surface, one gets $C = \mathbf{CP}^1 \setminus \{z_1, ..., z_{24}\}$, where $\{z_1, ..., z_{24}\}$ is a set of distinct 24 points in $\mathbf{CP}^1$.

Returning to the general case, we can restate a portion of our conjectures by saying that the smooth part of the Gromov-Hausdorff limit of a maximally degenerate family of Calabi-Yau manifolds is a Monge-Ampère manifold with an integral affine structure.

There is a well-known duality on local solutions of the Monge-Ampère equation.

Lemma 2 *Let $U \subset \mathbf{R}^n$ be a convex open domain in $\mathbf{R}^n$ equipped with the standard affine coordinates $(x_1, ..., x_n)$, and $K : U \to \mathbf{R}$ be a convex function satisfying the Monge-Ampère equation. Then the Legendre transform $\widehat{K}(y_1, ..., y_n) = max_{x \in U}(\sum_i x_i y_i - K(x_1, ..., x_n))$ also satisfies the Monge-Ampère equation.*

Proof. The graph of $L = dK$ is a Lagrangian submanifold in $T^*\mathbf{R}^n = \mathbf{R}^n \oplus (\mathbf{R}^n)^*$. Let p_1 and p_2 be the natural projections to the direct summands. They are local diffeomorphisms. Since K is defined up to the adding of an affine function, the graph itself is defined up to translations. The Monge-Ampère equation corresponds to the condition $p_1^*(vol_{\mathbf{R}^n}) = p_2^*(vol_{\mathbf{R}^{*n}})$, where $vol_{\mathbf{R}^n}$ (resp. $vol_{\mathbf{R}^{*n}}$) denotes the standard volume form on $\mathbf{R}^n$ (resp. $\mathbf{R}^{*n}$). The manifold L can be considered as a graph of $d\widehat{K}$. Thus $\widehat{K}$ satisfies the Monge-Ampère equation as well. The Lemma is proved. ■

The manifold L carries a Riemannian metric g_L induced by the indefinite metric $\sum_i dx_i\, dy_i$ on $\mathbf{R}^n \oplus (\mathbf{R}^n)^*$. This metric is given by the matrix $(\partial^2 K/\partial x_i \partial x_j)$ in coordinates $(x_1, ..., x_n)$, and by the matrix $(\partial^2 \widehat{K}/\partial y_i \partial y_j)$ in the dual coordinates. Thus on L we have a metric, and two affine structures (pullbacks of the standard affine structures on the coordinate spaces). Hence we have two structures of the Monge-Ampère manifold on L. It is easy to see that the local pictures can be glued together. This leads to the following result.

Proposition 1 *For a given Monge-Ampère manifold (Y, g_Y, ∇_Y) there is a canonically defined dual Monge-Ampère manifold $(Y^\vee, g_Y^\vee, \nabla_Y^\vee)$ such that (Y, g_Y) is identified with $(Y^\vee, g_Y^\vee)$ as Riemannian manifolds, and the local system $(T_{Y^\vee}, \nabla_Y^\vee)$ is naturally isomorphic to the local system dual to (T_Y, ∇_Y).*

Corollary 1 *If ∇_Y defines an integral affine structure on Y (i.e. the holonomy of ∇_Y belongs to $GL(n, \mathbf{Z})$), then $\nabla_Y^\vee$ defines an integral affine structure on $Y^\vee$. As the dual covariant lattice one takes the lattice $(T_Y^{\mathbf{Z}})^\vee$, which is dual to $T_Y^{\mathbf{Z}}$ with respect to the metric g_Y.*

Now we can state the geometric counterpart of the mirror symmetry conjecture.

Conjecture 3 *Smooth parts of maximal degenerations of dual families of Calabi-Yau manifolds are dual Monge-Ampère manifolds with dual integral affine structures.*

Monge-Ampère manifolds with integral affine structures are real analogs of Calabi-Yau manifolds. In fact the mirror duality in the sense of this section holds for a larger class of manifolds. We define an *AK-manifold* (AK stands for *affine* and *Kähler*) as in the Definition 2, but dropping the condition c) (Monge-Ampère equation), see also [CY]. The reader can check easily that all constructions of this section, including the duality of torus fibrations hold for AK-manifolds as well.

Remark 6 *The idea to use the Legendre transform for the purposes of mirror symmetry was around for some time (see for example [H], [Le]).*

Remark 7 *In our description of geometric mirror symmetry we ignore the B-fields. In what follows we will always assume that $B = 0$.*

3.3 Speculations about relations with non-archimedean geometry

Considerations from CFT and from differential geometry indicate that the integral affine structure on Y does not depend on the choice of the Kähler class of Calabi-Yau metrics. Thus, we obtain a "combinatorial" invariant $(Y, T_Y^{\mathbf{Z}})$ of (maximally degenerating) Calabi-Yau variety over the local field $K = \mathbf{C}((q))$. One can argue that in this case there will be a canonical atlas of coordinate charts such that the transition maps belong to the group $SAff(n, \mathbf{Z}) := SL(n, \mathbf{Z}) \ltimes \mathbf{Z}^n$. The natural question arises whether one can define and calculate it purely algebraically, without the use of transcendental methods and Calabi-Yau metrics. We expect that the answer to this question is positive. In other words there exists a canonical way to associate the data $(Y, T_Y^{\mathbf{Z}})$ with arbitrary smooth projective variety X, $c_1(T_X) = 0$ having "maximal degeneration" over an arbitrary field K with a discrete valuation.

The conjectural answer (only for the compactification $\overline{Y}$ of Y) is the following: let us choose (after an extension of the field K) a model with stable reduction. Call an irreducible component D of the special fiber X_0 *essential* if the order of pole at D of the global volume element on X is maximal among all components of X_0. We define topological space $\overline{Y}(X_0)$ as the Clemens complex spanned by essential divisors (see [LTY]). Roughly speaking, k-cells of $\overline{Y}$ correspond to irreducible components of $(k+1)$-fold intersections of essential divisors. Recently one of us (M.K.) proved, using ideas from motivic integration and from Berkovich theory of non-archimedean analytic spaces (see [Be]), that for different choices of models with stable reduction spaces $\overline{Y}(X_0)$ can be canonically identified . In examples coming from toric geometry the space $\overline{Y} = \overline{Y}(X_0)$ is always a manifold.

It is not clear yet what is the origin of the smooth part $Y \subset \overline{Y}$, and of the affine structure on it. Conjecturally, all this comes from a map $\pi : X(\overline{K}) \to \overline{Y}$ where $\overline{K}$ is the algebraic closure of K. In the differential-geometric picture of torus fibrations (when $K = \mathbf{C}_{mer}$) the map π is obvious: it associates with a meromorphic (finitely ramified) family of points $x_q \in X_q$ the limit point $lim_{q \to 0} x_q \in Y$ in the metric sense. Also, the differential-geometric picture suggests that the closure of the image $\pi(Z(\overline{K}))$ where $Z \subset K$ is an algebraic subvariety, should be a piecewise linear closed subset of Y, and linear pieces of it have rational directions. In particular, if Z is a curve then $\pi(Z(\overline{K}))$ is a graph in Y. This opens a way to express Gromov-Witten invariants of X in terms of the Feynman expansion for certain quantum field theory on Y.

Also, we expect that the choice of an ample class in $NS(X)\otimes\mathbf{R}$ on X gives rise to the *dual* integral affine structure on Y defined again in some purely algebro-geometric way. If the ample class is the first Chern class of a line bundle, then there should be also a canonical reduction of the dual integral affine structure to a $SAff(n,\mathbf{Z})$ -structure.

4 A_∞-algebras and A_∞-categories

4.1 Two problems with the general definition

The purpose of this section is to describe the framework in which the results concerning A_∞-categories will be formulated. We would like to make few comments even before recalling a definition of the Fukaya category. There are two main problems with the definition. First, morphisms can be defined only for transversal Lagrangian submanifolds (in particular, the identity morphism is never defined). Second, since there are pseudo-holomorphic discs with the boundary on a given Lagrangian submanifold, one has to add a composition m_0 to the set of compositions $m_n, n\geq 1$. As a result, the spaces of morphisms are not complexes: $m_1^2\neq 0$. On the other hand, the derived category of coherent sheaves arises from an A_∞-category without m_0 and with the condition $m_1^2=0$. Hence one should explain in which sense two A_∞-categories in question are equivalent.

The above-mentioned problems can be resolved by an appropriate generalization of the notion of A_∞-category. This generalization involves numerous preparations and will be given elsewhere (see [KoS]). On the other hand, the problem with m_0 does not appear in the case of abelian varieties, which is the main application of the approach offered in this paper. Hence, for the purposes of present paper it is sufficient to work with A_∞-pre-categories (or A_∞-categories with transversal structure, cf. [P1]). This gives a partial solution to the transversality problem, and provides a solution to the problem with the identity morphisms.

Using A_∞-pre-categories we formulate and prove a variant of the homological mirror symmetry conjecture. It can be applied to the case of abelian varieties. In particular, one can obtain certain formulas for Massey products for abelian varieties in terms of partial theta-sums similar to those considered in [P1].

4.2 Non-unital A_∞-algebras and A_∞-categories

Let $A = \oplus_{i \in Z} A^i$ be a **Z**-graded module over a **Z**-graded commutative associative algebra k. As usual, we will denote by $A[n]$ the graded k-module such that $(A[n])^i = A^{i+n}$ for all i.

Definition 3 *A structure of non-unital A_∞-algebra on A is given by a codifferential d of degree +1 on the cofree tensor coalgebra $T_+(A[1]) = \oplus_{n \geq 1}(A[1])^{\otimes n}$.*

The codifferential d is by definition a coderivation, such that $d^2 = 0$. It is uniquely determined by its "Taylor coefficients" $m_n : A^{\otimes n} \to A[2-n], n \geq 1$. The condition $d^2 = 0$ can be rewritten as a sequence of quadratic equations

$$\sum_{i+j=n+1} \sum_{0 \leq l \leq i} \epsilon(l,j) m_i(a_0, ..., a_{l-1}, m_j(a_l, ..., a_{l+j-1}), a_{l+j}, ..., a_n) = 0$$

where $a_m \in A$, and $\epsilon(l,j) = (-1)^{j \sum_{0 \leq s \leq l-1} deg(a_s)}$. In particular, $m_1^2 = 0$.

Definition 4 *A morphism of non-unital A_∞-algebras (A_∞-morphism for short) $(V, d_V) \to (W, d_W)$ is a morphism of tensor coalgebras $T_+(V[1]) \to T_+(W[1])$ of degree zero, which commutes with the codifferentials.*

A morphism f of non-unital A_∞-algebras is determined by its "Taylor coefficients" $f_n : V^{\otimes n} \to W[1-n], n \geq 1$ satisfying the system of equations
$\sum_{1 \leq l_1 < ... < l_i = n} \pm m_i^W(f_{l_1}(a_1, ..., a_{l_1}),$
$f_{l_2 - l_1}(a_{l_1+1}, ..., a_{l_2}), ..., f_{n-l_{i-1}}(a_{n-l_{i-1}+1}, ..., a_n)) =$
$\sum_{s+r=n+1} \sum_{1 \leq j \leq s} \pm f_s(a_1, ..., a_{j-1}, m_r^V(a_j, ..., a_{j+r-1}), a_{j+r}, ..., a_n)$.
We leave to the reader as an exercise to write down the formulas for the signs in terms of degrees of a_i and f_i.

Definition 5 *A non-unital A_∞-category $\mathcal{C}$ over k is given by the following data:*

1) A class of objects $Ob(\mathcal{C})$.

*2) For any two objects X_1 and X_2 a **Z**-graded k-module of morphisms $Hom(X_1, X_2)$.*

3) For any sequence of objects $X_0, ..., X_n$, $n \geq 1$, a morphism of k-modules (called a composition map) $m_n : \otimes_{0 \leq i \leq n-1} Hom(X_i, X_{i+1}) \to Hom(X_0, X_n)[2-n]$.

It is required that for any sequence of objects $X_0, ..., X_N$, $N \geq 0$ the graded k-module $A = A(X_0, ..., X_N) := \oplus_{i,j} Hom(X_i, X_j)$, equipped with the direct sum of the compositions $m_n, n \geq 1$, is a non-unital A_∞-algebra.

The class of objects $Ob(\mathcal{C})$ will be often denoted by $\mathcal{C}$. We hope it will not lead to a confusion.

Remark 8 *A non-unital A_∞-algebra A can be considered as a non-unital A_∞-category with one object X such that $Hom(X,X) = A$.*

Definition 6 *A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ between non-unital A_∞-categories is given by the following data:*

1) A map of classes of objects $\phi : \mathcal{C}_1 \to \mathcal{C}_2$.

2) For any finite sequence of objects $X_0, ..., X_n$, $n \geq 0$, a morphism of graded k-modules $f_n : \otimes_{0 \leq i \leq n-1} Hom_{\mathcal{C}_1}(X_i, X_{i+1}) \to Hom_{\mathcal{C}_2}(\phi(X_0), \phi(X_n))[1-n]$.

The following condition holds for any $X_1, ..., X_N \in \mathcal{C}_1$: the sequence $f_n, n \geq 1$ defines an A_∞-morphism

$$\oplus_{i,j} Hom_{\mathcal{C}_1}(X_i, X_j) \to \oplus_{i,j} Hom_{\mathcal{C}_2}(\phi(X_i), \phi(X_j)).$$

Remark 9 *Let $\mathcal{C}$ be a non-unital A_∞-category. Let us replace spaces of morphisms by their cohomology with respect to m_1. In other words, we define $Hom_{H(\mathcal{C})}(X,Y) := \{Ker\, m_1\}/\{Im\, m_1\}$, where $m_1 : Hom_{\mathcal{C}}(X,Y) \to Hom_{\mathcal{C}}(X,Y)[1]$ is the composition map. Then $H(\mathcal{C}) = (\mathcal{C}, Hom_{H(\mathcal{C})}(\cdot,\cdot))$ gives rise to a "non-unital" category structure with the class of objects $\mathcal{C}$ and composition of morphisms induced by m_2. We write "non-unital" because there are no identity morphisms $id_X \in Hom_{H(\mathcal{C})}(X,X)$.*

4.3 A_∞-pre-categories

We start with the notion of *non-unital* A_∞-pre-category. It allows us to work with "transversal" sequences of objects. [3] Then we will introduce the notion of A_∞-pre-category. It provides us with a replacement of the identity morphisms. Roughly speaking, we will have the identity morphism up to homotopy.

Definition 7 *Let k be a $\mathbf{Z}$-graded commutative associative ring as before. A non-unital A_∞-pre-category over k is defined by the following data:*

a) A class of objects $\mathcal{C}$.

[3] The notion of "transversality" is purely formal in this section. The choice of the name will become clear after concrete applications in the geometric context, see next sections.

b) For any $n \geq 1$ a subclass $\mathcal{C}_n^{tr}$ of $\mathcal{C}^n$, $\mathcal{C}_1^{tr} = \mathcal{C}$, called the class of transversal sequences.

*c) For $(X_1, X_2) \in \mathcal{C}_2^{tr}$ a **Z**-graded k-module of morphisms $Hom(X_1, X_2)$.*

d) For a transversal sequence of objects $(X_0, ..., X_n)$, $n \geq 0$, a morphism of k-modules (composition map) $m_n : \otimes_{0 \leq i \leq n-1} Hom(X_i, X_{i+1}) \to Hom(X_0, X_n)[2-n]$.

It is required that a subsequence $(X_{i_1}, ..., X_{i_l}), i_1 < i_2 < ... < i_l$ of a transversal sequence $(X_1, ..., X_n)$ is transversal, and that the composition maps satisfy the same system of equations as for non-unital A_∞-categories. Explicitly:

$\sum_{i+j=n+1} \sum_{0 \leq l \leq i} \epsilon(l,j) m_i(a_0, ..., a_{l-1}, m_j(a_l, ..., a_{l+j}), a_{l+j+1}, ..., a_n) = 0$, *where $a_m \in Hom(X_m, X_{m+1})$, and $\epsilon(l,j) = (-1)^{j \sum_{0 \leq s \leq l-1} deg(a_s)}$.*

Definition 8 *A functor $F : \mathcal{C} \to \mathcal{D}$ between non-unital A_∞-pre-categories is given by the following data:*

1) A map of classes of objects $\phi : \mathcal{C} \to \mathcal{D}$, such that $\phi^n(\mathcal{C}_n^{tr}) \subset \mathcal{D}_n^{tr}$.

2) For any transversal sequence of objects $(X_0, ..., X_n)$, $n \geq 1$ in $\mathcal{C}$, a morphism of graded k-modules

$$f_n : \otimes_{0 \leq i \leq n-1} Hom_{\mathcal{C}}(X_i, X_{i+1}) \to Hom_{\mathcal{D}}(\phi(X_0), \phi(X_n))[1-n].$$

These data satisfy the following property: the sequence $f_n, n \geq 1$ defines an A_∞-morphism $\oplus_{i<j} Hom_{\mathcal{C}}(X_i, X_j) \to \oplus_{i<j} Hom_{\mathcal{D}}(\phi(X_i), \phi(X_j))$.

The reader have noticed that we use the summation only over the increasing pairs of indices $i < j$. It differs from the case of non-unital A_∞-pre-categories. The reason is that we do not require the transversality to be a symmetric relation on objects. It is possible that $Hom(X_0, X_1)$ exists, but $Hom(X_1, X_0)$ does not. In the case when all $Hom's$ are defined, two discussed definitions agree. In particular, a non-unital A_∞-category is the same as a non-unital A_∞-pre-category such that $\mathcal{C}_n^{tr} = \mathcal{C}^n$ for any $n \geq 1$.

Definition 9 *Let $\mathcal{C}$ be a non-unital A_∞-pre-category, $(X_1, X_2) \in \mathcal{C}_2^{tr}$. We say that $f \in Hom^0(X_1, X_2)$ (zero stands for degree) is a quasi-isomorphism if $m_1(f) = 0$, and for any objects X_0 and X_3 such that $(X_0, X_1, X_2) \in \mathcal{C}_3^{tr}$ and $(X_1, X_2, X_3) \in \mathcal{C}_3^{tr}$ one has: $m_2(f, \cdot) : Hom(X_0, X_1) \to Hom(X_0, X_2)$ and $m_2(\cdot, f) : Hom(X_2, X_3) \to Hom(X_1, X_3)$ are quasi-isomorphisms of complexes.*

Definition 10 *An A_∞-pre-category is a non-unital A_∞-pre-category $\mathcal{C}$, satisfying the following extension property:*

For any finite collection of transversal sequences $S_1, ..., S_m$ in $\mathcal{C}$ and an object X there exist objects X_+ and X_- and quasi-isomorphisms $f_- : X_- \to X$, $f_+ : X \to X_+$ such that extended sequences $(X_-, S_1, ..., S_m, X_+), 1 \leq i \leq m$ are transversal.

Remark 10 *Let $\mathcal{C}$ be an A_∞-pre-category. Then partially defined on $H(\mathcal{C}) = (\mathcal{C}, Hom_{H(\mathcal{C})}(\cdot,\cdot))$ composition m_2 extends uniquely, so that it defines a structure of a category on $H(\mathcal{C})$.*

Definition 11 *Let $\mathcal{C}$ and $\mathcal{D}$ be A_∞-pre-categories over k. An A_∞-functor $F : \mathcal{C} \to \mathcal{D}$ is a functor between the corresponding non-unital A_∞-pre-categories such that F takes quasi-isomorphisms in $\mathcal{C}$ to quasi-isomorphisms in $\mathcal{D}$.*

There is an important notion of equivalence of A_∞-pre-categories (and A_∞-categories). We are planning to provide all the details elsewhere (see [KoS]). For the purposes of present paper we will be using the following definition (which is in fact a theorem in the more general framework).

Definition 12 *An A_∞-functor $F : \mathcal{C} \to \mathcal{D}$ between A_∞-pre-categories is called an A_∞-equivalence functor if:*

a) Every object $Y \in \mathcal{D}$ is quasi-isomorphic to an object $\phi(X), X \in \mathcal{C}$.

b) The functor induces quasi-isomorphisms of non-unital A_∞-algebras of morphisms, corresponding to all transversal sequences of objects.

Definition 13 *Two A-pre-categories $\mathcal{C}$ and $\mathcal{D}$ are called equivalent if there exists a finite sequence of A_∞-pre-categories $(\mathcal{C}_0, \ldots, \mathcal{C}_n)$, $\mathcal{C}_0 = \mathcal{C}$, $\mathcal{C}_0 = \mathcal{D}$ such that for every i, $0 \leq i \leq k-1$ there exists an A_∞-equivalence functor from $\mathcal{C}_i$ to $\mathcal{C}_{i+1}$ or vice versa.*

We suggest the language of A_∞-pre-categories in order to replace more conventional A_∞-categories with strict identity morphisms.

Definition 14 *An A_∞-category with strict identity morphisms is a non-unital A_∞-category $\mathcal{C}$, such that for any object X there exists an element $1 = 1_X \in Hom^0(X, X)$ (identity morphism) such that $m_2(1, f) = m_2(f, 1) = f$ and $m_n(f_1, ..., 1, ..., f_n) = 0, n \neq 2$ for any morphisms $f, f_1, ..., f_n$.*

An A_∞-category $\mathcal{C}$ with strict identity morphisms is an A_∞-pre-category, because (in the previous notation) we can extend a transversal sequence S to (X, S, X), and set $X_+ = X_- = X$, $f_\pm = 1_X$. Another remark is that if $\mathcal{C}$ has only one object, it is an A_∞-algebra with the strict unit. One can try to develop the deformation theory of such algebras along the lines of [KoS1]. The problem is that the corresponding operad is not free, and the standard theory becomes complicated. We hope that the framework of A_∞-pre-categories is appropriate for the purposes of deformation theory of A_∞-categories. The following conjecture gives another evidence in favor of such a generalization of A_∞-categories.

Conjecture 4 *Let us define the notion of equivalent A_∞-categories with strict identity morphisms) similarly to the case of A_∞-pre-categories (see above). Then the equivalence classes of A_∞-pre-categories are in one-to-one correspondence with the equivalence classes of A_∞-categories with strict identity morphisms.*

4.4 Example: directed A_∞-pre-categories

There is a useful special case of the notion of A_∞-pre-category (independently a similar notion was suggested in [Se]).

Definition 15 *A directed A_∞-pre-category is an A_∞-pre-category such that*

a) There is bijection of the class of objects and the set integer numbers: $\mathcal{C} \simeq \mathbf{Z}$. We denote by X_i the object corresponding to $i \in \mathbf{Z}$.

b) Transversal sequences are $(X_{i_1}, ..., X_{i_n}), i_1 < i_2 < ... < i_n$.

The extension property is equivalent to the following one: for any object X_i there are exist objects $X_j, j < i$ and $X_m, m > i$ which are quasi-isomorphic to X. Then one can formulate the following version of the previous conjecture.

Conjecture 5 *Equivalence classes of directed A_∞-pre-categories are in one-to-one correspondence with the equivalences classes of A_∞-categories with strict identity morphisms and countable class of objects.*

Having an A_∞-category $\mathcal{C}$ with strict identity morphisms, and countable class of objects, one can construct an infinite sequence of objects $(X_i)_{i \in \mathbf{Z}}$ such that each objects appears infinitely many times for positive and negative i. Then a directed A_∞-pre-category $\mathcal{C}'$ is defined by setting $Hom_{\mathcal{C}'}(X_i, X_j) = Hom_{\mathcal{C}}(X_i, X_j)$ for $i < j$. All other $Hom's$ are not defined.

5 Fukaya category and its degeneration

5.1 Fukaya category

Fukaya category (of a compact symplectic manifold) in the approach presented here will be in fact an A_∞-pre-category. Our definition is not given in the maximal generality, but it will be sufficient for the main application to abelian varieties. For more elaborated definitions see [Fu1], [Ko].

Let (V,ω) be a compact symplectic manifold of dimension $2n$, such that $c_1(T_V) = 0 \in H^2(V,\mathbf{Z})$. The Fukaya category (with the trivial B-field) associated with (V,ω) depends on some additional data, which we are going to describe below.

We fix an almost complex structure J compatible with ω and a smooth everywhere non-vanishing differential form Ω, which is $(n,0)$-form with respect to J. Let L be an oriented Lagrangian submanifold. Then one has a map $Arg_L := Arg_{\Omega_{|L}} : L \to \mathbf{R}/2\pi\mathbf{Z}$, where $Arg_{\Omega_{|L}}(x)$ is the argument of the non-zero complex number $\Omega(e_1 \wedge ... \wedge e_n)$, and $e_1, ..., e_n$ is an oriented basis of $T_xL, x \in L$.

Definition 16 *Objects of the Fukaya category $F(V,\omega,J,\Omega)$ are triples $(L,\rho,\widetilde{Arg}_L)$, where L is a compact oriented Lagrangian submanifold of V (called the support of the object), ρ is a local system on L (i.e. a complex vector bundle with flat connection), and $\widetilde{Arg}_L : L \to \mathbf{R}$ a continuous lift of Arg_L.*

We require that for any element $\beta \in \pi_2^{free}(V,L) := \pi_0(Maps((D^2,\partial D^2),(V,L))$, the pairing $([\omega],\beta)$ is equal to zero.

We will sometimes denote the Fukaya category by $F(V,\omega)$, or simply by $F(V)$. We will also often omit from the notation the lifted argument function, thus denoting an object simply by (L,ρ).

Let $\mathbf{C}_\varepsilon$ be the field consisting of formal series $f = \sum_{i\geq 0} c_i e^{-\lambda_i/\varepsilon}$, such that $c_i \in \mathbf{C}, \lambda_i \in \mathbf{R}, \lambda_0 < \lambda_1 < ..., \lambda_i \to +\infty$. In the case when $[\omega] \in H^2(V,\mathbf{Z})$, one can in fact work over the field $\mathbf{C}((q))$, where $q = exp(-\frac{1}{\varepsilon})$. In general we equip $\mathbf{C}_\varepsilon$ with the adic topology: a fundamental system of neighborhoods of zero consists of sets $U_x = \{f = \sum_{i\geq 0} c_i e^{-\lambda_i/\varepsilon} | \lambda_i \geq x, i \geq 0\}, x \in \mathbf{R}$.

Definition 17 *For two objects with transversal supports we define the space of morphisms such as follows*

$$Hom_{F(V,\omega)}((L_1,\rho_1,\widetilde{Arg}_1),(L_2,\rho_2,\widetilde{Arg}_2)) := (\oplus_{x\in L_1\cap L_2} Hom(\rho_{1x},\rho_{2x}))\otimes \mathbf{C}_\varepsilon.$$

Thus morphisms form a finite-dimensional vector space over the field $\mathbf{C}_\varepsilon$. There is a $\mathbf{Z}$-grading of the space of morphisms given in terms of Maslov index $deg : L_1 \cap L_2 \to \mathbf{Z}$ (see [Fu2], [Ko], [Se]).

Remark 11 *The condition $([\omega],\beta) = 0$ is introduced for convenience only. It helps to avoid the problem with the composition m_0 we mentioned before. The condition holds in the case when V is a torus with the constant symplectic form, and L is a Lagrangian subtorus. This is our main application in present paper. In general there is a way to work with non-trivial m_0, if it is small in the adic topology.*

Now we are going to describe the A_∞-structure. It is defined by means of a collection of maps (higher compositions) of graded vector spaces $m_k^{F(V)}$: $\otimes_{0\le i\le k-1} Hom_{F(V)}((L_i,\rho_i),(L_{i+1},\rho_{i+1})) \to Hom_{F(V)}((L_0,\rho_0),(L_k,\rho_k))[2-k]$, where $k \ge 1$ and the sequence $(L_0,...,L_k)$ corresponds to a transversal sequence of objects (the latter notion will be defined below).

In the case, when all local systems are trivial of rank one, the map m_k is defined such as follows. Let D be a standard disc $D = \{z \in \mathbf{C} |\, |z| \le 1\}$. Let us fix a sequence $(L_0,...,L_k)$ of supports of objects with pairwise transversal intersections, intersection points $x_i \in L_i \cap L_{i+1}, 0 \le i \le k-1$, $x_k \in L_0 \cap L_k$, and $\beta \in \pi_2^{free}(V, \cup_{0\le i\le k} L_i)$. We denote by $\mathcal{M}(L_0,...,L_k;x_0,...,x_k;\beta)$ the set of collections $(y_0,...,y_k;\psi)$, where $y_i, 0 \le i \le k$ are cyclically ordered pairwise distinct points on the boundary ∂D, and $\psi : D \to (V,J)$ a pseudo-holomorphic map such that $\psi(y_i) = x_i, \psi(\overline{y_iy_{i+1}}) \subset L_i, 0 \le i \le k, y_0 = y_k$, $[\phi] = \beta$. Here $\overline{y_iy_{i+1}}$ denotes the arc between y_i and y_{i+1}. There is a natural action of $PSL(2,\mathbf{R})$ on $\mathcal{M}(L_0,...,L_k;x_0,...,x_k;\beta)$ arising from the holomorphic action on D by fractional linear transformations. The action is free except of the case $k = 1, x_0 = x_1, \beta = 0$, which is not relevant for our purposes.

Let $x_i \in L_i \cap L_{i+1}, 0 \le i \le k-1, x_k \in L_0 \cap L_k$ satisfy the condition $deg\, x_k = \sum_{0\le i\le k-1} deg\, x_i + 2 - k$. Then the matrix element $(m_k(x_0,x_1,...,x_{k-1}),x_k)$ is given by the formula $(m_k(x_0,x_1,...,x_{k-1}),x_k) = \sum \pm q^{(\beta,[\omega])}$, where sum is taken over all $PSL(2,\mathbf{R})$-orbits of points in $\mathcal{M}(L_0,...,L_k;x_0,...,x_k;\beta)$. Signs are derived from orientations of certain cycles in the moduli space $\mathcal{M} =$

$\mathcal{M}(L_0, ..., L_k; x_0, ..., x_k; \beta)/PSL(2, \mathbf{R})$. We will comment on them below (see [Fu1], [Ko] for more details). In the case of non-trivial local systems there is an additional factor for each summand. It corresponds to the holonomies of local system along the arcs.

Now we will describe the transversality condition. Assume that we are given a sequence of objects $(L_i, \rho_i), 0 \leq i \leq k$ of the Fukaya category. We say that they are transversal if the following conditions hold:

1) There are only pairwise intersections $L_i \cap L_j$, and they are transversal.

2) For any subsequence $(L_{i_0}, ..., L_{i_m})$, $m \geq 1$, $i_0 < i_1 < ... < i_m$, any choice of intersection points $x_{i_m} \in L_{i_0} \cap L_{i_m}$, $x_{i_p} \in L_{i_p} \cap L_{i_{p+1}}, 0 \leq p \leq m-1$ such that $deg\, x_{i_m} - (\sum_{0 \leq i \leq m-1} deg\, x_{i_p} + 2 - m) = 0$, and any $\beta \in \pi_2^{free}(V, \cup_{0 \leq p \leq m} L_{i_p})$, the corresponding component of the moduli space $\mathcal{M}(L_{i_0}, ..., L_{i_m}; x_{i_0}, ..., x_{i_m}; \beta)/PSL(2, \mathbf{R})$ contains only smooth points, and is zero-dimensional.

3) If $deg\, x_{i_m} - (\sum_{0 \leq i \leq m-1} deg\, x_{i_p} + 2 - m) < 0$ then the corresponding component is empty.

Let us comment on these conditions. The first one is needed to define morphisms. The quotient set $\mathcal{M} = \mathcal{M}(L_{i_0}, ..., L_{i_m}; x_{i_0}, ..., x_{i_m}; \beta)/PSL(2, \mathbf{R})$ which appears in the second condition locally can be identified with the space of solutions of a non-linear elliptic problem. For the linearized problem the corresponding Fredholm operator has index $deg\, x_{i_m} - (\sum_{0 \leq i \leq m-1} deg\, x_{i_p} + 2 - m)$. We define smooth points $\mathcal{M}^{sm}$ of $\mathcal{M}$ as such points where the cokernel of the Fredholm operator is trivial. Then $\mathcal{M}^{sm}$ is a smooth manifold of the dimension equal to the index. Moreover, one checks that the spaces $\mathcal{M}^{sm}$ carry natural orientations given by the determinants of the corresponding Fredholm operators. It follows that in the zero-dimensional case what we get is a set of points with multiplicities ± 1 (in particular, the multiplicities are integer numbers). Multiple covers and stable maps which appear in the definition of Gromov-Witten invariants and produce non-trivial denominators, do not appear in our framework for the Fukaya category. Therefore one can define the Fukaya category over the ring $\mathbf{Z}_\varepsilon$ (the integral version of $\mathbf{C}_\varepsilon$). The number of points counted with signs gives a tensor coefficient of m_k.

Composition maps satisfy a system of quadratic equations, thus making $F(V, \omega)$ into a non-unital A_∞-pre-category. One can check that it is in fact an A_∞-pre-category. Proof of the extension property is based on the following result of Fukaya (see [Fu2], [Se]).

Proposition 2 *Let (L_t, ρ_t) be an object obtained by a small Hamiltonian* *deformation of an object (L,ρ) of $F(V)$. Then (L_t,ρ_t) and (L,ρ) are quasi-isomorphic.*

For example, a sequence consisting of one object (L,ρ) can be extended to a transversal sequence $((L_{t_1},\rho_{t_1}),(L,\rho))$. Similarly, one can extend any finite set of transversal sequences.

It is easy to see that the set of connected components of the space of pairs (J,Ω) (equipped with the natural topology) is a principal homogeneous space over the lattice $H^1(V,\mathbf{Z})$. Namely, $f: V \to U(1)$ acts on (J,Ω) such as follows: $(J,\Omega) \mapsto (J, f\Omega)$. The following theorem can be derived from [Fu2].

Theorem 1 *There exists a set Σ of the second category (in the sense of Baire) in the space of almost complex structures compatible with ω such that Fukaya categories $F(V,\omega,J_1,\Omega_1)$ and $F(V,\omega,J_2,\Omega_2)$ are equivalent as long as $J_1, J_2 \in \Sigma$, and (J_1,Ω_1) is homotopic to (J_2,Ω_2).*

Therefore the equivalence class of the Fukaya category depends on the connected component of the space of pairs.

5.2 Fukaya-Oh category for torus fibration

Let (Y, g_Y, ∇) be an AK-manifold with integral affine structure. The covariant lattice is denoted by $T_Y^{\mathbf{Z}}$, as before. From now on we will assume that Y is *compact*. This is a severe restriction. It was proven in [CY] that in this case a finite cover of space Y is a torus with the standard affine structure. It appears in the collapse of complex abelian varieties.

The manifold $X^\vee = T_Y^*/(T_Y^{\mathbf{Z}})^\vee$ is the total space of the torus bundle $p^\vee : X^\vee \to Y$. It carries a natural symplectic form $\omega = \omega_{X^\vee}$ induced from the standard one on T^*Y. We endow $X^\vee$ with a 1-parameter family of complex structures $J_\eta, \eta \to 0$ compatible with ω. Indeed, the manifold $X_\eta^\vee := T_Y^*/\eta(T_Y^{\mathbf{Z}})^\vee$ carries a canonical complex structure described before. We identify $X^\vee$ and $X_\eta^\vee$ by the map $(y,v) \mapsto (y, \eta v)$, where $y \in Y, v \in T_{Y,y}^*$. Using this identification, we pull back to $X^\vee$ the complex structure and the metric. The fibers of $p^\vee : X^\vee \to Y$ are flat Lagrangian tori for all values of η.

We define on $(X^\vee, J_\eta)$ a nowhere vanishing $(n,0)$-form Ω_η such as follows. Let us fix an oriented orthonormal basis $e_1, ..., e_n$ in $T_{Y,y}^*, y \in Y$. We define Ω_η as the n-form on $X^\vee$, which is invariant with respect to the $T_{Y,y}^*/(T_{Y,y}^{\mathbf{Z}})^\vee$-action, and is equal to $\bigwedge_{1 \le j \le n}((p^\vee)^* e_j + \sqrt{-1} J_\eta (p^\vee)^* e_j)$.

Let L be a compact oriented Lagrangian submanifold of $X^\vee$ such that $p^\vee_{|L}$ is an unramified covering, and the orientation of L is induced from the orientation of Y. We claim that there is a canonical choice $\widetilde{Arg}_L^{can} : L \to \mathbf{R}$ for the function $\widetilde{Arg}_L : L \to \mathbf{R}$. Indeed, for any point $x \in X^\vee$ the space of Lagrangian subspaces in $T_{X^\vee,x}$, which are transversal to the vertical tangent space $T_x^{vert} = Ker(p^\vee)_*$ is contractible. Let us consider the space $\mathcal{L}$ of pairs (x,l) such that $x \in X^\vee$ and $l \subset T_{X^\vee,x}$ is a Lagrangian subspace, which is transversal to T_x^{vert}, and endowed with the orientation induced from Y. Then the function $(x,l) \mapsto Arg_{(\Omega_\eta)_{|l}}(x) \in \mathbf{R}/2\pi\mathbf{Z}$ admits a unique continuous lifting $\widetilde{Arg} : \mathcal{L} \to \mathbf{R}$, vanishing at $(x, J_\eta(T_x^{vert})), x \in X^\vee$. Restricting this function to L we obtain $\widetilde{Arg}_L^{can}$.

We will denote by $F^\eta(X^\vee)$ the Fukaya category $F(X^\vee, \omega, J_\eta, \Omega_\eta)$, and by $F^\eta_{unram}(X^\vee)$ its full A_∞-pre-subcategory with objects $(L, \rho, \widetilde{Arg}_L^{can})$ such that L is a compact Lagrangian submanifold with the orientation induced from Y, $p^\vee_{|L}$ is an unramified covering, and $\widetilde{Arg}_L^{can}$ was described above. To simplify the notations we will denote objects of these categories by (L,ρ).

Remark 12 *One can check that for transversal Lagrangian submanifolds L_1 and L_2 as above, the Maslov index at any $x \in L_1 \cap L_2$ is equal to the Morse index at $p^\vee(x)$ of the smooth Morse function $f_1 - f_2 : Y \to \mathbf{R}$ such that locally near x one has $L_i = graph\,(df_i)\,(mod(T_Y^{\mathbf{Z}})^\vee), i = 1,2$.*

It follows from the results of [FuO] that there exists a limit of the family of A_∞-pre-categories $F^\eta_{unram}(X^\vee)$, $\eta \to 0$ in the following sense. Objects and morphisms of $F^\eta_{unram}(X^\vee)$ do not depend on η and remain the same in the limit. The compositions $m_k^{F^\eta_{unram}(X^\vee)}$ have limits as $\eta \to 0$ in the adic topology of $\mathbf{C}_\varepsilon$. They will be explicitly described below.

The following result can be derived from [FuO].

Proposition 3 *The limiting A_∞-pre-category is equivalent to $F^\eta_{unram}(X^\vee)$ for all sufficiently small η.*

We will denote this A_∞-pre-category by $FO(X^\vee)$ and call it the *Fukaya-Oh category* of $X^\vee$ (or *degenerate Fukaya category* of $X^\vee$).

Remark 13 *In what follows we will assume that $dim\,Y > 1$. The case $dim\,Y = 1$ is somewhat different, but also it is much more simple (see for example [P1]). In particular, $F^\eta_{unram}(X^\vee)$ does not depend on η in this case.*

As we said before, the objects and morphisms for $FO(X^\vee)$ are the same as for $F^{\eta}_{unram}(X^\vee)$. In order to define the composition map

$$m_k : \otimes_{0\leq i\leq k-1} Hom((L_i,\rho_i),(L_{i+1},\rho_{i+1})) \to Hom((L_0,\rho_0),(L_k,\rho_k))[2-k]$$

one uses the standard formulas, but the sum runs over certain two-dimensional surfaces in $X^\vee$ described below. For a sequence $((L_0,\rho_0),...,(L_k,\rho_k)), k\geq 1$ of objects in $FO(X^\vee)$ we consider immersed two-dimensional surfaces $S \to X^\vee$ such that:

a) Boundary of S belongs to $L_0\cup ...\cup L_k$.

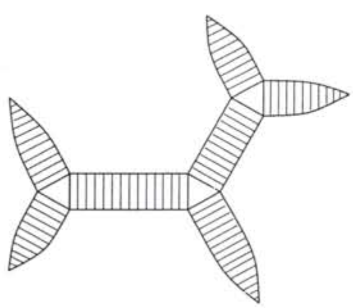
b) $S = (\cup_\alpha T_\alpha)\cup(\cup_\beta S_\beta)$ where and T_α are geodesic triangles in fibers of $p^\vee$, hence they are projected to points in Y.

c) Each S_β is a union of 1-parameter families of geodesic intervals contained in fibers of $p^\vee$ (i.e. a "strip"). Moreover, $p^\vee_{|S_\beta} : S_\beta \to Y$ is a fibration over a connected interval $I_\beta := p^\vee(S_\beta)$ immersed in Y. Fibers of S_β over the interior points of I_β are geodesic intervals of strictly positive length. Fibers of S_β over the boundary points of I_β are either edges of triangles T_α or intersection points $x_i\in L_i\cap L_{i+1}, 0\leq i\leq k-1$, $x_k\in L_0\cap L_k$.

d) Intervals I_β are edges of an immersed planar trivalent tree $\Gamma\subset Y$. Points $p^\vee(T_\alpha)$ are internal vertices of Γ. Tail vertices of Γ are projections of the intersection points $x_0,\dots,x_k$.

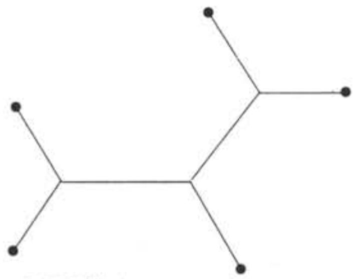
e) Let $r : T^*_Y \to X^\vee$ be the natural fiberwise universal covering. If the Lagrangian manifolds $r^{-1}(L_i), i=0,...,k$ are locally given by differentials of smooth functions $f_i, i=0,...,k$ on Y, then the edges of Γ must be gradient lines of $f_i - f_j$. Intersection points of $r^{-1}(L_i)$ and $r^{-1}(L_j)$ correspond to critical points of $f_i - f_j$.

We depict a typical surface below:



The projection of surface S to Y is a gradient tree, with tail vertices being critical points of $f_i - f_{i+1}$ or of $f_0 - f_k$, and edges $p^\vee(S_\beta)$ being the gradient lines of functions $f_i - f_j$, where $i=i(\beta), j=j(\beta), i<j$. The triangles are

mapped into the internal vertices of the tree. Here is the picture of $\Gamma = p^\vee(S)$ for surface S as above:



Compositions $m_k = m_k^{FO(X^\vee,\omega)}$ are given by the standard formulas, but now we are counting surfaces S described in a)-d). The weight $q^{\langle [S],[\omega]\rangle}$ can be written as $exp(-\frac{1}{\varepsilon}\sum_\beta var_{p^\vee(S_\beta)} f_\beta)$, where $f_\beta = f_{i(\beta)} - f_{j(\beta)}$, and var is the (positive) variation of the function along the gradient line.

The transversality condition for a sequence of objects of Fukaya-Oh category can be formulated similarly to the case of Fukaya category.

The reader can compare our considerations with those from [FuO]. The fibers of $p^\vee$ are "small" tori (of the size $O(\eta)$). The base Y is "large" (of the size of $O(1)$). Hence, the Lagrangian manifolds are close to the zero section of $p^\vee$. This is similar to the situation considered in [FuO]. Indeed, in [FuO] the authors study the A_∞-subcategory of $F(T_Y^*)$ (where Y is an arbitrary smooth compact manifold), with the objects (L, ρ) such that $L = \eta\, graph(df)$, $f : Y \to \mathbf{R}$ is a smooth function. In other words, they considered Lagrangian sections of the natural projection $T_Y^* \to Y$, which are close to the zero section. When $\eta \to 0$, pseudo-holomorphic discs get "stretched" along the fibers of $p^\vee$. Thus they look like the surfaces S described above. Then the higher compositions of the Fukaya category "approach" the compositions $m_k^{FO(X^\vee,\omega)}$. This was proved in [FuO] in the case when $X^\vee$ was replaced by T_Y^*. Considerations from [FuO] apply in our case as well.

Remark 14 *One can extend the Fukaya-Oh category considering Lagrangian submanifolds in $X^\vee$ which are not necessarily unramified coverings of Y. For example, one can try to add to $FO(X^\vee)$ new objects which are local systems on Lagrangian tori which are fibers of the projection $p^\vee : X^\vee \to Y$. It seems that with these objects one can go much further than with transversal ones. For example, in the general case of torus fibrations with singular fibers, one can argue that for almost any $y \in Y$ there is no limiting holomorphic discs with the boundary in the torus $(p^\vee)^{-1}(y)$. The set of such points y is the complement to a countable union Z of hypersurfaces in Y (this follows from* *the fact that $dim(Y^{sing}) = dim(Y) - 2$). Thus, we get a large collection of honest objects without the parasitic composition m_0. The total picture seems to be quite intricate, as examples show that the subset Z is everywhere dense. Presumably, it is related with some mysterious non-abelian 1-cocycle which we will discuss later in the remark in section 7.1.*

6 Morse-Smale complex and the category of Morse functions

6.1 Notations from Morse theory

Let (Y, g_Y) be a compact oriented Riemannian manifold of dimension n, $f : Y \to \mathbf{R}$ be a smooth Morse function. We will denote the set of critical points of f by $Cr(f)$. If $x \in Cr(f)$, we will denote by U_x (resp. S_x) the unstable (resp. stable) submanifolds associated with x. Namely, $U_x = \{y \in Y \,|\, lim_{t\to+\infty} e^{-t\, grad(f)} y = x\}$, and $S_x = \{y \in Y \,|\, lim_{t\to+\infty} e^{t\, grad(f)} y = x\}$.

Let $ind(x)$ be the Morse index of x, i.e. the negative rank of the quadratic form $(\partial^2 f)_{|T_x Y}, x \in Cr(f)$. The manifolds S_x and U_x are diffeomorphic to open balls of dimensions $ind(x)$ and $n - ind(x)$ respectively. It follows that the cohomology of S_x with compact support is a graded vector space with the only non-zero 1-dimensional component in degree $ind(x)$: $H_c^*(S_x) \simeq \mathbf{Z}[-ind(x)]$. A choice of generator defines an orientation of S_x. If the function f satisfies Morse-Smale transversality condition, i.e. for any $x, y \in Cr(f)$ the manifolds S_x and U_y intersect transversally, then $Y = \sqcup_{x \in Cr(f)} S_x$ is a cell decomposition of Y. The cohomology $H^*(Y, \mathbf{Z})$ can be computed as the cohomology of the Morse complex $(M^*(Y, f), \partial)$, with the components $M^i(Y, f) = \sum_{x \in Cr(f), ind(x) = i} H_c^i(S_x)$. Let us choose orientations of manifolds S_x for all $x \in Cr(f)$. We endow U_x with the dual orientations. The graded module $M^i(Y, f)$ can be identified with $\oplus_{0 \le i \le n} \mathbf{Z}^{Cr_i}[-i]$ where Cr_i is the set of critical points of f of index i. The choice of orientations gives a basis $([x])_{x \in Cr(f)}$ of $M^*(Y, f)$.

The differential ∂ is the standard Morse differential:

$$\partial([x]) = \sum_{y \in Cr(f), ind(y) = ind(x)+1} deg((U_x \cap S_y)/\mathbf{R}) \cdot [y],$$

where $(U_x \cap S_y)/\mathbf{R}$ an oriented 0-dimensional manifold (a set of points with signs), and $deg(\cdot) \in \mathbf{Z}$ denotes the total number of points counted with signs.

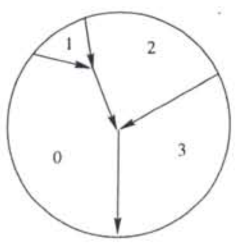
The action of $\mathbf{R}$ arises from the natural reparametrization $x \mapsto x + t$ of the gradient trajectories.

There is also a generalization $M^*(Y, f, \rho)$ of the Morse complex for a flat vector bundle ρ (see [BZ], [HL]).

6.2 Morse A_∞-category of smooth functions

Here we will define the Morse category of smooth functions $M(Y)$ following [FuO]. It will be an A_∞-pre-category over $\mathbf{C}$. Objects of $M(Y)$ are pairs (f, ρ), where $f : Y \to \mathbf{R}$ is a smooth function, and ρ is a local system of finite-dimensional complex vector spaces on Y. Before defining the transversality of objects, we will define the transversality of functions.

Suppose we are given a sequence of smooth functions $(f_0, ..., f_k), k \geq 2$ such that all $f_i - f_j, i \neq j$ are Morse functions, and a sequence of critical points $x_i \in Cr(f_i - f_{i+1}), 0 \leq i \leq k-1, x_k \in Cr(f_0 - f_k)$. We will use oriented binary planar trees in order to describe certain moduli spaces associated with such sequences. Let us fix a planar trivalent tree T with $k+1$ tails vertices. Among the tail vertices we choose one and call it the *root* vertex. Let us orient edges of T along the shortest paths towards the root. Thus, T becomes a binary tree considered as an *oriented* tree . We depict T inside of the standard unit disc $D \subset \mathbf{R}^2$ in such a way that tail vertices of T belong to ∂D, and connected components of $D \setminus T$ are cyclically numbered from 0 to k in the clockwise order. We assume that numbers attached to two regions near the root vertex are 0 and k.



We define a gradient immersion of T into Y as a continuous map $j : T \to Y$ such that:

1) The restriction $j|_e$ is an orientation preserving homeomorphism of the edge e onto an interval in the gradient line of $f_{l(e)} - f_{r(e)}$, where the label $l(e)$ (resp. $r(e)$) corresponds to the region of $D \setminus T$ which is left (resp. right) to e.

2) Each tail vertex v is mapped to the point $x_v \in Cr(f_{l_v} - f_{r_v})$, where l_v

(resp. r_v) is the label of the region which is left (resp. right) to the only tail edge containing v.

We will need immersed binary trees (let us call them *gradient* trees) in order to define compositions and transversal sequences in the Morse A_∞-pre-category. These structures can be defined in terms of certain varieties, which we are going to describe now.

Suppose that we are given a sequence of functions $(f_0, ..., f_k), k \geq 2$ and critical points $(x_0, \ldots, x_k)$ as above, and a binary planar tree T. Let us consider the manifold $Y(T) = Y^{V_i(T)}$, where $V_i(T)$ is the set of internal vertices of T. We are going to define several submanifolds in $Y(T)$. For each tail vertex $v_m, 0 \leq m \leq k-1$ we define $Z_{v_m} = \pi^{-1}_{\hat{v}_m}(U_{x_m})$ and for $m = k$ we define $Z_{v_k} = \pi^{-1}_{\hat{v}_k}(S_{x_k})$. Here $\hat{v}_l$ denotes the second endpoint of the edge of T containing v_l, and $\pi_v : Y(T) \to Y$ is the canonical projection on the factor corresponding to $v \in V_i(T)$.

For pair (f_i, f_j) we define a subset $Z_{i,j} \subset Y \times Y$, consisting of pairs (y_1, y_2) such that $y_1 \neq y_2$ and $y_2 = e^{t\, grad(f_i - f_j)} y_1$ for some $t > 0$. Then $Z_{i,j}$ is a non-compact submanifold of $Y \times Y$.

An edge e of T we call internal if both endpoints of it are internal vertices. The set of internal edges we denote by $E_i(T)$. For each internal edge $e \in E_i(T)$, which separates two regions labeled by $l(e)$ (left) and $r(e)$ (right), we define a submanifold $Z_e = \pi_e^{-1}(Z_{l(e),r(e)})$, where $\pi_e : Y(T) \to Y \times Y$ is the natural projection.

It follows from the definitions that the space of gradient immersions of a given T as above, up to homeomorphisms preserving tails, can be identified with $\mathcal{M}(T; f_0, ..., f_k; x_0, ..., x_k) := (\cap_{0 \leq m \leq k} Z_{v_m}) \cap (\cap_{e \in E_i(T)} Z_e) \subset Y^{V_i(T)}$.

Definition 18 *We say that a sequence $(f_0, \ldots, f_k)$, $k \geq 2$ is T-transversal for a given tree T, if for any sequence of intersection points $(x_0, \ldots, x_k)$ such that*

$$\sum_{i=0}^{k-1} ind(x_i) - ind(x_k) \leq k-2$$

the collection of submanifolds $\left((Z_{v_m})_{0 \leq m \leq k}, (Z_e)_{e \in E_i(T)}\right)$ is transversal in $Y(T)$ (i.e. intersection of any subcollection is transversal). For $k = 1$, we say that (f_0, f_1) is T-transversal (there is only one tree T in this case) if $f_0 - f_1$ is a Morse function, satisfying the Morse-Smale transversality condition.

Remark 15 *As in the case of Fukaya category we consider here only spaces $\mathcal{M}(T; f_0, ..., f_k; x_0, ..., x_k)$ of (virtual) dimensions less or equal than zero. Our* *condition in the case of strictly negative dimension means that the moduli space is empty.*

It can be proven (see [Fu1]) that there exists a subset of second Baire category in $(C^\infty(Y))^{\mathbf{Z}}$ such that for any element $(f_i)_{i\in\mathbf{Z}}$ of this set and for any strictly increasing sequence of integers $i_0 < \cdots < i_k)$ and for any planar tree T with $k+1$ tails, the sequence $(f_{i_0}, \ldots, f_{i_k})$ is T-transversal.

Definition 19 *A sequence of objects $(f_0, \rho_0), ..., (f_k, \rho_k)$ is called transversal if for any $m \geq 1$ and any binary tree T with $m+1$ tails, an arbitrary subsequence $(f_{i_0}, ..., f_{i_m}), i_0 < ... < i_m$ is T-transversal.*

For any two transversal objects $W_0 = (f_0, \rho_0)$ and $W_1 = (f_1, \rho_1)$ we define the space of morphisms $Hom_{M(Y)}(W_0, W_1)$ as the Morse complex $M^*(Y, f_0 - f_1, \rho_0^* \otimes \rho_1)$. Now we define the A_∞-structure on $M(Y)$.

The map $m_1 : Hom((f_0, \rho_0), (f_1, \rho_1)) \to Hom((f_0, \rho_0), (f_1, \rho_1))[1]$ is the standard differential in the Morse-Smale complex. Higher compositions m_k where $k \geq 2$ for transversal sequences of objects are linear maps

$$m_k : \otimes_{0\leq i\leq k-1} Hom((f_i, \rho_i), (f_{i+1}, \rho_{i+1})) \to Hom((f_0, \rho_0), (f_k, \rho_k))[2-k]$$

Each m_k is defined as a sum $m_k = \sum \pm m_{k,T}$ where T runs through the set of isomorphism classes of oriented binary planar trees with $(k+1)$ tails. Let us describe the summands $m_{k,T}$. For simplicity we will give the formulas in the case when all local systems are trivial of rank one.

Let us fix critical points $x_i \in Cr(f_i - f_{i+1}), 0 \leq i \leq k-1, y_k \in Cr(f_0 - f_k)$, such that $\sum_{0\leq i\leq k-1} ind(x_i) = ind(y_k) + 2 - k$, and orientations of manifolds $S_{x_i}, 0 \leq i \leq k$. It follows from the definition of a transversal sequence that the moduli space of gradient trees $\mathcal{M}(T; f_0, ..., f_k; y_0, ..., y_k)$ is an oriented compact zero-dimensional manifold.

Definition 20 *We define compositions $m_k, k \geq 2$ by the formula*

$$m_k([y_0], ..., [y_{k-1}]) = \sum_{[T]} \sum_{y_k \in Cr(f_0 - f_k)} deg(\mathcal{M}(T; f_0, ..., f_k; y_0, ..., y_k)) \cdot [y_k]$$

where $[T]$ is the equivalence class of T as an abstract oriented planar tree, and $deg(\cdot) \in \mathbf{Z}$ is the total number of points counted with signs, as before.

For local systems of higher ranks one proceeds as in the case of Fukaya categories, using flat connections in order to define an analog of the holonomy of local systems.

One can obtain slightly different formulas for m_k in the following way. For any point $\gamma \in \mathcal{M}(T; f_0, ..., f_k; y_0, ..., y_k)$ we define the weight

$$w_\gamma = exp(-\frac{1}{\varepsilon} \sum_{e \in E(T)} var_\gamma(f_{l(e)} - f_{r(e)})) \in \mathbf{C}_\varepsilon.$$

Here $var_\gamma(f_{l(e)} - f_{r(e)}) > 0$ is a variation of $f_{l(e)} - f_{r(e)}$ along the gradient line $\gamma(e)$, which is defined such as follows: $var_\gamma(f_{l(e)} - f_{r(e)}) = (f_{l(e)} - f_{r(e)})(y_{max}) - (f_{l(e)} - f_{r(e)})(y_{min})$, where y_{max} and y_{min} are the endpoints of $\gamma(e)$, such that $(f_{l(e)} - f_{r(e)})(y_{max}) - (f_{l(e)} - f_{r(e)})(y_{min}) > 0$. After extension of scalars to $\mathbf{C}_\varepsilon$ one can choose another basis in $Hom_{M(Y)}(W_0, W_1)$, namely $[y]_{new} = [y]exp(\frac{(f_0(y) - f_1(y))}{\varepsilon})$ for $y \in Cr(f_0 - f_1)$. Then the formulas for m_k will be modified. The contribution of each γ will be multiplied by w_γ. The formulas will be similar to those for the Fukaya-Oh category (see Section 5.2).

6.3 De Rham A_∞-category of smooth functions

The other A_∞-pre-category we are interested in will be a differential-graded category (dg-category for short). In other words, it is an A_∞-category with strict identity morphisms and vanishing compositions $m_n, n \geq 3$. We will call it *de Rham category* of Y and denote by $DR(Y)$. Objects of $DR(Y)$ are same as for $M(Y)$. They are pairs (f, ρ), where $f : Y \to \mathbf{R}$ is a smooth function and ρ is a local system on Y. Morphisms are complexes defined by the formula

$$Hom_{DR(Y)}((f_0, \rho_0), (f_1, \rho_1)) = \Gamma(Y, \wedge^* T_Y^* \otimes Hom(\rho_0, \rho_1)).$$

Notice that the space of morphisms does not depend on f_0 and f_1. The composition of morphisms is defined in the obvious way: in a local trivialization of ρ_0 and ρ_1 it is given by the product of matrices with the coefficients in $\Omega^*(Y)$.

Now we can formulate the main result of this section.

Theorem 2 *A_∞-pre-categories $M(Y)$ and $DR(Y)$ are equivalent.*

The proof of the theorem will occupy the rest of the section. First, we will discuss a version of formulas from "homological perturbation theory" (see [GS], [Me]). They will give an A_∞-structure on a subcomplex of a dg-algebra. Then we will discuss an approach to the proof based on the ideas of [HL]. It seems plausible that an alternative proof (but, presumably, much more difficult) can be obtained within the framework of Witten complex, using methods of [BZ].

6.4 A_∞-structure on a subcomplex

In this section we are going to restate in a convenient form some results from [GS] and [Me].

Let $(A, m_n), n \geq 1$ be a non-unital A_∞-algebra, $\Pi : A \to A$ be an idempotent which commutes with the differential $d = m_1$. In other words, Π is a linear map of degree zero such that $d\Pi = \Pi d, \Pi^2 = \Pi$. Assume that we are given an homotopy $H : A \to A[-1]$, $1 - \Pi = dH + Hd$. Let us denote the image of Π by B. Then we have an embedding $i : B \to A$ and a projection $p : A \to B$, such that $\Pi = i \circ p$.

Let us introduce a sequence of linear operations $m_n^B : B^{\otimes n} \to B[2-n]$ in the following way:

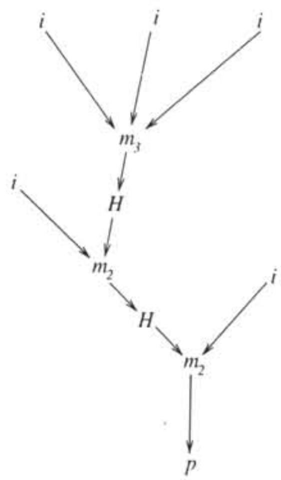
a) $m_1^B := d^B = p \circ m_1 \circ i$;

b) $m_2^B = p \circ m_2 \circ (i \otimes i)$;

c) $m_n^B = \sum_T \pm m_{n,T}, n \geq 3$.

Here the summation is taken over all oriented planar trees T with $n+1$ tails vertices (including the root vertex), such that the (oriented) valency $|v|$ (the number of ingoing edges) of every internal vertex of T is at least 2. In order to describe the linear map $m_{n,T} : B^{\otimes n} \to B[2-n]$ we need to make some preparations. Let us consider another tree $\bar{T}$ which is obtained from T by the insertion of a new vertex into every internal edge. As a result, there will be two types of internal vertices in $\bar{T}$: the "old" vertices, which coincide with the internal vertices of T, and the "new" ones, which can be thought geometrically as the midpoints of the internal edges of T.

To every tail vertex of $\bar{T}$ we assign the embedding i. To every "old" vertex v we assign m_k with $k = |v|$. To every "new" vertex we assign the homotopy operator H. To the root we assign the projector p. Then moving along the tree down to the root one reads off the map $m_{n,T}$ as the composition of maps assigned to vertices of $\bar{T}$. Here is an example of T and $\bar{T}$:



Proposition 4 *The linear map* m_1^B *defines a differential in* B.

Proof. Clear. ■

Theorem 3 *The sequence* $m_n^B, n \geq 1$ *gives rise to a structure of an* A_∞-*algebra on* B.

Sketch of the proof. The proof is quite straightforward, so we just briefly show main steps of computations.

First, one observes that p and i are homomorphisms of complexes. In order to prove the theorem we will replace for a given $n \geq 2$ each summand $m_{n,T}$ by a different one, and then compute the result in two different ways. Let us consider a collection of trees $\{\bar{T}_e\}_{e \in E(\bar{T})}$ such that $\bar{T}_e$ is obtained from $\bar{T}$ in the following way:

a) we split the edge e into two edges by inserting a new vertex w_e inside e;

b) the remaining part of $\bar{T}$ is unchanged.

We assign $d = m_1$ to the vertex w_e edge, and keep all other assignments untouched. In this way we obtain a map $m_{n,\bar{T}_e} : B^{\otimes n} \to B[3-n]$.

Let us consider the following sum (with appropriate signs):

$$\hat{m}_n^B = \sum_T \sum_{e \in E(\bar{T})} \pm m_{n,\bar{T}_e}.$$

We can compute it in two different ways: using the relation $1 - \Pi = dH + Hd$, and using the formulas for $d(m_j)$, $j \geq 2$ given by the A_∞-structure on A. The case of the relation $1 - \Pi = dH + Hd =: d(H)$ gives

$$\hat{m}_n^B = d(m_n^B) - m_n^{B,\Pi} + m_n^{B,1}$$

where $m_n^{B,\Pi}$ is defined analogously to m_n^B, with the only difference that we assign to a new vertex operator Π instead of H for some edge $e \in E_i(T)$. Similarly, the summand $m_n^{B,1}$ is defined if we assign to a new vertex operator $1 = id_A$ instead of H. Formulas for $d(m_j)$ are quadratic expressions in m_l, $l < j$. This gives us another identity

$$\hat{m}_n^B = m_n^{B,1}$$

Thus we have $d(m_n^B) = m_n^{B,\Pi}$, and it is exactly the A_∞-constraint for the collection $(m_n^B)_{n \geq 1}$. ■

Moreover, using similar technique, one can prove the following result.

Proposition 5 *There is a canonical A_∞-morphism $g : B \to A$, which defines a quasi-isomorphism of A_∞-algebras.*

For the convenience fo the reader we give an explicit formula for a canonical choice of g. The operator $g_1 : B \to A$ is defined as the inclusion i. For $n \geq 2$ we define g_n as the sum of terms $g_{n,T}$ over all planar trees T with $n+1$ tails. Each term $g_{n,T}$ is similar to the term $m_{n,T}$ defined above, the only difference is that we insert operator H instead of p into the root vertex.

One can also construct an explicit A_∞-quasi-isomorphism $A \to B$.

Remark 16 *a) Similar construction works in the case of an arbitrary non-unital A_∞-category. In that case one needs projectors $\Pi_{X,Y}$ and homotopies*

$H_{X,Y}$ for every graded space of morphisms $Hom(X,Y)$. All formulas remain the same as in the case of A_∞-algebras. The resulting A_∞-category with the spaces of morphisms given by $\Pi_{X,Y}(Hom(X,Y))$ is equivalent to the original one. We will use this fact later.

b) Propositions 4 and 5 should hold in a much more general case of algebras over operads (see e.g. [M]).

6.5 Projectors and homotopies in Morse theory

We would like to apply formulas for the A_∞-structure on a subcomplex to the proof of the Theorem 2. In order to do that we need to identify the Morse complex with a direct summand of the de Rham complex. Our approach is based on the ideas of Harvey and Lawson (see [HL]).

Let Y be a compact oriented smooth manifold, $dim\, Y = n$. The space of currents $D'(Y)$ we will identify with the space of distribution-valued differential forms. Continuous linear operators $\Omega^*(Y) \to D'(Y)$ are given by their Schwartz kernels, which are elements of $D'(Y \times Y)$. Smoothening operators $D'(Y) \to \Omega^*(Y)$ have kernels in $\Omega^*(Y \times Y) \subset D'(Y \times Y)$.

With any oriented submanifold $Z \subset Y$, $\dim Z = k$ of finite volume we associate a canonical current $[Z]$ of degree $n-k$ (namely, we can integrate smooth k-forms over Z).

Let g_Y be a Riemannian metric on Y, and f be a Morse-Smale function. The gradient flow $exp(t\, grad(f)), t \geq 0$ gives rise to a 1-parameter semigroup acting on $\Omega^*(Y)$: $\psi^t(\alpha) = exp(t\, grad(f))_*(\alpha)$. Schwartz kernel of ψ^t is $[G_t]$ where manifold $G_t \subset Y \times Y$ is given by $G_t := graph(exp(t\, grad(f)))$. We also have the identity

$$id - \psi^t = dH^t + H^t d,$$

where $H^t : \Omega^*(Y) \to \Omega^*(Y) \subset D'(Y)$ is a linear operator of degree -1 defined by the distributional kernel $[Z_t]$, $Z_t := \cup_{0 \leq t' \leq t} graph(exp(t'\, grad(f)))$.

It is checked in [HL] that this picture has a limit (in certain sense) as $t \to +\infty$. Namely, there exist limits of currents $[G_t]$ and $[Z_t]$:

$$[G_\infty] = lim_{t \to +\infty}[G_t] = \sum_{x \in Cr(f)} [S_x] \times [U_x]$$

$$[Z_\infty] = lim_{t \to +\infty}[Z_t] = [\cup_{0 \leq t < +\infty} G_t]$$

Linear operators ψ^∞ (of degree zero) and H^∞ (of degree -1), corresponding to these kernels, map $\Omega^*(Y)$ to $D'(Y)$ and satisfy the identity

$$i - \psi^\infty = dH^\infty + H^\infty d,$$

where $i : \Omega^*(Y) \to D'(Y)$ is the natural inclusion. According to the de Rham theorem this inclusion is a quasi-isomorphism of complexes, therefore ψ^∞ is. Morally, $\Pi_\infty := \psi^\infty$ should be thought of as a projector. The image $\Pi_\infty(\Omega^*(Y)) \subset D'(Y)$ coincides with $\oplus_{x\in Cr(f)}\mathbf{R}\cdot[U_x]$. We have

$$\Pi_\infty(\alpha) = \sum_{x\in Cr(f)} (\int_{S_x} \alpha)\cdot[U_x] = \sum_{x\in Cr(f)} \int_Y (\alpha\wedge[S_x])\cdot[U_x].$$

Moreover, the operator Π_∞ commutes with the differentials. Hence the complex $\Pi_\infty(\Omega^*(Y))$ is a finite-dimensional subcomplex of $D'(Y)$ isomorphic to the Morse complex $M^*(Y, f)$. In fact it is quasi-isomorphic to both complexes $\Omega^*(Y)$ and $D'(Y)$. In this way Harvey and Lawson prove that the de Rham cohomology is isomorphic to the cohomology of Morse complex.

In order to construct actual projectors and homotopies we will proceed as follows. Let $\rho_\delta, \delta \to 0$ be a family of smooth closed differential n-forms on $Y\times Y$ such that $supp(\rho_\delta)$ belongs to the open δ-neighborhood N_δ of the diagonal $diag \subset Y\times Y$, and the cohomology class of ρ_δ in $H^n_c(N_\delta, \mathbf{R})$ is the same as of $[diag]$.

We define $R_\delta : D'(Y) \to \Omega^*(Y)$ as the integral operator given by the kernel ρ_δ.

Lemma 3 *1) The operator R_δ is a homomorphism of complexes.*

2) If $Z_1, Z_2 \in Y$ are two oriented submanifolds of finite volume such that they intersect transversally at finitely many points, and $dim\, Z_1 + dim\, Z_2 = dim\, Y$, $\overline{Z}_1 \cap \overline{Z}_2 = Z_1 \cap Z_2$, then for sufficiently small δ one has:

$$\int_Y R_\delta([Z_1]) \wedge R_\delta([Z_2]) = deg(Z_1 \cap Z_2) \in \mathbf{Z}$$

3) There exists a linear operator $h_\delta : \Omega^*(Y) \to \Omega^*(Y)$ such that its kernel has support in N_δ, the wave front $WF(h_\delta)$ is the conormal bundle of $diag \subset Y\times Y$, and

$$dh_\delta + h_\delta d = id - (R_\delta)_{|\Omega^*(Y)}.$$

Proof. Part 1) follows from the fact that ρ_δ is a closed current. Part 2) follows from the fact that R_δ changes the supports of $Z_i, i = 1, 2$ by $O(\delta)$. To prove part 3) one observes that the operators id and $(R_\delta)_{|\Omega^*(Y)}$ preserve the space of smooth forms $\Omega^*(Y)$, and ρ_δ is cohomologous to $[diag]$. ■

Let $x, y \in Cr(f)$ be two critical points of the same Morse index. Then $deg(S_x \cap U_y) = \delta_{xy}$ (the Kronecker symbol). By the part 2) of the Lemma, for sufficiently small δ we obtain the identity

$$\int_Y R_\delta([S_x]) \wedge R_\delta([U_y]) = \delta_{xy}$$

This implies the following result.

Proposition 6 *Let us define for a sufficiently small δ a linear operator $D'(Y) \to \Omega^*(Y)$ by the formula $\Pi_\delta(\alpha) = \sum_{x \in Cr(f)} (\int_Y \alpha \wedge R_\delta([S_x])) \cdot R_\delta([U_x])$. Then*

1) $\Pi_\delta^2(\alpha) = \Pi_\delta(\alpha)$ if $\alpha \in \Omega^*(Y)$, and $\Pi_\delta d = d\Pi_\delta$.

2) The image $\Pi_\delta(M^*(Y, f))$ is a subcomplex in $\Omega^*(Y)$ which is canonically isomorphic to the Morse complex $M^*(Y, f)$.

We define a homotopy operator $H_\delta : \Omega^*(Y) \to \Omega^*(Y)[-1]$ as an integral operator given by the kernel $(R_\delta \boxtimes R_\delta)[Z_\infty] + (h_\delta \boxtimes h_\delta)([diag])$. (The last summand is well-defined because of the condition on the wave front of h_δ). It is easy to check that the following identity holds:

$$id - \Pi_\delta = dH_\delta + H_\delta d.$$

Thus we have a family of homotopies and projectors parametrized by δ.

Remark 17 *One can define the projector Π_δ using another canonical element $\sum_{x \in Cr(f)} [S_x] \otimes R_\delta([U_x])$, instead of $\sum_{x \in Cr(f)} R_\delta([S_x]) \otimes R_\delta([U_x])$, as we did. The above Proposition holds for the new canonical element as well.*

There is a version of the previous construction, which will be useful in the next subsection. Namely, we start with a differential $n + 1$-form ρ on $Y \times Y \times (0, 1)$ such that for the support of $supp(\rho)$ belongs to $\sqcup_{\delta > 0} (N_\delta, \delta)$ for all sufficiently small $\delta \in (0, 1)$, and ρ defines the same cohomology class in $H_c^n(Y \times Y \times (0, 1))$ as $[diag] \times (0, 1)$.

Let us consider now the spaces $\Omega_0^*(Y) := \varinjlim_{\delta \to 0} \Omega^*(Y \times (0, \delta))$ and $D_0'(Y) := \varinjlim_{\delta \to 0} \Omega^*(0, \delta) \widehat{\otimes} D'(Y)$. It is easy to see that both complexes $\Omega_0^*(Y)$ and $D_0'(Y)$ are quasi-isomorphic to $\Omega^*(Y)$.

We define a linear operator $R : D'_0(Y) \to \Omega^*_0(Y)$ similarly to the definition of R_δ. Then the Lemma and the Proposition hold with obvious changes. We will denote the corresponding objects by the same letters as before, skipping the subscript δ (like H for the homotopy and Π for the projector). Morally, they are obtained from the old objects by extending them as differential forms "in the direction of δ".

6.6 Proof of the theorem

For simplicity we will assume that all local systems are trivial and have rank one. The general case is completely similar.

We are going to construct the following chain of A_∞-equivalences connecting $DR(Y)$ and $M(Y)$:

$$DR(Y) \hookrightarrow DR_0(Y) \hookleftarrow DR_0^{tr}(Y) \hookleftarrow DR_0^{tr,\Pi}(Y)) \leftarrow M(Y)$$

Classes of objects of all these categories will be the same, and all functors will be identical on objects.

The A_∞-pre-category $DR_0(Y)$ is in fact a dg-category, i.e. all sequences of objects are transversal, compositions m_k vanish for $k \geq 3$ and it has strict identity morphisms. The space $Hom_{DR_0(Y)}(f_0, f_1)$ is defined as $\varinjlim_{\delta\to 0}\Omega^*(Y \times (0,\delta)) = \Omega^*_0(Y)$. Clearly the space of morphisms does not depend on objects. Using the wedge product of differential forms we make $DR_0(Y)$ into a dg-category over the field $\mathbf{C}$. There is a natural functor $DR(Y) \to DR_0(Y)$, which is the identity map on objects. On morphisms it is the natural embedding of $\Omega^*(Y)$ as the subspace of forms on $Y \times (0,\delta)$, which are pullbacks of forms on Y. Clearly it establishes an equivalence of A_∞-categories.

The A_∞-pre-category $DR_0^{tr}(Y)$ is defined as the full subcategory of $DR_0(Y)$, and it differs from the latter only by the choice of transversal sequences. Namely, we use the same notion of transversality in $DR_0^{tr}(Y)$ as in the Morse category.

The next A_∞-pre-category $DR_0^{tr,\Pi}(Y)$ is obtained from $DR_0^{tr}(Y)$ by applying homological perturbation theory. For any two transversal objects f_0, f_1 of $DR_0^{tr}(Y)$ we define $Hom_{DR_0^{tr,\Pi}(Y)}(f_0, f_1)$ as $\Pi_{f_0,f_1}(\Omega^*_0(Y))$. Here Π_{f_0,f_1} is the projector Π corresponding to the Morse function $f_0 - f_1$, it was described at the end of the previous subsection. We also have homotopies H_{f_0,f_1} associated with $f_0 - f_1$. Then formulas of homological perturbation theory (summation over trees) give rise to an A_∞-pre-category $DR_0^{tr,\Pi}(Y)$ and an equivalence $DR_0^{tr,\Pi}(Y) \to DR_0^{tr}(Y)$.

The last functor $\Psi : M(Y) \to DR_0^{tr,\Pi}(Y))$ will have no non-trivial higher components Ψ_n for $n \geq 2$. The first component Ψ_1 of it is a linear map

$$\Psi_1 : Hom_{M(Y)}(f_0, f_1) \to Hom_{DR_0^{tr,\Pi}(Y)}(f_0, f_1)$$

for every transversal pair (f_0, f_1). Recall that $Hom_{M(Y)}(f_0, f_1)$ has a basis $\{[x]\}$ labeled by critical points $x \in Cr(f_0 - f_1)$. We define $\Psi_1([x])$ as $R([S_x])$. It is clear that Ψ_1 gives a quasi-isomorphism of complexes for every transversal pair (f_0, f_1).

Now, we claim that Ψ is an A_∞-functor. This means that Ψ_1 maps all higher compositions in $M(Y)$ to higher compositions in $DR_0^{tr,\Pi}(Y)$. This follows directly from the descriptions of higher compositions in both categories in terms of planar trees and the lemma in the previous subsection. Indeed, the number of functions in any given sequence is finite. For all sufficiently small δ every summand in the formula for $m_k^{M(Y)}$, corresponding to a binary tree T, coincides with the summand for $m_k^{M(Y)}$ corresponding to the same T (we can assume that δ is so small that the part 2) of the Lemma can be applied). The theorem is proved. ■

7 A_∞-structure for the derived category of coherent sheaves

7.1 Rigid analytic space

It will be helpful (although not necessary) for the reader of this section to be familiar with basic facts of non-archimedean analysis (see [BGR]). For any smooth manifold Y with integral affine structure we will construct a sheaf $\mathcal{O}_Y$ of $\mathbf{C}_\varepsilon$-algebras on Y. Stalks $\mathcal{O}_{Y,y}$ of this sheaf are noetherian algebras, and one can define the notion of coherent sheaves of $\mathcal{O}_Y$-modules. If $Y = \mathbf{R}^n/\mathbf{Z}^n$ is the torus with the standard integral affine structure then the category of coherent $\mathcal{O}_Y$-modules will be equivalent (by a non-archimedean version of GAGA) to the category of coherent sheaves on an abelian variety over the field $\mathbf{C}_\varepsilon$.

We start with the local picture. We denote by $v : \mathbf{C}_\varepsilon \to \mathbf{R} \cup \{+\infty\}$ a (non-discrete) valuation defined by $v(\sum_{\lambda_1<\lambda_2<\dots} c_i e^{-\lambda_i/\varepsilon}) = -\lambda_1$ if $c_1 \neq 0$ and $v(0) = +\infty$.

Definition 21 *Let $U \subset \mathbf{R}^n$ be an open subset of the standard vector space $\mathbf{R}^n$. We define $\mathcal{O}_{\mathbf{R}^n}(U)$ as the vector space over $\mathbf{C}_\varepsilon$ consisting of formal Laurent series*

$$f = \sum_{k_1,...,k_n \in \mathbf{Z}^n} a_{k_1...k_n} z_1^{k_1} ... z_n^{k_n},$$

where $z_1, ..., z_n$ are formal variables, $a_{k_1...k_n} \in \mathbf{C}_\varepsilon$, and for any $(y_1, ..., y_n) \in U$ we have: $lim_{\sum_i |k_i| \to \infty}(v(a_{k_1...k_n}) + \sum_i k_i y_i) = +\infty$.

It follows from the definition that if $f \in \mathcal{O}_{\mathbf{R}^n}(U)$ and $(z_1, ..., z_n) \in (\mathbf{C}_\varepsilon^*)^n$ then the series $\sum_{k_1,...,k_n} a_{k_1...k_n} z_1^{k_1} ... z_n^{k_n}$ converges in the adic topology as long as $(v(z_1), ..., v(z_n)) \in U$.

We introduce an action of the group $GL(n, \mathbf{Z}) \ltimes \mathbf{R}^n$ on $(\mathbf{R}^n, \mathcal{O}_{\mathbf{R}^n})$ such as follows:

a) $GL(n, \mathbf{Z})$ acts simultaneously by the linear change of coordinates and linear transformation of indices $(k_1, ..., k_n)$ in the series;

b) translations $(t_1, ..., t_n) \in \mathbf{R}^n$ act on the coordinates $(y_1, ..., y_n)$ by the shift $(y_1, ..., y_n) \mapsto (y_1 + t_1, ..., y_n + t_n)$, and on the series by the rescaling of coefficients

$$\sum_{k_1,...,k_n} a_{k_1...k_n} z_1^{k_1} ... z_n^{k_n} \mapsto \sum_{k_1,...,k_n} (a_{k_1...k_n} e^{-\sum_i t_i k_i/\varepsilon}) z_1^{k_1} ... z_n^{k_n}.$$

Using this action we define the sheaf $\mathcal{O}_Y$ for an arbitrary smooth manifold Y with integral affine structure.

We claim that there is a canonically associated to Y a rigid analytic space Y^{an} defined over $\mathbf{C}_\varepsilon$. Here is the construction. Let us consider a covering of Y by open subsets U_i such that all non-empty intersections $U_{i_1 i_2 ... i_k} := U_{i_1} \cap ... \cap U_{i_k}$ in some local affine coordinates are convex polyhedra whose faces have rational slopes. Every $U_{i_1 i_2 ... i_k}$ can be identified with the intersection of finitely many half-spaces, such that their pre-images under the map $v^n : (\mathbf{C}_\varepsilon^*)^n \to \mathbf{R}^n$ are sets of the type $\{(z_1, \ldots, z_n) | \, v(z_1^{k_1} ... z_n^{k_n}) \geq C\}$ for some rational $C > 0$. It is known after Tate that such a system of inequalities defines an affinoid domain (i.e. a local model for a rigid analytic space over $\mathbf{C}_\varepsilon$).

Definition 22 *We define Y^{an} as the rigid analytic space over $\mathbf{C}_\varepsilon$ obtained by gluing the local data $(U_i, \mathcal{O}_{U_i})$ by means of the action of $GL(n, \mathbf{Z}) \ltimes \mathbf{R}^n$.*

It is easy to see that Y^{an} is canonically defined, and that the category $Coh(Y^{an})$ of coherent analytic sheaves on Y^{an} (in the sense on analytic geometry) is equivalent to the category of coherent $\mathcal{O}_Y$-modules (i.e. locally finitely generated $\mathcal{O}_Y$-modules).

To every algebraic variety $\mathcal{Y}$ over $\mathbf{C}_\varepsilon$ one can associate canonically a rigid analytic space $\mathcal{Y}^{an}$. If $\mathcal{Y}$ is projective then the category $Coh(\mathcal{Y}^{an})$ is equivalent to the category $Coh(\mathcal{Y})$ of algebraic coherent sheaves on $\mathcal{Y}$ (GAGA theorem).

Assume that $Y = \mathbf{R}^n/\Lambda$ is an n-dimensional torus equipped with the standard integral affine structure induced by $\mathbf{Z}^n \subset \mathbf{R}^n$, and Λ is a lattice commensurable with $\mathbf{Z}^n$. The following result can be derived from [Mum].

Proposition 7 *In the previous notation one has $Y^{an} \simeq \mathcal{Y}^{an}$ where $\mathcal{Y}$ is an abelian variety over $\mathbf{C}_\varepsilon$.*

Let us return to the picture of metric collapse in the case of abelian varieties. Since the collapse was defined by rescaling of the lattice (see Section 2) one can prove that $\mathcal{Y}$ is isomorphic to the original abelian variety over $\mathbf{C}_\varepsilon$. Therefore in the case of abelian varieties we have two equivalent descriptions of the collapse: the one in terms of Riemannian geometry and the one in terms of analytic non-archimedean geometry.

Remark 18 *For the case of collapse with singular fibers, the rigid analytic space Y^{an} constructed as above, seems to be a "wrong" one. First of all, it is not compact because Y is not compact. But there is also a more fundamental problem. It seems that Y^{an} can not be embedded into a compact analytic space associated with a projective algebraic variety. There are several indications that there exists another sheaf of algebras $\mathcal{O}'_Y$ which is (locally on Y) isomorphic to $\mathcal{O}_Y$, and the rigid analytic space $(Y^{an})'$ associated with $(Y, \mathcal{O}'_Y)$ admits an algebraic compactification. In general, sheaves $\mathcal{O}'_Y$ which are twisted versions of $\mathcal{O}_Y$ are classified by the first non-abelian cohomology $H^1(Y, \underline{Aut}(\mathcal{O}_Y))$ where $\underline{Aut}(\mathcal{O}_Y)$ is the sheaf of groups of automorphisms of $\mathcal{O}_Y$. Thus, in the mirror symmetry for Calabi-Yau manifolds which are not abelian varieties, we expect a new ingredient, the cohomology class $[\mathcal{O}'_Y]$.*

7.2 A_∞-structure on the derived category of coherent sheaves

There is a sheaf of abelian groups Af_Y on Y given by locally affine functions with integral slopes (such functions locally are given by $l = c + \sum_{1\le i\le n} m_i y_i$ where $m_i \in \mathbf{Z}, c \in \mathbf{R}$). There is a morphism of sheaves $exp : Af_Y \to \mathcal{O}_Y^*$ given by $l \mapsto exp(l) := e^{-c/\varepsilon} \prod_{1\le i\le n} z_i^{m_i}$.

Let (Y, g) be an AK-manifold (see Section 3.2). We are going to define a characteristic class $[g]$ of the metric, which will be an analog of the cohomology class of a Kähler form in complex geometry. Let $Af_Y \otimes \mathbf{R}$ be a sheaf of all real-valued locally affine functions. For a cover by convex sets $(U_i)_{i\in I}$ one can choose smooth functions K_i such that $g_{|U_i} = \partial^2 K_i$. Then $K_i - K_j \in Af_Y \otimes \mathbf{R}(U_i \cap U_j)$, defines a 1-cocycle whose cohomology class we denote by $[g] \in H^1(Y, Af_Y \otimes \mathbf{R})$. If the *dual* affine structure (see Section 3) is integral, we get a class $[g]$ in the subgroup $H^1(Y, Af_Y)/torsion \subset H^1(Y, Af_Y \otimes \mathbf{R})$. We will call such classes *integral*. In this case $exp([g]) \in H^1(Y^{an}, \mathcal{O}_Y^*)$ is the first Chern class of a line bundle on Y^{an}. By analogy with the Kähler geometry we expect that this line bundle is ample. In the case when (Y, g) is a flat torus, the ampleness can be proven directly (see [BL]).

From now on we assume that $[g]$ is integral. Then by GAGA the category of analytic coherent sheaves on Y^{an} is equivalent to the category of algebraic coherent sheaves on the corresponding algebraic projective variety $\mathcal{Y}$.

The sheaf $\mathcal{O}_Y$ admits a resolution $\widehat{\Omega}_Y^*$ by a soft sheaf of dg-algebras. Locally, for a small open $U \subset Y$, sections of $\widehat{\Omega}_Y^*$ are given by sums $\alpha = \sum_{i_1,\dots,i_n} c_{i_1\dots i_n} z_1^{i_1} \dots z_n^{i_n}$ where $c_{i_1\dots i_n} = \sum_j c_{j,i_1\dots i_n} e^{-\lambda_{j,i_1\dots i_n}/\varepsilon}, c_{j,i_1\dots i_n} \in \Omega^*(U)$ with the same convergence conditions as for the sheaf $\mathcal{O}_Y$. Differential is given by the de Rham differential acting on the coefficients $c_{j,i_1\dots i_n}$.

We define a dg-category $\mathcal{C}(Y)$ such as follows. Objects are finite complexes of locally free $\mathcal{O}_Y$-modules of finite rank. For any two such complexes E_1 and E_2 we define the space of morphisms as

$$Hom_{\mathcal{C}(Y)}(E_1, E_2) = \Gamma(Y, Hom_{\mathcal{O}_Y}(E_1, E_2) \widehat{\otimes}_{\mathcal{O}_Y} \widehat{\Omega}_Y^*),$$

where we use the completed tensor product in the r.h.s. Differential and grading on the spaces of morphisms are induced by those on $E_1, E_2, \widehat{\Omega}_Y^*$. We will treat $\mathcal{C}(Y)$ as an A_∞-pre-category in which all sequences of objects are transversal and there is no higher compositions except m_1 and m_2.

For a given projective algebraic variety V over a field, one can define canonically an equivalence class of A_∞-categories $D^b(V)$. It is obtained by the following enhancement of the bounded derived category of coherent sheaves on V. Objects of this A_∞-category are the same as of the derived category of coherent sheaves. In order to define the space of morphisms between two objects, one replaces them by arbitrary chosen acyclic resolutions by locally free sheaves (e.g. the Godement resolutions) and then takes the global sections of the space of morphisms between resolutions in the category of complexes of sheaves. In this way one obtains a dg-category. In the case of projective varieties over complex numbers, there is an alternative construction in terms of complexes of holomorphic vector bundles and Dolbeault forms. Different choices of resolutions lead to A_∞-equivalent categories. We will loosely denote the (A_∞-equivalence) class of these categories by $D^b(V)$.

Using the fact that spaces of morphisms of $\mathcal{C}(Y)$ are resolutions of the corresponding spaces of sheaves of $\mathcal{O}_Y$-modules, as well as GAGA theorem, one can prove the following result.

Proposition 8 *The category $\mathcal{C}(Y)$ is A_∞-equivalent to $D^b(\mathcal{Y})$, where $\mathcal{Y}$ is the projective algebraic variety corresponding to the analytic space Y^{an} assigned to Y.*

8 Homological mirror conjecture

In the previous section we constructed an A_∞-category $\mathcal{C}(Y)$ which is A_∞-equivalent to the derived category of coherent sheaves on a Calabi-Yau manifold over the field $\mathbf{C}_\varepsilon$. In this section we are going to construct a chain of A_∞-pre-categories and A_∞-equivalences (as in the Morse case)

$$\mathcal{C}_{unram}(Y) \hookrightarrow \mathcal{C}_{unram,0}(Y) \hookleftarrow \mathcal{C}^{tr}_{unram,0}(Y) \hookleftarrow \mathcal{C}^{tr,\Pi}_{unram,0}(Y) \leftarrow FO(X^\vee)$$

and a functor $F : \mathcal{C}_{unram}(Y) \to \mathcal{C}(Y)$ which establishes an equivalence between $\mathcal{C}_{unram}(Y)$ and a full subcategory of $\mathcal{C}(Y)$. Recall that the Fukaya-Oh category $FO(X^\vee)$, as defined in this paper, is also equivalent to a full subcategory of the Fukaya category $F(X^\vee)$. Thus, we establish an A_∞-equivalence between full subcategories of the Fukaya category $F(X^\vee)$ and of $\mathcal{C}(Y)$.

The approach here is similar to the one we used in the case of Morse theory. All the categories in our chain of A_∞-equivalence functors from above will have the same class of objects, i.e. the same as the Fukaya-Oh category.

8.1 Mirror symmetry functor on objects

Here we will define dg-category $\mathcal{C}_{unram}(Y)$ and the fully faithful embedding F of this category to $\mathcal{C}(Y)$.

In the Appendix we will explain the conventional picture for the mirror symmetry functor in case of complex numbers. There we will use a kind of Fourier-Mukai transform along fibers of the torus fibration. The kernel of this transform is an analog of Poincaré bundle. If one starts with a local system on a Lagrangian section of $p^\vee : X^\vee \to Y$ then the transform makes from it a smooth bundle on X with the connection which is flat in the anti-holomorphic directions. In other words, one gets a holomorphic bundle on X.

These considerations cannot be literally repeated in the non-archimedean case, because "holomorphic" considerations do not work. One can obtain the same result in the following way. Let (L, ρ) be an object of the category $FO(X^\vee)$ such that $rank(\rho) = 1$ and the projection $L \to Y$ is one-to-one map. The manifold L is locally given by the graph of $df\,(mod\,(T_Y^{\mathbf{Z}})^\vee)$, where f is a smooth function on Y. To such an object we assign a sheaf of rank one $\mathcal{O}_Y$-modules $F(L, \rho)$. For sufficiently small open $U \subset Y$ and chosen $f \in C^\infty(U)$ the sheaf $F(L, \rho)_{|U}$ is identified with $\mathcal{O}_{Y|U}$. Change $f \mapsto f + l$, where $l \in Af_Y(U)$ leads to the change of the trivialization of $F(L, \rho)_{|U}$ as $1_U \mapsto exp(l)\,1_U$ (here $1_U \in \mathcal{O}_Y(U)$ is the identity function). If $rank(\rho)$ is greater than one, we decompose $\rho_{|U}$ for small $U \subset Y$ into the sum of rank one local systems and then apply the construction. Analogously, if the covering $L \to Y$ has more than one leaf, we apply the previous construction to each leaf of the covering and then take the direct sum.

We will loosely call F the mirror symmetry functor on objects. The category $\mathcal{C}_{unram}(Y)$ is defined as the dg-category whose class of objects is $Ob(FO(X^\vee))$, and the spaces of morphisms are

$$Hom_{\mathcal{C}_{unram}(Y)}((L_1, \rho_1), (L_2, \rho_2)) := Hom_{\mathcal{C}(Y)}(F((L_1, \rho_1), F(L_2, \rho_2))$$

The functor F on morphisms is defined in the obvious way, as the identity map.

8.2 Spectrum of a morphism and the semigroup

Let $E_i = F(L_i, \rho_i), i = 1, 2$ be locally free $\mathcal{O}_Y$-modules (i.e. vector bundles) corresponding to objects $(L_i, \rho_i) \in FO(X^\vee), i = 1, 2$. For any $\alpha \in$

$Hom_{\mathcal{C}(Y)}(E_1, E_2)$ and a point $y \in Y$ we will define the spectrum of α at y as a certain (at most countable) discrete set of real numbers with finite multiplicities.

Let us assume first that $\rho_i, i = 1,2$ are trivial rank one local systems on $L_i, i = 1,2$, and $L_i, i = 1,2$ are unramified coverings of Y. For a sufficiently small open set U containing y we can write in local coordinates $L_i = graph(df_i)\,(mod\,(T_Y^{\mathbf{Z}})^{\vee}), i = 1,2$ for smooth functions $f_i : Y \to \mathbf{R}, i = 1,2$. Restriction to a small open set U of a morphism $\alpha \in Hom_{\mathcal{C}(Y)}(E_1, E_2)(U) = \widehat{\Omega}_Y^*(U)$ can be identified with the infinite series $\alpha = \sum_{i_1,\dots,i_n} c_{i_1\dots i_n} z_1^{i_1} \dots z_n^{i_n}$, where $c_{i_1\dots i_n} = \sum_j c_{j,i_1\dots i_n} e^{-\lambda_{j,i_1\dots i_n}/\varepsilon}$ and $c_{j,i_1\dots i_n} \in \Omega_Y^*(U)$.

We define the *spectrum of α at $y \in U$* as the set of real numbers (with multiplicities)

$$Sp_y(\alpha) = \{-\lambda_{j,i_1\dots i_n} + \sum_{1 \le k \le n} i_k y_k + f_1(y) - f_2(y)\},$$

where the germ of $c_{j,i_1\dots i_n}$ at y is not equal to zero. One can check that $Sp_y(\alpha)$ is well-defined (i.e. does not depend on the local trivialization), and has the only limiting point at $s = -\infty$.

In the general case of higher rank local systems and Lagrangian manifolds which are unramified coverings of Y, we decompose $E_i, i = 1,2$ locally near $y \in Y$ into the direct sum of trivial rank one $\mathcal{O}_Y$-modules. The spectrum of a morphism at the point y is then defined as the union of the spectra of morphisms between corresponding line bundles.

Remark 19 *One can use instead of the spectrum an* $\mathbf{R}$*-filtration* $Hom_{\mathcal{C}(Y)}(E_1, E_2)^{\le s}$ *on the space of morphisms. It comes from the filtration on the stalks of sheaves of morphisms* $\underline{Hom}_{\mathcal{O}_Y}(E_1, E_2)\widehat{\otimes}\widehat{\Omega}_Y^*$ *(completed tensor product) defined by the condition* $\{-\lambda_{j,i_1\dots i_n} + \sum_{1\le k\le n} i_k y_k + f_1(y) - f_2(y)\} \le s$. *It is easy to see that* α *belongs to* $Hom_{\mathcal{C}(Y)}(E_1, E_2)^{\le s}$ *iff for all* $y \in Y$ *one has* $Sp_y(\alpha) \subset (-\infty, s]$.

Let us consider a subspace $Hom^{alg}_{\mathcal{C}(Y)}(E_1, E_2) \subset Hom_{\mathcal{C}(Y)}(E_1, E_2)$ of algebraic morphisms. It consists of finite sums (both in z_i and $e^{-\lambda_{j,i_1\dots i_n}/\varepsilon}$). It is dense in the space of all morphisms (analytic functions can be approximated by Laurent polynomials). Moreover, the space $Hom_{\mathcal{C}(Y)}(E_1, E_2)$ coincides with the completion of $Hom^{alg}_{\mathcal{C}(Y)}(E_1, E_2)$ with respect to the $\mathbf{R}$-filtration introduced above.

There is a 1-parameter semigroup $\phi^t, t \geq 0$ acting on $Hom^{alg}_{C(Y)}(E_1, E_2)$. In local coordinates ϕ^t acts on the coefficients $c_{j,i_1...i_n}$ by moving them along the gradient flow of $f_1 - f_2$. In order to define it globally we need to describe the space $Hom^{alg}_{C(Y)}(E_1, E_2)$ in geometric terms. It will be done below.

Given two Lagrangian submanifolds $L_i \subset X^\vee, i = 1, 2$ as above, a point $y \in Y$, two points $x_i \in L_i, i = 1, 2$ such that $p^\vee(x_i) = y$, we define a set $P(L_1, L_2, y)$ of homotopy classes of paths $\gamma \in (p^\vee)^{-1}(y)$ starting at x_1 and ending at x_2. Each homotopy class contains a unique geodesic in the flat metric on the torus. We define the space $P(L_1, L_2) = \sqcup_{y \in Y} P(L_1, L_2, y)$. It carries an obvious topology such that the natural projection $\pi : P(L_1, L_2) \to Y$ is an unramified covering with countable fibers. Using the symplectic form ω on $X^\vee$ we define a closed 1-form μ on $P(L_1, L_2)$ by the formula $\mu = \int_\gamma \omega$. Locally on Y we have: $L_i = df_i(mod(T_Y^{\mathbf{Z}})^\vee), i = 1, 2$ where $f_i : Y \to \mathbf{R}$ are smooth functions. Then locally on $P(L_1, L_2)$ we have: $\mu = d(f_1 - f_2 + l)$, where l is a local section of the pullback of the sheaf Aff_Y. Clearly the function l is defined up the adding of a real constant. Thus obtain an $\mathbf{R}$-torsor on $P(L_1, L_2)$. Using the embedding $\mathbf{R} \to \mathbf{C}_\varepsilon^*$, $\lambda \mapsto exp(\lambda/\varepsilon)$ we get a $\mathbf{C}_\varepsilon^*$-torsor, which defines a local system $\mathbf{C}_\varepsilon^{tw}$ of 1-dimensional $\mathbf{C}_\varepsilon$-modules over $P(L_1, L_2)$. Fibers of $\mathbf{C}_\varepsilon^{tw}$ carry natural filtrations. Indeed, in a neighborhood of a point $(x_1, x_2, \gamma, y) \in P(L_1, L_2)$ we can choose a smooth function $f = f_1 - f_2 + l$ such that $\mu = df$. It defines a local trivialization of $\mathbf{C}_\varepsilon^{tw}$. In this trivialization the filtration is defined for $h \in \mathbf{C}_\varepsilon$ by the condition $v(h)(y) + f(y) \leq s, s \in \mathbf{R}$, where v is the valuation. We define a subsheaf $\mathbf{C}_\varepsilon^{tw,alg}$ of $\mathbf{C}_\varepsilon^{tw}$ by the requirement that in a local trivialization it is a subsheaf of finite sums of exponents.

Notice that there are natural projections $pr_i : P(L_1, L_2) \to L_i, i = 1, 2$. Having local systems ρ_i on $L_i, i = 1, 2$ we define local systems $\widehat{\rho}_i, i = 1, 2$ on $P(L_1, L_2)$ as pullbacks with respect to $pr_i, i = 1, 2$.

On $P(L_1, L_2)$ we define a sheaf $\underline{Hom}^{alg}(E_1, E_2)$ ($E_i, i = 1, 2$ were defined previously) such as follows: $\underline{Hom}^{alg}(E_1, E_2) = \mathbf{C}_\varepsilon^{tw,alg} \otimes (\widehat{\rho}_1)^* \otimes \widehat{\rho}_2 \otimes \underline{\Omega}^*_{P(L_1,L_2)}$, where $\underline{\Omega}^*_{P(L_1,L_2)}$ is the sheaf of differential forms. We endow stalks of $\underline{Hom}^{alg}(E_1, E_2)$ with $\mathbf{R}$-filtrations induced by the filtration on $\mathbf{C}_\varepsilon^{tw}$ and trivial filtrations on the other tensor factors.

Let $\pi_!$ denotes the functor of direct image with compact support. Then $\pi_!(\underline{Hom}^{alg}(E_1, E_2)) = \pi_!(\mathbf{C}_\varepsilon^{tw} \otimes \widehat{\rho}_1^* \otimes \widehat{\rho}_2) \otimes \underline{\Omega}_Y^*$, where the last tensor factor is the sheaf of de Rham differential forms on Y.

We can identify $\mathbf{Z}^n$ with $H_1(T^n, \mathbf{Z})$, and the latter group naturally acts on homotopy classes of paths γ. On the other hand, the group ring of $\mathbf{Z}^n$ over $\mathbf{C}_\varepsilon$ can be identified with the ring of Laurent polynomials $\mathbf{C}_\varepsilon[z_1^{\pm 1}, ..., z_n^{\pm 1}]$. Let $\mathbf{C}_\varepsilon^{alg} \subset \mathbf{C}_\varepsilon$ be the subring of finite sums of exponents. It is easy to see that the structure of $\mathbf{C}_\varepsilon^{alg}[\mathbf{Z}^n]$-module on the sections of $\underline{Hom}^{alg}(E_1, E_2)$ corresponds to the structure of $\mathbf{C}_\varepsilon^{alg}[z_1^{\pm 1}, ..., z_n^{\pm 1}]$-module on its image under $\pi_!$. Using this observation one can prove that

$$Hom_{C(Y)}^{alg}(E_1, E_2) \simeq \Gamma(Y, \pi_!(\underline{Hom}^{alg}(E_1, E_2))) = \Gamma_c(P(L_1, L_2), \underline{Hom}^{alg}(E_1, E_2)),$$

where the isomorphism is induced by the natural morphism of sheaves

$$\pi_!(\underline{Hom}^{alg}(E_1, E_2)) \to \underline{Hom}_{C(Y)}^{alg}(E_1, E_2)\,.$$

Here Γ_c refers to the functor of sections with compact support.

Using the metric on Y we assign to the 1-form μ a vector field ξ on $P(L_1, L_2)$. Locally ξ is the generator of the gradient flow of $f_1 - f_2 + l$. It is not difficult to show that there is no trajectory of the flow which goes to infinity for a finite time. Therefore the vector field ξ generates a 1-parameter semigroup ψ^t acting on $P(L_1, L_2)$. The following result is easy to prove.

Proposition 9 *The 1-parameter semigroup ψ^t decreases the filtration on stalks of points which do not belong to $L_1 \cap L_2$. More precisely,*

$$\psi^t(\underline{Hom}_p^{alg}(E_1, E_2)^s) \subset \underline{Hom}_{\psi^t(p)}^{alg}(E_1, E_2)^{s - \int_0^t \mu},$$

where $p \in P(L_1, L_2)$ is an arbitrary point.

Functor $\pi_!$ is compatible with the filtrations on the stalks of sheaves $\underline{Hom}_{C(Y)}^{alg}(E_1, E_2)$ and $\underline{Hom}^{alg}(E_1, E_2)$. It is easy to see that the completion of stalks of the former with respect to the filtration induced from the one on $\underline{Hom}^{alg}(E_1, E_2)$ coincides with $Hom_{C(Y)}(E_1, E_2)$. Since the semigroup ψ^t decreases the filtration, the semigroup ϕ^t extends continuously to the completion with respect to the filtration. Thus the following proposition holds.

Proposition 10 *The action of ϕ^t extends continuously from $Hom_{C(Y)}^{alg}(E_1, E_2)$ to $Hom_{C(Y)}(E_1, E_2)$.*

8.3 Homological mirror symmetry for abelian varieties

The whole approach here is parallel to the one from Section 6, so we will omit the details. In the previous subsection we defined the semigroup ψ^t acting on the sections with compact support $\Gamma_c(P(L_1, L_2), \underline{Hom}^{alg}(E_1, E_2))$. This action corresponds to the action of the semigroup ϕ^t on the space of morphisms $Hom_{\mathcal{C}(Y)}(E_1, E_2)$. Similarly to the case of Morse theory (Section 6) one proves the following result.

Proposition 11 *For any $\beta \in \Gamma_c(P(L_1, L_2), \underline{Hom}^{alg}(E_1, E_2))$ there exists a limit in the sense of distributions*

$$\psi^{\infty}(\beta) = lim_{t\to+\infty}\psi^t(\beta) \in \Gamma(P(L_1, L_2), \mathbf{C}_{\varepsilon}^{tw} \otimes (\widehat{\rho_1^*} \otimes \widehat{\rho_2}) \otimes \underline{D}'_{P(L_1,L_2)}),$$

where $\underline{D}'_{P(L_1,L_2)}$ is the sheaf of distribution-valued differential forms on $P(L_1, L_2)$.

The limit is not difficult to describe in terms of the gradient flow generating ψ^t. Using the fact that ψ^t moves the spectrum of a morphism to $-\infty$, one can prove similarly to the Section 6 that the limit $\psi^{\infty}(\beta)$ belongs to a finite-dimensional $\mathbf{C}_{\varepsilon}$-vector space generated by the distributions corresponding the unstable manifolds $U_x \subset P(L_1, L_2), x \in L_1 \cap L_2$. Clearly, the map $\beta \mapsto \psi^{\infty}(\beta)$ extends to the completion with respect to the filtration. It descends to the map $\alpha \mapsto \phi^{\infty}(\alpha)$, where $\alpha \in Hom_{\mathcal{C}(Y)}(E_1, E_2)$. The image of ϕ^{∞} belongs to the space isomorphic to $Hom_{FO(X^{\vee})}((L_1, \rho_1), (L_2, \rho_2))$.

We can repeat the arguments from the Morse theory (see Section 6). We define the A_{∞}-pre-category $\mathcal{C}_{unram,0}(Y)$ similarly to the category $DR_0(Y)$ from Section 6. It is A_{∞}-equivalent to $\mathcal{C}_{unram}(Y)$. By definition the spaces of morphisms of $\mathcal{C}_{unram,0}(Y)$ are dg-modules over the dg-algebra $\mathbf{C}_{\varepsilon}\widehat{\otimes}\Omega_0^*$, where Ω_0^* is the dg-algebra of germs of differential forms at $0 \in \mathbf{R}_{\geq 0}$. Compositions of morphisms in $\mathcal{C}_0(Y)$ are linear with respect to the dg-module structure. Imposing transversality conditions on $\mathcal{C}_{unram,0}(Y)$ to be the same as in $FO(X^{\vee})$, we obtain an A_{∞}-equivalent A_{∞}-pre-category $\mathcal{C}^{tr}_{unram,0}(Y)$.

Using homological perturbation theory (projectors and homotopies are defined by means of the semigroup) similarly to Section 6, we construct an analog of the category $DR_0^{tr,\Pi}(Y)$. It is an A_{∞}-pre-category denoted by $\mathcal{C}^{tr,\Pi}_{unrum,0}(Y)$, with the spaces of morphisms which are completed tensor products of Ω_0^* with finite-dimensional $\mathbf{C}_{\varepsilon}$-vector spaces, spanned by the "smoothenings" of the unstable currents $[U_x]$ (cf. Section 6). By definition, it has the same transversality conditions as the category $FO(X^{\vee})$, and the spaces of morphisms are naturally quasi-isomorphic to the corresponding spaces of morphisms in $FO(X^\vee)$ (compare with the Section 6.6). Similarly to the Section 6 we see that the A_∞-structure on $\mathcal{C}^{tr,\Pi}_{unram,0}(Y)$ is equivalent to the one on $FO(X^\vee)$. More precisely, we have a natural map from the space $Hom_{FO(X^\vee)}((L_1,\rho_1),(L_2,\rho_2))$ (it is defined in terms of the Morse theory) to the space $Hom_{\mathcal{C}^{tr,\Pi}_{unram,0}(Y)}(F(L_1,\rho_1),F(L_2,\rho_2))$ (it is defined in terms of de Rham differential forms on Y). Thus we have defined the mirror symmetry functor F on morphisms. Let us call the corresponding map ν_{X_1,X_2} for $X_i=(L_i,\rho_i), i=1,2$. The proof of the following proposition is similar to its analog from Section 6.6.

Proposition 12 *Let $E_i = F(X_i), 0\le i\le k, k\ge 1$ be locally free rank one $\mathcal{O}_Y$-modules (vector bundles) corresponding to objects $X_i=(L_i,\rho_i)\in FO(X^\vee), 0\le i\le k$. Then the formulas for*

$$m_k^{FO(X^\vee)}:\otimes_{0\le i\le k}Hom(E_i,E_{i+1})\to Hom(E_0,E_k)[2-k]$$

coincide (after the extension of scalars from $\mathbf{C}_\varepsilon$ to $\mathbf{C}_\varepsilon\widehat{\otimes}\Omega_0^*$) with the formulas for

$$m_k^{\mathcal{C}^{tr,\Pi}_{unram,0}(Y)}:\otimes_{0\le i\le k}Hom(X_i,X_{i+1})\to Hom(X_0,X_k)[2-k]$$

when the spaces of morphisms are identified via the maps $\nu(X_i,X_j)$.

Thus, A_∞-pre-categories $\mathcal{C}^{tr,\Pi}_{unram,0}(Y)$ and $FO(X^\vee)$ are equivalent. By the same arguments as in the Morse theory section we see that $\mathcal{C}_{unram}(Y)$ and $\mathcal{C}^{tr,\Pi}_{unram,0}(Y)$ are also equivalent. Finally, applying functor F, we get our main result.

Theorem 4 *The full subcategory $F(\mathcal{C}_{unram}(Y))$ of $C(Y)$ is A_∞-equivalent to $FO(X^\vee)$.*

This is the version of homological mirror symmetry we promised to prove.

Remark 20 *If we endow the torus $Y=\mathbf{R}^n/\mathbf{Z}^n$ with a flat metric and consider only flat Lagrangian subtori in $X^\vee$ then all higher compositions in the A_∞-pre-category $FO(X^\vee)$ can be written in terms of explicit "truncated theta series" analogous to those considered in [Ko] and [P1] in the case of elliptic curves.*

9 Appendix: constructions in the case of complex numbers

In the previous section we considered algebraic and analytic varieties over the complete local non-archimedean field $\mathbf{C}_\varepsilon$. In this section we explain our approach in the case of complex numbers (i.e. we will assume that ε is a fixed positive number). We should warn the reader that it is not yet clear how to obtain rigorous proofs in this case. In particular, it is not known how to prove convergence of the series defining compositions in the Fukaya category. Nevertheless we will discuss the complex case because the geometry is more transparent. One should treat the Appendix as a kind of geometric motivation for the results of the main part of the paper. For that reason we will not stress that X is an abelian variety, but will be using our conjectures about the collapse, and the assumption that the base Y of the torus fibration is a smooth manifold with integral affine structure and Kähler potential. We will be using the notation from Section 2.

9.1 Mirror symmetry functor on objects over C

In the case of complex numbers the mirror symmetry functor assigns a holomorphic vector bundle $F(L,\rho)$ on $X = X_\varepsilon$ to a pair (L,ρ), where $L \subset X^\vee$ is a Lagrangian submanifold, such that the projection $p^\vee_{|L} : L \to Y$ is an unramified covering, and ρ is a local system on L. If L is a section of $p^\vee$, and $rank(\rho) = 1$, then $E = F(L,\rho)$ is a line bundle. In general, E can be locally represented as a sum $E \simeq \oplus_{\alpha\in A} E_\alpha$ where A is the set of leaves (i.e. connected components) of the covering $L \to Y$, and E_α is a holomorphic vector bundle of the rank equal to the rank of ρ at the leaf α.

The following explicit construction of the mirror symmetry functor on objects is not new, see e.g. [AP]. We start with the remark that there is a canonical $U(1)$-bundle on $X \times_Y X^\vee$ (Poincaré line bundle). It will be denoted by P. It admits a canonical connection, which will be described below . Let us fix $y \in Y$. Then $p^{-1}(y) \simeq T_{Y,y}/\varepsilon T^{\mathbf{Z}}_{Y,y}$ and $(p^\vee)^{-1}(y) \simeq T^*_{Y,y}/(T^{\mathbf{Z}}_{Y,y})^\vee$. We identify torus $(p^\vee)^{-1}(y)$ with the moduli space of $U(1)$-local systems on the torus $p^{-1}(y)$ trivialized over a point $0 \in p^{-1}(y)$. We define $U(1)$-bundle P to be the tautological bundle on $X \times_Y X^\vee$ corresponding to this description.

In order to describe the connection on P let us consider the fiberwise universal coverings $r : T_Y \to T_Y/T^{\mathbf{Z}}_Y$ and $r^\vee : T^*_Y \to T^*_Y/(T^{\mathbf{Z}}_Y)^\vee$. Then the pullback $\bar{P}$ of P to $T_Y \times_Y T_Y^*$ is canonically trivialized. Thus we can work in coordinates. Let $y = (y_1, ..., y_n)$ be coordinates on Y, $x = (x_1, ..., x_n)$ and $x^\vee = (x_1^\vee, ..., x_n^\vee)$ be coordinates on the fibers of $T_Y \to Y$ and $T_Y^* \to Y$ respectively. Deck transformations $x_j \mapsto x_j + \varepsilon n_j$, $n_j \in \mathbf{Z}$ act on $\bar{P}$ preserving the trivialization, and transformations $x_j^\vee \mapsto x_j^\vee + n_j^\vee$, $n_j^\vee \in \mathbf{Z}$ act on $\bar{P}$ by the multiplication by $exp(2\pi i/\varepsilon \sum_j n_j^\vee x_j)$.

Let ∇_0 be the trivial connection on $\bar{P}$. We consider the connection $\bar{\nabla}$ on $\bar{P}$ which is given by the following formula

$$\bar{\nabla} = \nabla_0 + 2\pi i/\varepsilon \sum_{1 \le j \le n} x_j^\vee dx_j.$$

Lemma 4 *The connection $\bar{\nabla}$ gives rise to a connection on P.*

Proof. Obviously, connection $\bar{\nabla}$ does not change under the transformation $x_j \mapsto x_j + \varepsilon n_j, n_j \in \mathbf{Z}$. The transformation $x_j^\vee \mapsto x_j^\vee + n_j^\vee, n_j^\vee \in \mathbf{Z}$ together with the gauge transformation of $\bar{\nabla}$ by $h = exp(2\pi i/\varepsilon \sum_j n_j^\vee x_j)$ also preserves $\bar{\nabla}$. This proves the Lemma. ∎

Let (L, ρ) be as above. The mirror symmetry functor assigns to it a holomorphic vector bundle $E = F(L, \rho)$ such that (in coordinates) its fiber over a point (y, x) is given by the formula $E(y, x) = \oplus_{\{x^\vee \in L, p^\vee(x^\vee) = y\}} \rho(x^\vee) \otimes P(x, x^\vee)$. This vector bundle carries the induced connection ∇_E. In the case of unitary ρ the bundle E carries also a natural hermitean metric.

Proposition 13 *The $(0,2)$-part of the curvature $curv(\nabla_E)$ is trivial. In particular, ∇_E is a holomorphic connection.*

Proof. It follows from the fact that L is Lagrangian. Indeed, let us lift L to T_Y^*. Then locally in a neighborhood of a connected component of L, one can find a smooth real function $f = f(y)$ such that $L = df$. We can write the local equation for L: $x_j^\vee = \partial f/\partial y_j, 1 \le j \le n$. The connection ∇_E can be locally written as $\nabla_{E,0} + id_E \otimes (2\pi i/\varepsilon \sum_j \partial f/\partial y_j dx_j)$, where $\nabla_{E,0}$ is the trivial flat connection on the vector bundle E. Since the holomorphic coordinates on T_Y are given by $z_j = y_j + i x_j, i = \sqrt{-1}$, one sees that the $(0,2)$-part of the curvature is equal to $curv(\nabla_E)^{(0,2)} = const \times (\sum_{j,k} \partial^2 f/\partial y_j \partial y_k d\bar{z}_j d\bar{z}_k) = 0$. The Proposition is proved. ∎

Definition 23 *For any two holomorphic vector bundles E_1 and E_2 on X, we define $Hom_{Dolb}(E_1, E_2) = \Omega^{0,*}(X, Hom(E_1, E_2))$.*

We consider the space of Dolbeault differential forms with values in the vector bundle $Hom(E_1, E_2)$ as a dg-algebra with respect to the $\bar{\partial}$-differential. In this way one gets a structure of A_∞-category (in fact a dg-category) on the derived category of coherent sheaves on X. One can show that this A_∞-structure is equivalent to the one mentioned in the main text.

9.2 Sectors in the space of Dolbeault forms

Let $E_i = F(L_i, \rho_i), i = 1, 2$ be holomorphic vector bundles as above. There is an analog of the dg-category $\mathcal{C}(Y)$ in the case of complex numbers. We will denote it by $\mathcal{A}(Y)$. Objects of $\mathcal{A}(Y)$ are holomorphic vector bundles on X of the type $E = F(L, \rho)$. Morphisms are sections of soft sheaves on Y. Namely, we define the sheaf $\underline{Hom}_{\mathcal{A}(Y)}(E_1, E_2)$ on Y as the direct image $p_*(\underline{Hom}_{Dolb}(E_1, E_2))$ (in the self-explained notation). Then $Hom_{\mathcal{A}(Y)}(E_1, E_2)$ are global sections of this sheaf. This sheaf corresponds to the sheaf $\underline{Hom}_{\mathcal{C}(Y)}(E_1, E_2)$ in the non-archimedean geometry. Let us choose an open affine chart $U \subset Y, U \simeq \mathbf{R}^n$. Then $\Gamma(U, \underline{Hom}_{\mathcal{A}(Y)}(E_1, E_2))$ contains a subsheaf of finite Fourier sums with respect to the natural action of the torus T^n on $\Gamma(U \times T^n, \underline{Hom}_{Dolb}(E_1, E_2))$. Thus we have the sheaf $\underline{Hom}^{alg}_{\mathcal{A}(Y)}(E_1, E_2)$ which is an analog of the sheaf $\underline{Hom}^{alg}_{\mathcal{C}(Y)}(E_1, E_2)$ considered in the non-archimedean case. Notice that there exists a natural homomorphism of sheaves $j : p^*(\underline{\Omega}^*_Y) \to \underline{\Omega}^{0,*}_X$. The image of j consists of Dolbeault forms on X which have coefficients locally constant along fibers of p. In local coordinates j is given by the formula $f_{i_1,\dots,i_n}(y)dy_{i_1} \wedge \dots \wedge dy_{i_n} \mapsto f_{i_1,\dots,i_n}(y)d\bar{z}_{i_1} \wedge \dots \wedge d\bar{z}_{i_n}$, where $z_k = y_k - \sqrt{-1}x_k, 1 \le k \le n$. It is easy to see that j is compatible with the structure of dg-algebras on de Rham and Dolbeault forms. Thus for a pair of holomorphic vector bundles E_1 and E_2 on X we have a canonical structure of dg-module over $\underline{\Omega}^*_Y$ on the sheaf $\underline{Hom}_{\mathcal{A}(Y)}(E_1, E_2)$. In the case when $E_i = F(L_i, \rho_i), i = 1, 2$ the subsheaf $\underline{Hom}^{alg}_{\mathcal{A}(Y)}(E_1, E_2)$ is also a sheaf of dg-modules over $\underline{\Omega}^*_Y$.

As in the non-archimedean case there is a canonical decomposition of the stalk $\underline{Hom}^{alg}_{\mathcal{A}(Y)}(E_1, E_2)_y, y \in Y$ into the direct sum of dg-modules of finite rank over $\underline{\Omega}^*_Y$. Summands are labeled by the homotopy classes $[\gamma] \in P(L_1, L_2, y)$ and called *sectors*. We will denote them by $\underline{Hom}^{alg,[\gamma]}_{\mathcal{A}(Y)}(E_1, E_2)_y$. Informally, sectors correspond to "Fourier components" of Dolbeault forms in $\underline{Hom}^{alg}_{\mathcal{A}(Y)}(E_1, E_2)_y$ in the direction of torus fibers. Let us describe them more explicitly. For simplicity we will assume that $\rho_i, i = 1, 2$ are rank one trivial local systems, and $L_i, i = 1,2$ intersect with each fiber of p at exactly one point. Then near $p^{-1}(y)$ we can write $L_i = graph(df_i)\,(mod(T_Y^{\mathbf{Z}})^{\vee}), i = 1,2$, where f_i are germs at y of smooth functions on Y. ¿From the description of the Poincaré bundle P we deduce that $\underline{Hom}^{alg}_{\mathcal{A}(Y)}(E_1, E_2)_y$ is canonically identified with the space of germs of $\overline{\partial}$-forms near $T^n = p^{-1}(y)$, endowed with the twisted differential $\overline{\partial}'\alpha = \overline{\partial}\alpha + \frac{i}{\varepsilon}\sum \partial f/\partial y_i d\overline{z}_i \wedge \alpha$, where $f = f_1 - f_2$. Then the sector corresponding to a path γ consists of Dolbeault forms $\alpha = \sum_{i_1,...,i_n} exp(i\langle m, x\rangle/\varepsilon) f_{i_1...i_n}(y) d\overline{z}_1 \wedge ... \wedge d\overline{z}_n$. Here vector $m = m(\gamma)$ is the homotopy class of the loop in $T^n = p^{-1}(y)$ which is the composition of three paths:

1) the path $[0,1] \to T^n, t \mapsto t(df_1)_y\, mod\,(T_Y^{\mathbf{Z}})^{\vee}$;

2) the path γ;

3) the path $[0,1] \to T^n, t \mapsto (1-t)(df_2)_y\, mod\,(T_Y^{\mathbf{Z}})^{\vee}$.

A choice of sector corresponds to the choice of monomial $z_1^{i_1}...z_n^{i_n}$ in the non-archimedean case. Homotopy classes of paths in non-archimedean approach correspond the summands of Fourier series. Locally each sector can be identified with the de Rham complex on Y. Namely, to a form $\alpha = \sum_{i_1,...,i_n} f_{i_1...i_n}(y) exp(\langle m, x\rangle) d\overline{z}_{i_1} \wedge ... \wedge d\overline{z}_{i_n}$ we assign the form $\alpha_m = \sum_{i_1,...,i_n} f_{i_1...i_n}(y) exp(\frac{1}{\varepsilon}\langle m, x\rangle) dy_{i_1} \wedge ... dy_{i_n}$, where $m = m(\gamma)$ defines the sector. It is easy to see that the differential $\overline{\partial}'$ on Dolbeault forms on X corresponds to the de Rham differential d on $\Omega^*(Y)$. In this way we obtain an isomorphism of complexes $\underline{Hom}^{alg,[\gamma]}_{\mathcal{A}(Y)}(E_1, E_2)_y \simeq \underline{\Omega}^*_{Y,y} \otimes \mathbf{C}$.

Remark 21 *When ε is not a fixed number, but a parameter $\varepsilon \to 0$, the coefficients $f_{i_1...i_n}(y)$ are asymptotic series in ε of the type $f_{i_1...i_n}(y,\varepsilon) = \sum_{j\geq 1} exp(-\lambda_j/\varepsilon) f_{j,i_1...i_n}$ where $\lambda_j \in \mathbf{R}$, $\lambda_1 < ... < \lambda_j < ...$, and $\lambda_j \to +\infty$.*

The set of exponents appearing in the expansion of α_m at y corresponds to the spectrum $Sp_y(\alpha)$ considered in the non-archimedean case.

9.3 Semigroup φ^t

Now we can define a semigroup $\varphi^t : Hom_{Dolb}(E_1, E_2) \to Hom_{Dolb}(E_1, E_2), 0 \leq t < +\infty$. This is an analog of the semigroup ϕ^t in the non-archimedean case. First, we identify the sector $\underline{Hom}^{alg,[\gamma]}_{\mathcal{A}(Y)}(E_1, E_2)_y$ with $\underline{\Omega}^*(Y, Hom(\rho_1, \rho_2))_y$ as above. Let us recall from the non-archimedean part, that to the homotopy class of a path γ we canonically associated a closed 1-form $\mu_\gamma = \int_\gamma \omega$, where

ω is the symplectic form on $X^\vee$. Using the Riemannian metric g_Y on Y we assign to μ a vector field ξ_γ on Y. In a local trivialization it is given by $grad((f_1 - f_2 + \langle m(\gamma), \cdot \rangle)/\varepsilon)$. Then the infinitesimal action of φ^t is defined as the Lie derivative Lie_{ξ_γ}. Different Fourier components (sectors) move on Y with with different speeds in different directions. Hence the picture is more complicated than in the case of Morse theory.

One can show that the generator $\Delta = \frac{d}{dt}|_{t=0}\varphi^t$ is a second order differential operator on $Hom_{Dolb}(E_1, E_2)$. When g_Y is a flat metric and $f_1 = f_2 = 0$ one can find the following explicit formula for Δ:

$$\Delta = i \sum_j \frac{\partial^2}{\partial x_j \partial y_j} - \frac{1}{\varepsilon} \sum_j \frac{\partial^2}{\partial x_j^2}.$$

It seems plausible that there is an extension of the semigroup $\varphi^t = e^{t\Delta}$, $t \geq 0$, from $Hom^{alg}_{\mathcal{A}(Y)}(E_1, E_2)$ to the whole space of morphisms $Hom_{\mathcal{A}(Y)}(E_1, E_2)$. Notice that Δ is not self-adjoint, and its real part is not elliptic. Nevertheless, we expect that the semigroup operator φ^t converges as $t \to +\infty$ to a "projector" as in the case of Morse theory.

References

[AM] P.S. Aspinwall, D. Morrison, String Theory on K3 surfaces, hep-th/9404151.

[AP] D. Arinkin, A. Polishchuk, Fukaya category and Fourier transform, math.AG/9811023.

[Be] V. Berkovich, Spectral theory and analytic geometry over non-Archimedean fields. Mathematical Surveys and Monographs, 33. Amer. Math. Soc., 1990.

[BGR] S. Bosch, U. Günter, R. Remmert, Non-archimedean analysis. Springer-Verlag, 1984.

[BL] S. Bosch, W. Lütkebohmert, Degenerating abelian varieties, Topology 30 (1991), no. 4, 653-698.

[BZ] J-M. Bismut, W. Zhang, An extension of a theorem by Cheeger and

Müller, Asterisque 205 (1992).

[CC] J. Cheeger, T.H. Colding, On the structure of spaces with Ricci curvature bounded below. I. J. Differential Geom. 46 (1997), no. 3, 406-480.

[CG] J. Cheeger, M. Gromov, Collapsing Riemannian manifolds while keeping their curvature bounded, Parts I and II, J. Diff. Geom., 23 (1986), 309-346 and 32 (1990), 269-298.

[CY] S.-Y. Cheng, S.-T. Yau, The real Monge-Ampere equation and affine flat structures, Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, vol.1, Science Press, Beijing 1982, 339-370.

[De] P. Deligne, Local behavior of Hodge structures at infinity. Mirror symmetry, II, 683-699, AMS/IP Stud. Adv. Math., 1, Amer. Math. Soc., Providence, RI, 1997

[Fu 1] K. Fukaya, Morse homotopy and its quantization, AMS/IP Studies in Adv. Math., 2:1 (1997), 409-440.

[Fu2] K. Fukaya, A_∞-category and Floer homologies. Proc. of GARC Workshop on Geometry and Topology'93 (Seoul 1993), p. 1-102.

[FuO] K. Fukaya, Y.G. Oh, Zero-loop open string in the cotangent bundle and Morse homotopy, Asian Journ. of Math., vol. 1(1998), p. 96-180).

[G] M. Gromov, Metric structures for Riemannian manifolds, (J. Lafontaine and P. Pansu, editors), Birkhäuser, 1999.

[Gaw] K. Gawedzki, Lectures on Conformal Field Theory, in Mathematical Aspects of String Theory, AMS, 2000.

[GS] V. Gugenheim, J. Stasheff, On perturbations and A_∞-structures, Bul. Soc. Math. Belg. A38(1987), 237-246.

[GW] M. Gross, P. Wilson, Large complex structure limits of K3 surfaces, math.DG/0008018.

[H] N. Hitchin, The moduli space of special Lagrangian submanifolds, math.DG/9711002.

[HL] F. Harvey, B. Lawson, Finite volume flows and Morse theory. Preprint IHES 00/04, 2000.

[Ko] M. Kontsevich, Homological algebra of mirror symmetry. Proc. ICM Zuerich, 1994, alg-geom/9411018.

[KoS] M. Kontsevich, Y. Soibelman, Deformation theory, (book n preparation).

[KoS1] M. Kontsevich, Y. Soibelman, Deformations of algebras over operads and Deligne conjecture, math.QA/0001151.

[Le] N. Leung, Mirror symmetry without corrections, math.DG/0009235.

[LTY] B. Lian, A. Todorov, S-T. Yau, Maximal Unipotent Monodromy for Complete Intersection CY Manifolds, math.AG/0008061.

[M] M. Markl, Homotopy Algebras are Homotopy Algebras, math.AT/9907138.

[Me] S. Merkulov, Strong homotopy algebras of a Kähler manifold, math. AG/9809172.

[Mo] D. Morrison, Compactifications of moduli spaces inspired by mirror symmetry, Journées de Géométrie Algébrique d'Orsay (Juillet 1992), Astérisque, vol. 218, 1993, pp. 243-271, also alg-geom/9304007.

[Mum] D. Mumford, An analytic construction of degenerating abelian varieties over complete rings. Compositio Math. 24:3 (1972), 239-272.

[P1] A. Polishchuk, A_∞-structures on an elliptic curve, math.AG/0001048.

[Se] P. Seidel, Vanishing cycles and mutations. math.SG/0007115.

[SYZ] A. Strominger, S-T. Yau, E. Zaslow, Mirror symmetry is T-duality, hep-th/9606040.

[W1] E. Witten, Supersymmetry and Morse theory, J. Diff. Geom., v. 17 (1982), 661-692.

Addresses:
M.K.: IHES, 35 route de Chartres, F-91440, France
maxim@ihes.fr

Y.S.: Department of Mathematics, KSU, Manhattan, KS 66506, USA
soibel@math.ksu.edu